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Symmetry and Positive Definiteness in Oriented Matroids

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We show the properties of a class of oriented matroids that properly generalizes the class of oriented matroids that can be represented by matrices of the form $(\mathbf{I}, -\mathbf{A})$, where $\mathbf{A}$ is a real $n \times n$ symmetric matrix and $\mathbf{I}$ is the $n \times n$ identity matrix. Several results from linear algebra about positive definiteness, symmetry, and eigenvalues are shown to have natural generalizations in the context of oriented matroids. The relationships among the oriented matroid generalizations of the linear algebraic concepts are seen to be analogous to the relationships among the original linear algebraic concepts.

INTRODUCTION

Let E be a finite set. A *signed set* in E is a pair $X = (X^+, X^-)$ with $X^+ \subseteq E$, $X^- \subseteq E$, and $X^+ \cap X^- = \emptyset$. The *opposite* of X is the signed set $-X = (X^-, X^+)$, and the set *underlying* X is $\underline{X} = X^+ \cup X^-$. A signed set X *contains* a signed set Y, $Y \subseteq X$, if $Y^+ \subseteq X^+$, $Y^- \subseteq X^-$, and X *contains* an unsigned set Z if $Z \subseteq \underline{X}$.

An oriented matroid $\mathcal{M}$ is a pair $(E, \mathcal{C})$, where E is a finite set and $\mathcal{C}$ is a collection of signed sets in E, called circuits, satisfying

$$C_1 \begin{cases} (\emptyset, \emptyset) \notin \mathcal{C} \\ C \in \mathcal{C} \Rightarrow -C \in \mathcal{C} \\ C_1, C_2 \in \mathcal{C} \text{ and } \underline{C_1} \subseteq \underline{C_2} \Rightarrow C_1 = C_2 \text{ or } C_1 = -C_2. \end{cases}$$

$$C_2 \begin{cases} C_1, C_2 \in \mathcal{C} \text{ and } e \in (C_1^+ \cap C_2^-) \cup (C_1^- \cap C_2^+) \text{ imply that there exists} \\ C_3 \in \mathcal{C} \text{ with } C_3^+ \subseteq (C_1^+ \cup C_2^+) \backslash \{e\},\ C_3^- \subseteq (C_1^- \cup C_2^-) \backslash \{e\}. \end{cases}$$

We write $\mathcal{C} = \mathcal{C}(\mathcal{M})$.

The collection $\underline{\mathcal{C}} = \{\underline{C}: C \in \mathcal{C}\}$ of underlying sets of circuits is the family of circuits of an (unoriented) matroid $\underline{\mathcal{M}}$ on E, called the underlying matroid of $\mathcal{M}$. We will assume that the reader is familiar with the basic definitions and properties of matroids (see [13]). The independent and dependent sets, bases, loops and coloops, and rank of $\mathcal{M}$ are those of the underlying matroid $\underline{\mathcal{M}}$.

A signed set X in E has a *conformal decomposition* into circuits of an oriented matroid if $X^+ = C_1^+ \cup \ldots \cup C_m^+$ and $X^- = C_1^- \cup \ldots \cup C_m^-$ for circuits $C_1, \ldots, C_m$ in $\mathcal{C}(\mathcal{M})$. We called signed sets having conformal decomposition into circuits of $\mathcal{M}$ *cycles* of $\mathcal{M}$. We write $\mathcal{K}(\mathcal{M})$ for the set of cycles of $\mathcal{M}$.

The main motivation for oriented matroid theory comes from the case when the cycles of an oriented matroid are signed supports of vectors in a subspace of $\mathbb{R}^n$. Let V be a subspace of $\mathbb{R}^n$, and let $E = \{e_1, \ldots, e_n\}$. For every $\boldsymbol{x} \in V$, define a signed set $K_{\boldsymbol{x}} = (\{e_i: x_i > 0\}, \{e_i: x_i < 0\})$, called the *signed support* of $\boldsymbol{x}$. Then the set $\{K_{\boldsymbol{x}}: \boldsymbol{x} \in V\}$ is the set of cycles of an oriented matroid. If V is the null space of an $m \times n$ matrix $\mathbf{A}$, we call this oriented matroid the oriented matroid *represented by* $\mathbf{A}$, written $\mathcal{M}(\mathbf{A})$. Thus $\mathcal{M}(\mathbf{A})$ captures sign properties of vectors in the null space of $\mathbf{A}$, but ignores their numerical values.

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Oriented matroids give an abstract combinatorial setting for studying the properties of matrices that represent them. On the other hand, since there are oriented matroids that cannot be represented by any matrix, the results obtained properly generalize those of linear algebra. Previously, notions of arrangements of hyperplanes and topological extensions, face lattices and polarity of polyhedra, convexity, and signs of determinants of matrices have been studied in the context of oriented matroids [1], [3], [7], [8], [9]. Linear and quadratic programming have been shown to have natural generalizations in the context of oriented matroids, which capture many of the important features of these problems [2], [5], [6], [12]. In the study of quadratic programming for oriented matroids, one would like to have generalizations of the properties of quadratic forms associated with symmetric matrices. This is the subject of the present paper.

We would like to have oriented matroid generalizations of the notions of symmetry, positive (semi)definiteness, and eigenvalues. These generalizations should ideally satisfy many properties that the original linear algebraic notions satisfy. In particular:

1. Oriented matroids represented by matrices of the form $(\mathbf{I}, -\mathbf{A})$, where $\mathbf{A}$ is square and symmetric positive (semi)definite are symmetric and positive (semi)definite. Note that $y = \mathbf{A}x$ iff (y, x) is in the null space of $(\mathbf{I}, -\mathbf{A})$. Therefore, to study the sign properties of y and x, we consider the oriented matroid $\mathcal{M}(\mathbf{I}, -\mathbf{A})$.
2. Certain minors of symmetric (positive (semi)definite) oriented matroids, corresponding to principal submatrices in the representable case, are symmetric (positive (semi)definite) oriented matroids.
3. The inverse of a symmetric (positive definite) oriented matroid is symmetric (positive definite) iff it exists.
4. A symmetric positive semidefinite oriented matroid is positive definite iff it is non-singular.
5. A symmetric positive definite oriented matroid has a positive eigencycle, and a symmetric positive semidefinite oriented matroid has a positive eigencycle iff it is not representable by the matrix $(\mathbf{I}, \mathbf{0})$.
6. A symmetric oriented matroid is positive semidefinite iff it has no negative eigencycles.

These properties will all be shown to hold for natural definitions of the concepts for oriented matroids.

2. Preliminaries

Let $\mathcal{M}$ be an oriented matroid on a set E. If $\mathcal{M}$ is represented by a matrix $\mathbf{A}$, then $\mathcal{M}^*$, the oriented matroid that has as its cycles the signed supports of vectors in the row space of $\mathbf{A}$, is called the *dual* of $\mathcal{M}$. The null space of $\mathbf{A}$ and the row space of $\mathbf{A}$ are orthogonal complements, so $\mathcal{M}$ and $\mathcal{M}^*$ must satisfy the following *orthogonality* property;

($\perp$) For cycles K of $\mathcal{M}$ and L of $\mathcal{M}^*$
$(K^+ \cap L^+) \cup (K^- \cap L^-) \neq \varnothing$ iff $(K^+ \cap L^-) \cup (K^- \cap L^+) \neq \varnothing$.

In general, for every oriented matroid $\mathcal{M}$ on E, there is a unique oriented matroid $\mathcal{M}^*$ on E such that the cycles of $\mathcal{M}^*$ are all the signed sets on E that satisfy ($\perp$) with respect to all the cycles of $\mathcal{M}$. $\mathcal{M}^*$ is the dual of $\mathcal{M}$ (see [3]).

Let $\mathcal{M}$ be an oriented matroid on E with $\mathcal{K} = \mathcal{K}(\mathcal{M})$ its set of cycles. Let F and G be disjoint subsets of E. Let $\mathcal{K}(F) = \{(K^+ \backslash F) \cup (K^- \cap F), (K^- \backslash F) \cup (K^+ \cap F)) : K \in \mathcal{K}\}$. Then $\mathcal{K}(F)$ is the family of cycles of an oriented matroid; we say that oriented matroid is obtained by *reversing signs on* F. For a signed set X on E, and any subset H of E, let the signed set $X \backslash H$ be $(X^+ \backslash H, X^- \backslash H)$. Let $\mathcal{K} \backslash F/G$ denote the collection of signed sets $\{K \backslash G : K \in \mathcal{K}, \underline{K} \cap F = \varnothing\}$. Then $\mathcal{K} \backslash F/G$ is the set of cycles of an oriented matroid

$\mathscr{M}\backslash F/G$ on $E\backslash(F \cup G)$ [3, p. 114]. For the sake of brevity, $\mathscr{M}\backslash F(\mathscr{M}/G)$ is written for $\mathscr{M}\backslash F/\varnothing(\mathscr{M}\backslash\varnothing/G)$. We say $\mathscr{M}\backslash F/G$ is obtained from $\mathscr{M}$ *by deleting* F *and contracting* G. If $\mathscr{M}$ is represented by a matrix $\mathbf{A}$, then $\mathscr{M}\backslash F/G$ is represented by the matrix $\mathbf{A}'$, obtained from $\mathbf{A}$ by deleting the columns corresponding to F, then pivoting on successive nonzero columns corresponding to G, followed by deleting the pivot row and column, and finally deleting zero columns corresponding to G.

If B is a base of $\mathscr{M}$ and $e \in E\backslash B$, then there is exactly one circuit C of $\mathscr{M}$ with $e \in C^+$ and $C \subseteq B \cup \{e\}$. It is called the *fundamental circuit* $C(B, e)$ associated with B and e. For every base B of $\mathscr{M}$, define the *B-tableau* to be the set of fundamental circuits of $\mathscr{M}$ associated with e_i and B for $e_i \notin B$.

If K_1, K_2 are two cycles of an oriented matroid $\mathscr{M}$, then $K_1 \circ K_2 \equiv (K_1^+ \cup (K_2^+\backslash K_1^-), K_1^- \cup (K_2^-\backslash K_1^+)) \in \mathscr{K}(\mathscr{M})$. This operator $\circ$, Bland's *composition* operator, corresponds in the representable case to taking the signed support of $x^1 + \varepsilon x^2$ for some suitably small $\varepsilon > 0$, where K_1 and K_2 are the signed supports of x^1 and x^2, respectively.

Henceforth, we make the following blanket assumption:

$\mathscr{M}$ is an oriented matroid on a set $E = S \cup T$, where $S = \{s_1, \ldots, s_n\}$, $T = \{t_1, \ldots, t_n\}$, $S \cap T = \varnothing$, and S is a base of $\mathscr{M}$. Such an oriented matroid is called *square*.

The particular ordering of S and T is significant. We define the *switch* of a signed set X on E to be the signed set $\mathrm{sw}X = (\{s_i\colon t_i \in X^-\} \cup \{t_i\colon s_i \in X^+\}, \{s_i\colon t_i \in X^+\} \cup \{t_i\colon s_i \in X^-\})$, obtained by reversing signs on T and then interchanging occurrences of s_i and t_i in X for all $i = 1, \ldots, n$. The switch of an oriented matroid, $\mathrm{sw}\mathscr{M}$, is the matroid that has as its cycles the switches of the cycles of $\mathscr{M}$. The (S, T)-transpose of $\mathscr{M}$, denoted $\mathscr{M}^{\mathrm{t}}_{ST}$, is the switch of $\mathscr{M}^*$. This is motivated by observing that if $\mathscr{M}$ is represented by the matrix $(\mathbf{I}, -\mathbf{A})$, where S corresponds to the columns of $\mathbf{I}$ and T to the columns of $-\mathbf{A}$, then $\mathscr{M}^*$ is represented by $(\mathbf{A}^{\mathrm{t}}, \mathbf{I})$ and $\mathscr{M}^{\mathrm{t}}_{ST}$ by $(\mathbf{I}, -\mathbf{A}^{\mathrm{t}})$.

DEFINITION. A square oriented matroid $\mathscr{M}$ is symmetric (with respect to S and T) iff $\mathscr{M} = \mathscr{M}^{\mathrm{t}}_{ST}$.

It is possible to replace this indirect definition with another one that does not explicitly involve the dual oriented matroid.

THEOREM 2.1. *A square oriented matroid* $\mathscr{M}$ *is symmetric (with respect to* S *and* T*) iff the following property holds:*
$(\perp)'$: *For any two cycles* K_1 *and* K_2 *of* $\mathscr{K}(\mathscr{M})$, *(I) holds iff (II) holds.*

(I)	*(II)*
There exists an i such that one of the cases below holds:	*There exists a j such that one of the cases below holds:*
$s_i \in K_1^+,\ t_i \in K_2^-$;	$s_j \in K_1^+,\ t_j \in K_2^+$;
$s_i \in K_1^-,\ t_i \in K_2^+$;	$s_j \in K_1^-,\ t_j \in K_2^-$;
$t_i \in K_1^+,\ s_i \in K_2^+$;	$t_j \in K_1^+,\ s_j \in K_2^-$;
$t_i \in K_1^-,\ s_i \in K_2^-$.	$t_j \in K_1^-,\ s_j \in K_2^+$.

PROOF. The condition $(\perp)'$ is equivalent to the requirement that $\mathscr{M}$ and $\mathrm{sw}\mathscr{M}$ satisfy $(\perp)$. If $\mathscr{M}$ and $\mathrm{sw}\mathscr{M}$ satisfy $(\perp)$, then the cycles of $\mathrm{sw}\mathscr{M}$ are cycles of $\mathscr{M}^*$, and independent sets of $\mathscr{M}^*$ are independent in $\mathrm{sw}\mathscr{M}$. In particular, bases of $\mathscr{M}^*$ are bases of $\mathrm{sw}\mathscr{M}$, since both oriented matroids are of rank n. $\mathscr{M}$ and $\mathrm{sw}\mathscr{M}$ have the same number of bases, as do $\mathscr{M}$ and $\mathscr{M}^*$. Therefore, the bases of $\mathscr{M}^*$ are exactly the bases of $\mathrm{sw}\mathscr{M}$, and $\mathscr{M}^*$ and $\mathrm{sw}\mathscr{M}$ have the same underlying matroid. From [3], we then get that $\mathrm{sw}\mathscr{M} = \mathscr{M}^*$, implying that

$\mathscr{M} = \mathrm{sw}\mathscr{M}^* = \mathscr{M}^{\iota}_{ST}$. Clearly, if $\mathscr{M} = \mathscr{M}^{\iota}_{ST}$, then $\mathrm{sw}\mathscr{M} = \mathscr{M}^*$, and thus $\mathscr{M}$ and $\mathrm{sw}\mathscr{M}$ satisfy $(\perp)$.

The following theorem is an easy application of $(\perp)'$. We call a subset F of $E = S \cup T$, where $S = \{s_1, \ldots, s_n\}$, $T = \{t_1, \ldots, t_n\}$, $S \cap T = \varnothing$, *complementary* iff $|F \cap \{s_i, t_i\}| \leqslant 1$ for $i = 1, \ldots, n$.

THEOREM 2.2. *Let B be a complementary base of $\mathscr{M}$, a symmetric oriented matroid. Then the following hold*:

(*a*) *If* $s_i, s_j \in B$, *then* $s_j \in C(B, t_i)^+$ *iff* $s_i \in C(B, t_j)^+$, *and*
$s_j \in C(B, t_i)^-$ *iff* $s_i \in C(B, t_j)^-$.

(*b*) *If* $s_i, t_j \in B$, *then* $s_i \in C(B, s_j)^+$ *iff* $t_j \in C(B, t_i)^-$, *and*
$s_i \in C(B, s_j)^-$ *iff* $t_j \in C(B, t_i)^+$.

(*c*) *If* $t_i, t_j \in B$, *then* $t_i \in C(B, s_j)^+$ *iff* $t_j \in C(B, s_i)^+$, *and*
$t_i \in C(B, s_j)^-$ *iff* $t_j \in C(B, s_i)^-$.

COROLLARY 2.2.1. *The S-tableau of a symmetric oriented matroid is symmetric*:

$$s_j \in C(S, t_i)^+ \text{ iff } s_i \in C(S, t_j)^+ \text{ and } s_j \in C(S, t_i)^- \text{ iff } s_i \in C(S, t_j)^-.$$

This is a special case of Theorem 2.2, with $B = S$. When $\mathscr{M}$ is represented by $(\mathbf{I}, -\mathbf{A})$, with $\mathbf{A}$ symmetric, the circuits of the S-tableau are the signed sets corresponding to the rows of the matrix $(\mathbf{A}^t, \mathbf{I})$. This corollary, then, generalizes the property that a_{ij} and a_{ji} agree in sign, for all i, j.

If T is a base of a square oriented matroid $\mathscr{M}$, then we will call $\mathscr{M}$ *nonsingular*. The observation that the matrices $(\mathbf{I}, -\mathbf{A})$ and $(-\mathbf{A}^{-1}, \mathbf{I})$ have the same null spaces, for nonsingular $\mathbf{A}$, prompts the following definition. Let $\mathscr{M}$ be a nonsingular oriented matroid. The (S, T) inverse of $\mathscr{M}$, $\mathscr{M}^{-1}_{ST}$, is the oriented matroid obtained from $\mathscr{M}$ by interchanging the labels s_i and t_i, for all i.

PROPOSITION 2.3. *Let $\mathscr{M}$ be a square nonsingular oriented matroid. Then $\mathscr{M}^{-1}_{ST}$ is symmetric iff $\mathscr{M}$ is.*

PROOF. It is easily seen that $\mathscr{M}^{-1}_{ST}$ satisfies $(\perp)'$ iff $\mathscr{M}$ does, and since T is a base of $\mathscr{M}$, we have that S is a base of $\mathscr{M}^{-1}_{ST}$.

Principal submatrices of symmetric oriented matrices are also symmetric. It is therefore natural to expect that the corresponding minors of symmetric oriented matroids would stay symmetric.

THEOREM 2.4. *Let $\mathscr{M}$ be a symmetric oriented matroid. For any i, $1 \leqslant i \leqslant n$, $\mathscr{M}\backslash t_i/s_i$ is a symmetric oriented matroid with respect to $S\backslash s_i$ and $T\backslash t_i$, and if $s_i \in \underline{C}(S, t_i)$, then so is $\mathscr{M}\backslash s_i/t_i$.*

PROOF. The set $S\backslash s_i$ is easily seen to be a base of $\mathscr{M}\backslash t_i/s_i$. Furthermore, we have $\mathrm{sw}(\mathscr{M}\backslash t_i/s_i)^* = [\mathrm{sw}(\mathscr{M}\backslash t_i/s_i)]^* = [(\mathrm{sw}\mathscr{M})\backslash s_i/t_i]^* = (\mathrm{sw}\mathscr{M})^*\backslash t_i/s_i = \mathscr{M}\backslash t_i/s_i$. This proves the first part. For the second, note that $T\backslash t_i$ is a base of $\mathscr{M}\backslash s_i/t_i$ iff $s_i \in \underline{C}(S, t_i)$. As before, we have $\mathrm{sw}(\mathscr{M}\backslash s_i/t_i)^* = [\mathrm{sw}(\mathscr{M}\backslash t_i/s_i)]^* = [(\mathrm{sw}\mathscr{M})\backslash s_i/t_i]^* = (\mathrm{sw}\mathscr{M})^*\backslash t_i/s_i = \mathscr{M}\backslash t_i/s_i$.

All oriented matroids that can be represented by matrices of the form $(\mathbf{I}, -\mathbf{A})$, where $\mathbf{A}$ is symmetric, are symmetric. The class of symmetric oriented matroids is much larger than this, though. As an example, we give a symmetric orientation of the nonrepresentable

Vamos Matroid. Consider the oriented matroid represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -4 & 5 \\ 0 & 0 & 0 & 1 & -1 & -1 & 5 & -6 \end{pmatrix}.$$

The circuits of this oriented matroid are those given below and their negatives.

$1\overline{2}\overline{5}6$	$1\bar{3}567$	$1\overline{2}\bar{3}67$	$\bar{1}\overline{2}\bar{3}57$	$\bar{1}24\overline{67}$	$\bar{3}4\overline{567}$	$\bar{3}5678$
$16\overline{78}$	$1\bar{3}56\bar{8}$	$1\overline{2}\bar{3}6\bar{8}$	$\bar{1}\overline{2}\bar{3}5\bar{8}$	$\bar{1}24\bar{6}8$	$\bar{3}4\overline{56}\bar{8}$	$\overline{2\bar{3}}678$
$2\bar{3}45$	$\bar{3}478$	$1\bar{3}4\overline{57}$	$\bar{1}\overline{3}578$	$1\bar{2}\overline{4\bar{5}7}$	$2\bar{3}4\overline{67}$	$\bar{2}4\overline{567}$
$25\overline{78}$	$1\bar{3}46$	$1\bar{3}4\bar{5}8$	$\overline{1\bar{2}\bar{3}}78$	$1\bar{2}4\bar{5}8$	$2\bar{3}4\bar{6}8$	$\bar{2}4\overline{56}\bar{8}$
		$12\bar{3}\bar{4}7$	$\bar{1}4\overline{567}$	$14\overline{578}$	$2\bar{3}567$	$4\overline{5678}$
		$12\bar{3}48$	$\bar{1}4\overline{56}\bar{8}$	$1247\overline{78}$	$2\bar{3}56\bar{8}$	$24\overline{678}$

Here a circuit $(\{1, 6\}, \{2, 5\})$ is written $1\overline{2}\overline{5}6$. A symmetric orientation of the Vamos matroid is then obtained by replacing the circuit $25\overline{78}$ with the circuits $\bar{1}25\overline{78}$, $235\overline{78}$, $256\overline{78}$, and $245\overline{78}$. The result is an orientation that is symmetric with respect to $S = (1, 2, 3, 4)$, $T = (5, 6, 7, 8)$. By relabeling the elements $(1, 2, 3, 4, 5, 6, 7, 8)$ as $(1, 3, 6, 4, 5, 2, 7, 8)$, one can see that the underlying matroid is the same as that given in [3, p. 111]. For a more detailed explanation, see [10].

3. Positive Definiteness and Eigenvalues

3.1. Positive Definiteness

All of the oriented matroids in this section will be symmetric unless otherwise noted. The definition of positive definiteness in oriented matroids given below, however, applies to any square oriented matroid. An oriented matroid $\mathcal{M}$, represented by $(\mathbf{I}, -\mathbf{A})$ where $\mathbf{A}$ is symmetric, will be positive (semi)definite iff $\mathbf{A}$ is. The correspondence is not as direct when $\mathbf{A}$ is not symmetric. For example, the same symmetric oriented matroid is represented by both $\left(\begin{smallmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & -4 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & -2 \end{smallmatrix}\right)$, even though $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is positive definite and $\left(\begin{smallmatrix} 5 & 1 \\ 4 & 1 \end{smallmatrix}\right)$ is not. The set of matrices $\mathbf{A}$ for which $\mathcal{M}(\mathbf{I}, -\mathbf{A})$ is symmetric and positive (semi)definite forms a proper subset of the set of P-matrices (P_0-matrices) [4]; a matrix is in $P(P_0)$ iff all its principal minors are positive (nonnegative).

If a vector space V is the null space of an $n \times 2n$ matrix $(\mathbf{I}, -\mathbf{A})$, then V is the space of solutions $(\boldsymbol{y}, \boldsymbol{x})$ to the equation $(\mathbf{I}, -\mathbf{A})(\boldsymbol{y}, \boldsymbol{x}) = 0$, which is $\boldsymbol{y} = \mathbf{A}\boldsymbol{x}$. The matrix $\mathbf{A}$ is positive definite iff $\boldsymbol{x}^t\boldsymbol{y} > 0$ for all nonzero $(\boldsymbol{y}, \boldsymbol{x}) \in V$, and positive semidefinite iff $\boldsymbol{x}^t\boldsymbol{y} \geqslant 0$ for all $(\boldsymbol{y}, \boldsymbol{x}) \in V$.

Definitions. A cycle K of a symmetric oriented matroid $\mathcal{M}$ is called *sign reversing* if $\{s_i, t_i\} \not\subseteq K^+$ and $\{s_i, t_i\} \not\subseteq K^-$ for every i. K is called *strictly sign reversing* if, in addition, $s_i \in K^+$, $t_i \in K^-$ or $s_i \in K^-$, $t_i \in K^+$ for some i. $\mathcal{M}$ is called *positive definite* if it has no sign reversing cycles. $\mathcal{M}$ is called *positive semidefinite* if it has no strictly sign reversing cycles.

Note that if $\mathcal{M}$ is represented by $(\mathbf{I}, -\mathbf{A})$, then a sign reversing cycle implies the existence of a vector $(\boldsymbol{y}, \boldsymbol{x})$ in the null space of $(\mathbf{I}, -\mathbf{A})$ such that $\boldsymbol{y}_i\boldsymbol{x}_i \leqslant 0$ for all i. A strictly sign reversing cycle implies that, in addition $\boldsymbol{y}_i\boldsymbol{x}_i < 0$ for some i. Thus, if $\mathbf{A}$ is positive (semi)definite, then $\mathcal{M}$ is positive (semi)definite. The converse holds when $\mathbf{A}$ is symmetric.

If a symmetric $\mathbf{A}$ is not positive (semi)definite, then it has an eigenvector $\boldsymbol{x}$ corresponding to a nonpositive (negative) eigenvalue. The signed support of the vector $(\mathbf{A}\boldsymbol{x}, \boldsymbol{x})$ will then be a (strictly) sign reversing cycle of $\mathcal{M}$.

LEMMA 3.1.1. *If $\mathcal{M}$ is a symmetric positive semidefinite oriented matroid, then $s_i \notin \underline{C}(S, t_i)$ implies that $s_j \notin \underline{C}(S, t_i)$ for all j, so that t_i is a loop.*

PROOF. Suppose $s_j \in C(S, t_i)^+$ and $s_i \notin \underline{C}(S, t_i)$ for some i, j. Then $s_i \in C(S, t_j)^+$ by the symmetry of the S-tableau. Hence $K = C(S, t_i) \circ -C(S, t_j)$ has $\{s_i, t_j\} \subseteq K^-$, $\{s_j, t_i\} \subseteq K^+$, and $\{s_k, t_k\} \nsubseteq K$ for all other k. Then K is strictly sign reversing, contradicting the assumption that $\mathcal{M}$ is positive semidefinite. A strictly sign reversing cycle can be constructed in an analogous way if $s_j \in C(S, t_i)^-$ for some j.

COROLLARY 3.1.2. *Let $\mathcal{M}$ be a symmetric, positive semidefinite, nonsingular oriented matroid. Then $s_i \in \underline{C}(S, t_i)$ and $t_i \in \underline{C}(T, s_i)$ for all i.*

PROOF. If $s_i \notin \underline{C}(S, t_i)$ for some i, then t_i is a loop, contradicting the assumption that T is a base. If $\mathcal{M}$ is nonsingular, it is immediate that $\mathcal{M}_{ST}^{-1}$ is a symmetric positive semidefinite oriented matroid. Thus, if $t_i \notin \underline{C}(T, s_i)$ in $\mathcal{M}$, then $s_i \notin \underline{C}(S, t_i)$ in $\mathcal{M}_{ST}^{-1}$, which is impossible.

LEMMA 3.1.3. *If $\mathcal{M}$ is a positive (semi)definite symmetric oriented matroid, then $\mathcal{M}\backslash t_i/s_i$ is a positive (semi)definite symmetric oriented matroid for all i, and if $s_i \in \underline{C}(S, t_i)$, then $\mathcal{M}\backslash s_i/t_i$ is a positive (semi)definite symmetric oriented matroid.*

PROOF. By Theorem 2.2.1, $\mathcal{M}\backslash t_i/s_i$ is a symmetric oriented matroid for any i, and if $s_i \in \underline{C}(S, t_i)$, then $\mathcal{M}\backslash s_i/t_i$ is a symmetric oriented matroid. Suppose that $\mathcal{M}\backslash t/s_i$ has a (strictly) sign reversing cycle K; then this cycle, together with a possible additional element s_i, is a (strictly) sign reversing cycle of $\mathcal{M}$. The same argument applies to (strictly) sign reversing cycles of $\mathcal{M}\backslash s_i/t_i$.

THEOREM 3.1.4. *A positive semidefinite symmetric oriented matroid is positive definite iff it is nonsingular.*

PROOF. One implication is trivial. If $\mathcal{M}$ has a circuit C contained in T, then $\{s_i, t_i\} \nsubseteq C$ for all i, so C is a sign reversing cycle. The opposite implication is proved by induction on $|S|$. Suppose that Theorem 3.1.4 is true for symmetric oriented matroids with $|S| \leqslant n - 1$. Suppose also that $\mathcal{M}$ is a nonsingular symmetric positive semidefinite oriented matroid with $|S| = n$. If $\mathcal{M}$ is not positive definite, then there exists a nonempty cycle K of $\mathcal{M}$ such that $\{s_i, t_i\} \nsubseteq \underline{K}$ for all i. T is a base of $\mathcal{M}$, so there is an i such that $t_i \notin \underline{K}$. The oriented matroid $\mathcal{M}\backslash t_i/s_i$ is symmetric and positive semidefinite, by Lemma 3.1.3. T is a base of $\mathcal{M}$, and by Corollary 3.1.2, we have $t_i \in \underline{C}(T, s_i)$. Therefore, there can be no circuit of $\mathcal{M}\backslash t_i/s_i$ in $T\backslash t_i$, so $T\backslash t_i$ is a base of $\mathcal{M}\backslash t_i/s_i$. By the induction hypothesis, $\mathcal{M}\backslash t_i/s_i$ is positive definite. However, $K \nsubseteq \{s_i\}$, since S is a base of $\mathcal{M}$, so $K\backslash\{s_i, t_i\}$ is a nonempty cycle of $\mathcal{M}\backslash t_i/s_i$. Moreover, $K\backslash\{s_i, t_i\}$ is sign reversing. Thus we have a contradiction. To establish the basis for the induction, note that there are three symmetric oriented matroids with $|S| = 1$. The oriented matroid represented by $(1, 0)$ is positive semidefinite, singular, and not positive definite. $\mathcal{M}(1, -1)$ is both positive definite and positive semidefinite, and it is nonsingular. $\mathcal{M}(1, 1)$ is neither positive definite nor positive semidefinite.

From this result we can also conclude that if $\mathcal{M}$ is a symmetric nonsingular oriented matroid, then $\mathcal{M}$ is positive definite iff $\mathcal{M}_{ST}^{-1}$ is positive definite.

The theorem and lemmas from this section correspond to well known theorems about symmetric matrices. If $\mathbf{A}$ is a positive semidefinite symmetric matrix and $a_{ii} = 0$ for some i, then $a_{ij} = a_{ji} = 0$ for all j (Lemma 3.1.1). If $\mathbf{A}$ is positive definite, then so is $\mathbf{A}^{-1}$, and the diagonal elements of both $\mathbf{A}$ and $\mathbf{A}^{-1}$ are positive (Lemma 3.1.2). Deleting row i and column i from a symmetric positive (semi)definite matrix $\mathbf{A}$ yields a symmetric positive (semi)definite matrix, and pivoting on a nonzero diagonal element of $\mathbf{A}$ followed by deleting the pivot row and column yields another symmetric positive (semi)definite matrix (Lemma 3.1.3). Finally, a symmetric positive (semi)definite matrix is positive definite iff it is nonsingular (Theorem 3.1.4).

3.2. Eigencycles

Let $\mathbf{A}$ be a real $n \times n$ matrix. A real number λ is an eigenvalue of A iff there is a nonzero $\boldsymbol{x} \in \mathbb{R}^n$ such that $\lambda \boldsymbol{x} = \mathbf{A}\boldsymbol{x}$, or equivalently, iff there is a nonzero $\boldsymbol{x}$ such that $(\mathbf{I}, -\mathbf{A})(\lambda \boldsymbol{x}, \boldsymbol{x}) = 0$. This fact motivates the following definitions. Let $\mathscr{M}$ be an oriented matroid on $E = S \cup T$, $S = \{s_1, \ldots, s_n\}$, $T = \{t_1, \ldots, t_n\}$, S a base of $\mathscr{M}$. A nonempty cycle K of $\mathscr{M}$ is called a *positive eigencycle* of $\mathscr{M}$ iff we have $s_i \in K^+ \Leftrightarrow t_i \in K^+$ and $s_i \in K^- \Leftrightarrow t_i \in K^-$ for all i. K is called a *negative eigencycle* iff we have $s_i \in K^+ \Leftrightarrow t_i \in K^-$ and $s_i \in K^- \Leftrightarrow t_i \in K^+$. K is a *zero eigencycle* iff $\underline{K} \subseteq T$. A positive (negative, zero) eigencycle K will be called a *minimal eigencycle* if there is no positive (negative, zero) eigencycle K' contained in K and different from it. In the representable case, where $\mathscr{M} = M(\mathbf{I}, -\mathbf{A})$, if $(\boldsymbol{y}, \boldsymbol{x})$ is in the null space of $(\mathbf{I}, -\mathbf{A})$, $\boldsymbol{x}$ must have the same signed support as $\boldsymbol{y}$ for $\boldsymbol{x}$ to be a positive eigenvector. This is, of course, not a sufficient condition. Consider, for example, the matrix

$$\mathbf{A} = \begin{pmatrix} 19 & 10 & 4 \\ 10 & 10 & -14 \\ 4 & -14 & 7 \end{pmatrix}$$

The eigenvectors of this matrix are $(1, -2, -2)$, $(-2, 1, -2)$, and $(-2, -2, 1)$, corresponding to the eigenvalues -9, $+18$, and $+27$, respectively. The vector $(52, 16, 1, 2, 1, 1)$ is in the null space of $(\mathbf{I}, -\mathbf{A})$. Thus $(\{s_1, s_2, s_3, t_1, t_2, t_3\}, \varnothing)$ is a positive eigencycle of $\mathscr{M}(\mathbf{I}, -\mathbf{A})$. There are, however, no eigenvectors of $\mathbf{A}$ that are positive on all coordinates. These qualifications must be kept in mind when one uses the term eigencycle.

Theorem 3.2.1. *If $\mathscr{M}$ is a symmetric positive definite oriented matroid, then it has a positive eigencycle.*

Proof. Consider the fundamental circuit $C(S, t_i)$. $\mathscr{M}$ is positive definite, so for each i, $s_i \in C(S, t_i)^+$ (otherwise $C(S, t_i)$ would be sign reversing). The following algorithm produces a positive eigencycle K:

(0) $K \leftarrow C(S, t_1)$, $i \leftarrow 1$
(1) If $i = n$, stop
  If $s_{i+1} \in K^-$, then $K \leftarrow K \circ -C(S, t_{i+1})$
    else $K \leftarrow K \circ C(S, t_{i+1})$
    $i \rightarrow i + 1$
    go to (1)

In step (1), K does not contain t_{i+1}, though it may contain s_{i+1}. The composition in step (1) ensures that $\{s_{i+1}, t_{i+1}\} \subseteq K^+$ or $\{s_{i+1}, t_{i+1}\} \subseteq K^-$. The algorithm will stop with a positive eigencycle K such that $\underline{K} = E$.

The proof must be modified for the positive semidefinite case, for it may happen that $s_i \notin \underline{C}(S, t_i)$ for some i.

THEOREM 3.2.2. *If $\mathscr{M}$ is a symmetric positive semidefinite oriented matroid, and there is an element of T that is not a loop of $\mathscr{M}$, then $\mathscr{M}$ has a positive eigencycle.*

PROOF. Let t_i be an element of T that is not a loop of $\mathscr{M}$. Then $s_i \in C(S, t_i)^+$ by Lemma 3.1.1. The following algorithm will produce a positive eigencycle:

(0) $K \leftarrow C(S, t_i)$
(1) If there is a j such that $s_j \in \underline{K}$ but $t_j \notin \underline{K}$ then do:
    If $s_j \in K^+$, let $K \leftarrow K \circ C(S, t_j)$
    If $s_j \in K^-$, let $K \leftarrow K \circ -C(S, t_j)$
    go to (1)
    Else stop

At each step of the algorithm, a t_j is admitted into K so that it has the same sign as s_j, which is already in K.

Note that if every element of T is a loop, then every cycle of $\mathscr{M}$ is in T, so there can be no positive eigencycle. In that case $\mathscr{M}$ is represented by $(\mathbf{I}, \mathbf{0})$.

From the definition, a symmetric oriented matroid with a negative eigencycle is not positive semidefinite. In fact, the converse is also true.

THEOREM 3.2.3. *A symmetric oriented matroid $\mathscr{M}$ is positive semidefinite iff it has no negative eigencycle.*

PROOF. Let $\mathscr{M}$ be symmetric and not positive semidefinite. It suffices to show that $\mathscr{M}$ contains a negative eigencycle. By definition, $\mathscr{M}$ contains a strictly sign reversing cycle K. If $|\{s_i, t_i\} \cap \underline{K}| = 1$ for some i, then the member of $\{s_i, t_i\}$ that is in K is called a violator in K. The following algorithm creates a negative eigencycle from a strictly sign reversing cycle K with all of its violators in S.

(0) Let K be any strictly sign reversing cycle with all of its violators in S.
(1) If there exists an i such that $s_i \in \underline{K}$ but $t_i \notin \underline{K}$, then
    if $s_i \in K^+$, let $K \leftarrow K \circ -C(S, t_i)$
    if $s_i \in K^-$, let $K \leftarrow K \circ C(S, t_i)$
    go to (1)
  If there is no such i, stop; K is a negative eigencycle.

After every step, a strictly sign reversing K is maintained with all of its violators in S. We need such a K before we can apply the above algorithm. First, note that if $s_i \in C(S, t_i)^-$ for any i, then $C(S, t_i)$ suffices. Next, suppose that $s_i \notin \underline{C}(S, t_i)$, $s_j \in \underline{C}(S, t_i)$ for some i, j. By the symmetry of the S-tableau, $s_i \in \underline{C}(S, t_j)$. If $s_j \in C(S, t_i)^+$, then $s_i \in C(S, t_j)^+$ and $C(S, t_i) \circ -C(S, t_j)$ will be a strictly sign reversing cycle with all of its violators in S, and if $s_j \in C(S, t_i)^-$, $s_i \in C(S, t_j)^-$ then $C(S, t_i) \circ C(S, t_j)$ will be one.

Suppose that Theorem 3.2.3 is true for symmetric oriented matroids with $|S| = n - 1$. Let K be a strictly sign reversing cycle of $\mathscr{M}$. If, for some i, we have $t_i \notin \underline{K}$, then we have a strictly sign reversing cycle $K \backslash \{s_i, t_i\}$ in the symmetric oriented matroid $\mathscr{M} \backslash t_i / s_i$. By the induction hypothesis, there must be a negative eigencycle of $\mathscr{M} \backslash t_i / s_i$ and therefore a strictly sign reversing cycle of $\mathscr{M}$ for which the only possible violator is s_i, which is in S. Apply the algorithm to this cycle.

Now suppose that $t_i \in \underline{K}$ for all i. If $s_i \in \underline{K}$ for all i, then K is a negative eigencycle. Suppose $s_i \notin \underline{K}$ for some i. As shown earlier, if $s_i \in C(S, t_i)^-$ or if $s_i \notin \underline{C}(S, t_i)$ when t_i is not a loop, we get a strictly sign reversing cycle with all of the violators in S. If t_i is a loop, then $K \backslash t_i$ is a strictly sign reversing cycle of $\mathscr{M}$ with $t_i \notin \underline{K}$; this was treated earlier. The last case is when $s_i \in C(S, t_i)^+$. In that case, $\mathscr{M} \backslash s_i / t_i$ is a symmetric oriented matroid containing a strictly sign reversing cycle $K \backslash t_i$. By the induction hypothesis, there then exists a negative eigencycle in $\mathscr{M} \backslash s_i / t_i$, implying the existence of a strictly sign reversing cycle $\bar{K}$ in $\mathscr{M}$ with t_i being its only possible violator. If $t_i \notin \underline{\bar{K}}$, then $\bar{K}$ is a negative eigencycle. If $t_i \in \underline{\bar{K}}$, then either $\bar{K} \circ C(S, t_i)$ or $\bar{K} \circ -C(S, t_i)$ is a strictly sign reversing cycle with all of its violators in S. Thus, one can always find such a strictly sign reversing cycle. For the ground case of the induction, note that when $|S| = |T| = 1$, any strictly sign-reversing cycle is necessarily a negative eigencycle.

From this result and Theorem 2.1.4, we obtain

COROLLARY 3.2.4. *A symmetric oriented matroid is positive definite iff it has no nonpositive eigencycle.*

The results in this section are again direct analogs to theorems of linear algebra. A symmetric positive definite matrix has a real positive eigenvalue (Theorem 3.2.1). A symmetric positive semidefinite matrix has a real positive eigenvalue iff it is not the zero matrix. A symmetric matrix is positive (semi)definite iff it has no nonpositive (negative) eigenvalues (Theorem 3.2.3 and Corollary 3.2.4).

As noted earlier, there are limitations to what one can say about eigenvectors from only looking at the sign patterns of vectors in a subspace of $\mathbb{R}^n$. There is no way yet known of finding out which eigencycles of $\mathscr{M}(\mathbf{I}, -\mathbf{A})$ correspond to real eigenvectors of $\mathbf{A}$ by looking only at the oriented matroid. However, the results given in this section demonstrate the possibility of extending some significant basic properties of quadratic forms to the combinatorial setting of oriented matroids.

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