NORTH-HOLLAND

Mixed Dominating Matrices

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ABSTRACT

We characterize the class of matrices for which the set of supports of nonnegative vectors in the null space can be determined by the signs of the entries of the matrix. This characterization is in terms of mixed dominating matrices, which are defined by the nonexistence of square submatrices that have nonzeros of opposite sign in each row. The class of mixed dominating matrices is contained in the class of L-matrices from the theory of sign-solvability, and generalizes the class of S-matrices. We give a polynomial-time algorithm to decide if a matrix is mixed dominating. We derive combinatorial conditions on the face lattice of a Gale transform of a matrix in this class. © 1998 Elsevier Science Inc.

1. INTRODUCTION

All entries of matrices in this paper are real numbers. A matrix is said to be *mixed* if every row contains nonzeros of opposite sign. A matrix is *dominating* if it does not contain a nonempty square mixed submatrix. Mixed dominating matrices have proved to be very important in the study of affine semigroups (see [6], [7], and [8]), in which case the entries of the matrices involved are integers. We have found that the mixed dominating property of a matrix by itself implies many interesting properties of the matrix that do not involve integrality of the entries. In this paper, we collect and present many of these properties.

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A great deal of research has gone into the study of properties of matrices that can be derived from simply looking at the signs of the matrix entries. A recent book, [3], gives an overview of the research on this subject. Matrices for which linear independence of the rows can be inferred from the signs of the entries are called L-matrices. (This follows the notation of [2]. Earlier papers defined L-matrices to be those for which linear independence of columns can be determined from the signs of the entries.) Determining if a matrix is an L-matrix has been shown to be an NP-complete problem [12]. Determining if a matrix has a nonnegative vector in its nullspace by looking at signs of entries has also been considered [1, 5]. An $r \times (r + 1)$ matrix such that one can determine if its null space is spanned by a positive vector by examining signs of entries is called an S-matrix. An $O(r^2)$ algorithm for recognizing an S-matrix was given by Klee [11].

We use a decomposition theorem for mixed dominating matrices, proved in [8], to derive a polynomial-time algorithm that determines if a matrix is mixed dominating. The mixed dominating property of a matrix has a very natural graph-theoretic interpretation. We investigate this interpretation thoroughly in Section 3, and use it to give a new proof of the decomposition theorem for mixed dominating matrices.

Our work can be seen as an extension of the work on S-matrices. We characterize the class of matrices for which the signs of the entries determine the set of supports of all nonnegative vectors in the null space. We show that the problem of determining if a matrix M is in this class reduces to the problem of finding, in a submatrix we call the *derived* submatrix of M, a nonempty square mixed submatrix.

The set of supports of nonnegative vectors in the null space of a matrix M, partially ordered by inclusion, is isomorphic to the face lattice of a cone generated by a set of points called a *Gale transform* of M. Using the decomposition theorem, we derive some properties of the cones of Gale transforms of mixed dominating matrices. We show that every k-dimensional face of such a cone is contained in at most $2(d - k) - 2(k + 1)$-dimensional faces if k is at most $d - 2$, where d is the dimension of the cone. We show that this is also true for every face of the cone. We also derive bounds on the diameters of the 2-skeleton and the dual 2-skeleton of the cone.

2. MIXED DOMINATING MATRICES

Throughout this section, M is an $r \times n$ matrix with real entries, where we will allow r to be zero. A vector in R^n is called *mixed* if it has a positive and a negative component. We say M is mixed if every row of M is mixed.

Thus, an empty (with $r = 0$) matrix is mixed. A matrix M is said to be *dominating* if M contains no nonempty square mixed submatrix. The motivation for this notation is given by the following proposition (Proposition 2.6 from [7]):

PROPOSITION 2.1. *Let M be an $r \times n$ matrix. The following are equivalent:*

(1) *M is dominating.*
(2) *For any subset $[k] \subset [r]$, nonzero numbers a_i, $i \in [k]$, and rows u_i, $i \in [k]$, there exists $j \in [k]$ such that $(\Sigma a_i u_i)^+ \geqslant (a_j u_j)^+$.*

Proposition 2.1 implies that the set of indices for the positive entries of any nontrivial linear combination of the rows of M contains the set of indices for the positive or negative entries of some row of M. This immediately implies the following corollaries (Corollary 2.7 and Corollary 2.8 of [7]).

COROLLARY 2.2. *Let M be an $r \times n$ mixed dominating matrix.*

(1) *Every nonzero vector in the row space of M is a mixed vector.*
(2) *The rows of M are linearly independent.*

A matrix M is an *L-matrix* (see [2]) if every matrix with the same sign pattern as M has linearly independent rows. Corollary 2.2 implies that mixed dominating matrices are L-matrices. We will return to these linear algebraic concepts in Section 4. The rest of the present section is devoted to combinatorial properties of mixed dominating matrices. The following proposition follows directly from Proposition 2.1, but we give an independent combinatorial proof.

PROPOSITION 2.3. *Let M be an $r \times n$ matrix with $r > 0$ in which every row has a positive entry and every column contains a negative entry. Then M contains a nonempty square mixed submatrix.*

Proof. The statement is vacuously true if $r = 1$. Suppose $r > 1$; then M has an $r \times k$ submatrix N with $k \leqslant r$, containing a positive entry in every row. If there is a negative entry in every row of N, then N and hence M contains a square mixed submatrix. If there is a row of N that contains no negative entry, we may delete this row, retaining the property that every column has a negative entry. By induction, the resulting matrix contains a nonempty square mixed submatrix. ■

COROLLARY 2.4. *Let M be a mixed dominating matrix. Then M has a nonnegative column.*

Note that multiplying any set of rows of M by -1 does not destroy the mixed dominating property of M. Hence, by Corollary 2.4, if M is an $r \times n$ mixed dominating matrix and x is a vector of length r, then M has a column for which no entry has nonzero sign opposite to the sign of the corresponding entry of x. This is also true of any mixed dominating submatrix of M. It is known (see Theorem 2.1.1 of [3]) that all L-matrices have this property.

COROLLARY 2.5. *Let M be an $r \times n$ mixed dominating matrix with $n > r + 1$. Then M has two columns i and j such that no row of M has nonzeros of opposite sign in columns i and j.*

Proof. We first give a greedy algorithm to find a maximal $k \times (k + 1)$ mixed submatrix J of M [i.e., one not contained in any $l \times (l + 1)$ mixed submatrix of M with $l > k$]. We will build the sets J_R and J_C of row and column indices of J. Initialize $J_R = \varnothing$ and $J_C = i$, where i is any column of M. Scan the columns of J_C. If a column of J_C contains a nonzero in a row s not in J_R, set $J_R = J_R \cup s$ and $J_C = J_C \cup j$, where j is a column of M in which row s has a nonzero of sign opposite to the nonzeros of row s that are in columns of J_C. Resume scanning the columns of J_C. If at some point the columns of J_C contain no nonzeros in rows of M not in J_R, let J be the submatrix of M with rows indexed by J_R and columns indexed by J_C, and stop. Suppose there is a mixed $l \times (l + 1)$ submatrix L of M containing J when the algorithm stops. If $l > k$, then the submatrix of L formed from the rows and columns that do not intersect J is a nonempty square mixed matrix, contradicting the mixed dominating property of M.

Now let c be a column of M not in J_C. By the discussion following Corollary 2.4, there must be a column j of J such that no row of J has a nonzero in column j of opposite sign to the corresponding entry of column c. Since there are no nonzeros in column j that are in rows not indexed by J_R, we see that no row has nonzeros of opposite sign in columns j and c. ■

Suppose that M is an $r \times n$ mixed matrix with no 2×2 mixed submatrix. If for every pair of columns of M there is a row of M containing nonzeros of opposite sign in those columns, then a theorem of Graham and Pollak [9] states that $n \leqslant r + 1$. Corollary 2.5 therefore follows from that theorem. We have included our proof of Corollary 2.5 because no purely combinatorial proof of Graham and Pollak's theorem is known.

We call a column of a mixed matrix *isolated* if it contains a nonzero entry that is the only entry of its sign in its row. The removal of a non-isolated column from a mixed matrix leaves a mixed matrix. We will see the importance of this in Sections 4 and 5.

PROPOSITION 2.6. *A nonempty mixed dominating $r \times n$ matrix has a row with no more than $n - r + 1$ nonzero entries.*

Proof. It was proved in [7] that the mixed dominating $r \times (r + 1)$ matrices are precisely the $r \times (r + 1)$ S-matrices, as defined in [11]. Thus, the proposition in the case $n = r + 1$ is well known from the theory of S-matrices. Suppose M is an $r \times n$ mixed dominating matrix with $n - r > 1$. Suppose every column of M is isolated. Since $n > r$, this implies that some row of M contains exactly two nonzero entries, and we are done. Otherwise, there is a column of m that is not isolated. This column may be deleted, leaving a mixed dominating submatrix of M. By induction on $n - r$, we can assume that this submatrix has a row with no more than $n - r$ nonzero entries. It follows that M has a row with no more than $n - r + 1$ nonzero entries. ∎

We say that M is *dense* if every 2×3 mixed submatrix of M has five nonzero entries. We say that M is *full* if it is not possible to change any zero entry of M into a nonzero without creating a nonempty square mixed submatrix in the resulting matrix. It is true, but not obvious, that every full mixed dominating matrix is dense.

PROPOSITION 2.7. *A full mixed dominating matrix is dense.*

Proposition 2.7, which is crucial to our proofs of Theorem 2.9, was proved in [8] (Lemma 2.3). A graph-theoretic proof of Proposition 2.7 is given in Section 3. The converse is false in general, but true in the following special case.

PROPOSITION 2.8. *A dense $r \times (r + 1)$ mixed dominating matrix is full.*

Proof. We prove by induction on r that if M is a dense $r \times (r + 1)$ mixed dominating matrix and i and j index two distinct columns of M, then there is a row of M containing nonzeros of opposite sign in these columns. This is vacuously true for $r = 0$. Suppose $r > 0$, and let i and j index two distinct columns of M. Suppose we run the algorithm from the proof of

Corollary 2.5, starting with $J_R = \varnothing$, $J_C = \{i\}$, to find a maximal $k \times (k+1)$ submatrix J of M. Since M is $r \times (r+1)$, we have $J = M$. We must therefore at some stage add j to J_C. When this happens, we have a $k \times (k+1)$ submatrix K of M with columns indexed by J_C and rows indexed by J_R, $j \notin J_C$, and a row $s \notin J_R$ that contains nonzeros of opposite sign in columns j and l for some $l \in J_C$. If $l = i$, then row s contains nonzeros of opposite sign in columns i and j. Otherwise, the inductive hypothesis applied to K implies that there is a row $t \in J_R$ with nonzeros of opposite sign in columns i and l. We therefore have a 2×3 mixed submatrix of M with rows indexed by $\{s, t\}$ and columns indexed by $\{i, j, l\}$. Since M is dense, either row s or row t contains nonzeros of opposite sign in columns i and j. It is now clear that changing a zero entry of a dense $r \times (r+1)$ mixed dominating matrix to a nonzero must create a 2×2 mixed submatrix. ■

Theorem 2.9 below is the key to our polynomial-time algorithm for recognizing mixed dominating matrices as well as to the results of Section 5. It generalizes a theorem of Delorme [4] on numerical semigroups that are complete intersections. Paper [7] shows that an affine semigroup which is a complete intersection has a mixed dominating matrix of relations and derives Delorme's theorem by proving Theorem 2.9 for the case $n = r + 1$. The equivalence of $r \times (r+1)$ mixed dominating matrices and S-matrices was also proved in [7]. The case $n = r + 1$ of Theorem 2.9 is therefore also implied by a result of [12] that S-matrices can be decomposed as in the theorem. Delorme's result was extended to affine semigroups of dimension at most three in [13]. Theorem 2.9 was proved and used to extend Delorme's theorem to affine semigroups of arbitrary dimension in [8]. Because of the importance of Theorem 2.9 for mixed dominating matrices, we give an alternative proof here and another in Section 3.

THEOREM 2.9. *Let M be a nonempty $r \times n$ mixed matrix. Then M is dominating if and only if the rows and columns of M can be rearranged so that the resulting matrix has the form*

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \\ \hline a & b \end{array}\right), \tag{1}$$

where A and B are dominating matrices and a and b are unmixed vectors.

Proof. Note that either A or B or both may be a matrix with no rows. Let M be a mixed matrix with the block structure of (1). Suppose that A and B are both dominating. If M contains a nonempty square mixed submatrix N, then let $\mathrm{nc}(A)$ be the number of columns of N that intersect A, and let $\mathrm{nr}(A)$ be the number of rows of N that intersect A. Define $\mathrm{nc}(B)$ and $\mathrm{nr}(B)$ similarly. If $\mathrm{nc}(A) = 0$, then N does not intersect the last row of M, so B contains N. It is impossible to have $\mathrm{nc}(A) = 1$ if the rows of N are to be mixed. If $\mathrm{nc}(A)$ and $\mathrm{nc}(B)$ are both at least 2, then since $\mathrm{nc}(A) + \mathrm{nc}(B) \leqslant \mathrm{nr}(A) + \mathrm{nr}(B) + 1$, either $\mathrm{nc}(A) \leqslant \mathrm{nr}(A)$ or $\mathrm{nc}(B) \leqslant \mathrm{nr}(B)$. Thus N contains a nonempty square mixed submatrix of A or B.

For every mixed dominating M, there is a full mixed dominating matrix M' obtained by turning zero entries of M into nonzeros. A decomposition of M' as in (1) would give a decomposition for M. Proposition 2.7 implies that M' is dense. For any row s of M', define $S(s)$ to be the set of indices of columns that contain nonzeros in row s. Suppose rows $S(s) \cap S(t) \neq \emptyset$ for rows s and t of M'. Denseness of M' implies that $S(s) \subset S(t)$ or $S(t) \subset S(s)$. If $S(s) \subseteq S(t)$, then the entries of row t indexed by $S(s)$ must all be of the same sign, for otherwise there would be a 2×2 mixed matrix contained in rows s and t. Let u be a row of M' so that $S(u)$ is not contained in $S(v)$ for any row v of M'. We may assume that u is the last row of M'. By an argument similar to that of the first half of this proof, we could change any zero entry of row u to a nonzero without creating a square mixed submatrix. Thus fullness of M' implies that u has no nonzero entries. Therefore M' (and also M) has the block structure of (1) if we place the rows of M' with support contained in set of indices of positive entries of row u first and also place the columns of M' containing positive entries in row u first. ■

We will say that a mixed dominating matrix M is *extendable* if it is possible to add a mixed row to M so that the resulting matrix is mixed dominating. The paper [8] proved that every $r \times n$ mixed dominating matrix M with $n > r + 1$ is extendable, and used this result to prove the decomposition theorem. Conversely, one can deduce the extendability of M, or the following sharper property, from the decomposition theorem.

PROPOSITION 2.10. *Let M be an $r \times n$ mixed dominating matrix. Then $[n]$ can be partitioned into $r + 1$ subsets, so that any mixed vector of length n with support in one of the parts of the partition can be appended as a new row of M, with the resulting enlarged matrix mixed dominating.*

Proof. The proof is by induction on r. If $r = 0$, we can obviously append any mixed n-vector. The partition of $[n]$ for a $0 \times n$ matrix thus has one part. Suppose $r > 0$, and M has the block structure of (1). We may assume by induction that the sets of column indices of A and B are partitioned, so together these partitions make up a partition of $[n]$ with $r+1$ parts. Suppose we add a mixed row to M with support in one of the parts of the partition, say a part indexing columns that meet A. By induction, the resulting enlargement of A has no nonempty square mixed submatrix, and then by Theorem 2.9, M also has no nonempty square mixed submatrix. ∎

We now apply Theorem 2.9 to obtain a polynomial-time algorithm to recognize mixed dominating matrices.

ALGORITHM I. Input: (s, M), where M is a nonempty mixed matrix and s is a row of M. Output: (α, A, B), where $\alpha = 1$ if there is a rearrangement of the rows and columns of M that has the form (1) with row s as the last row, and $\alpha = 0$ if there is no such rearrangement.

Let $G(s, M)$ be a graph with a vertex for every column of M and an edge between two vertices if there is a row of M other than s with nonzero entries of opposite sign in these columns. Determine the connected components of $G(s, M)$. If a connected component of $G(s, M)$ has two vertices for which the corresponding columns of M contain nonzero entries of opposite sign in row s, set $\alpha = 0$. Otherwise, set $\alpha = 1$, and let A be the submatrix of M with columns that correspond to vertices of $G(s, M)$ that are in components for which the corresponding columns do not contain negative entries in row s, and the rows of A are the rows of M that have negative and positive entries in the column set of A. Let the columns of B be columns of M that are not in the column set of A, and let the row set of B be the rows of M other than s that are not in the row set of A.

In the case that $\alpha = 1$, the rearrangement of the rows and columns of M that puts the rows and columns of A first and row s last is of the form (1).

We will refer to the (α, A, B) returned by Algorithm I with input (s, M) as $(\alpha(s, M), A(s, M), B(s, M))$. Similarly the (β, M') returned by Algorithm II below will be referred to as $(\beta(M), M'(M))$.

ALGORITHM II. Input: A mixed matrix M. Output: (β, M'), where $\beta = 0$ if M is not dominating, and $\beta = 1$ and M' is a matrix that arises from a rearrangement of the rows and columns of M if M is dominating.

Apply Algorithm I to (s, M) for each row s of M until either $\alpha(s, M) = 1$ for some row s or $\alpha = 0$ for all rows s.

If $\alpha(s, M) = 1$, apply Algorithm II to $A(s, M)$ and $B(s, M)$ if they are nonempty. If $A(s, M)$ is empty or $\beta(A(s, M)) = 1$, and $B(s, M)$ is empty or $\beta(B(s, M)) = 1$, then set $\beta = 1$ and form M' as follows: Let s be the last row of M', and let the first rows and columns of M' contain $M'(A)$ (or the empty matrix if A is empty) and let the last rows and columns of M' other than row s contain $M'(B)$ (or the empty matrix if B is empty.) If one of $\beta(A(s, M)) = 0$ or $\beta(B(s, M)) = 0$, then set $\beta = 0$.

If $\alpha(s, M) = 0$ for all rows s of M, then set $\beta = 0$.

THEOREM 2.1. *Algorithm II finds that an $r \times n$ mixed matrix M is dominating or determines that M is not dominating in $O(r^2 n^3)$ steps.*

Proof. Algorithm I takes $O(n^3)$ steps to determine the connected components of the graph $G(s, M)$, which has $O(n)$ vertices. Algorithm II calls Algorithm I $O(r)$ times before calling itself or returning β. We show by induction that Algorithm II is called at most r times. Algorithm II applied to a matrix with one row will not call itself, so it is applied once for a matrix with one row. If Algorithm II calls itself, it does so for one or two matrices, with disjoint row sets of total cardinality $r - 1$. By induction, the total number of times Algorithm II is called for these matrices is at most $r - 1$. Therefore Algorithm II is called at most r times. ■

Note that Algorithm II does not return a square mixed submatrix if M is not dominating. In order to remedy this, we propose the following algorithm.

ALGORITHM III. Input: A mixed matrix M. Output: A determination that M is dominating or a nonempty square mixed submatrix of M.

Run Algorithm II on matrix M. If Algorithm II returns $\beta = 0$, then for each column c of M, run Algorithm II on the matrix $M \setminus c$ obtained from M by deleting column c and any unmixed rows created by deleting column c, if this matrix is not square and not empty. If a column c is found for which $\beta(M \setminus c) = 0$, run Algorithm III on the matrix $M \setminus c$.

THEOREM 2.12. *Algorithm III finds a nonempty square mixed submatrix of an $r \times n$ mixed matrix M or determines that M is dominating in $O(r^3 n^4)$ steps.*

Proof. Suppose that the first call to Algorithm II returns $\beta = 0$. If $M \setminus c$ is square and nonempty, we are done. If $M \setminus c$ is empty or $\beta(M \setminus c) = 1$, then the decomposition theorem implies that every square mixed submatrix of M meets column c. This can happen for at most r columns of M. Since $n > r$, $\beta(M \setminus c) = 0$. Thus Algorithm III calls itself at most n times, and for each call it runs Algorithm II at most r times. ■

3. A GRAPH-THEORETIC INTERPRETATION

Given an $r \times n$ matrix $M = (m_{ij})$, one may associate to it a colored multigraph G whose vertices correspond to the columns of M with an edge of color s between vertices v_j and v_k if there exists a row s such that $m_{sk} m_{sj} < 0$. It is clear that for each row of M, the subgraph of G induced by the edges of color s is a complete bipartite graph. If M is a mixed matrix that contains no 2×2 mixed submatrices, then G is a graph. We will call any subgraph of G with all edges of distinct colors *multicolored*. It follows from Lemma 2.5 of [7] that G contains no multicolored circuits if and only if M is mixed dominating. This has also been proved for the case $n = r + 1$ in [3, Theorem 4.4.4].

By definition, a mixed dominating matrix M is dense if and only if any two vertices of G connected by a multicolored path of length two are connected by an edge of one of the two colors in the multicolored path. Therefore, if M is dense, it follows that any two vertices connected by a multicolored path are adjacent.

In this section we give graph-theoretic proofs of Proposition 2.7 and Theorem 2.9. We will call a vertex v_j a source for color s if $m_{sj} > 0$, and a sink for color s if $m_{sj} < 0$. Note that every source for color s is connected to every sink for color s by an edge of color s.

LEMMA 3.1. *Let M be a mixed dominating matrix and let G be its graph. Suppose that vertices u, w, and y are vertices of G with edges of color t between u and w and between y and w. Suppose that p is a multicolored path from u to y not containing edges of the color t. Let q be a multicolored path starting at w, also not containing edges of color t. Then the sets of edge colors of paths p and q must be disjoint.*

Proof. Starting from w, let e be the first edge in the path q whose color s also appears for an edge f in the path p. We may assume that the vertex a of edge e closest to w is a source for color s. If vertex b of edge f is a sink for color s and if b is closest to u (respectively y), then there exists a multicolored path between a and b going through u (respectively y) not containing edges of color s. Since a is a source and b is a sink for color s, this creates a multicolored cycle and contradicts the assumption that M is mixed dominating. ■

The following lemma proves that in a mixed dominating matrix, every 2×3 mixed submatrix with four nonzero entries may be modified by changing a zero into a nonzero, keeping the matrix mixed dominating. This will therefore prove Proposition 2.7.

LEMMA 3.2. *Let M be a mixed dominating matrix, and let G be its graph. Suppose that vertices x and y of G are connected by an edge of color s and that vertices y and z are connected by an edge of color t distinct from s. If the vertices x and z are not adjacent, then a zero entry of M can be made nonzero so that x and z are connected by a new edge of color s or t, and the resulting matrix is mixed dominating.*

Proof. We may assume that x is a source for color s and that z is a sink for color t. If we are prevented from changing M so that z is a sink for color s, then there must exist a vertex v that is a source for color s and a multicolored path between vertices v and z that does not contain the color s. Since vertices v and y are connected by an edge of color s, this path must contain an edge of color t, for otherwise we would have a multicolored cycle. Since vertex z is a sink for color t, we can assume that the first edge from z must be of color t, since otherwise the path can be shortened. Hence, there exists a vertex u that is a source for color t and adjacent to z. Call the path from vertex u to y, through v, path p. Notice that path p is multicolored and does not contain an edge of color t, but contains an edge of color s.

A similar argument shows that if vertex x cannot be made into a source for color t, it is due to the existence of a vertex w that is a sink for color t and the existence of a multicolored path q from w to x that does not contain edges of color t but must of necessity contain an edge of color s. But then vertices w and y and also vertices w and u are connected by edges of color t. Since the multicolored path p between vertices y and u does not contain the color t, we have a contradiction to Lemma 3.1. ■

We say that a row u of a mixed matrix M is *distinguished* for M if the columns of M can be partitioned into two nonempty sets X and Y so that X contains the negative support of u, Y contains the positive support of u, and the support of any other row of M is contained in one of X or Y. The decomposition of a mixed dominating matrix in Theorem 2.9 shows that the bottom row is distinguished. If G is the multigraph of M and u is distinguished for M, then removal of the edges of color u from G leaves a multigraph in which there is no path from a vertex that was a source for color u to a vertex that was a sink for color u. We will give another proof of Theorem 2.9 now by showing that every G coming from a mixed dominating matrix must have such a color corresponding to a distinguished row.

THEOREM 3.3. *Suppose that M is mixed dominating. Then M must contain a distinguished row.*

Proof. We may assume that M is dense. If a row s is not a distinguished row for M, then there must exist a cycle of G containing precisely one edge of color s. Since M is dense, we may assume that, in fact, there exists a cycle of length three with one edge of color s and the remaining two edges of a color t distinct from s. Denote such a cycle by $Z(s, t)$.

Suppose that there is such a cycle $Z(s, t)$ for every row s. There must then be a sequence $(Z(s_1, s_2), Z(s_2, s_3), \ldots, Z(s_{k-1}, s_k))$ with $s_k = s_1$ and with $s_1, s_2, \ldots, s_{k-1}$ distinct. For each $i < k - 1$ we have an edge of color s_{i+1} from $Z(s_i, s_{i+1})$ to $Z(s_{i+1}, s_{s+2})$, and an edge of color s_{k-1} from $Z(s_{k-1}, s_k)$ to $Z(s_1, s_2)$. From this sequence of edges one gets a multicolored cycle using colors $s_1, s_2, \ldots, s_{k-1}$. ■

Suppose that M is mixed and G is its multigraph. The theorem of Graham and Pollak states that if G is a graph (contains no multicolored circuit of length two) and $n > r + 1$, then there are two vertices of G that are not adjacent. The following theorem seems to be a close relative, although the exact relationship is unclear to us.

THEOREM 3.4. *Let M be an $r \times n$ mixed dominating matrix with $n > r + 1$, and let G be its graph. Then there are two vertices of G that are not connected by a multicolored path.*

Proof. We may assume that M is full, since turning zeros of M into nonzeros adds edges to G. The matrix M is dense by Proposition 2.7. Since $n > r + 1$, Corollary 2.5 says that M contains two vertices that are not adjacent. Therefore, the two nonadjacent vertices of G are not connected by a multicolored path. ■

We note that Proposition 2.10 implies that if M is an $r \times n$ mixed dominating matrix and G is its graph, then the vertex set of G can be partitioned into $r + 1$ parts so that no two vertices in the same part are connected by a multicolored path. This proposition appears to be a close relative of a conjecture of Alon, Saks, and Seymour (Problem 9.12 in [10]).

4. SUPPORTS OF NONNEGATIVE VECTORS IN THE NULL SPACE OF M

We will prove an alternative characterization of mixed dominating matrices in terms of sign patterns of vectors in the null space of M. By null space we will always mean right null space. The null space of a $0 \times n$ matrix will be considered to be R^n.

PROPOSITION 4.1. *If M is $r \times n$ mixed dominating and $n > 0$, then there is a positive vector in the null space of M.*

Proof. The proof is by induction on r. If $r = 0$, then the null space of M is R^n, which contains a positive vector. If $r > 0$, we may assume that M has the form (1). By the inductive hypothesis, there are positive vectors x and y respectively in the null spaces of A and B. We may replace y by a positive multiple so that $a^T x + b^T y = 0$. Thus, the concatenation of x and y is a positive vector in the null space of M. ■

Note that Corollary 2.2 may be proved from the decomposition theorem in a similar way.

An $r \times (r + 1)$ matrix M is called an S-matrix if every matrix with the same sign pattern as M has its null space spanned by a positive vector. Corollary 2.2 and Proposition 4.1 imply that an $r \times (r + 1)$ mixed dominating matrix is an S-matrix. That every S-matrix is mixed dominating was proved in [7]. This is also implied by Proposition 4.3 below. Another proof of the equivalence of $r \times (r + 1)$ mixed dominating matrices and S-matrices is in [3, Theorem 4.4.4]. The main result of this section, Theorem 4.4, is a generalization of this equivalence of S-matrices and $r \times (r + 1)$ mixed dominating matrices.

If M is an $r \times n$ mixed dominating matrix and E is a set of columns of M, then $M \setminus E$ will be the matrix obtained from M by deleting the columns in E and then deleting any zero rows that are created. The *support* of a vector y in the null space of M is $\{i \in [n] : y_i \neq 0\}$.

PROPOSITION 4.2. *Let M be an $r \times n$ mixed dominating matrix. Let I be a subset of $[n]$, and let E be the set of columns of M indexed by I. Then $[n] \setminus I$ is the support of a nonnegative vector in the null space of M if and only if $M \setminus E$ is a mixed dominating matrix.*

Proof. Suppose $[n] \setminus I$ is the support of a nonnegative vector in the null space of M. Then $M \setminus E$ is clearly dominating, because M was dominating. An unmixed row of $M \setminus E$ would contradict the existence of a nonnegative vector y in the null space of M with support equal to $[n] \setminus I$. Thus $M \setminus E$ is mixed. Conversely, if $M \setminus E$ is mixed dominating and I is not $[n]$, then there is a positive vector in the null space of $M \setminus E$, and extending this vector to a vector in the null space of M by adding zeros for the entries indexed by I shows that $[n] \setminus I$ is the support of a nonnegative vector in the null space of M. If I is $[n]$, then $[n] \setminus I = \varnothing$ is the support of the zero vector in the null space of M. ∎

We say that the set of supports of nonnegative vectors of the null space of M is *determined* by the sign pattern of M if, for any matrix N with the same sign pattern as M and any subset I of $[n]$, $[n] \setminus I$ is the support of a nonnegative vector in the null space of M if and only if $[n] \setminus I$ is the support of a nonnegative vector in the null space of N. Proposition 4.2 implies that the set of supports of nonnegative vectors in the null space of M is determined by the sign pattern of M if M is mixed dominating.

PROPOSITION 4.3. *Let M be an $r \times n$ mixed matrix. If the set of supports of nonnegative vectors in the null space of M is determined by the sign pattern of M, then M is mixed dominating.*

Proof. Suppose M is not dominating. Let B be a square mixed submatrix of maximum size. Suppose that B is $k \times k$. Let N be the $r \times k$ submatrix of M consisting of the columns that meet B. Without loss of generality, we may assume that B consists of the first k rows of N. We will show that there exist $r \times k$ matrices A and C with the same sign pattern as N such that there exists a positive vector y in the null space of A, but there is no positive vector in the null space of C. This will imply that the set of supports of nonnegative vectors in the null space of M is not determined by the sign pattern of M. The maximality of B implies that each row of N is mixed or zero. Hence, one can easily find positive numbers with which to scale the entries of a row of N so that it is orthogonal to the k-tuple of all ones. If the rows of A are created this way, then the k-tuple of ones is in the null space of A. The construction of C follows. Let the first row of C be the first row of B. Suppose we have constructed l rows of C from the corresponding rows of B, where $l < k$. Suppose the matrix of these l rows of C has a positive vector y_l in its null space. One can then scale the entries of row $l + 1$ of B by positive numbers so that the resulting vector is not orthogonal to y_l, and make this row the $(l + 1)$th row of C. Note that the dimension of the null space of the matrix of the first $l + 1$ rows of C is one less than the dimension of the null space of the matrix of the first l rows of C. If we create k rows of C in this way, then there will be no positive vector in the null space of C. If the matrix of the first l rows of C has no positive vector in its null space, then the remaining rows of C can be the corresponding rows of N. ■

An alternative, more combinatorial construction of C from the preceding proof uses Koenig's theorem from matrix theory and is similar to the proof of Theorem 3.2 of [1]. We prefer the argument above for its simplicity.

In order to completely characterize matrices M for which the sign pattern determines the set of supports of nonnegative vectors in the null space of M (i.e. for matrices that are not necessarily mixed), we need the concept of the *derived* submatrix of M. If M is a matrix, we can create a new matrix by deleting the columns of M that contain nonzero components of unmixed rows, and then deleting any zero rows. This process may be repeated until we have a mixed matrix H. This mixed matrix is called the derived submatrix of M. It is clear that any nonnegative vector y in the null space of M must have $y_i = 0$ if column i of M does not meet H. Conversely, any nonnegative vector in the null space of H can be extended to a nonnegative vector in the null space of M by adding zeros in the components corresponding to columns of M deleted to create H. Thus, the following theorem follows from Propositions 4.2 and 4.3.

THEOREM 4.4. *Let M be a matrix. The sign pattern of M determines the set of supports of nonnegative vectors in the null space of M if and only if the derived submatrix of M is mixed dominating.*

5. FACES OF A GALE TRANSFORM OF M

In this section we give a geometric interpretation of the previous section. We start will recalling some basic facts about Gale transforms. Let M be an $r \times n$ matrix and let $b_1, b_2, \ldots, b_k$ be a basis for the null space of M. Let N be the $k \times n$ matrix for which the ith row is b_i, $i = 1, 2, \ldots, k$. Let $S = \{s_1, s_2, \ldots, s_n\}$ in R^k be the set of columns of N. Then S is called a *Gale transform* of M. We will also call a set $T \subseteq R^l$ a Gale transform of M if T is the image of S under a nonsingular linear transformation from R^k to R^l. The *cone* of S, denoted $C(S)$, is defined by $C(S) = \{Nx : x \geqslant 0, x \in R^n\}$. A subset F of R^k is called a *face* of $C(S)$ if there is a vector $z \in R^k$ such that $F = \{w \in R^k : z^T w = 0\} \cap C(S)$, and $z^T w \geqslant 0$ for all $w \in C(S)$. In this case, the hyperplane orthogonal to z is said to be a *supporting hyperplane* of $C(S)$. The *dimension* of a face F of $C(S)$ is the dimension of the smallest subspace containing F. This subspace is the span of $\{s_i : s_i \in F\}$. For each face F of $C(S)$, define $I(F) = \{i \in [n] : s_i \in F\}$. The null space of M is the row space of N, so we see that $[n] \setminus I(F)$ is the support of a nonnegative vector in the null space of M for every face F of $C(S)$. Also, in $[n] \setminus I$ is the support of a nonnegative vector in the null space of M, then there is a face F of $C(S)$ such that $I = I(F)$. If E is the set of columns of M induced by $I(F)$, then the dimension of F is $\operatorname{rank}(M \setminus E) + k + |I| - n$.

If M is a mixed dominating matrix, then its rows are linearly independent, so S is a subset of R^{n-r}. Furthermore, we see that $C(S)$ is an $(n-r)$-dimensional face of itself. Since a mixed dominating matrix has a positive vector in the null space, we see that the origin in R^{n-r} is a 0-dimensional face of $C(S)$, i.e. $C(S)$ is pointed. A general characterization of the faces of $C(S)$ is given by the next proposition.

PROPOSITION 5.1. *Let M be an $r \times n$ mixed dominating matrix. Let S be a Gale transform of M. A subset I of $[n]$ is $I(F)$ for a k-dimensional face F of $C(S)$ if and only if the rows and columns of M can be rearranged as*

$$\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right), \tag{2}$$

where A is an $(r - |I| + k) \times (n - |I|)$ mixed matrix, and B is an $(|I| - k) \times |I|$ mixed matrix, and the columns containing B are the rearranged columns indexed by I.

Proof. Suppose M has the form in the proposition, and let I be the columns containing B. Since A is mixed dominating, there is, by Corollary 4.1, a positive vector in the null space of A. This vector can be extended to a vector in the null space of M by appending zeros in the entries indexed by I. Thus there is a face F of $C(S)$ with $I = I(F)$. The matrix A is the matrix obtained by deleting the columns indexed by I from M followed by deleting any zero rows created. The rank of A is $r - |I| + k$, since the rows of a mixed dominating matrix are linearly independent. Thus the dimension of F is k.

Conversely, suppose F is a k-dimensional face of $C(S)$. Arrange the columns of M so that the columns indexed by $I(F)$ are last, and arrange the rows so that the rows that become zero when the columns indexed by $I(F)$ are deleted are the last rows. The resulting matrix has the block structure required, and clearly B is mixed. A is mixed by Theorem 4.2. Because A is also dominating, its rows are linearly independent, and so it has $r - |I| + k$ rows. Then B must have the remaining $|I| - k$ rows. ■

LEMMA 5.2. *Suppose M is a full $r \times n$ mixed dominating matrix, and that J is a maximal $k \times (k+1)$ mixed submatrix of M. Then J is an $r \times (r+1)$ matrix.*

Proof. We again use induction on r, and note that the case $r = 0$ is clear. Suppose that M is a full mixed dominating matrix with at least one row. We can assume that M has the block structure of (1). Let J be a maximal $k \times (k+1)$ mixed submatrix of M, and let J_A and J_B be the intersections of J with A and B respectively. By an argument similar to that in the proof of Theorem 2.9, we see that J_A and J_B both have one more column than they have rows. By induction, J_A and J_B meet all rows of A and B. J must also meet the last row of M, because it has only one more column than it has rows. Thus J has r rows. ■

COROLLARY 5.3. *If M is a full mixed dominating matrix, then every facet of $C(S)$ is a simplicial cone.*

Proof. Suppose that S is a Gale transform of M, and that F is a facet of $C(S)$. Suppose also that M is $r \times n$ and has the block structure of (2), where $s_i \in F$ if and only if column i of M meets B. Then A must be a maximal $k \times (k+1)$ mixed submatrix of M. Lemma 5.2 implies that A is $r \times (r+1)$, since A is also full. We can therefore delete any subset of the columns of M that meet B (which has 0 rows), and the resulting matrix will still be mixed. This implies that every subset of $\{s_i : s_i \in F\}$ spans a face of $C(S)$, which proves that F is a simplicial cone. ■

Let M be an $r \times n$ mixed dominating matrix and let $i \in [n]$. Let S be a Gale transform of M. If there is no entry of M in column i that is the only nonzero entry of its sign in its row, then s_i is on an extreme ray of $C(S)$ and no other element of S is on this extreme ray. In this case, the matrix obtained by deleting column i from M is still mixed dominating, so there is a nonnegative vector y in the null space of M that has support $[n] \setminus \{i\}$. This observation is useful for understanding the examples that follow.

EXAMPLE 5.4. Let M be a 2×6 matrix with entries having the following sign pattern:

$$\begin{pmatrix} + & - & + & - & 0 & 0 \\ 0 & 0 & + & + & - & - \end{pmatrix}.$$

Let S be a Gale transform of M. Then each s_i lies on a distinct extreme ray of the 4-dimensional cone $C(S)$. There are eleven 2-dimensional faces of $C(S)$: the cones of the pairs of elements of S not in $\{\{s_1, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, \{s_5, s_6\}\}$. There are seven 3-dimensional faces of $C(S)$: the cones of the sets $\{s_1, s_2, s_5\}$, $\{s_1, s_2, s_6\}$, $\{s_1, s_4, s_5\}$, $\{s_1, s_4, s_6\}$, $\{s_2, s_3, s_5\}$, $\{s_2, s_3, s_6\}$, and the "degenerate" face $\{s_3, s_4, s_5, s_6\}$.

EXAMPLE 5.5. Let M be an $r \times (2r + 2)$ matrix with entries having the following sign pattern:

$$m_{ij} = \begin{cases} 0 & \text{if } j \leqslant 2i - 2 \text{ or } j \geqslant 2i + 3, \\ + & \text{if } j = \operatorname{sign} 2i - 1,\ 2i, \\ - & \text{if } j = 2i + 1,\ 2i + 2. \end{cases}$$

Let S be a Gale transform of M. Let d be a nonnegative integer less than $r + 2$. The d-dimensional faces F are the cones of the d-element subsets of S that do not contain $\{s_i, s_{i+1}\}$ for any odd i. Those familiar with polytope theory will see that the lattice of faces of $C(S)$ is isomorphic to that of the $(r + 1)$-dimensional *cross-polytope*. In particular, the number of facets [$(r + 1)$-dimensional faces] of $C(S)$ is 2^{r+1}.

THEOREM 5.6. *A set S of n points in R^k is the Gale transform of a mixed dominating matrix if and only if $d = \dim C(S) = n$ or $d < n$ and S can be partitioned into two parts S_A and S_B so that $C(S_A) \cap C(S_B)$ is a one-dimensional pointed cone and S_A and S_B are Gale transforms of mixed dominating matrices and* $\dim C(S_A) + \dim C(S_B) = d + 1$.

Theorem 5.6 follows from the decomposition theorem and the linear independence of the rows of a mixed dominating matrix. For the details, see the proof of Theorem 3.1 of [8]. The sets S_A and S_B are Gale transforms of the matrices A and B when M is in the form (1). The following proposition bounds the number of extreme rays (one-dimensional faces) that a Gale transform of a mixed dominating matrix may have.

COROLLARY 5.7. *Suppose S is a Gale transform of a mixed dominating matrix, with $d = \dim C(S) > 1$. Then $C(S)$ contains at most $2d - 2$ extreme rays.*

Proof. The proof is by induction on $r = |S| - d$. If $r = 0$, then the points of S are linearly independent and each one determines an extreme ray of $C(S)$. The condition $d > 1$ implies that $|S| = d \leqslant 2d - 2$. If $r > 0$, then $d < |S|$, so we can partition S into S_A and S_B as in Theorem 5.6. If $\dim C(S_A) = 1$, then $\dim C(S_B) = d > 1$ and $|S_B| - d < |S| - d$. In this case, $C(S) = C(S_B)$, and by induction, $C(S_B)$ has at most $2d - 2$ extreme rays. If both $\dim C(S_A)$ and $\dim C(S_B)$ are greater than one, the inductive hypothesis implies that $C(S_A)$ has at most $2 \dim C(S_A) - 2$ extreme rays and $C(S_B)$ has at most $2 \dim C(S_B) - 2$ extreme rays. Every extreme ray of $C(S)$ is an extreme ray of $C(S_A)$ or of $C(S_B)$, so $C(S)$ can have at most $2d - 2$ extreme rays. ■

COROLLARY 5.8. *Suppose S is a Gale transform of a mixed dominating matrix, with $d = \dim C(S)$. Let F be a k-dimensional face of $C(S)$, with $k < d - 1$. Then F is contained in at most $2(d - k) - 2$ faces of $C(S)$ of dimension $k + 1$.*

Proof. The case $k = 0$ is Corollary 5.7, as the origin is the only zero-dimensional face of $C(S)$, and the faces one dimension higher containing it are the extreme rays. Assume that S is a Gale transform of a matrix M of the form (2), where column i of M meets B if and only if $s_i \in F$. Let T be a Gale transform of the matrix A. Then for any $k + 1$-dimensional face G of $C(S)$ containing F, G must contain s_i for every column i of M that meets B, and a minimal nonempty set of s_i for which the corresponding columns of M meet A and are deletable from A. Therefore, these faces are in 1-1 correspondence with the extreme rays of T. Since $\dim C(T) = d - k$, the result follows from Corollary 5.7. ■

COROLLARY 5.9. *Suppose S is a Gale transform of a mixed dominating matrix. Let F be a k-dimensional face of $C(S)$, with $k > 1$. Then F has at most $2k - 2$ extreme rays.*

Proof. We may assume that S is a Gale transform of a matrix M of the form (2), where column i of M meets B if and only if $s_i \in F$. Then $\{s_i : i \in F\}$ is a Gale transform of the matrix B. ■

The next proposition and lemma show that when we prove results on the facial structure of $C(S)$, we may assume that the points of S all lie on distinct extreme rays of $C(S)$. That is, we may delete an element s_i from S if s_i is in a cone generated by other members of S, so that the resulting set is also the Gale transform of a mixed dominating matrix.

PROPOSITION 5.10. *Let M be a mixed dominating matrix, and suppose the entry m_{ij} is the only nonzero entry of its sign in its row. Let N be obtained from M by adding multiples of row i to other rows in order to make the entries in column j other than m_{ij} zero. Then N is mixed dominating.*

Proof. We can assume without loss of generality that $m_{ij} < 0$ and that the entries in row i and column j other than m_{ij} are nonnegative. Suppose that N contains a square mixed submatrix, and let N' be one of minimal size. The minimality of N' assures that N' does not meet column j or row i. Let M' be the submatrix of M with the same row and column indices as N', and let M'' be the submatrix of M obtained by adding i and j respectively to the sets of row and column indices of M'. Since M is mixed dominating, there is a mixed row of N', say that contained in row k of N, for which the corresponding row of M' is unmixed. This implies that the row of M'' contained in row k of M is mixed. Arguing this way, one can show that all of the rows of M'' other than that contained in row i of M are mixed. But the row of M'' contained in row i of M is also mixed, so M'' is square mixed, contradicting the mixed dominating property of M. ■

We will say that a matrix obtained from M by performing the row operations of Proposition 5.10 and then deleting row i and column j is obtained by *contracting* column j on row i. In the case that row i of M has exactly two nonzero entries, a matrix obtained from M by contracting either of the columns containing the nonzeros of row i on row i is what is called a *conformal contraction* in [11]. In this case, one has the stronger result that the matrix N obtained as in Proposition 5.10 is mixed dominating if and only if M is mixed dominating. This is unfortunately false if row i has more than two nonzero entries, as one can see by very simple counterexamples.

LEMMA 5.11. *Suppose S is a Gale transform of a mixed dominating matrix M, and that $s_i \in C(T)$ for some $s_i \in S$ and $T \subseteq S \setminus s_i$. Then $S \setminus s_i$ is also a Gale transform of a mixed dominating matrix.*

Proof. If $s_i \in C(T)$, then there is a vector x in the row space of M that has a single negative entry x_i and has all its positive entries in $\{x_j : s_j \in T\}$. Proposition 2.1 then implies that there is a row of M that contains a single negative entry, in column i. Let N be the matrix obtained from M by contracting column i on this row. By Proposition 5.10, N is mixed dominating. Furthermore, $S \setminus s_i$ is a Gale transform of N. ■

Note that the Gale transforms of Example 5.5 satisfy the conditions of Corollary 5.8 at equality, for all faces. Thus the face lattice of the cross-polytope is in a sense the "worst" face lattice for Gale transforms of mixed dominating matrices. The following argument gives an alternative view of why this is so. Suppose S is a Gale transform of a full $r \times n$ mixed dominating matrix M, and S also has the property that every point of S lies on a distinct extreme ray of $C(S)$. Let Π be the partition of $[n]$ given by Proposition 2.10. This partition is unique because M is full. Furthermore, the assumption that all points are on distinct extreme rays implies that no part is of size one. If every part of Π has two elements, then it is easy to see that the face lattice of $C(S)$ must be isomorphic to that of the $(n - r - 1)$-dimensional cross-polytope. If some part p of Π has more than two elements, then we enlarge M as follows. Suppose $i \in p$. Append a copy of column i to M to get M'. Then append a row with positive entries in columns indexed by $p \setminus i$, negative entries in columns i and $n + 1$, and zeros otherwise, to M' to get M''. Then M'' is full, and its partition is the same as Π except that p is replaced by the two sets $p \setminus i$ and $\{i, n + 1\}$. Let T be a Gale transform of M''. One can show that if F is a facet of $C(S)$ that contains s_i, then cone$(\{t_j : s_j \in F\})$ is a facet of $C(T)$, while if F is a facet of $C(S)$ that does not contain s_i, then for each $k \in p \setminus i$ there is a facet of $C(T)$ of the form cone$((\{t_j : s_j \in F\} \cup t_{n+1}) \setminus t_k)$. Geometrically, one can get a $C(T)$ from $C(S)$ by placing a new point t_{n+1} in the interior of cone$(\{s_j : j \in p \setminus i\})$, pulling t_{n+1} away from s_i, and then relabeling s_j by t_j for all $j \in [n]$. We can repeat this process until every part of the partition has two elements, and we then have a cone with face lattice isomorphic to that of the cross-polytope.

The *2-skeleton* of a pointed cone C is a graph with a vertex for every extreme ray of C and an edge joining two vertices iff the corresponding extreme rays are contained in a two-dimensional face of C.

PROPOSITION 5.12. *Suppose S is a Gale transform of a mixed dominating matrix. The diameter of the 2-skeleton of $C(S)$ is at most two.*

Proof. We can assume, by Lemma 5.11, that the elements of S are on distinct extreme rays of $C(S)$. If $|S| = \dim C(S)$, then every pair of vertices of the 2-skeleton is adjacent, so the diameter is at most one. Suppose then that $|S| > \dim C(S)$. Let S_A and S_B partition S as in Theorem 5.6. Let M be a mixed dominating matrix in the form of Theorem 2.9, so that S is a Gale transform of M and S_A and S_B are Gale transforms of A and B, respectively. Let $s_i \in S_A$ and $s_j \in S_B$. We claim that s_i and s_j are on a face of $C(S)$ of dimension two. The assumption that the elements of S are on distinct extreme rays implies that the last row of M has at least two nonzeros of each sign, so that deleting columns i and j of M would not make the last row of M unmixed. No row of A becomes unmixed by deletion of column i, since s_i was assumed to define an extreme ray. Similarly, no row of B becomes unmixed by the deletion of column j. Thus the claim is proved. If s_i and s_j are in S_B, then there is an extreme ray of S_A that is in a two-dimensional face of $C(S)$ with each of s_i and s_j. ■

The *dual 2-skeleton* of a pointed cone C is a graph with a vertex for every facet (face of C of dimension $\dim C - 1$) of C and an edge connecting two vertices if the corresponding facets share a face of dimension $d - 2$.

PROPOSITION 5.13. *Suppose that S is a Gale transform of a mixed dominating matrix. The dual 2-skeleton of $C(S)$ has diameter at most $\dim C(S) - 1$.*

Proof. Let $d = \dim C(S)$. Suppose that F and G are facets of $C(S)$. We will show by induction on $k = d - 1 - \dim(F \cap G)$ that the vertices of the dual 2-skeleton corresponding to F and G are connected by a path of at most k edges. If $k = 0$, then $F = G$. Suppose that $k > 0$ and that a mixed dominating matrix M for which S is a Gale transform has the form (2), with $\{i : s_i \in F \cap G\}$ indexing the columns of B. Let T be a Gale transform of the matrix A of (2). Then $T_F = \{t_i : s_i \in F \setminus G\}$ and $T_G = \{t_i : s_i \in G \setminus F\}$ span disjoint $(d - k - 1)$-dimensional faces F_T and F_G of the $(d - k)$-dimensional cone $C(T)$. By Corollary 5.9, F_T and G_T contain all of the extreme rays of $C(T)$. Let H_T be a facet of $C(T)$ for which $\dim(H_T \cap F_T) = d - k - 2$. Then H_T must contain an extreme ray not on F_T, that is, has nonempty intersection with G_T. Now let H be the face of $C(S)$ spanned by $\{s_i : t_i \in H_T\}$. Then H is a facet of $C(S)$ for which the corresponding vertex of the dual 2-skeleton is adjacent to that corresponding to F. Also, $\dim(H \cap G) > \dim(F \cap G)$, so by induction the vertices of the dual 2-skeleton are connected by a path of at most $k - 1$ edges. ■

6. CONCLUSIONS AND QUESTIONS

We have determined many properties of mixed dominating matrices and their Gale transforms. We have shown that one can determine in polynomial time if a matrix is mixed dominating. We feel that it is unlikely that our Algorithms I–III are the most efficient or the most natural ways to show this, so that an interesting problem is to find an algorithm with better complexity and a more transparent proof of correctness.

One could define an MD^* matrix to be any matrix of the form MD, where M is a mixed dominating matrix and D is a diagonal matrix with no zeros on the diagonal. Every MD^* matrix is an L-matrix, but no MD^* matrix is a barely L matrix as defined in [2]. Also, every totally L (see [3]) matrix is an MD^* matrix. We do not know if there is a polynomial-time algorithm to recognize MD^* matrices.

The paper [7] showed that matrices that do not contain any $s \times t$ mixed submatrices with $s > t$ are also very important for the study of affine semigroups. It would be interesting to know what one can prove about such matrices or their Gale transforms.

Given a set S of n points in R^d, we would like to know if there is a polynomial-time algorithm to determine if S is a Gale transform of some $(n-d) \times n$ mixed dominating matrix M. We note that the combinatorial structure of $C(S)$ does not give enough information to answer this question. Consider a set S of six points in linearly general position in R^4, so that the face lattice of $C(S)$ is isomorphic to that of the octahedron. The linearly general position property of S implies that S cannot satisfy the conditions of Theorem 5.6. On the other hand, the cone of Example 2, with $r = 2$, has its face lattice isomorphic to that of the octahedron.

The problem of finding a mixed dominating M so that a given set S of points is a Gale transform of M appears in the theory of affine semigroups (see [7] and [8]). In that case, the points of S have integer coordinates and we want a matrix M that is not only mixed dominating, but has integer entries and is such that the columns of M span the integer lattice in R^{n-d}.

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