The Connected Components of the Set of R_0-Matrices

Walter Morris
Mathematical Sciences
George Mason University
Fairfax, Virginia, 22030

Submitted by David H. Carlson

ABSTRACT

A matrix M is an R_0-matrix if it satisfies a nondegeneracy property that is often assumed for the linear complementarity problem. This property can be verified from the incidence structure of a polytope defined by M. It is shown that a 3×3 matrix in R_0 can be perturbed within R_0 so that the associated polytope is one of seven types, each corresponding to a different component of R_0.

1. INTRODUCTION

Given a matrix $m \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the *linear complementarity problem*, abbreviated as LCP(M, q), is to find x, $y \in \mathbb{R}^n$ with $x \geqslant 0$, $y \geqslant 0$, $Mx + q = y$, and $x^T y = 0$. Much research in LCP theory has gone into classification of matrices according to properties of the solution sets of the corresponding LCPs. In many of these classifications a certain nondegeneracy property, R_0, plays a part. The determination of the connected components of the set of matrices satisfying this property is useful for LCP theory and could also be useful in homotopy theory.

A matrix $M \in \mathbb{R}^{n \times n}$ is said to have property R_0 or to be in the matrix class R_0 if there is no nonzero $x \in \mathbb{R}^n$ satisfying $x \geqslant 0$, $Mx \geqslant 0$, $x^T Mx = 0$. An equivalent way to say this is to say that the homogeneous LCP(M, 0) has no nontrivial solution. This class of matrices plays a prominent role in many earlier investigations, such as [1, 7, 6, 5]. The class of matrices in R_0 can be written explicitly as the complement of the union of solution sets to a large number of systems of polynomial equations and inequalities in the entries of M. No one has yet been able to make use of such a direct characterization, so

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

attention has usually been focused on geometric methods to study these matrices.

In this paper, we use a geometric approach introduced in [8] to study the class of R_0-matrices when $n = 3$. The main result is that for $n = 3$ the class of R_0-matrices has seven connected components. Representatives are given for each of these components. The papers [7] and [4] showed that one could assign to each connected component of the set of R_0-matrices an integer which is the degree of certain mappings associated with matrices in the component. In particular, positive definite matrices or more generally P-matrices, as well as strictly semimonotone matrices, all belong to the class of matrices associated with maps of degree one. Howe and Stone [7] point out that a homotopy algorithm that traces a continuous path from one matrix to another would benefit from having all matrices along the path in R_0. Motivated by this, they ask if the set of matrices associated with maps of degree one is connected. Our research introduces a methodology for resolving this question, and for the case of 3×3 matrices shows that there is one component for each of the possible degrees -2, -1, 0, and 1, while there are three connected components containing matrices that yield maps of degree 2.

2. GEOMETRIC FRAMEWORK

Let M be in $\mathbb{R}^{n \times n}$. The geometric object that we will study is the polytope $\mathscr{P}(M) = \{x \in \mathbb{R}^n : x \geqslant 0,\ Mx \geqslant 0,\ e^T x = 1\}$, where e is the n-vector with 1 in each component. This polytope was also studied in [8]. The polytope $\mathscr{P}(M)$ is bounded and of dimension at most $n - 1$, since it is in the $(n-1)$-simplex given by $\{x \in \mathbb{R}^n : x \geqslant 0,\ e^T x = 0\}$. First we show that one can tell if a matrix has property R_0 by looking at the incidence structure of $\mathscr{P}(M)$.

LEMMA 2.1. *$M \in \mathbb{R}^{n \times n}$ is in R_0 if and only if there is no vertex v of $\mathscr{P}(M)$ for which the set $\{i : v_i = 0$ or $(Mv)_i = 0\}$ is equal to $\{1, 2, \dots, n\}$.*

Proof. If there is a vertex v of $\mathscr{P}(M)$ for which $\{i : v_i = 0$ or $(Mv)_i = 0\}$ is equal to $\{1, 2, \dots, n\}$, then $v^T M v = 0$. Furthermore, v is nonzero and satisfies $v \geqslant 0$, $Mv \geqslant 0$, as do all vectors in $\mathscr{P}(M)$. Conversely, if a nonzero $x \in \mathbb{R}^n$ satisfies $x \geqslant 0$, $Mx \geqslant 0$, then there exists a real number $\alpha > 0$ such that $\alpha x \in \mathscr{P}(M)$. Since $x^T M x = 0$, we have $\alpha x^T M \alpha x = 0$, so that $\{i : \alpha x_i = 0$ or $(M\alpha x)_i = 0\} = \{1, 2, \dots, n\}$. Finally, if v is a vertex of $\mathscr{P}(M)$ on the smallest dimensional face of $\mathscr{P}(M)$ containing αx, then v must satisfy all of the equalities $v_i = 0$ and $(Mv)_i = 0$ that αx does, so $\{i : v_i = 0$ or $(Mv)_i = 0\} = \{1, 2, \dots, n\}$. ■

Call a vertex v of $\mathscr{P}(M)$ *complementary* if $\{i : v_i = 0 \text{ or } (Mv)_i = 0\} = \{1, 2, \dots, n\}$. (This differs from the notation used in [8].) In the following, our goal will be to perturb M continuously so that $\mathscr{P}(M)$ is simplified, but so that $\mathscr{P}(M)$ never gains any complementary vertices. The remainder of the paper concentrates on the case $n = 3$.

3. ELEMENTARY DEFORMATIONS

Let M be a 3×3 matrix in R_0. Note for $n = 3$ that $\mathscr{P}(M)$ is at worst a convex polygon. We first introduce perturbations of M that resolve degeneracy.

LEMMA 3.1. *Let M be in R_0. If a vertex v of $\mathscr{P}(M)$ satisfies three or more equations $v_i = 0$ or $(Mv)_i = 0$, then at least one of these equations must be of the type $(Mv)_i = 0$. Suppose in that case that $(Mv)_i = 0$, and denote by M^i the ith row of M. (Note that $M^i v = (Mv)_i$.) Let $z = \frac{1}{2}[v + (M^i)^T]$. Let $\varepsilon > 0$ be such that for any vertex v' of $\mathscr{P}(M)$ for which $(Mv')_i > 0$ we also have $(M^i - tz)v' > 0$ for $0 \leqslant t \leqslant \varepsilon$. Then the row M^i may be replaced by $M^i + tz$ for $0 \leqslant t \leqslant \varepsilon$, creating matrices M_t for which $\mathscr{P}(M_t)$ has fewer complementary vertices. If there is another equation $x_j = 0$ or $(Mx)_j = 0$ with the same solution set as $(Mx)_i = 0$, then $\mathscr{P}(M_t)$ will be less degenerate than $\mathscr{P}(M)$. If there are no two facets of the form $x_j = 0$ or $(Mx)_i = 0$ that have the same solution set, then $\mathscr{P}(M_t)$ will be less degenerate than $\mathscr{P}(M)$.*

Proof. Let x be any point in $\mathscr{P}(M)$ for which $(Mx)_i > 0$. Then $(M^i + tz)x = (Mx)_i - \frac{1}{2}t[(Mx)_i + \mathrm{v}^{\mathrm{T}}\mathrm{x}]$, which is > 0 for sufficiently small values of t. Thus ε exists. Note that $v \notin \mathscr{P}(M_t)$, since $M_t^i v = -\frac{1}{2}v^T v < 0$ for $0 < t \leqslant \varepsilon$. If there is another equation $x_j = 0$ or $(Mx)_j = 0$ having the same solution set as $(Mx)_i = 0$, then this other equation will meet $\mathscr{P}(M_t)$ in fewer vertices than it meets $\mathscr{P}(M)$, since $v \notin \mathscr{P}(M_t)$. If there is no such pair of equations having the same solution set, then the vertices of $\mathscr{P}(M_t)$ which were not in $\mathscr{P}(M)$ will be on exactly two facets of $\mathscr{P}(M_t)$. In both cases, all newly created vertices are on subsets of the sets of equations that old vertices were on, so no new complementary vertices can be created. ■

Now assume that every vertex of $\mathscr{P}(M)$ satisfies exactly two equations of the form $x_i = 0$ or $(Mx)_i = 0$.

LEMMA 3.2. *The following elementary operations perturb M continuously but do not introduce complementary vertices into $\mathscr{P}(M)$. These operations reduce the number of vertices of $\mathscr{P}(M)$, so they can only be applied a finite number of times.*

(A) *If there is an i such that* $\{x : x_i = 0\}$ *is a facet of* $\mathscr{P}(M)$ *but* $\{x : (Mx)_i = 0\}$ *is not a facet of* $\mathscr{P}(M)$, *then change M to* M_1 *by changing the row* M^i *of M to* $M_t^i = te_i + (1-t)M^i$, *where* e_i *is the ith coordinate vector. Follow this by applying Lemma 3.1 until* $\mathscr{P}(M_i)$ *is nondegenerate.*

(B) *If there is an index i such that* $\{x : x_i = 0\}$ *and* $\{x : (Mx)_i = 0\}$ *share a vertex v of* $\mathscr{P}(M)$, *let* v^1 *be the other vertex of* $\mathscr{P}(M)$ *on* $\{x : x_i = 0\}$ *and let* v^2 *be the other vertex of* $\mathscr{P}(M)$ *on* $\{x : (Mx)_i = 0\}$. *Let* $z \in \mathbb{R}^n$ *satisfy* $z^Tv^1 = z^Tv^2 = 0$, $z^Tv < 0$. *Change M to* $M_{1+\varepsilon}$ *by changing row i of M to* $M_t^i = tz + (1-t)M^i$ *for* $0 \leqslant t \leqslant 1+\varepsilon$, *where* $\varepsilon > 0$ *is small enough so that* $M_{1+\varepsilon}^i w > 0$ *for all vertices w of* $\mathscr{P}(M)$ *other than v,* v^1,*and* v^2.

(C) *If there is an index i such that* $\{x : x_i = 0\}$ *and* $\{x : (Mx)_i = 0\}$ *are both facets of* $\mathscr{P}(M)$ *and there is exactly one facet between them, then let* v^1 *be the vertex of* $\mathscr{P}(M)$ *on* $\{x : x_i = 0\}$ *and not on the intermediate facet, and let* v^2 *be the vertex of* $\mathscr{P}(M)$ *on* $\{x : (Mx)_i = 0\}$ *and not on the intermediate facet. Let* $z \in \mathbb{R}^n$ *satisfy* $z^Tv^1 = z^Tv^2 = 0$ *and* $z^Tx < 0$ *for all x on the intermediate facet. Change M to* $M_{1+\varepsilon}$ *by changing the row* M^i *to* $M_t^i = tz + (1-t)M^i$ *for* $0 \leqslant t \leqslant 1+\varepsilon$, *where* $\varepsilon > 0$ *is small enough so that* $M_{1+\varepsilon}^i w > 0$ *for all vertices w of* $\mathscr{P}(M)$ *other than* v^1, v^2, *and the vertices on the intermediate facet.*

Proof. Let $z \in \mathscr{P}(M)$. Then in case (A), we have $M^iz > 0$. For $0 \leqslant t < 1$, $M_t^iz = tz^Te_i + (1-t)M^iz > 0$, so no new complementary vertices are introduced. For $t = 1$, $M_t^i = e_i$, so again no new complementary vertices are introduced. In case (B), changing M to M_1 rotates the facet of $\mathscr{P}(M)$ that is $\{x : (Mx)_i = 0\}$, keeping the end v^2 fixed and moving the other end until it hits vertex v^1. When $t = 1$, vertex v^1 satisfies three of the inequalities, but two of them involve the subscript i. Finally, in case (C), changing M to M_1 rotates the facet of $\mathscr{P}(M)$ corresponding to $\{x : (Mx)_i = 0\}$, keeping vertex v^2 fixed and moving the other end until it hits v^1. Here there are two values of t, corresponding to the two vertices of $\mathscr{P}(M)$ for which $x_i = 0$, such that a vertex of $\mathscr{P}(M_t)$ will satisfy three of the equations. But as before, two of these equations will involve the subscript i. Thus no new complementary vertices are introduced. See Figure 1 for examples of deformations (A), (B), and (C). ∎

4. FINAL STATES

There are three possible situations in which none of the deformations (A), (B), (C) or the degeneracy-resolving Lemma 3.1 may be applied. Either $\mathscr{P}(M)$ is empty, or it is a triangle with the three sides given by equations involving different subscripts, or it is a hexagon in which pairs of opposite sides are given by equations involving the same subscript. This observation leads to a complete classification of the connected components of R_0 when $n = 3$.

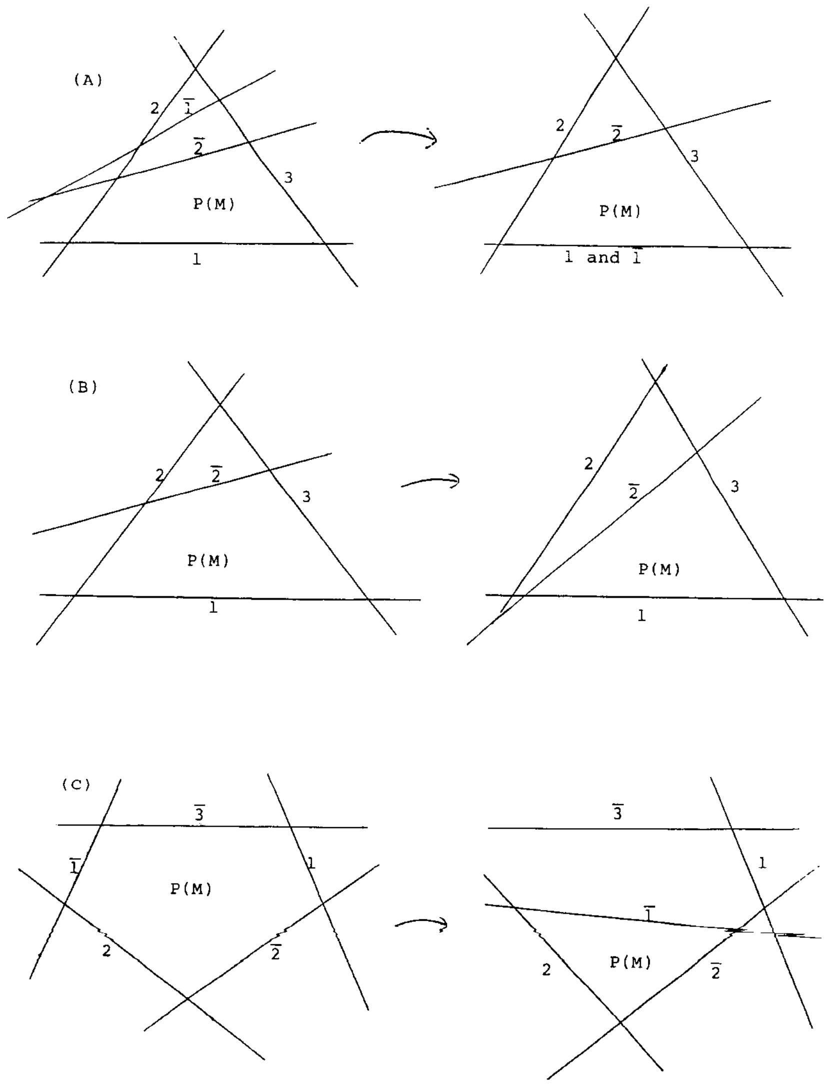

FIG. 1. i denotes $\{x : x_i = 0\}$, and $\bar{\imath}$ denotes $\{x : (Mx)_i = 0\}$.

LEMMA 4.1. *If $\mathscr{P}(M)$ is empty, then M is in the same connected component of R_0 as*

$$M^{-} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$

Proof. Suppose $z \geqslant 0$. Since $\mathscr{P}(M)$ is empty, there is an i such that $M^i z < 0$. Then for $0 \leqslant t \leqslant 1$, $[tM + (1-t)M^-]^i z < 0$. Thus $\mathscr{P}(M_t)$ stays empty for all $0 \leqslant t \leqslant 1$, and no new complementary vertices are introduced.

LEMMA 4.2. *If $\mathscr{P}(M)$ is a triangle, the three sides given by the equations $(Mx)_i = 0$ for $i = 1, 2, 3$, and the triangle $\mathscr{P}(M)$ is oriented coherently with the positive orthant, then M is in the same component of R_0 as the matrix*

$$P = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

Proof. The polytope $\mathscr{P}(P)$ is drawn in Figure 2, which indicates what is meant by coherent orientation with respect to the positive orthant. The facets $\{x : x_1 = 0\}$, $\{x : x_2 = 0\}$, $\{x : x_3 = 0\}$ of the positive orthant appear in a clockwise order around the positive orthant, and the facets $\{x : (Px)_1 = 0\}$, $\{x : (Px)_2 = 0\}$, $\{x : (Px)_3 = 0\}$ of $\mathscr{P}(P)$ also appear in a clockwise order around $\mathscr{P}(P)$. It should be clear that any triangle $\mathscr{P}(M)$ with facets of the form $\{x : (Mx)_i = 0\}$ for $i = 1, 2, 3$ that is coherently oriented with respect to the positive orthant can be shrunk, rotated, and aligned with $\mathscr{P}(P)$ in a continuous manner without having $\mathscr{P}(M)$ touch the boundary of the positive orthant, and thus without creating any complementary vertices. ∎

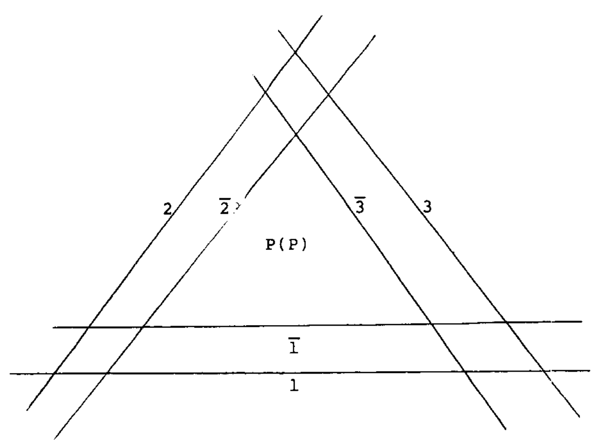

FIG. 2.

LEMMA 4.3. *If $\mathscr{P}(M)$ is a triangle with facets of the form $\{x : (Mx)_i = 0\}$ for $i = 1, 2, 3$, and $\mathscr{P}(M)$ is oriented oppositely to the positive orthant, then M is in the same component of R_0 as the matrix*

$$P^- = \begin{pmatrix} -1 & 3 & -1 \\ 3 & -1 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

Proof. See Figure 3. The argument is essentially the same as that for Lemma 4.2. Note that $\mathscr{P}(P)$ and $\mathscr{P}(P^-)$ differ only in that two facets have been traded, changing the orientation of the triangle. ∎

LEMMA 4.4. *If $\mathscr{P}(M)$ is a hexagon with pairs of opposite sides given by equations involving the same subscript, then $\mathscr{P}(M)$ is in the same connected component of R_0 as one of the following four matrices:*

$$M_{21} = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 3 & 6 \\ -2 & 6 & 3 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 3 & -2 & 6 \\ 2 & -1 & 2 \\ 6 & -2 & 3 \end{pmatrix},$$

$$M_{23} = \begin{pmatrix} 3 & 6 & -2 \\ 6 & 3 & -2 \\ 2 & 2 & -1 \end{pmatrix}, \quad or \quad MU = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

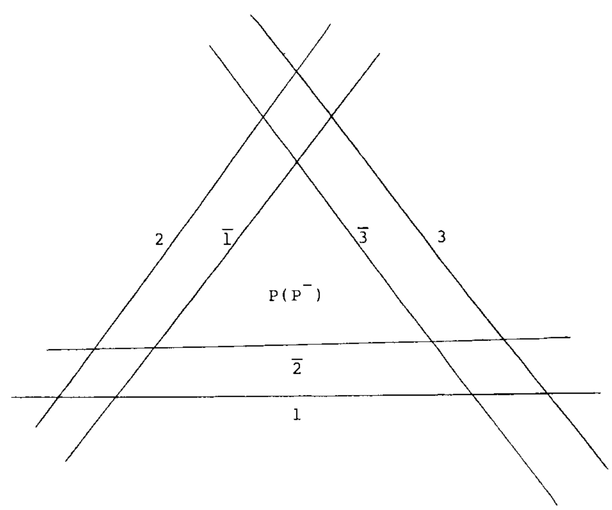

FIG. 3.

Proof. See Figure 4 for drawings of $\mathscr{P}(M)$ for the four different matrices. These four drawings correspond to the four different ways to order the six facets of $\mathscr{P}(M)$ subject to a given orientation of the positive orthant and the restriction that opposite faces correspond to equations involving the same subscript. It should be clear from the drawing that any two polytopes $\mathscr{P}(M)$ with the same ordering of the faces can be obtained from each other by moving the faces corresponding to inequalities of the form $\{x:(Mx)_i = 0\}$, without changing the ordering of the faces. ∎

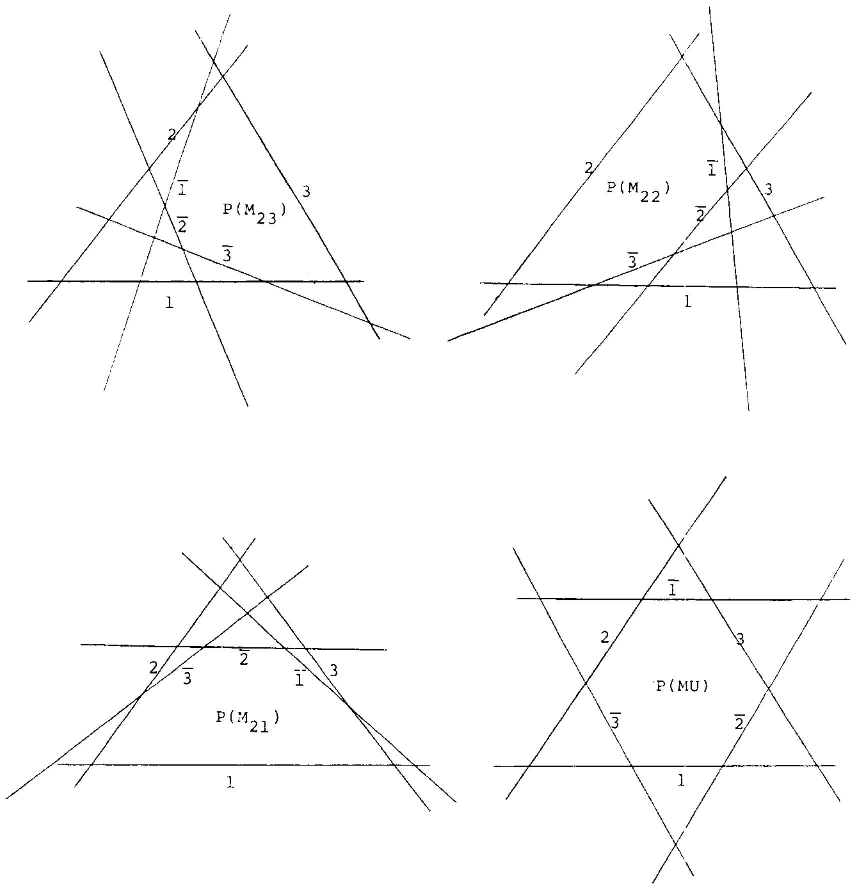

FIG. 4.

LEMMA 4.5. *The four matrices from Lemma 4.4 lie in different components of R_0.*

Proof. An arrangement of lines in the plane containing a hexagon is given in Figure 5. From the enumeration of arrangements of six lines in the plane in [3], we see that such an arrangement with a hexagon always has exactly six triangles (including perhaps triangles at infinity), and these are located on the six edges of the hexagon. Passing from one arrangement of lines to another, which would be necessary in order to change $\mathscr{P}(M)$, must involve collapsing one of these triangles to a point. In each of the arrangements corresponding to the matrices of Lemma 4.4, the six triangles all have facets defined by

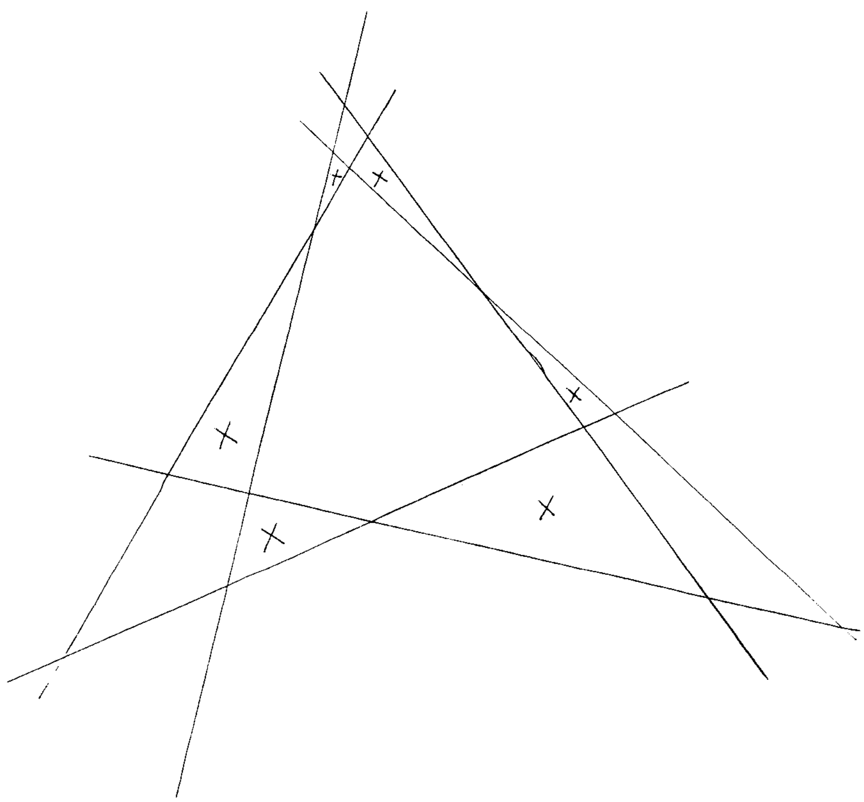

FIG. 5. Arrangement of six lines with a hexagon.

inequalities of three different subscripts. Thus collapsing one of them introduces a complementary vertex on $\mathscr{P}(M)$. ■

THEOREM. *The set of* 3×3 *matrices in* R_0 *has seven connected components, represented by the seven matrices given in Lemmas* 4.1–4.4.

Proof. From Lemma 4.5, we see that the four matrices of Lemma 4.4 lie in separate components of R_0 that are also separate from the components containing the matrices M^-, P, and P^-. Using techniques from [4] or [7], one can calculate the degree of the mappings associated to these three matrices. For M^- the answer is 0, for P it is 1, and for P^- it is -1. Thus these three matrices represent separate components of R_0. For the record, the matrices M_{21}, M_{22}, and M_{23} are associated with maps of degree 2, while MU, given by Murty in [9], gives a map of degree -2. ■

5. BEYOND $n = 3$

The techniques used in this paper give some idea of the complexity of the set of matrices in R_0. It seems possible that for all n, the set of matrices in R_0 for given small values of the degree might be connected. The techniques used in this paper could probably be elaborated on to obtain results for 4×4 matrices. In the 4×4 case one would have to look at three-dimensional polytopes, and the analysis might be similar to the proof of Steinitz's theorem for three-polytopes. It would be interesting to find out if the negative results on the isotopy problem (see [2]) would pose problems for $n \geqslant 5$.

REFERENCES

1 M. Aganagic and R. W. Cottle, A note on Q-matrices, *Math. Programming* 16:374–377, (1979).
2 J. Bokowski and A. G. de Oliveira, Isotopy Does Not Hold for Simplicial Convex 4-Polytopes, Manuscript, Fachbereich Mathematik, Technische Hochschule Darmstadt, D-6100 Darmstadt, F.R. Germany, 1988.
3 B. Grunbaum, *Arrangements and Spreads*, Regional Conf. Ser. in Math. 10, Amer. Math. Soc., 1972.
4 C. B. Garcia, F. J. Gould, and T. R. Turnbull, Relationships between PL maps, complementary cones, and degree in linear complementarity problems, in *Homotopy Methods and Global Convergence* (B. C. Eaves, F. J. Gould, H. O. Peitgen, and M. J. Todd, Eds.), Plenum, New York, 1983, pp. 91–144.

5 M. S. Gowda and J.-S. Pang, On Solution Stability of the Linear Complementarity Problem, Manuscript, Dept. of Mathematics and Statistics, Univ. of Maryland—Baltimore County, Baltimore, Md. 21228, 1989. *Mathematics of Operations Research*, to appear.

6 C. D. Ha, Application of degree theory in stability of the complementarity problem, *Math. Oper. Res.* 12:368–376 (1987).

7 R. Howe and R. Stone, Linear complementarity and the degree of mappings, in *Homotopy Methods and Global Convergence* (B. C. Eaves, F. J. Gould, H. O. Peitgen, and M. J. Todd, Eds.), Plenum, New York, 1983, pp. 179–223.

8 W. Morris, On the maximum degree of an LCP mapping, *Math. Oper. Res.*, 15: 423–429 (1990).

9 K. G. Murty, On the number of solutions to complementarity problems and the spanning properties of complementary cones, *Linear Algebra Appl.* 5:65–108 (1972).

Received 25 July 1990; final manuscript accepted 7 August 1991