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The P-matrix problem is co-NP-complete

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Abstract

Recently Rohn and Poljak proved that for interval matrices with rank-one radius matrices testing singularity is NP-complete. This paper will show that given any matrix family belonging to the class of matrix polytopes with hypercube domains and rank-one perturbation matrices, a class which contains the interval matrices, testing singularity reduces to testing whether a certain matrix is not a P-matrix. It follows from this result that the problem of testing whether a given matrix is a P-matrix is co-NP-complete.

Keywords: P-matrix; Linear complementarity problem; Interval matrix; NP-complete

1. Notation and terminology

Suppose the matrices $A_-, A_+ \in \mathbb{R}^{n\times n}$ satisfy $(A_-)_{ij} \leqslant (A_+)_{ij}$ for $i, j = 1, \ldots, n$. Then A_- and A_+ may be used to define an interval matrix

$$A_\mathrm{I} := [A_-, A_+] \\ = \{A \in \mathbb{R}^{n\times n}\colon (A_-)_{ij} \leqslant A_{ij} \leqslant (A_+)_{ij},\ i, j = 1, \ldots, n\}\,.$$

Another representation often used for an interval matrix A_I is

$$A_\mathrm{I} = [A_\mathrm{c} - \Delta, A_\mathrm{c} + \Delta]$$

where

$$A_\mathrm{c} := \tfrac{1}{2}(A_- + A_+), \qquad \Delta := A_+ - A_\mathrm{c} \geqslant 0\,.$$

The matrix A_c is referred to as the center matrix, and Δ as the radius matrix, of interval matrix A_I.

Consider a matrix family of the form

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$$A(p) := A_0 + \sum_{i=1}^{k} p_i A_i, \tag{1}$$

$p \in Q_k := [0, 1]^k$, $A_0 \in \mathbb{R}^{n\times n}$, and $A_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, k$. Since the family of matrices defined in this way is the convex hull of a finite set of points in $\mathbb{R}^{n\times n}$, this type of matrix family will be called a matrix polytope.

Suppose further that for a matrix polytope described by (1) the matrix A_i has rank 1 for each $i = 1, \ldots, k$. The nature of dependence of the matrix family on each variable p_i is described by a rank-one matrix. A matrix family satisfying this condition will be referred to as a rank-one matrix polytope. Interval matrices are rank-one matrix polytopes.

For any vector $v \in \mathbb{R}^n$, let $D(v)$ denote the $n \times n$ diagonal matrix with the elements of v in order along the diagonal. Any real $n \times n$ rank-one matrix polytope $A(p)$, $p \in Q_k$, may be represented by

$$\begin{aligned} A(p) &:= A_0 + \sum_{i=1}^{k} p_i r_i s_i^{\mathrm{T}} \\ &= A_0 + RD(p)S^{\mathrm{T}} \end{aligned}$$

over Q_k, where $A_0 \in \mathbb{R}^{n\times n}$, $\rho(A_0) = n$, $r_i, s_i \in \mathbb{R}^n$ for $i = 1, \ldots, k$ and R and S are the two $n \times k$ matrices

$$R := (r_1\ r_2\ \ldots\ r_k), \qquad S := (s_1\ s_2\ \ldots\ s_k).$$

Matrix polytope $A(Q_k)$ is said to be regular if $A(p)$ is nonsingular for each $p \in Q_k$. Otherwise it is said to be singular.

Let $N := \{1, 2, \ldots, k\}$, $\Gamma := \{J: J \subset N\}$, and let $p_J \in \{0, 1\}^k$ be defined, for $J \in \Gamma$, by

$$p_j := \begin{cases} 1 & \text{if } j \in J, \\ 0 & \text{otherwise}. \end{cases}$$

Then the set $\{p_J: J \in \Gamma\}$ is the set of 2^k vertices of the hypercube Q_k. For any $J \in \Gamma$, $A(p_J)$ will be called the vertex of the matrix polytope $A(Q_k)$ associated with J.

For a given integer k, let I_k denote the k-dimensional identity matrix. Associated with any matrix polytope $A(p)$ is a family of characteristic polynomials

$$\begin{aligned} \psi(\lambda, p) &:= \det(\lambda I_n - A(p)) \\ &:= \lambda^n + \sum_{i=1}^{n} \psi_i(p) \lambda^{n-i}, \end{aligned}$$

$p \in Q_k$. For any rank-one matrix polytope, the coefficient functions that define the polynomial family are multiaffine. In particular, $\psi_n(p) = (-1)^n \det(A(p))$ is a multiaffine function of p.

2. Background and motivation

In [1], Rohn derives a list of regularity conditions for interval matrices. For the special case of an interval matrix with a rank-one radius matrix, the conditions simplify greatly. One of the conditions is given in the following proposition.

Proposition 1. *Suppose that* $A_{\mathrm{I}} = [A_c - rs^{\mathrm{T}}, A_c + rs^{\mathrm{T}}]$ *specifies an interval matrix, for nonsingular* $A_c \in \mathbb{R}^{n\times n}$ *and nonnegative vectors* $r, s \in \mathbb{R}^n$. *Then* A_{I} *is regular if and only if* $z^{\mathrm{T}} D(s) A_c^{-1} D(r) y < 1$ *for each* $y, z \in \{-1, 1\}^n$.

Recently Rohn and Poljak [2] (see also [7]) have shown that the NP-complete decision problem SIMPLE MAX CUT [3] reduces in polynomial time to the problem of testing whether the regularity condition of Proposition 1 is violated. Consider, then, a decision problem formulation for testing whether this condition is violated for a given interval matrix with rank-one radius matrix.

Decision Problem: Singularity of an Interval Matrix With Rank-One Radius (SING-INT-RK1R).
Instance: Nonsingular matrix $A_c \in \mathbb{R}^{n\times n}$ and nonnegative vectors $r, s \in \mathbb{R}^n$ specifying an interval matrix $[A_c - rs^{\mathrm{T}}, A_c + rs^{\mathrm{T}}]$.
Question: Is

$$z^{\mathrm{T}} D(s) A_c^{-1} D(r) y \geqslant 1 \tag{2}$$

for some pair of vectors y and z in $\{-1, 1\}^n$?

It is straightforward to establish that SING-INT-RK1R belongs to NP. Indeed, once a guess is made of a pair of vectors from $\{-1, 1\}^n$ satisfying condition (2), checking the guess requires at most a number of multiplications polynomial in n and one logical operation. By Rohn and Poljak's reduction, SING-INT-RK1R is NP-complete.

P-matrices are defined as matrices all of whose principal minors are positive [5]. This class of matrices is one of several which arise in the study of the Linear Complementarity Problem (LCP), a generalization of both Linear Programming and Quadratic Programming. A particular case of the LCP is specified by a matrix M and a vector q. Whenever M is a P-matrix the LCP is guaranteed to have a unique solution for all q. Murty [6] has described as an open question in LCP theory the issue of whether the problem of determining a given matrix to be a P-matrix is NP-complete.

This paper will show that the problem of testing singularity for the class of rank-one matrix polytopes, for which SING-INT-RK1R is a special case, reduces to testing whether a matrix formulated from the data defining the matrix polytope is not a P-matrix. This will establish that the problem of testing whether a given matrix is a P-matrix is co-NP-complete, thereby settling Murty's open question.

3. Main result

Lemma 1. *Suppose that* $F \in \mathbb{R}^{k\times n}$ *and* $G \in \mathbb{R}^{n\times k}$. *Then* $\det(I_k + FG) = \det(I_n + GF)$.

Proof. This is a corollary of Theorem 3 in Section 2.5 of Gantmacher [4]. □

By Lemma 1, if $A(p) = A_0 + RD(p)S^{\mathrm{T}}$ is a rank-one matrix polytope over hypercube Q_k and if A_0 is nonsingular, then for any $J \in \Gamma$,

$$\det(A_0 + RD(p_J)S^{\mathrm{T}}) = \det(A_0)\det(I_n + A_0^{-1}RD(p_J)S^{\mathrm{T}})$$
$$= \det(A_0)\det(I_k + D(p_J)S^{\mathrm{T}}A_0^{-1}R).$$

This last equation equates, for each $J \in \Gamma$, the determinant of the vertex matrix associated with J and the product of the determinant of A_0 with the determinant of the principal submatrix of $I_k + S^{\mathrm{T}}A_0^{-1}R$ formed from the rows and columns whose indices are elements of J.

Theorem 1. *The* $n \times n$ *rank-one matrix polytope*

$$A(p) = A_0 + RD(p)S^{\mathrm{T}},$$

$p \in Q_k$, *is regular if and only if* A_0 *is nonsingular and* $I_k + S^{\mathrm{T}}A_0^{-1}R$ *is a P-matrix.*

Proof. Assume, without loss of generality, that A_0 is nonsingular.

($\Leftarrow$). For each $J \in \Gamma$, the determinant of the principal submatrix of $I_k + S^{\mathrm{T}}A_0^{-1}R$ formed from rows and columns with indices in J is

$$\det(I_k + D(p_J)S^{\mathrm{T}}A_0^{-1}R) = \det(I_n + A_0^{-1}RD(p_J)S^{\mathrm{T}}),$$

making use of Lemma 1. If $I_k + S^{\mathrm{T}}A_0^{-1}R$ is a P-matrix, then

$$\det(A_0 + RD(p_J)S^{\mathrm{T}}) = \det(A_0)\ \det(I_k + D(p_J)S^{\mathrm{T}}A_0^{-1}R)$$

has the same sign as $\det(A_0) \neq 0$ for each $J \in \Gamma$. Since for each J this is the determinant of the unique vertex matrix associated with J, it follows that every vertex matrix has a determinant with the same sign as that of A_0. In particular, this implies that the determinant of the matrix polytope $A(p)$ over Q_k, a multiaffine function of p, cannot change sign or vanish as p varies along any edge of Q_k.

Any multiaffine function has the property that over a hyperrectangular domain any extreme value attained at an interior point of the domain must also be attained at a vertex. Since over vertices of Q_k the determinant of $A(p)$ never changes sign or vanishes, the upper and lower bounds over Q_k of the determinant have the same sign and it follows that $A(Q_k)$ is regular.

($\Rightarrow$). Assume that $I_k + S^{\mathrm{T}}A_0^{-1}R$ is not a P-matrix. Then one of its principal minors is nonpositive. Equivalently, by the above discussion, there must exist some vertex p_J of Q_k such that

$$\det(I_n + A_0^{-1}RD(p_J)S^{\mathrm{T}}) \leqslant 0.$$

However, when $p_J = 0$, or when $J = \emptyset \subset \Gamma$,

$$\det(I_n + A_0^{-1}RD(p_J)S^{\mathrm{T}}) = \det(I_n) = 1 > 0,$$

so the determinant function is positive at one vertex and nonpositive at another. In traversing a ray from the first vertex to the second, this determinant must pass through zero. It follows that $A(Q_k)$ is singular. □

Suppose that the P-matrix problem is formulated as a decision problem:

Decision Problem: P-Matrix (PMAT).
Instance: $M \in \mathbb{R}^{n\times n}$.
Question: Are all of the principal minors of M positive?

and that the complement of the decision problem is defined as well:

Decision Problem: Not a P-Matrix (NOT-PMAT).
Instance: $M \in \mathbb{R}^{n\times n}$.
Question: Does M possess a nonpositive principal minor?

Then we have the following corollary.

Corollary 1. *NOT-PMAT is NP-Complete.*

Proof. Consider a nondeterministic machine which guesses a set of row indices element by element and then checks whether the principal minor of M determined by the set of indices is nonpositive. Such a machine would solve NOT-PMAT nondeterministically in polynomial time, and therefore NOT-PMAT belongs to NP.

It remains to provide an NP-complete problem which reduces to NOT-PMAT in polynomial-time. Such a problem is the decision problem SING-INT-RK1R defined earlier in the paper. Since instances of SING-INT-RK1R are interval matrices with rank-one radius matrices, and interval matrices are rank-one matrix polytopes, Theorem 1 provides a polynomial-time reformulation of any instance of SING-INT-RK1R as an instance of NOT-PMAT. □

Finally, Corollary 1 can be restated as a complexity classification of the problem PMAT using the complementarity of PMAT and NOT-PMAT.

Corollary 2. *PMAT is co-NP-Complete.*

4. Conclusion

In this paper we have shown that the problem of testing whether a given matrix is not a P-matrix, formulated as a decision problem, is NP-complete. This result settles an open problem in the theory of the Linear Complementarity Problem. In the process, we have shown that testing regularity of any rank-one matrix polytope is equivalent to testing whether a certain matrix is a P-matrix.

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A criterion for time aggregation in intertemporal dynamic models

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Abstract

Nonlinear intertemporal general equilibrium models are hard to solve because of the dimensionality of the optimization problem involved. The computation of intertemporal general equilibria therefore calls for time-aggregation assumptions. A question then immediately arises: what criterion should one use to choose a sequence of possibly unequal time intervals in order to reduce the dimensionality of the optimization problem, yet keep under control the errors resulting from the numerical approximation of a continuous time process by a discrete time process? We propose one such criterion based on the current value of capital, which exploits near steady-state optimal dynamics. We show, using a parameterized version of the standard Ramsey–Koopmans–Cass model of optimal growth, that it outperforms alternative criterions used in the literature.

Keywords: Dynamic aggregation; Intertemporal dynamics; Optimal growth; General equilibrium

1. Introduction

Nonlinear, continuous-time, intertemporal general equilibrium models are hard to solve because of the dimensionality of the optimization problem involved.[1] Although some methods—known as two-point boundary value techniques—have proved to be well adapted for

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[1] For a broad introduction to the formulation and solution of general equilibrium models, see Manne (1985).

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