NORTH-HOLLAND

Extended *P*-Pairs

Walter D. Morris, Jr.
Department of Mathematical Sciences
George Mason University
Fairfax, Virginia 22030

Submitted by Richard W. Cottle

ABSTRACT

A P-matrix is a square matrix with positive principal minors. We introduce a natural extension of the class of P-matrices, the class of extended P-pairs. A pair $\{I, M\}$, for a P-matrix M, is shown to be contained in an extended P-pair iff SMS has an n-step vector for some sign matrix S. Such a matrix M is called *extendable*, and an example of a P-matrix that is not extendable is given. Applications to the linear complementarity problem are discussed. A polynomial time algorithm to determine if a pair of matrices is an extended P-pair is given. © Elsevier Science Inc., 1997

1. INTRODUCTION

A square matrix with positive principal minors is called a P-matrix. This class of matrices has been widely studied, for a wide variety of reasons. One motivation for studying this class comes from the elegant characterization from linear complementarity theory, which states that a matrix M is a P-matirx if the linear complementarity problem (LCP) with matrix M and vector q has a unique solution for all q. A discussion of this theorem, originally proved in [11], can be found in [1]. Even though the P-property of a matrix M guarantees that a linear complementarity problem with matrix M has a unique solution, it does not guarantee that the solution can be found quickly. Furthermore, the problem of determining if a matrix is a P-matrix was shown in [2] to be Co-NP complete.

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For these reasons, subclasses of the class of P-matrices are often studied. One such subclass is the class of positive definite matrices. Linear complementarity problems with positive definite matrices can be solved in polynomial time, as shown in [5]. One can also determine quickly if a matrix is positive definite. Other subclasses for which the LCPs can be solved and for which membership can be tested in polynomial time are the class of hidden Minkowski matrices and the class of matrices for which the transpose is hidden Minkowski. These classes are studied in [4], [6], [8], [9], and [10]. The class of *extendable* matrices, introduced in this paper, is a subclass of the class of P-matrices that properly includes the matrices for which the transpose is hidden Minkowski.

The extendable matrices are introduced as matrices M for which the pair $\{I, M\}$ is part of what we call an *extended P-pair*. This will imply that there must be a diagonal matrix S with diagonal entries in $\{-1, 1\}$ such that SMS has an n-step vector. Because [8] and [10] proved that a P-matrix has an n-step vector iff its transpose is a hidden Minkowski matrix, we see that M is extendable iff the transpose of SMS is a hidden Minkowski matrix for some sign matrix S. The applicability of this class to the linear complementarity problem is argued by proving that a P-matrix is extendable iff a principal pivot transform of the matrix has an n-step vector. We give a polynomial time algorithm to determine if a pair of $n \times (n + 1)$ matrices is an extended P-pair, and we show that the problem of recognizing an extendable P-matrix is in NP.

The author's motivation for studying this class of matrices came from his work on Lemke paths. There was presented, in [7], a sequence of d-dimensional polytopes, one for each positive integer d, with the property that the shortest of the d Lemke paths, defined in [7], of the d-dimensional polytope grew exponentially with d. Examination of these polytopes revealed that certain submatrices of the matrices representing these polytopes could not be P-matrices. The case in which all of these submatrices are P-matrices is the case of extended P-pairs. We show that the Lemke paths in this case can have length no more than d. The variable d of Reference [7] corresponds to $n + 1$ of our paper.

2. EXTENDED P-PAIRS

A square matrix with real entries is called a *P-matrix* if all of its principal minors are positive. All matrices will be assumed to have real entries, except in Section 6, where they are assumed to be rational. Every positive definite matrix is a P-matrix, but the class of P-matrices contains many matrices that are not positive definite. The P-property of an $n \times n$ matrix M can be stated in terms of determinants of $n \times n$ matrices with columns taken from the pair of matrices $\{I, M\}$. Such a restatement will make our extension more natural.

DEFINITION 2.1. Let $\{A, B\}$ be a pair of $n \times n$ matrices. A *complementary submatrix* of $\{A, B\}$ is an $n \times n$ matrix C for which the column C_i is either A_i or B_i, for $i = 1, \dots, n$. A pair of columns $\{A_i, B_i\}$ is a *complementary pair* of columns.

DEFINITION 2.2. A pair $\{A, B\}$ of $n \times n$ matrices is called a *P-pair* if the determinants of the complementary submatrices of $\{A, B\}$ all have the same nonzero sign.

Clearly, $\{I, M\}$ is a P-pair iff M is a P-matrix. In general, $\{A, B\}$ will be a P-pair iff A^{-1} exists and $A^{-1}B$ is a P-matrix. For the next definition, we will say that if A is an $n \times m$ matrix and $K \subseteq \{1, \dots, m\}$, then A_K is the submatrix of A formed from the columns indexed by K.

DEFINITION 2.3. A pair $\{A, B\}$ of $n \times m$ matrices, with $m > n$, is called an *extended P-pair* if for every n-element set $K \subseteq \{1, \dots, m\}$ the pair $\{A_K, B_K\}$ is a P-pair.

Sznajder and Gowda [12] use the term column-W to describe pairs of matrices that are P-pairs or extended P-pairs. In the following, we will be interested in the case $m = n + 1$. Before proceeding to this, we would like to show that extended P-pairs exist for all positive integers $m > n$.

EXAMPLE 2.4. Let $m > n$, and let C be the $n \times 2m$ matrix with, for $i = 1, \dots, 2m$, column $C_i = (1, i, i^2, \dots, i^{n-1})^T$. Then C is known to be a *totally positive* matrix, that is, all of its square submatrices have positive determinant. Define $\{A, B\}$ by $A_i = C_{2i-1}$ and $B_i = C_{2i}$ for $i = 1, \dots, m$. Then $\{A, B\}$ is an extended P-pair.

The pair $\{A, B\}$ of Example 2.4 satisfies a stronger property than that of being an extended P-pair: For every n-element set $K \subseteq \{1, \dots, m\}$, the complementary submatrices of $\{A_K, B_K\}$ all have positive determinants. For general P-pairs, these determinants would not all have to be positive, but for a given K, they would have to have the same nonzero sign.

3. N-STEP VECTORS

We will begin this section by reviewing the definition of n-step vectors. For a set $J \subseteq \{1, \dots, n\}$, the principal submatrix with rows and columns in J of an $n \times n$ matrix M is denoted by M_{JJ}, and the subvector with indices in J of an n-vector d is denoted d_J.

DEFINITION 3.1. Let M be an $n \times n$ P-matrix. A vector $d > 0$ in $\mathfrak{R}^n$ is called an *n-step vector* for M if for each nonempty $J \subseteq \{1, \dots, n\}$, $M_{JJ}^{-1} d_J > 0$.

The n-step property of d was shown in [8] to have an equivalent geometrical formulation:

PROPOSITION 3.2. *Let M be an $n \times n$ P-matrix. A vector $d \in \mathfrak{R}^n$ is an n-step vector for M iff for every complementary submatrix C of the P-pair $\{I, M\}$, the solution x to $Cx = d$ is positive.*

NOTE. One can show that Proposition 3.2 implies that the polyhedron of solutions to $w + Mz = d$, $w \geqslant 0$, $z \geqslant 0$ is combinatorially equivalent to a cube, with vertices satisfying $w^T z = 0$, $w + z > 0$. The fact that all of these vertices are "strictly complementary" is the property exploited by Mangasarian in [6] to show that the optimal solution to the linear program "Minimize $d^T x$ subject to the constraints $y - M^T x = q$, $y \geqslant 0$, $x \geqslant 0$" will have all optimal solutions satisfying $y^T x = 0$, regardless of q.

In view of Proposition 3.2, the following definition is natural.

DEFINITION 3.3. Let $\{A, B\}$ be a P-pair of $n \times n$ matrices. A vector $d \in \mathfrak{R}^n$ is an n-step vector for $\{A, B\}$ if for every complementary submatrix C of $\{A, B\}$, the solution x to $Cx = d$ is positive.

For an $n \times n$ matrix A and a vector $a \in \mathfrak{R}^n$, we will denote by (A, a) the $n \times (n + 1)$ matrix formed by appending a to the end of A.

THEOREM 3.4. *Suppose $\{A, B\}$ is a P-pair of $n \times n$ matrices, and let a and b be n-step vectors, not necessarily distinct, for $\{A, B\}$. Then $\{(A, a), (B, b)\}$ is an extended P-pair.*

Proof. Without loss of generality, assume that the complementary submatrices of $\{A, B\}$ have positive determinants. Then each matrix obtained from a complementary submatrix C of $\{A, B\}$ by replacing a column of C by a or b will have positive determinant, by Cramer's rule. For a given index i, then, the matrices formed from complementary submatrices of $\{A, B\}$ by replacing column k by column $k + 1$, for $k = i, \dots, n - 1$, and placing a or b in column n, will have all determinants of the same nonzero sign. These matrices are the complementary submatrices of $\{(A, a)_K, (B, b)_K\}$, where $K = \{1, \dots, n + 1\} \setminus \{i\}$. ■

Note that if $\{A, B\}$ is an extended P-pair of $n \times m$ matrices and π is a permutation of $\{1, \dots, m\}$, then applying π to the columns of A and to the columns of B yields another extended P-pair. Therefore we could, in Theorem 3.4, insert columns a and b between columns i and $i + 1$ of A and B or before the first columns of these matrices and still end up with an extended P-pair.

The converse of Theorem 3.4 is not true, that is, it is not true that any column of a matrix in an extended P-pair $\{(A, a), (B, b)\}$ of $n \times (n + 1)$ matrices is an n-step vector for the P-pair obtained by removing the complementary pair containing the column. The following observations are crucial to deriving a partial converse to Theorem 3.4.

PROPOSITION 3.5. *Suppose that $\{(A, a), (B, b)\}$ is an extended P-pair of $n \times (n + 1)$ matrices. The system $Ax = a$ has a unique solution, and no component of this solution is* 0.

Proof. Because $\{(A, a), (B, b)\}$ is an extended P-pair, every set of n columns of (A, a) is linearly independent. ■

PROPOSITION 3.6. *Suppose that $\{(A, a), (B, b)\}$ is an extended P-pair of $n \times (n + 1)$ matrices. Let x be the solution to $Ax = a$. If C is any complementary submatrix of $\{A, B\}$ and d is either a or b, then the solution to $Cy = d$ is unique and satisfies $x_i y_i > 0$ for $i = 1, \dots, n$.*

Proof. The determinants of A and C will have the same nonzero sign, because $\{(A, a), (B, b)\}$ is an extended P-pair. For $i = 1, \dots, n$, let A^i be formed from (A, a) by deleting column i, and let C^i be formed from (C, d) by deleting column i. The determinants of A^i and C^i have the same nonzero sign. Cramer's rule then implies that $x_i y_i > 0$. ■

THEOREM 3.7. *Suppose that* $\{(A, a), (B, b)\}$ *is an extended P-pair of* $n \times (n+1)$ *matrices. There exists a sign matrix S such that a and b are n-step vectors for the P-pair* $\{AS, BS\}$.

Proof. Let x solve $Ax = a$. Define $S_{ii} = 1$ if $x_i > 0$, $S_{ii} = -1$ if $x_i < 0$, and $S_{ij} = 0$ if $i \neq j$. Then $\{AS, BS\}$ is a P-pair. Furthermore, the solution to $ASy = a$ is positive. Proposition 3.6 then tells us that if D is a complementary submatrix of $\{AS, BS\}$ and d is a or b, then the solution to $Dz = d$ will also be positive. Thus a and b are n-step vectors for $\{AS, BS\}$. ∎

4. EXTENDABILITY

We now define a class of matrices that is contained in the class of P-matrices and contains the P-matrices for which there exists an n-step vector.

DEFINITION 4.1. A P-pair $\{A, B\}$ of $n \times n$ matrices is *extendable* if there exist vectors a and b, not necessarily distinct, in $\Re^n$ such that $\{(A, a), (B, b)\}$ is an extended P-pair. A P-matrix M is extendable if $\{I, M\}$ is extendable.

The following proposition follows from Theorem 3.7.

PROPOSITION 4.2. *A P-matrix M is extendable iff there is a sign matrix S such that there is an n-step vector for SMS.*

We would like to show that the extendable P-matrices form a proper subclass of the class of P-matrices, and that the P-matrices for which there is an n-step vector form a proper subclass of the class of extendable P-matrices.

EXAMPLE 4.3. Let

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

M is a P-matrix, but there is no n-step vector for M. If we examine the systems $M_{JJ}^{-1}d_J > 0$ for all of the two-element subsets of $\{1, 2, 3\}$, we get the contradiction $d_3 > d_2 > d_1 > d_3$. On the other hand, if

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$SMS = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and any positive d with $d_3 > d_1 + d_2$ is an n-step vector for SMS.

EXAMPLE 4.4. Let

$$M = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Again M is a P-matrix and there is no n-step vector for M. If we examine the systems $M_{JJ}^{-1}d_J > 0$ for all of the two-element subsets of $\{1, 2, 3\}$, we get the inequalities $2d_1 > 3d_2$, $2d_2 > 3d_3$, and $2d_3 > 3d_1$, which cannot all hold. Now let S be as in Example 4.3, so that

$$SMS = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & 3 \\ -3 & 1 & 2 \end{pmatrix}.$$

For $J = \{2, 3\}$, the system $(SMS)_{JJ}^{-1}d_J > 0$ gives the inequality $2d_3 > d_2$, whereas for $J = \{1, 2, 3\}$, the system $(SMS)_{JJ}^{-1}d_J > 0$ gives the inequality $d_2 > 7d_1 + 5d_3$, and these two inequalities cannot both hold for positive d. Because M is a circulant matrix, we will similarly find that SMS has no n-step vector for other matrices S with one -1 on the diagonal. Note next that for any sign matrix S, $SMS = (-S)M(-S)$, so SMS will not have an n-step vector if S has two negatives on the diagonal, because such an S is the negative of one with one -1 on the diagonal. Finally, if the diagonal of S is all negative, then $SMS = M$, so again SMS will have no n-step vector. We have therefore shown that M is a P-matrix that is not extendable.

5. APPLICATIONS TO THE LINEAR COMPLEMENTARITY PROBLEM

Given an $n \times n$ matrix M and a vector q in $\mathfrak{R}^n$, the *linear complementarity problem* (LCP) is to find nonnegative vectors x and y in $\mathfrak{R}^n$ for which $y - Mx = q$ and $x^T y = 0$. A fundamental result of LCP theory is that M is a P-matrix if and only if the LCP with matrix M and vector q has a unique solution for all $q \in \mathfrak{R}^n$. It is shown in [10] that if M has an n-step vector d, then Lemke's pivoting algorithm with artificial vector d will find a solution in at most $n + 1$ pivots. This is the origin of the name n-step vector. Pang gives a polynomial time algorithm in [10] to find an n-step vector for M if there is one. Solving an LCP when the matrix M has an n-step vector can therefore be done in polynomial time.

We now consider LCPs in which M is an extendable P-matrix. Recall that this means that there is a sign matrix S such that SMS has an n-step vector. We will show that SMS has an n-step vector if and only if a related principal pivot transform of M does. A principal pivot transform can be thought of as a way to preprocess an LCP. One starts with a matrix $[I, -M, q]$ and a subset J of $\{1, \dots, n\}$. One then interchanges column j of I with column j of $-M$, for each $j \in J$, and then pivots on the element in row j and column j of the resulting matrix, for each $j \in J$. The result is a matrix $[I, -P_J(M), R_J(q)]$. The matrix $P_J(M)$ is called the principal pivot transform of M determined by J. It is easy to see that $y - Mx = q$ iff $y' - [P_J(M)]x' = R_J(q)$, where $x_i = x'_i$ and $y_i = y'_i$ for $i \notin J$, and $x_i = y'_i$ and $y_i = x'_i$ for $i \in J$. One can thus recover a solution to the original problem if one has a solution to the transformed problem.

THEOREM 5.1. *Let M be an $n \times n$ P-matrix, and let J be a subset of $\{1, \dots, n\}$. Let S be an $n \times n$ sign matrix with $S_{JJ} = -1$ iff $j \in J$. Then SMS has an n-step vector iff $P_J(M)$ has an n-step vector.*

Proof. Suppose that SMS has an n-step vector d, where $S_{jj} = -1$ iff $j \in J$. If C is then a complementary submatrix of $\{I, -M\}$, the solution z to $Cz = Sd$ will satisfy $z_i > 0$ if column i of C comes from I and $i \notin J$, $z_i < 0$ if column i of C comes from I and $i \in J$, $z_i > 0$ if column i of C comes from $-M$ and $i \in J$, and $z_i < 0$ if column i of C comes form $-M$ nand $i \notin J$. We see from this that if D is a complementary submatrix of $\{I, -P_J(M)\}$, the solution w to $Dw = R_J(Sd)$ will satisfy $w_i > 0$ if column i of D comes from I, and $w_i < 0$ if column i of D comes from $-P_J(M)$. This implies that $R_J(Sd)$ is an n-step vector for $P_J(M)$. The argument is reversible, so if $R_J(Sd)$ is an n-step vector for $P_J(M)$, then d is an n-step vector for SMS. ■

Suppose one wants to solve an LCP with M an extendable P-matrix and with vector q, and one is told that d is an n-step vector for SMS. One can then create the transformed problem above in $|J|$ pivot steps to obtain a preprocessed problem in which the artificial vector $R_J(Sd)$ is an n-step vector and find the solution to the preprocessed problem in at most $n + 1$ pivots. The total number of pivots required is therefore at most $2n + 1$.

One would like to be able to solve the problem quickly even if the appropriate S and d are not known, and one is only told that M is an extendable P-matrix. We do not know how to do this for a general extendable P-matrix M. However, a class of linear complementarity problems that can easily be attacked using Theorem 5.1 is the class of P-matrices for which there is a sign matrix S such that SMS is a *Z-matrix*, i.e., the off-diagonal elements of SMS are all nonpositive. It was shown in [10] that any positive d is an n-step vector for such a matrix SMS. Thus these matrices are all extendable. The following efficient algorithm can find an S such that SMS is a Z-matrix or prove that no such S exists.

ALGORITHM 5.2. First, check to see if $m_{ij}m_{ji} < 0$ for any i, j. If so, then SMS cannot be a Z-matrix for any S. If $m_{ij}m_{ji} \geqslant 0$ for all i, j, then we color the edges of the complete graph with vertex set $1, \dots, n$ as follows. Edge $\{i, j\}$ is red if $m_{ij} + m_{ji} < 0$, edge $\{i, j\}$ is blue if $m_{ij} + m_{ji} > 0$, and edge $\{i,j\}$ is white if $m_{ij} + m_{ji} = 0$. Let $H_1, \dots, H_t$ be the components of the graph with vertex set $\{1, \dots, n\}$ and the red edges defined earlier. If there is a blue edge of the complete graph connecting two vertices in the same H_k, then there is no S for which SMS is a Z-matrix. Define a new graph G with vertex set $\{1, \dots, t\}$ and an edge $\{k, l\}$ whenever there is a blue edge $\{i, j\}$ in the complete graph with $i \in H_k$ and $j \in H_l$. If G is not bipartite, then there is no S for which S is a Z-matrix. If G has a bipartition $\{X, Y\}$, then SMS will be a Z-matrix if we let $S_{jj} = -1$ iff $j \in H_k$ for some $k \in X$.

The correctness and efficiency of Algorithm 5.2 are easy to verify. A classical result of Fiedler and Pták [3] is that a Z-matrix SMS is a P-matrix if and only if $SMSx > 0$ for some positive vector x. This can be verified by a linear program. Algorithm 5.2 therefore gives an efficient algorithm for recognizing P-matrices M for which SMS is a Z-matrix for some sign matrix S.

6. COMPLEXITY CLASSIFICATION

We want to give a polynomial time algorithm for determining if a pair $\{(A, a), (B, b)\}$ of $n \times (n+1)$ matrices is an extended P-pair. This is a contrast to the problem of determining if an $n \times n$ matrix M is a P-matrix, which was shown in [2] to be Co-NP complete. We will also show that the problem of determining if an $n \times n$ matrix M is an extendable P-matrix is in NP.

We know from Theorems 3.4 and 3.7 and Propositions 3.5 and 3.6 that $\{(A, a), (B, b)\}$ is an extended P-pair if and only if $\{A, B\}$ is a P-pair, the solution x to $Ax = a$ has no components equal to zero, and a and b are n-step vectors for the P-pair $\{AS, BS\}$, where S is the sign matrix for which Sx is positive. This is equivalent to the assumption that $SA^{-1}BS$ is a P-matrix and that $SA^{-1}a$ and $SA^{-1}b$ are n-step vectors for this P-matrix. Despite the fact that this property includes the assumption that $SA^{-1}BS$ is a P-matrix, the property is known to be checkable in polynomial time. We review the essential ingredients of such a check.

THEOREM 6.1. *Let M be a P-matrix. A vector $d > 0$ is an n-step vector for M if and only if there exists a Z-matrix X such that M^TX is a Z-matrix and $X^Td > 0$.*

The "if" part of this theorem was proved in [10], and the "only if" in [8]. The theorem states that a P-matrix has an n-step vector if and only if the transpose of the matrix is a "hidden Minkowski" matrix. The matrix X given by the theorem is both a P-matrix and a Z-matrix, i.e. a Minkowski matrix. The problem with Theorem 6.1 is that it assumes that M is a P-matrix, which is not easy to check. A result of [9] lets us avoid this. An S-matrix is a matrix M for which there exists a positive vector x with $Mx > 0$. It is well known (see [1]) that every P-matrix is an S-matrix. The vector e is the vector of 1's.

THEOREM 6.2 [9]. *Suppose that M is a square matrix for which there exists a Z-matrix X such that M^TX is a Z-matrix. If $X^Te > 0$ and M^T is an S-matrix, then M is a P-matrix.*

An analysis of the proof of this theorem reveals that one may use any positive vector d in place of e. It is also pointed out in [1] that one may test for the existence of such an X by solving a linear program.

ALGORITHM 6.3. Let $\{(A, a), (B, b)\}$ be a pair of $n \times (n+1)$ matrices. If A is singular, or if A is nonsingular and the solution to $Ax = a$ has a component equal to 0, then $\{(A, a), (B, b)\}$ is not a P-pair. Otherwise, let S be the sign matrix for which the solution x to $ASx = a$ is positive, and let $M = SA^{-1}BS$. Solve a linear program to determine if M^T is an S-matrix. If it is not, then $\{A, B\}$ is not a P-pair. Otherwise, solve a linear program for the coefficients of X to determine if there is a Z-matrix X for which $X^T(SA^{-1}a) > 0$, $X^T(SA^{-1}b) > 0$, and M^TX is a Z-matrix. If this linear program is infeasible, then $SA^{-1}a$ and $SA^{-1}b$ would not be n-step vectors for M if M happened to be a P-matrix. If the linear program is feasible, then Theorem 6.2 implies that M is a P-matrix, and Theorem 6.1 implies that $SA^{-1}a$ and $SA^{-1}b$ are n-step vectors for M. This in turn implies that $\{(A, a), (B, b)\}$ is an extended P-pair.

In order to show that a square matrix M is an extendable P-matrix, one needs a sign matrix S, a Z-matrix X, and a vector $x \geqslant 0$ such that $X^Te > 0$, $(SMS)^TX$ is a Z-matrix, and $SM^TSx > 0$. Performing these multiplications and checking the signs of the matrix entries can clearly be done quickly. Thus the problem of determining if a matrix is extendable is in NP. Finding an appropriate S or showing that there is none seems to be more difficult. The efficiency of Algorithms 5.2 and 6.3 gives some hope of finding a polynomial time algorithm to produce an S for which SMS has an n-step vector or showing that no such S exists. We will be pessimistic, however, and conjecture that the problem is NP-complete. If this turns out to be true, one would be tempted to call the transpose of an extendable P-matrix a *lost* Minkowski matrix.

7. LEMKE PATHS AND EXTENDED P-PAIRS

One of the ways to find a solution to an LCP with matrix M and vector q is Lemke's algorithm. This algorithm traces a path on the polytope of solutions to $y - Mx = \mu q + \lambda d$, $x_1 + \cdots + x_n + \mu + \lambda = 1$, $y, x, \mu, \lambda \geqslant 0$. The vector d is a positive constant vector, often taken to be e. μ and λ are scalar variables. We denote this polytope by $P(M, q, d)$. (Often one takes $\mu = 1$ and drops the requirement on the sum of the variables.) We will be interested here in the case where $\{(I, d), (M, -q)\}$ is an extended P-pair of $n \times (n+1)$ matrices. We state our results without the proofs, which are not hard.

PROPOSITION 7.1. *Suppose that $\{(I, d), (M, -q)\}$ is an extended P-pair of $n \times (n+1)$ matrices. Then the LCP $y - Mx = q$, $y \geqslant 0$, $x \geqslant 0$, $x^Ty = 0$ has a unique solution (y^1, x^1) and the LCP $y - Mx = d$, $y \geqslant 0$, $x \geqslant 0$,*

$x^T y = 0$ *has a unique solution* (y^2, x^2). *The solutions satisfy* $(x^1)^T x^2 = 0$, $(y^1)^T y^2 = 0$, $x^1 + x^2 > 0$, $y^1 + y^2 > 0$.

Proposition 7.1 is a straightforward consequence of Proposition 3.6. It follows that the polytope $P(M, q, d)$ has two complementary vertices that are not on any common facets.

PROPOSITION 7.2. *Let* $\{(I, d), (M, -q)\}$ *be an extended* P*-pair of* $n \times (n + 1)$ *matrices. Let* (y^1, x^1) *and* (y^2, x^2) *be as in Proposition* 7.1. *In the polytope* $P(M, q, d)$, *each of the* $n + 1$ *Lemke paths connecting the vertex corresponding to* (y^1, x^1) *to the vertex corresponding to* (y^2, x^2) *has length at most* $n + 1$.

Proposition 7.2 follows from the fact that there is a sign matrix S such that Sd and $-Sq$ are n-step vectors for the P-pair $\{I, SMS\}$.

Proposition 7.2 represents an ideal situation. In general, one would not expect a pair $\{(A, a), (B, b)\}$ to be an extended P-pair even if $\{A, B\}$ were a P-pair. An interesting line of research is to examine the lengths of the Lemke paths for pairs $\{(A, a), (B, b)\}$ when some, but not all, of the pairs of matrices obtained by deleting a complementary pair of columns from $\{(A, a), (B, b)\}$ are P-pairs.

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