Counterexamples to Q-Matrix Conjectures

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ABSTRACT

A matrix $M \in \mathbb{R}^{n \times n}$ is in the class Q if for all $q \in \mathbb{R}^n$ there exist $w, z \in \mathbb{R}^n_+$ such that $w - Mz = q$, $w^T z = 0$. It has been conjectured that it is possible to tell if a matrix M is in Q solely by considering the signs of subdeterminants of M. We present two matrices that have the same signs of corresponding subdeterminants, but are such that one is in Q and the other is not. The second of these two matrices has the property that every column of the matrix $[I, -M]$ is in the interior of the union of the complementary cones associated with $[I, -M]$.

1. INTRODUCTION

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the *linear complementarity problem*, denoted by LCP(q, M), is to find $w, z \in \mathbb{R}^n_+$ such that $w - Mz = q$, $w^T z = 0$. A matrix M is in the class Q (is a Q-matrix) if the LCP(q, M) has a solution for all q. The only known finite test for determining if a matrix is in Q is due to Gale (see [1]). It involves in the worst case determining the feasibility of n^{2^n} linear programs, each with 2^n constraints. Much work has gone into attempting to find a more efficient characterization of the class Q, as it is hoped that a better understanding of the class Q would motivate better algorithms for the LCP.

Some of the best known subclasses of Q can be characterized by signs of certain subdeterminants of M or, equivalently, in terms of sign patterns of vectors in the nullspace of the matrix $(I, -M)$. Along these are the classes P (matrices with positive principal minors) and $\underline{Q}$ (matrices M such that for all $0 \neq x \geqslant 0$ there is an i such that $x_i(Mx)_i > 0$; see [2].) It was conjectured in

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[3] that the class Q has such a characterization in terms of signs of subdeterminants. It was shown in [8] that such a characterization would have to involve more than just the signs of the principal minors of M. Such a characterization would have an analog in the more general setting of oriented matroids, and it was in an attempt to prove such an analog (see [7]) that the counterexamples of this paper were found.

Kelly and Watson proved in [6] that the set of matrices in Q is neither open nor closed in $\mathbb{R}^{n \times n}$. Their proof produced matrices $M, D \in \mathbb{R}^{n \times n}$ such that M is in Q but $M + \epsilon D$ is not in Q for all sufficiently small $\epsilon > 0$. Their matrix M has a zero subdeterminant for which the corresponding subdeterminant of $M + \epsilon D$ is nonzero. The authors of [3] refer to an unpublished claim by Kelly and Watson that M can be perturbed to be in Q and have all nonzero minors while still admitting a matrix D such that $M + \epsilon D$ is not in Q for all sufficiently small $\epsilon > 0$, and they point out that if this were correct, it would give a counterexample to their conjecture. This paper verifies Kelly and Watson's claim.

Our perturbed matrix M not in Q will also have the property that each of the columns of $[I, -M]$ is in the interior of the set of vectors in q in $\mathbb{R}^n$ for which the LCP(q, M) has a solution, which disproves a conjecture of [4].

2. COMPLEMENTARY MATRICES AND VISIBILITY SETS

Let R be a matrix in $\mathbb{R}^{n \times 2n}$, with columns $(s_1, \dots, s_n, t_1, \dots, t_n)$. Call an $n \times n$ submatrix C of R *complementary* if it contains exactly one of the columns s_i and t_i for $i = 1, \dots, n$. If $R = [I, -M]$, with $(s_1, \dots, s_n)$ the columns of I and $(t_1, \dots, t_n)$ the columns of $-M$, then the LCP(q, M) for a vector $q \in \mathbb{R}^n$ has a solution iff q is in the convex cone of the columns of a complementary submatrix of $[I, -M]$. Call R a *Q-arrangement* if for all $q \in \mathbb{R}^n$ there is a complementary submatrix C of R with $x \geqslant 0$ such that $q = Cx$. Then M is a Q-matrix iff $[I, -M]$ is a Q-arrangement. Assume in the following that all of the complementary submatrices of R are nonsingular.

Definition. An element q of $\mathbb{R}^n$ is visible from $-c_i \in \{s_i, t_i\}$ if the line segment $\{x \in \mathbb{R}^n : \lambda(-c_i) + (1-\lambda)q,\ 0 \leqslant \lambda < 1\}$ does not intersect any of the cones cone$(\{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n\})$ for complementary submatrices C containing c_i.

Clearly, $\lambda(-c_i) + (1-\lambda)q \in \text{cone}(\{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n\})$ for some $0 \leqslant \lambda < 1$ iff $q \in \text{cone}(\{c_1, \dots, c_n\})$. An element q of $\mathbb{R}^n$ is therefore visible from $-c_i$ if there is no complementary submatrix C containing c_i with $x \geqslant 0$ such that $Cx = q$. Let Vis$(-c_i)$ be the set of points visible from $-c_i$.

THEOREM 1 [6, Theorem 3]. *Assume that all of the complementary submatrices of R are nonsingular. R is a Q-arrangement iff* $\mathrm{Vis}(-s_i)\cap \mathrm{Vis}(-t_i)=\varnothing$ *for some $i=1,\dots,n$.*

Finally, note that if R' is obtained from R by premultiplying R by a nonsingular matrix in $\mathbb{R}^{n\times n}$, followed by positive rescaling of the columns, then R' is a Q-arrangement iff R is, and any $n\times n$ submatrix of R' is nonsingular iff the corresponding submatrix of R is.

3. KELLY AND WATSON'S EXAMPLE

The counterexample presented in this paper is a perturbation of Kelly and Watson's example from [6], which shows that the set of Q-matrices is neither open nor closed in $\mathbb{R}^{n\times n}$. The features of Kelly and Watson's arrangement that are crucial to our argument are pointed out in this section. For a more detailed discussion, see [6].

Consider the matrix

$$M=\begin{bmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{bmatrix}.$$

The matrix $[I,\ -M]$ can be transformed by premultiplication by a nonsingular matrix followed by positive scaling of the columns into the matrix

$$R=\begin{bmatrix} 2 & -2 & 0 & -\frac{3}{4} & 1 & -1 & 0 & \frac{3}{4} \\ 2 & 2 & -2 & -\frac{1}{3} & -2 & -2 & 2 & -\frac{1}{3} \\ 0 & 0 & 2 & -\frac{1}{2} & 0 & 0 & 2 & -\frac{1}{2} \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

Denote the columns of this matrix by $(s_1,s_2,s_3,s_4,t_1,t_2,t_3,t_4)$. The points $s_1,s_2,s_3,-s_4,t_1,t_2,t_3,-t_4$ all lie in the affine space $F=\{x\in\mathbb{R}^4 : x_4=1\}$. We identify this space with $\mathbb{R}^3$. The points in F corresponding to $s_1,s_2,s_3,-s_4,t_1,t_2,t_3,-t_4$ are shown in Figure 1. The cones associated with complementary submatrices of R are represented by convex hulls in F. A triangle $\triangle c_1c_2c_3$ in F, where $c_i\in\{s_i,t_i\}$ for $i=1,2,3$ is the intersection of F with the convex cone of $\{c_1,c_2,c_3\}$ in $\mathbb{R}^4$. Kelly and Watson observe that if the visibility sets of $-s_4$ and $-t_4$ in F are bounded, then R is a

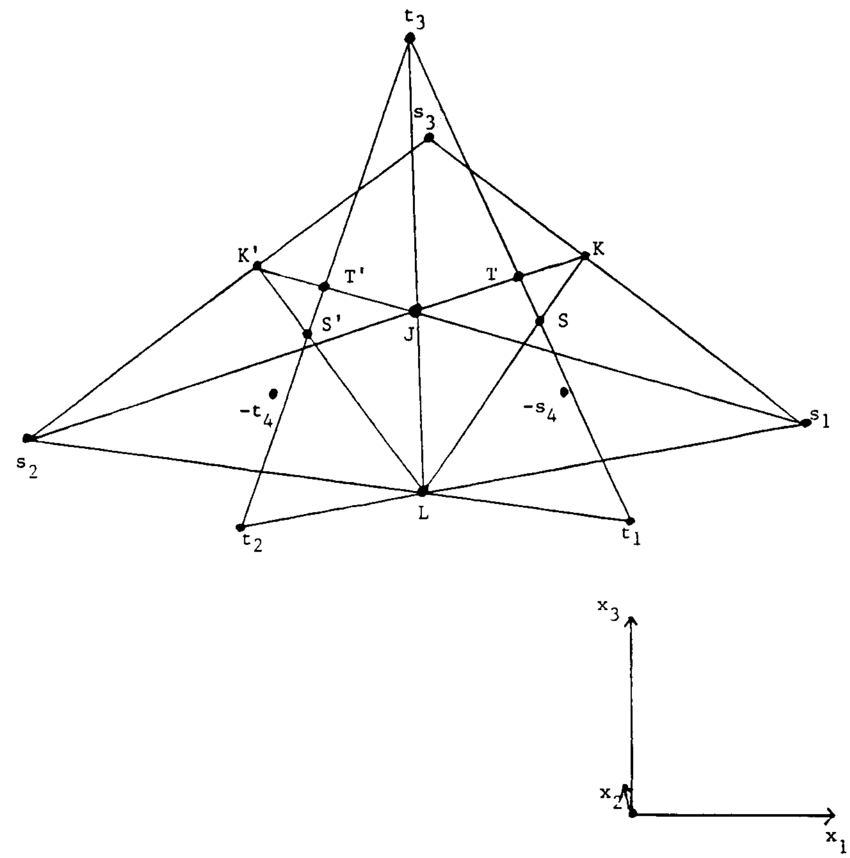

FIG. 1. Kelly and Watson's configuration.

Q-arrangement iff these visibility sets do not intersect. The idea, then, is to enclose the points $-s_4$ and $-t_4$ in "boxes" with sides formed by triangles of the form $\triangle c_1c_2c_3$, where $c_i \in \{s_i, t_i\}$ for $i = 1,2,3$.

The coordinates of the points J, K, L, S, T in Figure 1 are

$$J = \begin{bmatrix} 0 \\ \frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} \frac{6}{7} \\ -\frac{2}{7} \\ \frac{8}{7} \\ 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}, \quad S = \begin{bmatrix} \frac{3}{5} \\ -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}, \quad T = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

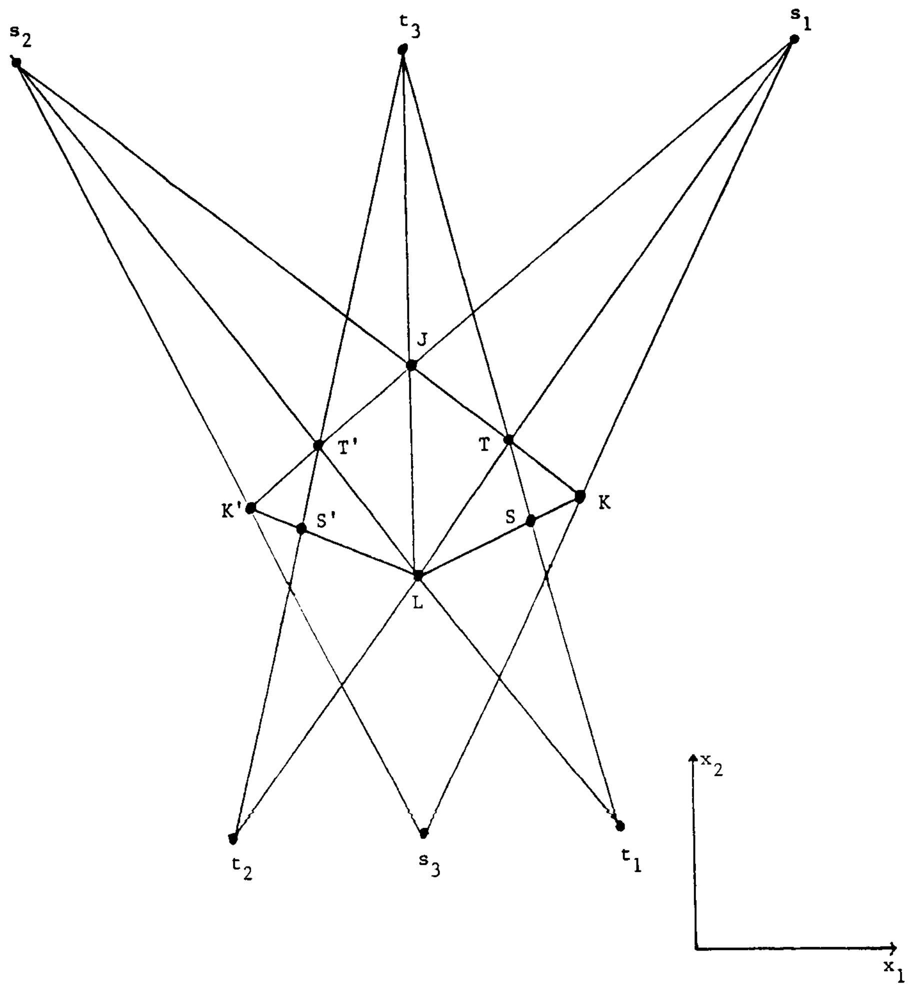

FIG. 2. Projection of K-W configuration onto x_1x_2 plane.

The points K', S', T' are obtained from K, S, T by multiplying the first coordinate by -1.

First, consider the point $-s_4 = \frac{5}{24}J + \frac{7}{24}K + \frac{1}{4}L + \frac{1}{4}s_1$. It is contained in the interior of the tetrahedron tet $JKLs_1 = \text{conv}(\{J, K, L, s_1\})$. The triangle $\triangle KLs_1$ is contained in the triangle $\triangle s_1t_2s_3$, $\triangle JLs_1$ is contained in $\triangle s_1t_2t_3$, and $\triangle JKs_1$ is contained in $\triangle s_1s_2s_3$. The remaining face $\triangle JKL$ is the union of conv($\{JLST\}$), which is contained in $\triangle t_1s_2t_3$, and the triangle $\triangle KST$, which is in no triangle $\triangle c_1c_2c_3$. The interior of $\triangle KST$ is thus a "window" for $-s_4$ to see through. The visibility set of $-s_4$ is further constrained by the triangle $\triangle s_1Ls_3$, which is in $\triangle s_1t_2s_3$, and the triangle $\triangle s_1Js_3$, which is in $\triangle s_1s_2s_3$. Also, $-s_4$ is coplanar with the points S, T, and s_3. This gives us the visibility set of s_4: $\text{Vis}(-s_4) = \text{int}\,\text{tet}\,JKLs_1 \cup \text{int}\,\text{tet}\,KSTs_3 \cup \text{relint}\,\triangle KST$, where $\text{int}\,\text{tet}\,JKLs_1$ is the interior of $\text{tet}\,JKLs_1$, and $\text{relint}\,\triangle KST$ is the relative interior of $\triangle KST$ in the plane containing $\{K, S, T\}$. The situation for $-t_4$ is symmetric: $\text{Vis}(-t_4) = \text{int tet } JK'Ls_2 \cup \text{int tet } K'S'T's_3 \cup \text{relint}\,\triangle K'S'T'$. The sets $\text{Vis}(-s_4)$ and $\text{Vis}(-t_4)$ are disjoint (see [5] for details), and thus R is a Q-arrangement.

Kelly and Watson use this example to show that the set of Q-matrices is not open. A slight perturbation of the point $-t_4$ will make $\text{Vis}(-t_4)$ and $\text{Vis}(-s_4)$ intersect. If $-t_4$ is moved off of the plane defined by S', T', and s_3 to the same side as s_2, the perturbed point $-t_4'$ will be able to "see" through the relative interiors of the triangles $\triangle S'T's_3$ and $\triangle STs_3$ into the interior of $\text{tet}\,STKs_3$, which is in $\text{Vis}(-s_4)$.

4. THE CLASS Q CANNOT BE CHARACTERIZED BY SIGNS OF SUBDETERMINANTS

Unfortunately, Kelly and Watson's peturbation of t_4 forces the point t_4 off of the plane defined by t_2, t_3, and s_3, so that the subdeterminant of M corresponding to rows 1, 2, and 4 and columns 2, 3, and 4 is no longer zero in the perturbed matrix. Thus this example does not show that Q cannot be characterized by signs of subdeterminants, as pointed out in [3]. In fact, it might seem that the coplanarity of the points t_2, t_3, t_4, and s_3 is crucial to the example. We will now show that it is possible to perturb M so that it has all nonzero subdeterminants, while still staying on the boundary of the set of Q-matrices.

In the matrix R, make the following changes:

$$t_2^\epsilon = \frac{1}{1+\epsilon}\begin{bmatrix} -1 \\ -2+2\epsilon \\ 2\epsilon \\ 1+\epsilon \end{bmatrix}, \qquad s_4^\epsilon = \begin{bmatrix} -\frac{3}{4} \\ \frac{8}{3}\epsilon - \frac{1}{3} \\ \frac{2}{3}\epsilon - \frac{1}{2} \\ -1 \end{bmatrix}, \qquad t_4^\epsilon = \begin{bmatrix} \frac{3}{4}+\epsilon \\ \frac{4}{3}\epsilon - \frac{1}{3} \\ -\frac{1}{2} \\ -1 \end{bmatrix}.$$

For $\epsilon > 0$, replace t_2 by t_2^ϵ, $-s_4$ by $-s_4^\epsilon$, and $-t_4$ by $-t_4^\epsilon$. Call the resulting matrix R^ϵ. Note that $t_2^\epsilon = [1/(1+\epsilon)](t_2 + \epsilon t_3)$. Let X be the point $[1/(1+\epsilon)](s_3 + \epsilon s_1)$. The perturbations from $-s_4$ to $-s_4^\epsilon$ and $-t_4$ to $-t_4^\epsilon$ are made so that the sets of points $\{X, S', T', -t_4^\epsilon\}$ and $\{X, S, T, -s_4^\epsilon\}$ will be coplanar.

LEMMA 1. *For sufficiently small $\epsilon > 0$, every set of four columns of R^{ϵ} is independent.*

Proof. For $\epsilon = 0$, we have $R^{\epsilon} = R$. For sufficiently small $\epsilon > 0$, every set of four independent columns of R will give a corresponding set of four independent columns of R^{ϵ}. There are five sets of four dependent columns of R. These are $\{t_1, t_2, s_1, s_2\}$, $\{t_1, t_3, s_3, s_4\}$, $\{s_1, s_2, s_4, t_4\}$, $\{s_3, t_2, t_3, t_4\}$, and $\{t_1, t_2, t_4, s_4\}$. It can be checked that for small $\epsilon > 0$, each of the corresponding sets of four columns of R^{ϵ} is independent. ∎

LEMMA 2. $\mathrm{Vis}(-t_4) \cap \mathrm{Vis}(-s_4) = \varnothing$ *when t_2 is changed to t_2^{ϵ}.*

Proof. Let L_1^{ϵ} be the point where the line from s_1 to t_2^{ϵ} hits the triangle $\triangle t_1 s_2 t_3$, and let L_2^{ϵ} be the point where it hits $\triangle t_1 s_2 s_3$. Since t_2^{ϵ} is on the line segment from t_2 to t_3, the point L_1^{ϵ} will be on the segment from L to J (see Figure 3). Let S^{ϵ} be the point where the line from L_1^{ϵ} to K hits the segment ST. The visibility set of $-s_4$ then becomes $\mathrm{int\,tet}\, JKL_1^{\epsilon}s_1 \cup \mathrm{int\,tet}\, S^{\epsilon}TKs_3 \cup$

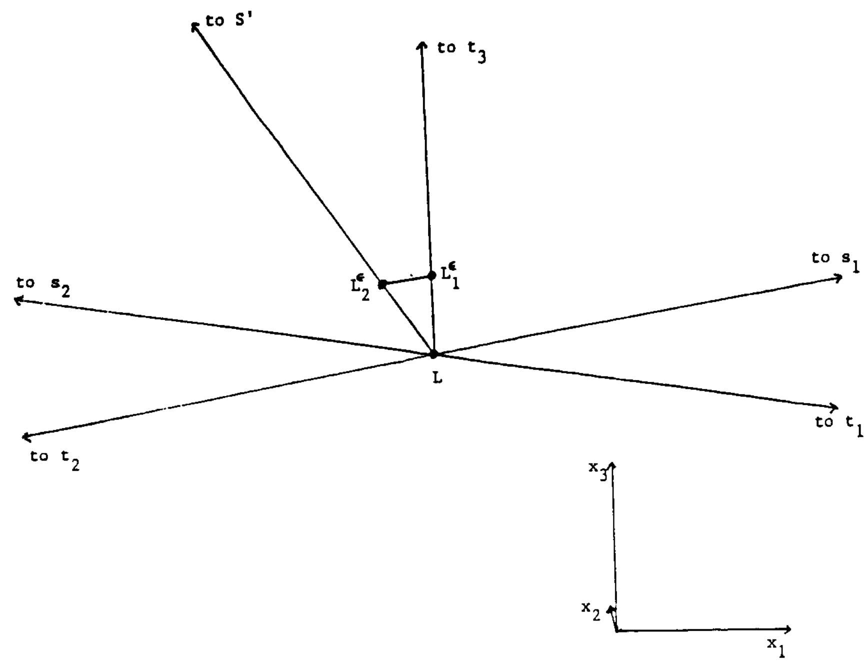

FIG. 3. The new hole in $\mathrm{Vis}(-t_4)$.

$\operatorname{relint} \triangle KS^{\epsilon}T$, which is contained in the original visibility set of $-s_4$. The visibility set of $-t_4$ stays the same, except that a second "hole" in the face of $\triangle JK'L$ of the tetrahedron tet $JK'Ls_2$ is given by $\operatorname{relint} \triangle L_1^{\epsilon}L_2^{\epsilon}L$. This hole does not let the visibility set of $-t_4$ intersect that of $-s_4$, however, since lines from $-t_4$ through the hole are blocked from Vis($-s_4$) by the triangles $\triangle t_1 s_2 t_3$ and $\triangle t_1 s_2 s_3$.

Thus Vis($-s_4$) is strictly contained in what it was before, and the new "hole" int tet $JK'Ls_2$ does not allow $-t_4$ to see into Vis($-s_4$), so Vis($-t_4$) $\cap$ Vis($-s_4$) $= \varnothing$. ■

LEMMA 3. *Assume that t_2 has been replaced by t_2^{ϵ}. Then* Vis($-t_4^{\epsilon}$) $\cap$ Vis($-s_4^{\epsilon}$) $= \varnothing$ *when t_4 and s_4 are changed to t_4^{ϵ} and s_4^{ϵ}.*

Proof. (See Figure 4). When s_4 is changed to s_4^{ϵ}, the visibility set of $-s_4^{\epsilon}$ is that of $-s_4$, except that the part in int tet $KS^{\epsilon}Ts_3$ becomes

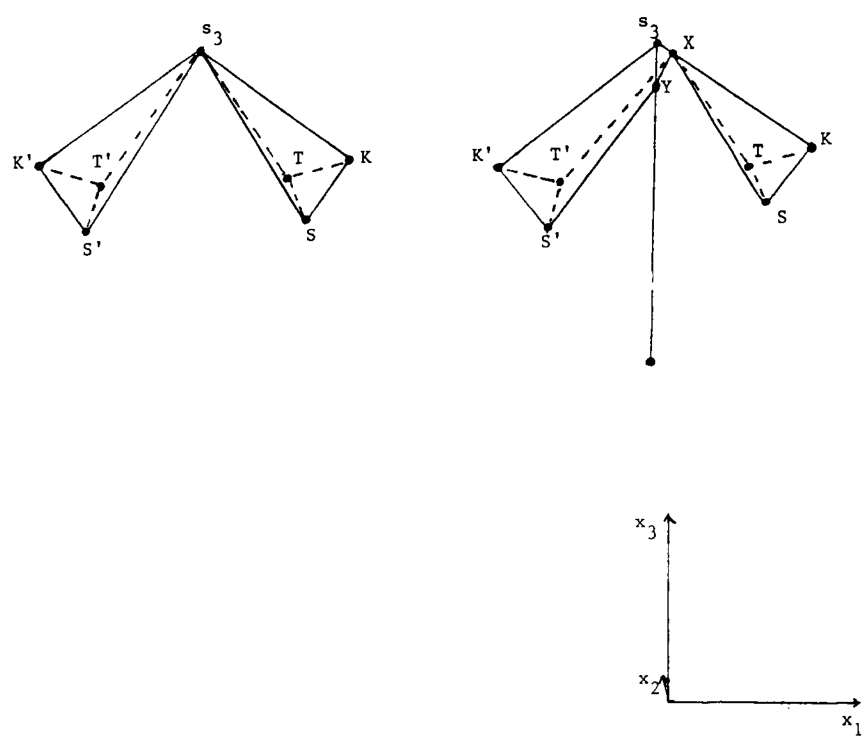

FIG. 4. Change in meeting point of Vis($-s_4$) and Vis($-t_4$).

int tet $KS^{\epsilon}TX$, since X is on the plane containing S^{ϵ}, T, and $-s_4^{\epsilon}$. Similarly, the set $\operatorname{int}\operatorname{conv}(\{S', T', X, s_3, Y\}) \cup \operatorname{relint} \triangle S'T's_3$ is added to the visibility set of $-t_4$, where Y is the intersection of the plane containing X, S', T', and the line segment from L_2^{ϵ} to s_3. Some modifications to Vis$(-t_4)$ are made around the hole $\operatorname{relint} \triangle L_1^{\epsilon} L_2^{\epsilon} L$, but, as before, lines from $-t_4^{\epsilon}$ through the hole are blocked from Vis$(-s_4^{\epsilon})$ by $\triangle t_1 s_2 t_3$ and $\triangle t_1 s_2 s_3$. ■

THEOREM 2. *The class Q cannot be characterized in terms of signs of subdeterminants.*

Proof. Note that the point X is on the boundary of both Vis$(-s_4^{\epsilon})$ and Vis$(-t_4^{\epsilon})$, so that in this regard it plays the role of the point s_3 of Kelly and Watson's example. Now we can perturb t_4^{ϵ} to $t_4^{\epsilon+\delta}$, so that $-t_4^{\epsilon+\delta}$ moves off of the plane containing $\triangle S'T'X$ to the same side as s_2. Then $-t_4^{\epsilon+\delta}$ can see into Vis$(-s_4^{\epsilon})$.

Pivot on the columns $s_1, s_2, s_3, s_4^{\epsilon}$ of the matrix R^{ϵ} to get a matrix $[I, -M^{\epsilon}]$. The perturbation of $-t_4^{\epsilon}$ in the matrix R^{ϵ} will yield a new matrix $R^{\epsilon,\delta}$. Pivot on the columns $s_1, s_2, s_3, s_4^{\epsilon}$ of $R^{\epsilon,\delta}$ to get a matrix $[I, -M^{\epsilon,\delta}]$. The signs of all of the subdeterminants of M^{ϵ} are nonzero by Lemma 1. Thus for small enough δ, the signs of the corresponding subdeterminants of M^{ϵ} and $M^{\epsilon,\delta}$ will agree. But M^{ϵ} is in Q, while $M^{\epsilon,\delta}$ is not. ■

For $\epsilon = \frac{1}{150}$, the matrix $M^{\epsilon,\delta}$ becomes

$$\begin{bmatrix} \frac{1561}{624} & \frac{278921}{94224} & -\frac{2029}{624} & -\frac{51011}{140400} - \frac{2497}{1872}\delta \\ \frac{523}{624} & \frac{30971}{94224} & -\frac{679}{624} & \frac{53407}{140400} - \frac{211}{1872}\delta \\ \frac{223}{156} & \frac{33071}{23556} & -\frac{379}{156} & \frac{301}{35100} - \frac{223}{468}\delta \\ \frac{75}{13} & \frac{11175}{1963} & -\frac{75}{13} & -\frac{38}{39} - \frac{25}{13}\delta \end{bmatrix}.$$

If $\delta = \frac{1}{600}$, then the point

$$q = \begin{bmatrix} \frac{10}{1411} \\ 0 \\ \frac{1400}{1411} \\ -\frac{1}{1411} \end{bmatrix}$$

is not in any of the complementary cones.

REMARK. To construct $M^{\epsilon,\delta}$ and M^{ϵ} from R so that M^{ϵ} is in Q but $M^{\epsilon,\delta}$ is not, it is not necessary to perturb the element t_2. This was only done to make the matrix M^{ϵ} totally nondegenerate, in the sense of [3].

THEOREM 3. *The property that all of the columns of* $[I, -M]$ *are in the interior of the union of the complementary cones associated with M is not a sufficient condition for M to be in Q.*

Proof. The sufficiency of this condition for totally nondegenerate $M \in \mathbb{R}^{3\times 3}$ was shown in [4]. Kelly and Watson's perturbed example has the column I_3 on the boundary of the union of the complementary cones associated with $M + \epsilon D$. Consider the perturbed configuration $R^{\epsilon,\delta}$. The set of uncovered points (not in the union of the complementary cones) in F is contained in an arbitrarily small neighborhood of the point X, for a given $\epsilon > 0$. Thus the point s_3 is in the interior of the union of the complementary cones. However, each of the columns of $R^{\epsilon,\delta}$ is in the interior of the union of the complementary cones. This disproves a conjecture of [4]. ■

5. CONCLUSIONS

Any characterization of the class Q must make use of more information than that which is in the signs of subdeterminants of matrices in Q. The example in this article also casts doubt on the possibility that a matrix M can be shown to be in Q by solving LCP(q, M)'s for a relatively small "test set" (based on M) of vectors q. This is possible for $M \in \mathbb{R}^{3\times 3}$, due to [5]. The set of points q in the affine space F for which the LCP$(q, -M^{\epsilon,\delta})$ has no solution is an arbitrarily small set in a neighborhood of the point X. The point X can be placed at various points on the line segment between s_3 and s_1, for various values of ϵ. It seems unlikely that any relatively small test set would include a point in this region of points that have no solution, for all choices of ϵ and δ.

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