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The maximum number of complementary facets of a simplicial polytope

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Abstract

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Let P be an $(n-1)$-dimensional simplicial polytope with $2n$ vertices labelled $s_1, \dots, s_n$, $t_1, \dots, t_n$. Call a face of P *complementary* if the vertices it contains all have different subscripts. We study the maximum number of complementary faces that P can have. This problem arose in the determination of the maximum possible degree of an LCP mapping. We give examples of polytopes achieving a conjectured bound, and give some results supporting the conjecture.

1. Introduction

Let P be an $(n-1)$-dimensional simplicial polytope with $2n$ vertices labelled $s_1, \dots, s_n$, $t_1, \dots, t_n$. Call a face of P *complementary* if the vertices it contains all have different subscripts. How many complementary facets can P have? In [6] it was shown that the determination of this number gives a bound on the degree of a mapping arising from the linear complementarity problem. The degree of this mapping gives information about the solutions to the linear complementarity problem.

In [6], a bound was conjectured on the maximum number of complementary facets. Examples of polytopes achieving this conjectured bound are given in Section 2. These polytopes are dual to certain sections of the n-cube. Rather than just counting the numbers of vertices of these polytopes, we calculate their h-vectors, which are interesting in their own right. For definitions of polytope concepts used in this paper, see [2, 4].

It then remains to show that one cannot find polytopes with more complementary facets than the conjectured bound. In Section 3, this is proved for two classes of simplicial polytopes: (a) polytopes for which every face is complementary, and (b) polytopes that have exactly one edge with vertices of the same subscript. The techniques used are those used by Stanley [7] to prove the upper bound theorem for spheres. This involves showing that the dimensions of parts of a certain ring are bounded by the components of the h-vector found in Section 2.

2. Polytopes achieving the bound

For nonnegative integers k, n, $k \leq n-1$, define $P(k,n)$ to be the polytope $\{x \in \mathbb{R}^n$: $0 \leq x_i \leq 1$, $i=1,\ldots,n$, $\sum_{i=1}^n x_i = k+\frac{1}{2}\}$. Then $P(k,n)$ is an $(n-1)$-dimensional section of the n-cube. $P(k,n)$ is a simple polytope because it misses all of the vertices of the cube. $P(0,n)$ and $P(n-1,n)$ are simplices. The symmetry of the cube implies that $P(k,n)$ and $P(n-k-1,n)$ are isomorphic. It is also clear that for $i=1,\ldots,n$ and $k \geq 1$, the facet $\{x \in P(k,n)$: $x_i = 1\}$ of $P(k,n)$ is of the same combinatorial type as $P(k-1,n-1)$, whereas for $i=1,\ldots,n$, $k \leq n-2$, the facet $\{x \in P(k,n)$: $x_i=0\}$ of $P(k,n)$ is of the same combinatorial type as $P(k,n-1)$. Finally, note that for $k=0,1,\ldots,n-1$, $P(k,n)$ has $n\binom{n-1}{k}$ vertices.

Let $P^*(k,n)$ be a dual polytope to $P(k,n)$ for all k, n. For $i=1,\ldots,n$ let s_i be the vertex of $P^*(k,n)$ that is the image of the facet $\{x \in P(k,n)$: $x_i=1\}$ of $P(k,n)$ under the duality map, and let t_i be the vertex that is the image of the facet $\{x \in P(k,n)$: $x_i = 0\}$. The next lemmas are devoted to determining the h-vector of $P^*(k,n)$. The h-vector of a d-dimensional polytope P is given by $f_j(P) = \sum_{i=0}^{j+1} \binom{d-i}{d-j-i} h_i(P)$, for $i=0,\ldots,d$, $j=-1,\ldots,d-1$, where $f_j(P)$ is the number of j-dimensional faces of P. This correspondence is invertible, and since the coefficients of the h_i above are nonnegative, bounds on the $h_i(P)$ imply corresponding bounds on the $f_j(P)$.

Lemma 2.1. *Subject to the boundary conditions $h_{-1}(P^*(k,n)) = h_n(P^*(k,n)) = 0$ for all k, n, and $h_i(P^*(n,n)) = h_i(P^*(-1,n)) = 0$, for all i, n, the h-vectors of the polytopes $P^*(k,n)$ satisfy*:

$$h_i(P^*(k,n)) = h_{i-1}(P^*(k,n-1)) + h_i(P^*(k-1,n-1)) + \delta(k,i)\binom{n-1}{k}. \tag{1}$$

Proof. Here $\delta(k,i)=1$ if $k=i$, $\delta(k,i)=0$ otherwise. The h-vector of $P^*(k,n)$ can be calculated using the straight-line shelling of [3]. This is easier to visualize in its dual version. Orient the edges of $P(k,n)$ so that the edge connecting vertices x and y is directed from x to y iff $\sum_{i=1}^n \varepsilon^i x_i > \sum_{i=1}^n \varepsilon^i y_i$, where $0 < \varepsilon < 1/n$ is small enough so that there are no ties. Then $h_i(P^*(k,n))$ counts the number of vertices of indegree i. The vertices of $P(k,n)$ can be divided into three sets. Let $A = \{$vertices x of $P(k,n)$ with $x_1 = 0\}$, $B = \{$vertices y of $P(k,n)$ with $y_1 = \frac{1}{2}\}$, and $C = \{$vertices z of $P(k,n)$ with $z_1 = 1\}$. For $x \in A$, $y \in B$, $z \in C$, we have $\sum_{i=1}^n \varepsilon^i x_i < \sum_{i=1}^n \varepsilon^i y_i < \sum_{i=1}^n \varepsilon^i z_i$. Each vertex $y \in B$ will have indegree k, and there are $\binom{n-1}{k}$ of these vertices. In A there will be $h_{i-1}(P^*(k, n-1))$ vertices of indegree i, because the facet $\{x \in P(k,n)\colon x_1 = 0\}$ is of the same combinatorial type as $P(k, n-1)$. (It is empty if $k = n-1$.) In B there will be $h_i(P^*(k-1, n-1))$ vertices of indegree i, since the facet $\{x \in P(k,n)\colon x_1 = 1\}$ is of the same combinatorial type as $P(k-1, n-1)$. (It is empty if $k = 0$.) This gives the lemma. □

Lemma 2.2. *For* $0 \le i, k \le \lfloor \frac{1}{2}(n-1) \rfloor$,

$$h_i(P^*(k,n)) = \sum_{j=0}^{\min(i,k)} \binom{n}{j}. \tag{2}$$

Proof. The proof is by induction on n. For $n = 1$, $h_0(P^*(0,1)) = \delta(0,0)\binom{0}{0}$. Next, assume that for $0 \le i, k \le \lfloor \frac{1}{2}(n-2) \rfloor$, that $h_i(P^*(k, n-1)) = \sum_{j=0}^{\min(i,k)} \binom{n-1}{j}$. The Dehn–Somerville equations imply that $h_i(P^*(k, n-1)) = h_{n-2-i}(P^*(k, n-1))$ for any k, and the symmetry of the cube implies that $P^*(k, n-1)$ and $P^*(n-k-2, n-1)$ have the same h-vector. Therefore, the inductive hypothesis determines $h_i(P^*(k, n-1))$ for all i and k. For $0 \le i, k \le \lfloor \frac{1}{2}(n-2) \rfloor$, the inductive hypothesis and (1) give

$$\begin{aligned} h_i(P^*(k,n)) &= \sum_{j=0}^{\min(i,k-1)} \binom{n-1}{j} + \sum_{j=0}^{\min(i-1,k)} \binom{n-1}{j} + \delta(i,k)\binom{n-1}{k} \\ &= \sum_{j=0}^{\min(i,k)} \binom{n}{j}. \end{aligned} \tag{3}$$

If n is odd and $i = \lfloor \frac{1}{2}(n-1) \rfloor$, then the term $h_i(P^*(k-1, n-1))$ of equation (1) can be replaced by $h_{i-1}(P^*(k-1, n-1))$ by the Dehn–Somerville equations. Note then that $\min(i-1, k-1) = \min(i, k-1)$ for $k \le \lfloor \frac{1}{2}(n-1) \rfloor$, so (3) is still valid. If n is odd and $k = \lfloor \frac{1}{2}(n-1) \rfloor$, then the term $h_{i-1}(P^*(k, n-1))$ of equation (1) can be replaced by $h_{i-1}(P^*(k-1, n-1))$ by the symmetry of the cube. As before, here $\min(i-1, k-1) = \min(i-1, k)$ for $i \le \lfloor \frac{1}{2}(n-1) \rfloor$, so that (3) holds. Thus (3) holds for $0 \le i, k \le \lfloor \frac{1}{2}(n-1) \rfloor$, and the lemma is proved. □

For $i = 1, \ldots, n$, the facets $\{x \in P(k,n)\colon x_i = 0\}$ and $\{x \in P(k,n)\colon x_i = 1\}$ never meet. This implies that every face of the simplicial polytope $P^*(k,n)$ is complementary, for all k, n. In particular, for $k = \lfloor \frac{1}{2}(n-1) \rfloor$, the number of facets of $P^*(k,n)$ is $n\binom{n-1}{\lfloor \frac{1}{2}(n-1) \rfloor}$, which is the conjectured bound on the number of complementary facets.

3. Proof of the bound for special cases

We start by proving that the conjectured bound holds for two classes of simplicial polytopes: (a) polytopes for which every face is complementary, and (b) polytopes that have exactly one edge with vertices of the same subscript.

Theorem 3.1. *Let P be a simplicial polytope of dimension $n-1$, with $2n$ vertices labelled $s_1,\dots,s_n$, $t_1,\dots,t_n$, and suppose that every face of P is complementary. Then $h_i(P)\le h_i(P^*(\lfloor\frac{1}{2}(n-1)\rfloor,n))$, $i=0,1,\dots,n-1$.*

Proof. The proof is a direct application of the techniques used by Stanley [7] to prove the upper bound theorem for spheres. The notation used will be as in the survey article by Billera [2].

Let $k[P]$ be the Stanley–Reisner ring of the $(n-2)$-dimensional simplicial complex determined by the faces of P, with k an infinite field. Define $\theta_1,\dots,\theta_{n-1}$ by $\theta_j=s_j+s_n+t_j+t_n$. Then $\theta_1,\dots,\theta_{n-1}$ is a homogeneous system of parameters for $k[P]$. This is because [8] the $n-1$ by $2n$ matrix of the transformation defining $\theta_1,\dots,\theta_{n-1}$ has the property that a subset of the columns is linearly independent iff it corresponds to a subset of $\{s_1,\dots,s_n,t_1,\dots,t_n\}$ with distinct subscripts, and has less than n columns.

Define the lexicographic ordering on monomials in the variables $s_1,\dots,s_n$, $t_1,\dots,t_n$ by $m<m'$ if $\deg m<\deg m'$ or if $\deg m=\deg m'$ and m comes before m' lexicographically with the ordering $s_1<s_2<\cdots<s_n<t_1<\cdots<t_n$. Define a collection $\eta_1,\eta_2,\dots,\eta_m$ of monomials in $k[P]$ by $\eta_1=1$, and for $r\ge 1$ let η_{r+1} be the first monomial in $k[P]$ (in the ordering defined above) that cannot be expressed as a polynomial $\sum_{j=1}^{r}\eta_j p_j(\theta_1,\dots,\theta_{n-1})$. Then for $i=0,1,\dots,n-1$, the number of η_r of degree i is equal to $h_i(P)$.

Lemma 3.2. *The monomials $\eta_1,\eta_2,\dots,\eta_m$ are of the form t_1^k or of the form $s_{i_1}^u s_{i_2}\cdots s_{i_v}$, $i_1<i_2<\cdots<i_v$.*

Proof. It is sufficient to show that the monomials t_j, $j>1$, s_it_1, $i=1,\dots,n$, and $s_{i_1}s_{i_2}^2$, $1\le i_1<i_2\le n$ are not in the collection, because if we exclude all of the monomials containing these, we are left with the ones given by the lemma.

The expression $\theta_1=s_1+s_n+t_1+t_n$ expresses t_n as a combination of monomials no greater than t_1. Also, $\theta_1-\theta_i=s_1-s_i+t_1-t_i$ expresses t_i as a combination of monomials less than or equal to t_1, for $i=2,\dots,n-1$. Next, note that s_1t_1 is not in $k[P]$, since it contains two variables with the same subscript. Thus s_1t_1 is not an η_i. For $i=2,\dots,n-1$, we have $\theta_1-\theta_i=s_1-s_i+t_1-t_i$, which implies that $s_i(\theta_1-\theta_i)=s_is_1-s_i^2+s_it_1-s_it_i$. Now s_it_i is not in $k[P]$, so $s_it_1=s_i(\theta_1-\theta_i)-s_is_1+s_i^2$ expresses s_it_1 as a combination of monomials no greater than $s_i^2<s_it_1$. Also, $\theta_1=s_1+s_n+t_1+t_n\Rightarrow s_n\theta_1=s_ns_i+s_n^2+s_nt_1+s_nt_n\Rightarrow s_nt_1=s_n\theta_1-s_ns_i-s_n^2$, expressing s_nt_1 as a combination of monomials at most equal to $s_n^2<s_nt_1$.

Finally, we need to exclude monomials $s_{i_1}s_{i_2}^2$, $1\le i_1<i_2\le n$. $\theta_i=s_i+s_n+t_i+t_n\Rightarrow s_is_n\theta_i=s_i^2s_n+s_is_n^2+s_it_is_n+s_is_nt_n=s_i^2s_n+s_is_n^2$, so $s_is_n^2=s_is_n\theta_i-s_i^2s_n$, expressing $s_is_n^2$ as a combination of monomials no greater than $s_i^2s_n<s_is_n^2$. Also, for $1\le i_1<i_2\le n-1$, we have $\theta_{i_1}-\theta_{i_2}=s_{i_1}-s_{i_2}+t_{i_1}-t_{i_2}\Rightarrow s_{i_1}s_{i_2}(\theta_{i_1}-\theta_{i_2})=s_{i_1}^2s_{i_2}-s_{i_1}s_{i_2}^2+s_{i_1}s_{i_2}t_{i_1}-s_{i_1}s_{i_2}t_{i_2}=s_{i_1}^2s_{i_2}-s_{i_1}s_{i_2}^2\Rightarrow s_{i_1}s_{i_2}^2=s_{i_1}^2s_{i_2}-s_{i_1}s_{i_2}(\theta_{i_1}-\theta_{i_2})$, expressing $s_{i_1}s_{i_2}^2$ as a combination of monomials less than or equal to $s_{i_1}^2s_{i_2}$. Thus the lemma is proved. □

Proof of Theorem 3.1 (continued). We now count the monomials of degree i given in the statement of the lemma. There is one ($=\binom{n}{0}$) of the form t_1^i. For $r=1,\ldots,i$, there are $\binom{n}{j}$ of the form $s_{i_1}^{(i-r+1)}s_{i_2}\cdots s_{i_j}$. Thus there are $\sum_{j=0}^{i}\binom{n}{j}$ monomials of degree i not excluded by the lemma. Therefore, $h_i(P)\le\sum_{j=0}^{i}\binom{n}{j}$. For $i\le\lfloor\frac{1}{2}(n-1)\rfloor$, $i=\min(i,\lfloor\frac{1}{2}(n-1)\rfloor)$, so $h_i(P)\le h_i(P^*(\lfloor\frac{1}{2}(n-1)\rfloor,n))$. The Dehn-Somerville equations then imply this inequality for the remaining i. □

Theorem 3.3. *Let P be a simplicial polytope of dimension $n-1$, with $2n$ vertices labelled $s_1,\ldots,s_n$, $t_1,\ldots,t_n$, and suppose that P has exactly one edge with both vertices of the same subscript. Let Δ be the simplicial complex obtained from the simplicial complex of the faces of P by deleting all faces that contain this edge. Then $h_i(\Delta)\le h_i(P^*(\lfloor\frac{1}{2}(n-1)\rfloor,n))$, $i=0,\ldots,n-1$.*

Proof. Assume that the edge with both vertices of the same subscript is the edge s_1t_1. The union of the faces of P that do not contain edge s_1t_1 is homeomorphic to an $(n-2)$-dimensional ball. Therefore, by Reisner's characterization of the Cohen-Macaulay complexes (see [2]), the arguments of the preceding proof apply and imply that $h_i(\Delta)\le h_i(P^*(\lfloor\frac{1}{2}(n-1)\rfloor,n))$, for $i=0,\ldots,\lfloor\frac{1}{2}(n-1)\rfloor$. Now, however, the Dehn-Somerville equations do not apply. In this case, [1] shows that the relationship $h_i(\Delta)\ge h_{n-1-i}(\Delta)$ holds. Thus the theorem is true. □

If P has more than one edge with both vertices of the same subscript, then it is not obvious how to apply these methods to the simplicial complex of complementary faces of P. The following property of the polytopes $P^*(\lfloor\frac{1}{2}(n-1)\rfloor,n)$ is reminiscent of the properties of polytopes that achieve the bound of the upper bound theorem for polytopes.

Proposition. *Let S be any complementary set of vertices of $P^*(\lfloor\frac{1}{2}(n-1)\rfloor,n)$ with $|S|\le\lfloor\frac{1}{2}(n-1)\rfloor$. Then the convex hull of S is a face of $P^*(\lfloor\frac{1}{2}(n-1)\rfloor,n)$.*

Proof. We need to show that the intersection of the facets $\{x\in P(\lfloor\frac{1}{2}(n-1)\rfloor,n)$: $x_i=0\}$ of $P(\lfloor\frac{1}{2}(n-1)\rfloor,n)$ for $t_i\in S$ and $\{x\in P(\lfloor\frac{1}{2}(n-1)\rfloor,n)$: $x_i=1\}$ for $s_i\in S$ is a face of $P(\lfloor\frac{1}{2}(n-1)\rfloor,n)$. This is clear from the definition of $P(\lfloor\frac{1}{2}(n-1)\rfloor,n)$. □

Finally, recall that the upper bound theorem for polytopes was originally proved by McMullen [5] using shelling. It would be interesting to know if Theorems 3.1 and 3.3 could be proved by such a method.

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