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# Toroidal Embeddings I

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embedding without self-intersection). Therefore $X_{\{\sigma_\alpha\}}$ has all the required properties.

## CHAPTER III

## Construction of nice polyhedral subdivisions

Finn F. Knudsen

I don't know whether from a combinatorial point of view the following question has ever been asked:

Given a polyhedron $\sigma \subset \mathbb{R}^n$ with integral vertices, find an integer $\nu \geq 1$ and a decomposition of $\sigma$ into simplices $\tau_\alpha$ such that for all $\alpha$:

1) vertices of $\tau_\alpha \subset \frac{1}{\nu}\mathbb{Z}^n$

2) volume $(\tau_\alpha) = \dfrac{1}{\nu^n n!}$ .

But the theory of the previous chapter makes it clear that this is the essential construction needed to carry out semi-stable reduction. The purpose of this Chapter is to study and solve this combinatorial problem with certain refinements ("projectivity of the subdivision in the case when $\sigma$ is convex and globalization") independent of algebraic geometry.

One of the key steps (4.2) is due to Alan Waterman. The rest is a truly joint effort by Mumford and me.

## §1. Definitions and Projective subdivisions

We continue to use the definition of a compact polyhedral complex X with an integral structure $M = \{M_\alpha\}$ (definitions 5 and 6 of §1 in the previous chapter) except that we call M a rational structure here. If $\mu$ is an integer $\geq 1$ such that the functions in M take values in $\frac{1}{\mu}\mathbb{Z}$ on the vertices, then we say that M is integral over $\frac{1}{\mu}\mathbb{Z}$. For such $\mu$, we have:

Definition 1.1. The number

$$m(\sigma_\alpha, M_\alpha, \mu) = \mu^{\dim(\sigma_\alpha)} \cdot (\dim \sigma_\alpha)! \cdot \mathrm{vol}\, \sigma_\alpha$$

is an integer, and we will call this integer the multiplicity of $\sigma_\alpha$ with respect to M and $\mu$.

Also we define

$$m(X,M,\mu) = \max_{\sigma_\alpha \subset |X|} \{m(\sigma_\alpha, M_\alpha, \mu)\} \quad .$$

Observation 1.2. Let $M \subset N$ be two rational structures on a polyhedral complex X, such that

$$M \cap \{\text{constants}\} = N \cap \{\text{constants}\} \ .$$

Let $\mu$ be an integer such that (X,M) and (X,N) are both integral over $\frac{1}{\mu}\mathbb{Z}$. Then for any polyhedron $\sigma_\alpha$ we have

$$m(\sigma_\alpha, M, \mu) = \#(N_\alpha/M_\alpha) \cdot m(\sigma_\alpha, N, \mu) \ .$$



Suppose M is integral over $\mathbb{Z}$ and contains the constant 1. Then

$$m(\sigma_\alpha, M, 1) = 1 \quad \text{if and only if :}$$

$\sigma_\alpha$ is a simplex, and $M_\alpha$ is generated by the functions $x_i$, where $x_i$ is the linear function on $\sigma_\alpha$ which is 1 on the vertex $P_i$ and 0 on all the other vertexes.

Definition 1.3. Let X and X' be polyhedral complexes. We say that X' is a subdivision of X if

i) X and X' have the same underlying topological space,

ii) Whenever $\tau$ is a polyhedron of X', there is a polyhedron $\sigma$ of X such that $\tau \subset \sigma$,

iii) If $\tau$ is in X' and $\sigma$ in X and $\tau \subset \sigma$, then

$$V'_\tau = \mathrm{res}_\tau\, V_\sigma .$$

Observation 1.4. For any $\sigma$ in X let $\sigma'$ be the topological space $\sigma$ together with the collection of polyhedra $\tau$ in X' such that $\tau \subset \sigma$. Then it follows that $\sigma'$ is a polyhedral complex and $\sigma'$ is a subdivision of $\sigma$.

Definition 1.5. Let X' be a subdivision of a polyhedral complex X. We say that X' is a projective subdivision of X if there exists a real-valued continuous function $f\colon X \longrightarrow \mathbb{R}$ such that

i) For each polyhedron $\sigma \subset X$ $f|_\tau$ is piecewise-linear and convex, i.e., $f|_\tau = \min_{1 \leq i \leq N} \ell_i$, $\ell_i \in V_\sigma$.

ii) If $\sigma$ is a polyhedron in X and $\ell$ a linear function on $\sigma$ such that $\ell \geq f|_\sigma$, then the set

$$\tau = \{x \in \sigma \,|\, f(x) = \ell(x)\}$$

is either empty or a polyhedron of X'.

Whenever X' is a subdivision of X and f is a continuous function which satisfies i) and ii) we will say that f is a good function for the subdivision X'.

Definition 1.6. Let X be a polyhedral complex with a rational structure M. We say that a subdivision X' of X is a rational subdivision if the functions in M take rational values on the vertexes of X'. When X' is a rational subdivision of X we restrict the rational structure to X'.

Definition 1.7. Let X be a polyhedral complex. Then we denote by PL(X) the set of real-valued continuous functions f on X such that $f|_\tau \in V_\tau$ for each polyhedron $\tau$.

Note that evaluation of f at the vertexes of f gives us a linear embedding:

$$PL(X) \longrightarrow \mathbb{R}^{X_o}$$

Next we will prove a numerical criterion for projectiveness.



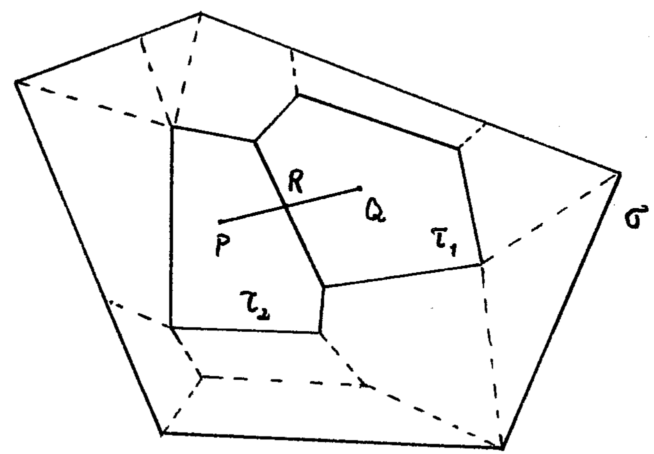
Lemma 1.8. Let X' be a subdivision of a polyhedral complex X. Then we can find a finite collection of linear functions:

$\Delta_i : PL(X')/PL(X) \longrightarrow \mathbb{R}$ with the property that if f is a function in PL(X'), then f is a good function for the subdivision X' of X if and only if $\Delta_i(t) > 0$ for all i.

Proof. Let $\sigma$ be a polyhedron in X, say of dimension n, and let $\tau$ be an n-1 polyhedron of X' such that $int(\tau) \subset int(\sigma)$. There are exactly two n-polyhedra $\tau_1$ and $\tau_2$ of X' such that $\tau_i \subset \sigma$, $i = 1,2$ and $\tau = \tau_1 \cap \tau_2$. Let P,Q be points in int $\tau_1$ and int $\tau_2$ respectively such that the line segment P,Q meets $\tau$ in, say, a point R. For any $f \in PL(X')$ we define

$$\Delta_\tau(f) = f(R) - g(R)$$

where g is the linear function on the line segment P,Q such that $g(P) = f(P)$ and $g(Q) = f(Q)$



Now choose such a function $\Delta_\tau$ for all $\tau$ with the above property, i.e., $\text{int}(\tau) \subset \text{int}(\sigma)$ where $\sigma$ is a top dimensional polyhedron of X and $\dim \tau = \dim(\sigma)-1$.

Claim: $f \in PL(X')$ is good if and only if $\Delta_\tau(f) > 0$ for all such $\tau$. We leave the proof of this to the reader.

We have two immediate corollaries.

Corollary 1.9. Let X' be a subdivision of a polyhedral complex X. Then the set of good functions for the subdivision X' form an open convex polyhedral cone in PL(X').

Corollary 1.10. Let X be a polyhedral complex with a rational structure M and let X' be a rational, projective subdivision of X. Then we can find a good function f for the subdivision X' such that for each polyhedron $\tau$ in X', $f|_\tau \in M_\tau$.

Proof. Let $C \subset PL(X')$ be the open cone of good functions. By assumption $C \neq \emptyset$.

Since X' is a rational subdivision it carries a rational structure and therefore PL(X') considered as a subspace of $\mathbb{R}^{X'_0}$ is defined over the rationals, i.e.,

$$PL(X') \cap \mathbb{Q}^{X'_0} \text{ is dense in } PL(X').$$

Let g be any element of $C \cap \mathbb{Q}^{X'_0}$. Then a suitable multiple $f = n.g$ will do. q.e.d.



The following follows from the proof of Lemma 1.8:

Lemma 1.11. Let X,X',X" be three polyhedral complexes such that X' is a subdivision of X and X" is a subdivision of X'. Then we can find two homogeneous convex functions $\Delta'$ and $\Delta''$ on PL(X")/PL(X) and PL(X")/PL(X') respectively such that:

| | | |
|---|---|---|
| $f \in PL(X')$ is a good function for the subdivision X' of X | $\Longleftrightarrow$ | $\Delta'(f) > 0$ |
| $g \in PL(X'')$ is a good function for the subdivision X" of X' | $\Longleftrightarrow$ | $\Delta''(g) > 0$ |
| $h \in PL(X'')$ is a good function for the subdivision X" of X | $\Longleftrightarrow$ | $\min\{\Delta'(h),\Delta''(h)\} > 0$ |

Corollary 1.12. (Transitivity of projective subdivisions) Let X,X', and X" be polyhedral complexes such that X' is a projective subdivision of X and X" is a projective subdivision of X'. Then X" is a projective subdivision of X.

Proof. Let $\Delta'$ and $\Delta''$ be as in Lemma 2.9. By assumption we can find functions f in PL(X'), g in PL(X") such that

$$\Delta'(f) = \Theta_1 > 0$$
$$|\Delta'(g)| = K < \infty$$
$$\Delta''(f) = 0$$
$$\Delta''(g) = \Theta_2 > 0 .$$

Let $\epsilon > 0$ be any real number such that $\Theta_1 - \epsilon K > 0$. Then

$$\min\left(\Delta'(f+\epsilon g),\ \Delta''(f+\epsilon g)\right) \geq \min\left(\Theta_1 - \epsilon K,\ \epsilon\Theta_2\right) > 0 .$$

Hence $f+\epsilon g$ is a good function for the subdivision X" of X.

q.e.d.



## §2. Some examples of projective subdivisions

### §2A

Example 2.1. (The barycentric subdivision)

Let X be a polyhedral complex of dimension n and let X' be the barycentric subdivision.

Let $X^{(k)}$, $0 \leq k \leq n-1$, be the subdivision of X obtained by taking the cones over the barycenters of the n-polyhedrons, n-1-polyhedrons,$\cdots$,n-k-polyhedrons taken in this order. We then have a succession of subdivisions.

$$X = X^{(0)}, X^{(1)}, \cdots, X^{(k)}, \cdots, X^{(n-1)} = X'.$$

Let $f_k$ be the function which has value 1 on all the barycenters of the n-k-polyhedra in X, 0 on all the vertexes of $X^{(k-1)}$ and linear on each polyhedron in $X^{(k)}$. Then clearly $f_k$ is a good function for the subdivision $X^{(k)}$ of $X^{(k-1)}$. By the transitivity of projective subdivisions we get

Lemma 2.2. The barycentric subdivision is projective.

### §2B

Example 2.3 (The regular subdivision of a simplex).

We will consider a simplex $\Delta$ with an ordered set of vertexes $P_1, \cdots, P_{n+1}$.

If $x_1,\cdots,x_{n+1}$ are the barycentric coordinates, we define the cumulative coordinates as follows.

$$y_o = 0$$
$$y_1 = x_1$$
$$y_2 = x_1+x_2$$
$$\vdots$$
$$y_k = x_1 + x_2 + \cdots + x_k$$
$$\vdots$$
$$y_{n+1} = 1$$

We then identify $\Delta$ with the set of n-tuples of real numbers $y_1,\cdots,y_n$ such that

$$0 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq 1.$$

Let $\mu$ be a positive integer and consider the hyperplanes

$$H_k^{i,j}: \quad y_i - y_j = k/\mu$$

where $0 \leq j < i \leq n$; $0 \leq k \leq \mu$.

Lemma 2.4. The hyperplanes $H_k^{ij}$ define a subdivision of $\Delta$ which we will call $\Delta^{(\mu)}$. $\Delta^{(\mu)}$ is a simplicial complex and if $\sigma$ is any n-simplex of $\Delta^{(\mu)}$ then volume $\sigma = 1/\mu^n \cdot$volume $\Delta$.

Proof. Let $y_1,\cdots,y_n$ be a point in $\Delta' = \text{int}\ \Delta - \bigcup H_k^{ij}$.

If we put $a_i = [\mu\cdot y_i]$ and $t_i = y_i - a_i/\mu$ it is clear that $t_i \neq t_j$ whenever $i \neq j$. Hence there is a unique permutation $\sigma$ such that:

$$0 < t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(n)} < 1/\mu .$$



Also it is clear that the connected component of $\Delta'$ determined by the point $y_1\cdots y_n$ is the set of points $y_1'\cdots y_n'$ such that

$$0 < y'_{\sigma(1)} - a_{\sigma(1)}/\mu < \cdots < y'_{\sigma(n)} - a_{\sigma(n)}/\mu < \frac{1}{\mu} .$$

And this we see is the interior of a simplex which modulo a permutation of the coordinate axes and a translation equals $1/\mu\cdot\Delta$, and hence its volume is $1/\mu^n$.volume $\Delta$. q.e.d.

The analogous subdivision of the entire $\mathbb{R}^n$ by the hyperplanes $H_k^{ij}$ (all $k \in \mathbb{Z}$) is just the decomposition into affine Weyl chambers of type $A_n$.

Lemma 2.4': The regular subdivision $\Delta^{(\mu)}$ is projective.

Proof. Let f be the function

$$f = \sum_{\substack{i_1,i_2,k \\ 1\leq i_1<i_2\leq n+1 \\ 1\leq k\leq \mu-1}} \left| \sum_{i_1\leq i\leq i_2} x_i - \frac{k}{\mu} \right| .$$

It is easy to see that f is a good function.

In order to treat the more difficult mixed $(\mu,\nu)$-subdivision discussed below, we need to analyze the regular subdivision in much greater detail, describing explicitly all its simplices, their vertices, and their faces. This will occupy us through (2.17) below.

Lemma 2.5. There is a natural 1-1 correspondence between n-simplexes of $\Delta^{(\mu)}$ and maps $\pi: \{1,\cdots,n\} \longrightarrow \{0,\cdots,\mu-1\}$.

Proof. Suppose first we are given a simplex in $\Delta^{(\mu)}$. Then as in the proof of the last lemma, for any interior point $y_1 \cdots y_n$ we get the numbers $a_i = [\mu \cdot y_i]$ and a permutation $\sigma$. This does not depend on the interior point chosen. The corresponding map will then be $i \rightsquigarrow a_{\sigma(i)}$.

Conversely, given a map $\pi: \{1,\cdots,n\} \longrightarrow \{0,\cdots,\mu-1\}$ there is a unique permutation $\sigma$ satisfying:

i) $\pi\sigma(i) \leq \pi\sigma(j)$ for $i \leq j$

ii) $\pi\sigma(i) = \pi\sigma(j) \Longrightarrow (i < j \Longleftrightarrow \sigma(i) < \sigma(j))$.

In fact we have the formula

$$\sigma^{-1}(k) = \#\left\{i \in \{1,\cdots,n\} \,\middle|\, \pi(i) < \pi(k) \text{ or } i \leq k \text{ and } \pi(i) = \pi(k)\right\}.$$

If $0 \leq t_{\sigma^{-1}(1)} \leq t_{\sigma^{-1}(2)} \leq \cdots \leq t_{\sigma^{-1}(n)} \leq \frac{1}{\mu}$, it follows by i) and ii) that the point $y_1,\cdots,y_n$ defined by

$$y_i = \frac{\pi\sigma(i)}{\mu} + t_i$$

lies in $\Delta$. By varying the t's we get a simplex of $\Delta^{(\mu)}$ which we denote by $\Delta^{\pi}$.

It is immediate from these definitions that the two maps are inverse to each other. q.e.d.



Definition 2.6. Let $\Delta,\mu,\pi,\sigma$ be as in the last lemma. We define

$$\delta_k(i) = \begin{cases} 1/\mu & \text{if } \sigma(i) \geq k \\ 0 & \text{if } \sigma(i) < k \end{cases}$$

$$\forall\ 1 \leq i \leq n, \qquad \forall\ 1 \leq k \leq n+1.$$

Clearly we have:

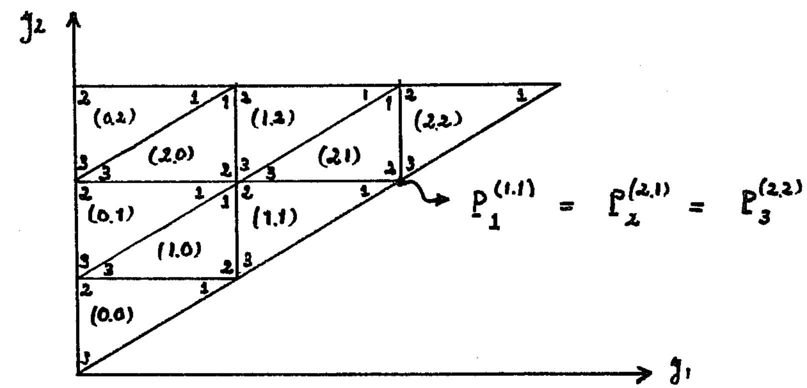
$$0 \leq \delta_k(\sigma^{-1}(1)) \leq \delta_k(\sigma^{-1}(2)) \leq \cdots \leq \delta_k(\sigma^{-1}(n)) \leq \frac{1}{\mu}.$$

Hence the point $P_k^{\pi}$ given by

$$y_i(P_k^{\pi}) = \frac{\pi\sigma(i)}{\mu} + \delta_k(i)$$

lies in $\Delta^{\pi}$. We call this particular ordering of the vertexes of $\Delta^{\pi}$ the canonical ordering.

Example of regular subdivision in the case $n = 2$, $\mu = 3$:




In each triangle we have indicated the function $\pi$ and the induced ordering of the vertices.

Lemma 2.7. Let $\Delta,\mu,\pi,\sigma$ as before, and let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be real negative numbers which are linearly independent over the rationals. The linear function $\ell = \sum \alpha_i y_i$ separates all the vertexes of $\Delta^{(\mu)}$ and hence defines a total ordering on the vertexes of $\Delta^{(\mu)}$. Moreover this ordering induces the canonical ordering on each simplex of $\Delta^{(\mu)}$.

Proof. Let $P_j^\pi$ and $P_k^\pi$ be two vertexes of $\Delta^\pi$ such that $j < k$. We then have

$$\ell(P_j^\pi) - \ell(P_k^\pi) =$$

$$\sum_i \alpha_i \frac{\pi\sigma(i)}{\mu} + \alpha_i\delta_j(i) - \alpha_i \frac{\pi\sigma(i)}{\mu} - \alpha_i\delta_k(i) =$$

$$\sum_i \alpha_i(\delta_j(i)-\delta_k(i)) =$$

$$\sum_i \alpha_{\sigma^{-1}(i)}(\delta_j(\sigma^{-1}(i)) - \delta_k(\sigma^{-1}(i))) =$$

$$\sum_{i=j}^{k-1} \alpha_{\sigma^{-1}(i)}/\mu < 0 \quad .$$

q.e.d.

Lemma 2.8. Let $\Delta,\mu,\pi,\sigma$ be as before. Then if $y_1',\cdots,y_n'$ are the cumulative coordinates of $\Delta^\pi$ with respect to the canonical ordering we have

$$y_i = \frac{\pi\sigma(i)}{\mu} + \frac{y'_{\sigma(i)}}{\mu}$$

Proof. $y_i$ is a linear function on $\Delta^\pi$ and hence is a linear combination of the barycentric coordinates $x_1',\cdots,x_{n+1}'$ on $\Delta^\pi$. Say

$$y_i = \alpha_1 x_1' + \cdots + \alpha_{n+1}x_{n+1}' \ .$$



But the coefficients are given by

$$\alpha_k = y_i(P_k^\pi) = \frac{\pi\sigma(i)}{\mu} + \delta_k(i)$$

and so

$$y_i = \sum_{k=1}^{n+1} x_k' \cdot \frac{\pi\sigma(i)}{\mu} + \sum_{k=1}^{n+1} x_k' \cdot \delta_k(i)$$

$$= \frac{\pi\sigma(i)}{\mu} + \sum_{k\leq\sigma(i)} \frac{x_k'}{\mu} = \frac{\pi\sigma(i)}{\mu} + \frac{y'_{\sigma(i)}}{\mu} \ .$$

q.e.d.

Corollary 2.9. Let $\nu$ be a positive integer and subdivide each simplex of $\Delta^{(\mu)}$ with respect to $\nu$ and the canonical ordering. By Lemma 2.7 these subdivisions patch up and give us a subdivision of $\Delta$ which we denote by $\Delta^{(\mu)(\nu)}$. We have

$$\Delta^{(\mu)(\nu)} = \Delta^{(\mu\cdot\nu)} \ .$$

Proof. Immediate by the formula of Lemma 2.8. q.e.d.

Next we want to study some particular properties of adjacent n-simplexes in $\Delta^{(\mu)}$. Let $\pi$ be a map $\{1,\cdots,n\} \longrightarrow \{0,\cdots,\mu-1\}$, and $\sigma$ the associated permutation. We may consider $\sigma$ as a permutation of the set $\{0,1,\cdots,n+1\}$ leaving $0$ and $n+1$ fixed. If $P_1^\pi,\cdots,P_{n+1}^\pi$ are the vertexes of $\Delta^\pi$ taken with the canonical ordering, the equation of the plane which goes through $P_1^\pi,\cdots,\hat{P}_k^\pi,\cdots,P_{n+1}^\pi$ is:

$$y_k' - y_{k-1}' = 0 \ .$$

Here the convention is $y'_o \equiv 0$, $y'_{n+1} \equiv 1$. By Lemma 2.8 this equation is:

$$y_{\sigma^{-1}(k)} - y_{\sigma^{-1}(k-1)} = \frac{1}{\mu}(\pi(k-\pi(k-1))) .$$

Here the convention is $\pi(o) \equiv 0$, $\pi(n+1) \equiv \mu-1$.

Lemma 2.10. If we denote the plane through the vertexes $P_1^\pi, P_2^\pi, \cdots, P_k^\pi, \cdots, P_{n+1}^\pi$ by $H_k^\pi$, $1 \leq k \leq n+1$, we have

$$H_k^\pi \text{ is a face of } \Delta \iff \pi(k) = \pi(k-1).$$

Proof. The equation of $H_k^\pi$ is

$$y_{\sigma^{-1}(k)} - y_{\sigma^{-1}(k-1)} = \frac{1}{\mu}(\pi(k)-\pi(k-1)) .$$

Since the equations of the faces of $\Delta$ are given by equations

$$y_i - y_{i-1} = 0$$

the if part is clear.

Conversely if $\pi(k) - \pi(k-1) = 0$

$$\sigma^{-1}(k) = \#\left\{i\in\{1,\cdots,n\} \,\middle|\, \pi(i) < \pi(k) \;\cup\; i\in\{1\cdots n\} \,\middle|\, \pi(i)=\pi(k),\ i\leq k\right\}$$

$$\sigma^{-1}(k-1) = \#\left\{i\in\{1\cdots n\} \,\middle|\, \pi(i) < \pi(k-1) \;\cup\; i\in\{1\cdots n\} \,\middle|\, \pi(i)=\pi(k-1),\ i<k-1\right\}$$

We see that the first set is the second set with the element k adjoined so

$$\sigma^{-1}(k) = \sigma^{-1}(k-1) + 1 .$$



In the next lemma we intend to write down the coordinates of the vertices of two adjacent simplexes. The reader is advised to work out a couple of low-dimensional examples instead of reading through this mess.

Let $\Delta, \mu, \pi, \sigma$ be as before and let k be an integer such that $2 \leq k \leq n$. Suppose moreover that $\pi(k) \neq \pi(k-1)$. By the last lemma this means that there is a simplex adjacent to $\Delta^\pi$ opposite the vertex $P_k^\pi$.

Let $\epsilon_k$ be the permutation of $\{0,\cdots,n+1\}$ obtained by interchanging k and k-1, i.e.,

$$\epsilon_k(i) = i \qquad \text{for} \quad i \notin \{k,k-1\}$$

$$\epsilon_k(k-1) = k$$

$$\epsilon_k(k) = k-1.$$

Let $\pi'$ be the function $\pi\cdot\epsilon_k$, and let $\sigma'$ be the permutation $\epsilon_k\cdot\sigma$. Then

Lemma 2.11. With the above notation, $\Delta^{\pi'}$ is the simplex adjacent to $\Delta^\pi$ opposite the vertex $P_k^\pi$. $\sigma'$ is the permutation associated to $\pi'$.

Proof. First we show that $\sigma'$ is the right permutation, i.e., has the property i) and ii) of Lemma 2.5. i) is immediate since $\pi'\sigma' \neq \pi\epsilon_k\epsilon_k\sigma = \pi\sigma$. Suppose then that $\pi'\sigma'(i) = \pi'\sigma'(j)$. But then $\pi\sigma(i) = \pi\sigma(j)$ so we have $i < j \iff \sigma(i) < \sigma(j)$. By our assumption $\pi(k) \neq \pi(k-1)$ hence the pair $\{\sigma(i),\sigma(j)\}$ is not equal the pair

$\{k,k-1\}$, so $\sigma(i) < \sigma(j) \iff \sigma'(j) < \sigma'(j)$.

To show that $\Delta^{\pi}$ and $\Delta^{\pi'}$ are adjacent we compute the coordinates of the vertexes.

Let $\delta$ be the function defined in Definition 2.6, and let $\delta'$ be the corresponding function for $\sigma'$. The vertexes are then given by:

$$y_i(P_j^{\pi}) = \frac{\pi\sigma(i)}{\mu} + \delta_j(i)$$

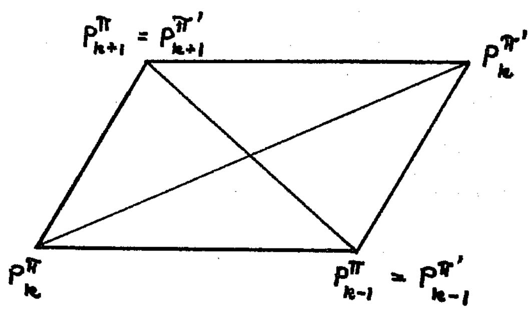
$$y_i(P_j^{\pi'}) = \frac{\pi'\sigma'(i)}{\mu} + \delta_j'(i) \ .$$

Since $\pi\sigma = \pi'\sigma'$ we just have to compare $\delta$ and $\delta'$. But if $j \neq k$ we have $\epsilon_k(i) \geq j \iff i \geq j$ and so $\delta_j(i) = \delta_j(i)$ for all $i$. It follows that $P_j^{\pi} = P_j^{\pi'}$ for $j \neq k$.

q.e.d.

Lemma 2.12 (Quadrilateral lemma in the case $2 \leq k \leq n$).

The four points $P_{k-1}^{\pi}, P_k^{\pi}, P_{k+1}^{\pi}, P_k^{\pi'}$ lie in a plane and the two line segments $P_k^{\pi}P_k^{\pi'}$ and $P_{k-1}^{\pi}P_{k+1}^{\pi}$ cut each other in half.






Proof. Clearly $\delta_{k-1}(i) = \delta_k(i) = \delta_k'(i) = \delta_{k+1}(i)$ for $i \notin \{\sigma^{-1}(k-1), \sigma^{-1}(k), \sigma^{-1}(k+1)\}$.

When $i = \sigma^{-1}(k-1)$ we have

$$\delta_{k-1}(i) = 1/\mu,\ \delta_k(i) = 0,\ \delta_k'(i) = 1/\mu,\ \delta_{k+1}(i) = 0.$$

When $i = \sigma^{-1}(k)$ we have

$$\delta_{k-1}(i) = 1/\mu,\ \delta_k(i) = 1/\mu,\ \delta_k'(i) = 0,\ \delta_{k+1}(i) = 0.$$

When $i = \sigma^{-1}(k+1)$ we have

$$\delta_{k-1}(i) = 1/\mu,\ \delta_k(i) = 1/\mu,\ \delta_k'(i) = 1/\mu,\ \delta_{k+1}(i) = 1/\mu.$$

In all cases

$$\delta_{k-1}(i) + \delta_{k+1}(i) = \delta_k(i) + \delta_k'(i) \ .$$

Hence for all $i$

$$y_i(P_{k-1}^{\pi}) + y_i(P_{k+1}^{\pi}) = y_i(P_k^{\pi}) + y_i(P_k^{\pi'}).$$

q.e.d.

Next we consider the case $k = 1$ and we assume that $\pi(1) \neq \pi(0) = 0$. By Lemma 2.10, there is a simplex adjacent to $\Delta^{\pi}$ opposite the vertex $P_1^{\pi}$.

Let $\pi'$ be the function $\{0,\cdots,n+1\} \longrightarrow \{0,\cdots,\mu\}$ defined by $\pi'(0) = 0$, $\pi'(n+1) = \mu-1$, $\pi'(n) = \pi(1)-1$ and $\pi'(i) = \pi(i+1)$ for $i \notin \{0,n,n+1\}$.

<u>Lemma</u> 2.13. <u>$\Delta^{\pi'}$ is the simplex adjacent to $\Delta^{\pi}$ opposite the vertex $P_1^{\pi}$</u>.

<u>Proof</u>. The proof is pure calculation. We calculate the coordinate functions of $\Delta^{\pi}$ and $\Delta^{\pi'}$ and compare. Recall that $\sigma'$ is given by the formula

$$\sigma'^{-1}(k) = \#\Big\{i\in\{1\cdots n\}\Big|\pi'(i)<\pi'(k)\Big\} + \#\Big\{i\in\{1,\cdots,k\}\Big|\pi'(i) = \pi'(k)\Big\} .$$

Hence $\sigma'^{-1}(0) = 0$, $\sigma'^{-1}(n+1) = n+1$

$$\begin{aligned}\sigma'^{-1}(n) &= \#\Big\{i\in\{1\cdots n\}\Big|\pi'(i)<\pi(1)-1\Big\} + \#\Big\{i\in\{1\cdots n\}\Big|\pi'(i) = \pi(1)-1\Big\}\\ &= \#\Big\{i \in \{1\cdots n\}\Big|\pi'(i) < \pi(1)\Big\}\\ &= \#\Big\{i \in \{1\cdots n-1\}\Big|\pi(i+1) < \pi(1)\Big\} + 1\\ &= \#\Big\{i \in \{1\cdots n\}\Big|\pi(i)<\pi(1)\Big\} + \#\Big\{i\in\{1\}\Big|\pi(i) = \pi(1)\Big\}\\ &= \underline{\sigma^{-1}(1)}.\end{aligned}$$

For $k \in \{1,\cdots,n-1\}$ we have

$$\begin{aligned}\sigma'^{-1}(k) &= \#\Big\{i\in\{1\cdots n\}\Big|\pi'(i)<\pi(k+1)\Big\} + \#\Big\{i\in\{1\cdots k\}\Big|\pi(i+1) = \pi(k+1)\Big\}\\ &= \#\Big\{i\in\{1\cdots n\}\Big|\pi(i) < \pi(k+1)\Big\} + \delta_{\pi(1),\pi(k+1)}\\ &\quad + \#\Big\{i\in\{1\cdots k+1\}\Big|\pi(i) = \pi(k+1)\Big\} - \delta_{\pi(1),\pi(k+1)}\\ &= \underline{\sigma^{-1}(k+1)} .\end{aligned}$$

Let $\epsilon$ be the permutation defined by

$$\epsilon(0) = 0, \quad \epsilon(n+1) = n+1, \quad \epsilon(n) = 1, \quad \epsilon(i) = i+1 \text{ if } i \in \{1,\cdots,n-1\}.$$

We then have



$$\underline{\pi'(i) = \pi \cdot \epsilon(i) - \delta_{n,i}}$$

$$\underline{\sigma'(i) = \epsilon^{-1} \cdot \sigma(i)}$$

$$\begin{aligned}\underline{\pi'\sigma(i)} &= \pi \epsilon (\epsilon^{-1}(\sigma(i))) - \delta_{n,\epsilon^{-1}\sigma(i)}\\ &= \pi\sigma(i) - \delta_{\epsilon(n),\sigma(i)} = \underline{\pi\sigma(i)-\delta_{1,\sigma(i)}} .\end{aligned}$$

As before, we define the functions

$$\delta_i(k) = \begin{Bmatrix} \frac{1}{\mu} \Longleftrightarrow \sigma(k) \geq i \\ 0 \Longleftrightarrow \sigma(k) < i \end{Bmatrix}$$

$$\delta_i'(k) = \begin{Bmatrix} \frac{1}{\mu} \Longleftrightarrow \sigma'(k) \geq i \\ 0 \Longleftrightarrow \sigma'(k) < i \end{Bmatrix}$$

Let $i \in \{1,\cdots,n\}$, and consider the two vertexes $P_{i+1}^{\pi}$ and $P_i^{\pi'}$. Coordinates are given by

$$y_k(P_{i+1}^{\pi}) = \frac{\pi\sigma(k)}{\mu} + \delta_{i+1}(k)$$

$$y_k(P_i^{\pi'}) = \frac{\pi'\sigma'(k)}{\mu} + \delta_i'(k).$$

<u>Case 1</u>. $k \neq \sigma^{-1}(1)$.

In this case we have $\pi\sigma(k) = \pi'\sigma'(k)$. Moreover: $k = \sigma^{-1}(1) \Longleftrightarrow \sigma'(k) = n$, so

$$\begin{aligned}\delta_{i+1}(k) = \tfrac{1}{\mu} &\Longleftrightarrow \sigma(k) \geq i+1 \Longleftrightarrow \epsilon\sigma'(k) \geq i+1\\ &\Longleftrightarrow \sigma'(k) \geq i \quad (\text{since } \sigma'(k) \neq n) \Longleftrightarrow \delta_i'(k) = \tfrac{1}{\mu} .\end{aligned}$$

Hence $P^{\pi}_{i+1}$ and $P^{\pi'}_{i}$ have the same $y_k$-coordinates for $k \neq \sigma^{-1}(1)$.

Case 2. $k = \sigma^{-1}(1) \Longleftrightarrow \sigma'(k) = n.$

We have $\sigma(k) = 1 < i+1$ for all i

and $\sigma'(k) = n \geq i$ for all i.

Hence:

$$y_k(P^{\pi}_{i+1}) = \frac{\pi\sigma(k)}{\mu}$$

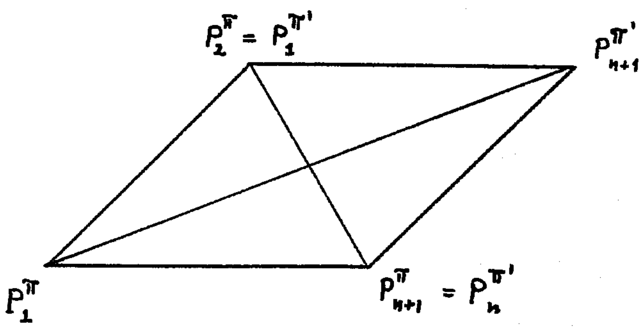
$$y_k(P^{\pi'}_{i}) = \frac{\pi'\sigma'(k)}{\mu} + \frac{1}{\mu}$$

$$= \frac{\pi\sigma(k)}{\mu} - \frac{\delta_{1,\sigma(k)}}{\mu} + \frac{1}{\mu} = \frac{\pi\sigma(k)}{\mu}.$$

This shows that $P^{\pi}_{i+1} = P^{\pi'}_{i}$ for $i \in \{1\cdots n\}$. q.e.d.

Lemma 2.14. (Quadrilateral lemma in the case k = 1).

With the notation as in Lemma 2.13 the four points $P^{\pi}_1, P^{\pi'}_{n+1}, P^{\pi}_2$ and $P^{\pi}_{n+1}$ lie in a plane and the two line segments $P^{\pi}_1P^{\pi'}_{n+1}$ and $P^{\pi}_2P^{\pi}_{n+1}$ cut each other in half.






Proof. $y_k(P^{\pi}_1) + y_k(P^{\pi'}_{n+1}) = \frac{\pi\sigma(k)}{\mu} + \delta_1(k) + \frac{\pi'\sigma'(k)}{\mu} + \delta'_{n+1}(k)$

$$= \frac{\pi\sigma(k)}{\mu} + \frac{\pi'\sigma'(k)}{\mu} + \frac{1}{\mu}$$

$$y_k(P^{\pi}_2) + y_k(P^{\pi}_{n+1}) = y_k(P^{\pi'}_1) + y_k(P^{\pi}_{n+1})$$

$$= \frac{\pi'\sigma'(k)}{\mu} + \delta'_1(k) + \frac{\pi\sigma(k)}{\mu} + \delta_{n+1}(k)$$

$$= \frac{\pi\sigma(k)}{\mu} + \frac{\pi'\sigma'(k)}{\mu} + \frac{1}{\mu}.$$ q.e.d.

Note that the case k = n+1 follows from this case by interchanging $\pi$ and $\pi'$.

Definition 2.15. (Good and bad hyperplanes)

Let $\Delta = \{P_1,\cdots,P_{n+1}\}$ be an n-simplex with an ordered set of vertexes, $y_1,\cdots,y_n$ the cumulative coordinates and $\mu$ a positive integer.

A hyperplane $H^{i,j}_k$ defined by the equation

$$y_j - y_i = k/\mu, \qquad 0 \leq i < j \leq n+1$$

is said to be a good hyperplane if it can be written in the form

$$y_\ell = k'/\mu, \qquad 1 \leq \ell \leq n .$$

Note that this is the case if and only if $j = n+1$ or $i = 0$.

The hyperplanes which are not of this form will be called bad.

We now state a souped-up version of the last 5 lemmas.

Lemma 2.16. Let again $\Delta$ be an n-simplex with ordered set of vertexes $\{P_1 \cdots P_{n+1}\}$, and let $\sigma$ and $\tau$ be two adjacent n-simplexes in the $\mu$-regular subdivision of $\Delta$. Let H be the hyperplane that separates $\sigma$ and $\tau$. If P is the vertex of $\sigma$ which does not meet $\tau$ and Q is the vertex of $\tau$ that does not meet $\sigma$, there are two vertexes A and B in $\sigma \cap \tau$ such that the four points P,Q,A and B lie in a plane and the line segments PQ and AB cut each other in half.

Moreover if $x_{n+1}$ is the n+1st barycentric coordinate and $x_{n+1}(P) = x_{n+1}(Q)$ then $x_{n+1}(A) = x_{n+1}(B)$ and the plane A,B,P,Q lies in the hyperplane $x_{n+1}$ = constant.

However if $x_{n+1}(P) \neq x_{n+1}(Q)$, then $x_{n+1}(A) \neq x_{n+1}(B)$ and the hyperplane H cuts the face opposite $P_{n+1}$ in a good hyperplane, or H is defined by $x_{n+1}$ = constant.

Proof. The proof of the last two assertions follows directly from the formulas developed so far and is left to the reader.

Observation 2.17. Let $\Delta$ be an n-simplex on our ordered set of vertexes $P_1 \cdots P_{n+1}$, $\mu$ an integer > 0 and k any integer $0 \leq k < \mu$.

Let H be the hyperplane defined by $x_{n+1} = k/\mu$, and let $P_i'$ be the intersection point of H with the line $P_i$, $P_{n+1}$, $1 \leq i \leq n$.

We denote by $\Delta'$ the truncated simplex with vertexes $\{P_1' \cdots P_n', P_{n+1}\}$.

In this case we have:

$$\Delta' \cap \Delta^{(\mu)} = \Delta'^{(\mu-k)}$$

i.e., the $\mu$-regular subdivision of $\Delta$ induces the $(\mu-k)$-regular subdivision of $\Delta'$.

Proof. Left to reader.



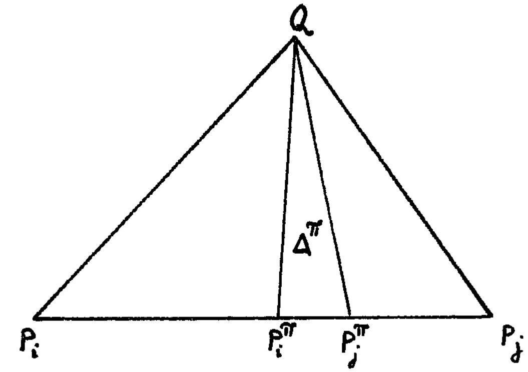
## §2C

Example 2.18. (The mixed $(\nu,\mu)$ subdivision)

We consider a simplex $\Delta$ with vertexes $\{P_1,\cdots,P_{n+1},Q\}$ taken in this order, and we denote the face opposite Q by $\Delta_Q = \{P_1,\cdots,P_{n+1}\}$.

If $\pi\colon \{1,\cdots n\} \longrightarrow \{0,\cdots \nu-1\}$ is any function we denote as before by $\Delta_Q^\pi$ the corresponding simplex in $\Delta_Q^{(\nu)}$.

Let $\Delta^\pi$ be the simplex $\{P_1^\pi,\cdots,P_{n+1}^\pi,Q\}$ with vertexes taken in this order:




If we now subdivide each $\Delta^\pi$ regularly with respect to $\mu$ and the given ordering, these subdivisions clearly patch up so as to give a subdivision of $\Delta$. We call this the mixed $(\nu,\mu)$-subdivision of $\Delta$ with respect to this ordering. It is easy to see using 1.12 and 2.4' that this subdivision is projective. The obvious good functions would not however extend to the global Example D which is our main goal so we do not stop to do this. Instead we want some facts about adjacent simplices in this subdivision ((2.20)) which are the key points for establishing projectivity in the general case.

Lemma 2.19. Let $\Delta,\pi,\mu,\nu$ be as above. If $y_1,\cdots,y_{n+1}$ are the cumulative coordinates on $\Delta$ and $y_1^{\pi},\cdots,y_{n+1}^{\pi}$ are the cumulative coordinates on $\Delta^{\pi}$ we have the formula

$$y_k = \frac{1}{\nu}(\pi\sigma(k)\cdot y_{n+1}^{\pi} + y_{\sigma(k)}^{\pi} \qquad 1 \le k \le n+1$$

$$y_{n+1} = y_{n+1}^{\pi}.$$

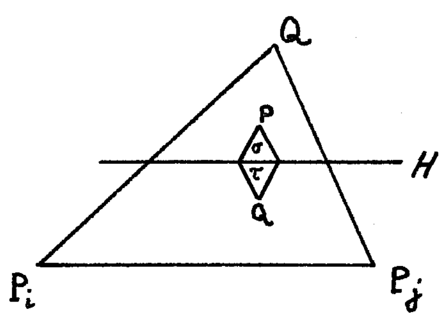
Proof. This follows immediately from Lemma 2.8.

Note that $y_{n+1} = y_{n+1}^{\pi} = 1-\ell$ where $\ell$ is the linear function which takes the value 1 at Q and 0 at $P_i$, $1 \le i \le n$. q.e.d.

Lemma 2.20 (Main Lemma)

Let $\Delta$ and $\Delta^{(\nu,\mu)}$ be as before and consider two adjacent n+1-simplexes $\sigma$ and $\tau$ in the mixed $(\nu,\mu)$-subdivision. Let P be the vertex of $\sigma$ which does not meet $\tau$ and let Q be the vertex of $\tau$ that does not meet $\sigma$. Let H be the hyperplane that separates $\sigma$ and $\tau$. Now three things may happen.

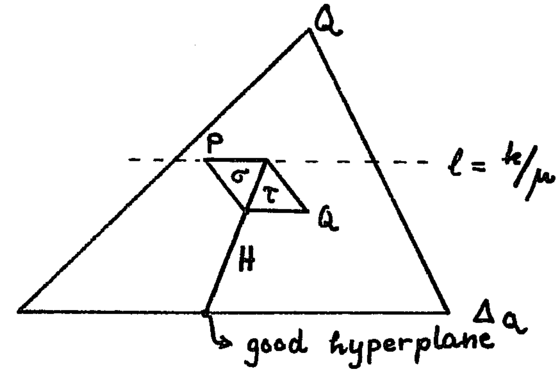
a) The hyperplane H is defined by the equation $\ell$ = constant






b) The hyperplane H instersects $\Delta_Q = \{P_1,\cdots,P_{n+1}\}$ in a good hyperplane.

In this case we may assume that $\ell(P) = k/\mu$ and $\ell(Q) = k-1/\mu$, $1 \le k \le \mu-1$. And the important fact is that the part of the hyperplane H which lies below $\ell = k/\mu$ does not meet the interior of any n+1-simplex in the subdivision.



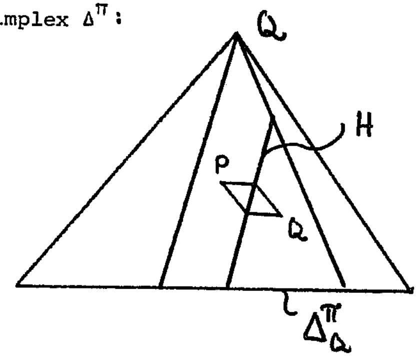

c) The hyperplane H intersects $\Delta_Q$ in a bad hyperplane.

[In this case H might cut through interiors of n+1-simplexes in the subdivision even in the level below $\ell = \max(\ell(Q),\ell(P))$. Hence the name bad.]

In this case however the following fortunate thing happens: There are two vertexes A and B of $\sigma\cap\tau$ such that the four points P,Q,A and B lie in a plane, the two line segments PQ and AB cut each other in half and moreover $\ell(P) = \ell(Q) = \ell(A) = \ell(B)$, i.e., the four points lie in the hyperplane defined by $\ell$ = constant.

Proof. We first consider the case where $\sigma$ and $\tau$ belong to the same simplex $\Delta^{\pi}$:

i.e.




By the coordinate transformation rules Lemma 2.8, we see that $H \cap \Delta_Q^{\pi}$ is a good hyperplane of $\Delta_Q^{\pi}$ if and only if $H \cap \Delta_Q$ is a good hyperplane of $\Delta_Q$, and so by Lemma 2.17 a) and c) follows.

Now for b) the assumption is that $H \cap \Delta_Q^{\pi}$ is a good hyperplane. Therefore the equation of H has to be of either of the following two forms.

(i) $$y_{n+1}^{\pi} - y_j^{\pi} = \frac{\mu-k}{\mu}$$

(ii) $$y_j^{\pi} = \frac{k}{\mu} \qquad 1 \leq k \leq \mu-1 .$$

Since $y_{n+1}^{\pi} = 1-\ell$ restricted to $\Delta_Q^{\pi}$ is identically equal to 1, both of these equations restrict to the equation

$$y_j^{\pi} = \frac{k}{\mu} \text{ on } \Delta_Q^{\pi} .$$



Using the coordinates transformation formula in Lemma 2.19 we get H defined by the following equation.

(i) $$\boxed{y_{\sigma^{-1}(j)} - \frac{\pi(j)+1}{\nu}(1-\ell) = \frac{k-\mu}{\mu\cdot\nu}}$$

(ii) $$\boxed{y_{\sigma^{-1}(j)} - \frac{\pi(j)}{\nu}(1-\ell) = \frac{k}{\mu\cdot\nu}}$$

Since $\ell = 0$ on $\Delta_Q$ both equations restrict to the equation

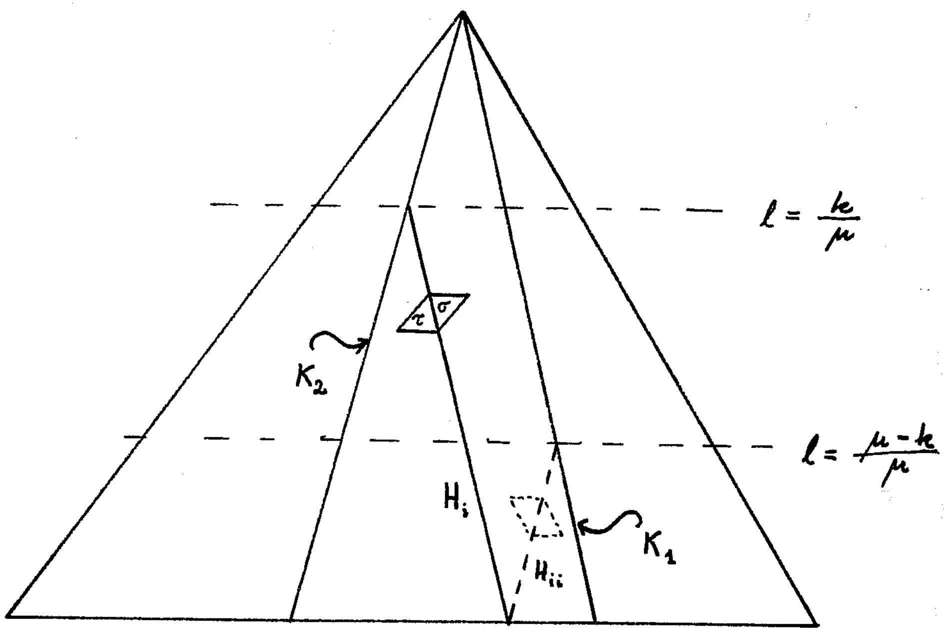
$$y_{\sigma^{-1}(j)} = \frac{k}{\mu\cdot\nu} + \frac{\pi(j)}{\nu} .$$

Consider the two hyperplanes $K_1$ and $K_2$ defined by

$$K_1: \quad y_{\sigma^{-1}(j)} - \frac{\pi(j)}{\nu}(1-\ell) = 0$$

$$K_2: \quad y_{\sigma^{-1}(j)} - \frac{\pi(j)+1}{\nu}(1-\ell) = 0$$

We have the following self explanatory picture:




b) will follow if we can prove that for any simplex of the form $\Delta^{\pi'}$ which is trapped between $K_1$ and $K_2$ the hyperplane $H \cap \Delta^{\pi'}$ is one of the hyperplanes which defines the $\mu$-regular subdivision of $\Delta^{\pi'}$, i.e., the equation of H in the accumulative $\pi'$-coordinates is of the form

$$y_j^{\pi'} - y_i^{\pi'} = \frac{k}{\mu} .$$



Now the condition that $\Delta^{\pi'}$ is trapped between $K_1$ and $K_2$ is the same as that of the simplex $\Delta_Q^{\pi'}$ lying betwen $K_1 \cap \Delta_Q$ and $K_2 \cap \Delta_Q$. Hence for each vertex $P_i^{\pi'}$ of $\Delta_Q^{\pi'}$ we must have

$$\frac{\pi(j)}{\nu} \leq y_{\sigma^{-1}(j)}(P_i^{\pi'}) \leq \frac{\pi(j)+1}{\nu} .$$

But by some old formula we have

$$y_{\sigma^{-1}(j)}(P_i^{\pi'}) = \frac{\pi'\sigma'(\sigma^{-1}(j))}{\nu} + \delta_i(\sigma^{-1}(j)) .$$

Since this holds for all i the trapping condition reads

$$\pi'\sigma'\sigma^{-1}(j) = \pi(j) .$$

Using this condition and Lemma 2.19 once again, a small calculation shows that the equation of H expressed in the $y^{\pi'}$ coordinates is:

(i) $$y_{n+1}^{\pi'} - y_{\sigma'(\sigma^{-1}(j))}^{\pi'} = \frac{\mu-k}{\mu}$$

or (ii) $$y_{\sigma'(\sigma^{-1}(j))}^{\pi'} = \frac{k}{\mu} .$$

Next we consider the case where $\sigma$ and $\tau$ belong to different simplexes say $\sigma \subset \Delta^{\pi'}{}_o$. Obviously $\Delta^{\pi}{}_o$ and $\Delta^{\pi'}{}_o$ are adjacent simplexes.

In this case the hyperplane H cuts "nicely" through the subdivision whether it is good or bad so we are only left with c), i.e., $H \cap \Delta_Q$ is a <u>bad</u> hyperplane.

If $\left\{P_1^{\pi_o},\cdots,P_{n+1}^{\pi_o},Q\right\}$ and $\left\{P_1^{\pi'_o},\cdots,P_{n+1}^{\pi'_o},Q\right\}$ are the vertexes of $\Delta^{\pi_o}$ and $\Delta^{\pi'_o}$ respectively it follows since H is bad that

$$P_i^{\pi_o} = P_i^{\pi'_o} \qquad \text{for all } i \neq k$$

where k is some integer $2 \leq k \leq n$, Lemma 2.11.

Since $\sigma$ has a face in common with $\Delta^{\pi_o}$ and $\tau$ has a face in common with $\Delta^{\pi'_o}$. This face has to be the common face of $\Delta^{\pi'_o}$ and $\Delta^{\pi_o}$, namely the face opposite $P_k^{\pi_o}$ (resp. $P_k^{\pi'_o}$). Now $\sigma$ is determined by a map, say

$$\pi: \ \{0,1,\cdots,n+1,n+2\} \longrightarrow \{0,1,\cdots,\mu-1\}$$

when $\pi(0) = 0$, $\pi(n+2) = \mu-1$ and $\tau$ is determined by <u>the same map</u>, i.e.,

$$\sigma = (\Delta^{\pi_o})^{\pi} \quad \text{and} \quad \tau = (\Delta^{\pi'_o})^{\pi} \ .$$

Let $\rho$ be the permutation associated to $\pi$. Then in the $y^{\pi_o}$ and $y^{\pi'_o}$ coordinates the vertexes of $\sigma$ and $\tau$ are given by

$$y_k^{\pi_o}(P_j^{\sigma}) = \frac{\pi\rho(k)}{\mu} + \delta_j(k)$$
$$y_k^{\pi'_o}(P_j^{\tau}) = \frac{\pi\rho(k)}{\mu} + \delta_j(k)$$



Suppose that $P_j^{\sigma}$ is the vertex of $\sigma$ that does not meet $\tau$. The equation of the face opposite $P_j^{\sigma}$ is then given by Lemma 2.10

$$y^{\pi_o}_{\rho^{-1}(j)} - y^{\pi_o}_{\rho^{-1}(j-1)} = \frac{1}{\mu}(\pi(j) - \pi(j-1)) = 0 \ .$$

Hence $\rho^{-1}(j) = k$, $\rho^{-1}(j-1) = k-1 \geq 1$ and

$$\pi(j) = \pi(j-1) \ .$$

Since $\rho^{-1}(j-1) \geq 1$ it follows that $j \geq 2$. Since $\rho^{-1}(j) = k \leq n$ it follows that $j \ngeq n+2$, i.e.,

$$2 \leq j \leq n+1 \ .$$

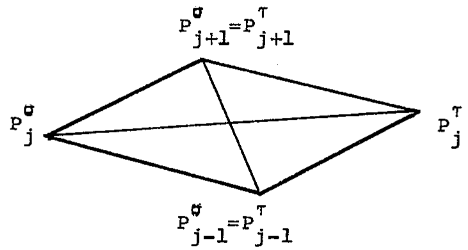
Using the formula 2.19 we get:

$$y_i(P_i^{\sigma}) = \frac{1}{\nu}\left[\pi_o\sigma_o(j)(1-\ell) + \left(\frac{\pi\rho\sigma_o(j)}{\mu} + \delta_i(\sigma_o(j))\right)\right]$$
$$y_j(P_i^{\tau}) = \frac{1}{\nu}\left[\pi'_o\sigma'_o(j)(1-\ell) + \left(\frac{\pi\rho\sigma'_o(j)}{\mu} + \delta_i(\sigma'_o(j))\right)\right] .$$

Since $\pi'_o\sigma'_o = \pi_o\sigma_o$ we have

$$y_j(P_i^{\sigma}) = \varphi(j) + \frac{1}{\nu}\left(\frac{\pi\rho\sigma_o(j)}{\mu} + \delta_i(\sigma_o(j))\right)$$
$$y_j(P_i^{\tau}) = \varphi(j) + \frac{1}{\nu}\left(\frac{\pi\rho\sigma'_o(j)}{\mu} + \delta_i(\sigma'_o(j))\right)$$

We want to show that the four points $P^{\sigma}_{j-1}\ P^{\sigma}_{j}\ P^{\tau}_{j}\ P^{\sigma}_{j+1}$ form a quadrilateral :




From these formulas it follows that

$$y_s(P^{\sigma}_{j-1}) = y_s(P^{\sigma}_{j}) = y_s(P^{\tau}_{j}) = y_s(P^{\sigma}_{j+1})$$

for all $s \notin \{\sigma_o^{-1}(k), \sigma_o^{-1}(k-1)\}$.

<u>Case $s = \sigma_o^{-1}(k)$.</u>

In this case we have

$$\underline{\sigma_o(s) = k.} \qquad \underline{\sigma_o'(s)} = \epsilon\sigma_o(s) = \epsilon(k) = \underline{k-1}$$

$$\underline{\rho\sigma_o(s) = j} \qquad \text{and} \qquad \underline{\rho\sigma_o'(s) = j-1} \quad .$$

Since $\pi(j) = \pi(j-1)$ we have



$$y_s(P^{\sigma}_{j-1}) = \beta + \frac{1}{\nu}\delta_{j-1}(\sigma_o(s)) = \beta + \frac{1}{\mu\nu}$$

$$y_s(P^{\sigma}_{j+1}) = \beta + \frac{1}{\nu}\delta_{j+1}(\sigma_o(s)) = \beta$$

$$y_s(P^{\sigma}_{j}) = \beta + \frac{1}{\nu}\,\delta_j(\sigma_o(s)) = \beta + \frac{1}{\mu\nu}$$

$$y_s(P^{\tau}_{j}) = \beta + \frac{1}{\nu}\,\delta_j(\sigma_o'(s)) = \beta \quad ,$$

for some constant $\beta$.

Hence $y_s(P^{\sigma}_{j-1}) + y_s(P^{\sigma}_{j+1}) = y_s(P^{\sigma}_{j}) + y_s(P^{\tau}_{j})$.

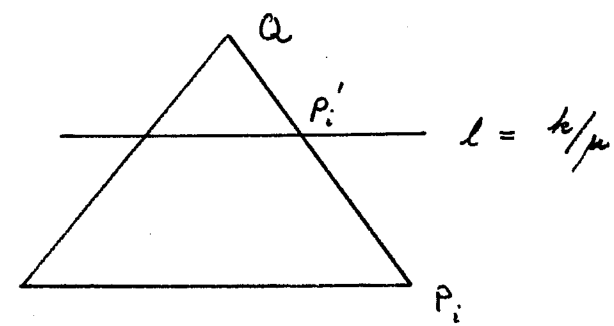
The case $s = \sigma_o^{-1}(k-1)$ we leave to the reader.

This completes the proof of c) since $n+1 \notin \{\sigma_o^{-1}(k), \sigma_o^{-1}(k-1)\}$.

q.e.d.

I believe that we have a quadrilateral also in the case where $\sigma$ and $\tau$ lie in two adjacent simplexes $\Delta^{\pi}$ and $\Delta^{\pi'}$ of type $k = 1$; computations however are more complicated.

<u>Observation</u> 2.21. <u>Let $\Delta$ be as in Lemma 2.20 and let H be the hyperplane defined by $\ell = k/\mu$. Denote by</u> $P_i'$ <u>the intersection of H with the line segment</u> $P_iQ$.



The mixed $(\nu,\mu)$-subdivision of $\Delta$ induces the mixed $(\nu,\mu\text{-}k)$-subdivision of $\Delta' = \{P_1',\cdots,P_{n+1}',Q\}$, and it induces the $\nu\cdot(\mu\text{-}k)$-regular subdivision on $\Delta_Q' = \{P_1',\cdots,P_{n+1}'\}$.

Proof. Observation 2.17 and Cor. 3.9.

§2D

Final Example: Global mixed $(\nu_i,\mu_i)$-subdivisions

We are now ready to prove the main result of this section. We consider a simplicial complex X with a totally ordered set of vertices $P_1,P_2,\cdots,P_N,Q_{N+1},\cdots,Q_{N+s}$ and such that the open stars of $Q_{N+1},\cdots,Q_{N+s}$ are disjoint. Let $\nu_1,\cdots,\nu_s$; $\mu_1,\cdots,\mu_s$ be a set of positive integers such that

$$\nu_i\mu_i = \nu_j\mu_j = \mu \quad \text{for all } i,j.$$

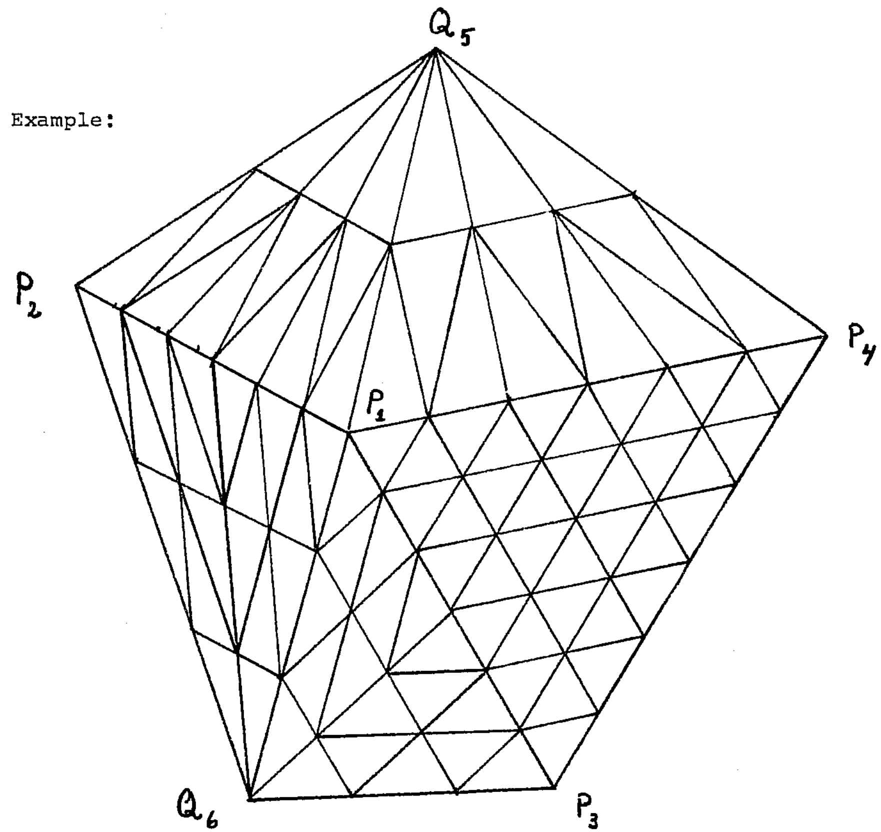
Let X' be the subdivision of X obtained as follows:

If $\Delta$ is a simplex in the open star of $Q_{N+i}$, we let $\Delta'$ be the mixed $(\nu_i,\mu_i)$ subdivision of $\Delta$ with respect to the given ordering.

If $\Delta$ does not contain any of the $Q_i$'s we let $\Delta'$ be the $\mu$-regular subdivision of $\Delta$ with respect to the given ordering.



Example:




$\nu_1 = 2,\ \mu_1 = 3, \qquad \nu_2 = 3,\ \mu_2 = 2, \qquad \mu = 6$

These subdivisions clearly patch together. Moreover:

Theorem 2.22. X' is a projective subdivision of X.

Proof. The idea of the proof is simply to write down a function and prove that it is good using the numerical criterion.

Let $x_i$ be the function defined on all of X, linear over each simplex of X having the property

$$x_i(P_j) = \delta_{ij} \qquad 1 \le j \le N$$

$$x_i(Q_j) = \delta_{ij} \qquad 1+N \le j \le N+s \; .$$

Sometimes we will write $\ell_i$ instead of $x_{N+i}$, $1 \le i \le s$.

Let f' be the function

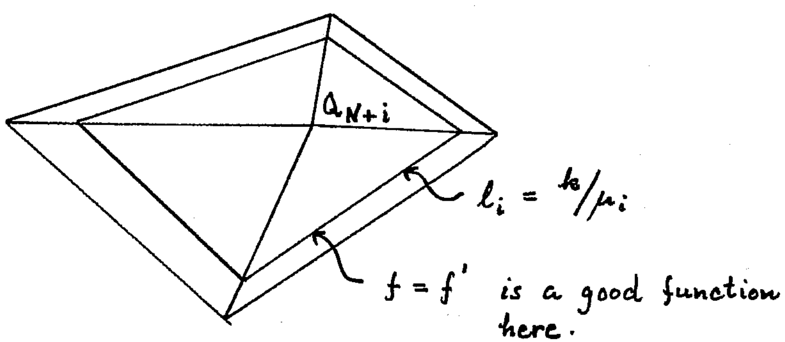
$$f' = -\sum_{\substack{i_1,i_2,k \\ 1\le i_1<i_2\le N+s \\ 1\le k\le \mu-1}} \left| \sum_{i_1\le i\le i_2} x_i - \frac{k}{\mu} \right|$$

Now f' is not a good function. What we do is: we restrict f' to the vertices of X' and extend by linearity over the simplexes of X'. The resulting function we call f,

i.e., $$f = \overline{f'_o}$$

Note that by Observation 2.21 $f = f'$ over the set defined by $\ell_i = \dfrac{k}{\mu_i}$ :

i.e.






For all pairs of integers m,k such that $1 \le m \le N$, $1 \le k \le \mu-1$, choose $\alpha_i$ and $\beta_i$, $1 \le i \le s$, such that

$$\frac{\alpha_i}{\nu_i} \le \frac{k}{\mu} \le \frac{\alpha_i+1}{\nu_i} \quad \text{and} \quad \frac{\alpha_i}{\nu_i} + \frac{\beta_i}{\mu} = \frac{k}{\mu} \; .$$

Define

$$g^1_{m,k} = \begin{cases} -\left| \sum_{j=1}^{m} x_j - \frac{k}{\mu} \right| & \text{if } \ell_i = 0, \; 1 \le i \le s \\ -\left| \sum_{j=1}^{m} x_j + \ell_i \frac{\alpha_i+1}{\nu_i} - \frac{k}{\mu} \right| & \text{if } 0 \le \ell_i \le \frac{\beta_i}{\mu_i}, \; \ell_j = 0 \;\; j \ne i \\ -\left| \sum_{j=1}^{m} x_j + \ell_i \frac{\alpha_i}{\nu_i} - \frac{\alpha_i}{\nu_i} \right| & \text{if } \frac{\beta_i}{\mu_i} \le \ell_i \le 1, \; \ell_j = 0 \;\; j \ne i \end{cases}$$

and

$$g^2_{m,k} = \begin{cases} -\left| \sum_{j=1}^{m} x_j - \frac{k}{\mu} \right| & \text{if } \ell_i = 0, \; 1 \le i \le s \\ -\left| \sum_{j=1}^{m} x_j + \ell_i \frac{\alpha_i}{\nu_i} - \frac{k}{\mu} \right| & \text{if } 0 \le \ell_i \le 1 - \frac{\beta_i}{\mu_i}, \; \ell_j = 0 \;\; j \ne i \\ -\left| \sum_{j=1}^{m} x_j + \ell_i \frac{\alpha_i+1}{\nu_i} - \frac{\alpha_i+1}{\nu_i} \right| & \text{if } 1 - \frac{\beta_i}{\mu_i} \le \ell_i \le 1, \; \ell_j = 0 \;\; j \ne i \end{cases}$$

Then put

$$g = \sum_{\substack{0\le m\le N \\ 1\le k\le \mu-1 \\ \alpha=1,2}} g^{\alpha}_{m,k}$$

Comparing this with the equations on page 137 we note that g "breaks" exactly along all the good hyperplanes of X, and possibly along hyperplanes of the form $\ell_i$ = constant.

Finally we define h by

$$h = -\sum_{i=1}^{s} \sum_{1\le k\le \mu_i-1} \left|\ell_i - \frac{k}{\mu_i}\right|$$

If $\epsilon_1$ and $\epsilon_2$ are positive real numbers put

$$F_{\epsilon_1\epsilon_2} = h + \epsilon_1 g + \epsilon_2 f \quad .$$

For each pair of adjacent (max dimensional) simplexes $\sigma$ and $\tau$ of X' such that $\sigma$ and $\tau$ belong to the same simplex of X, we choose a little line segment through the common face and define

$$\Delta_{\sigma\tau}(f)$$

as in the proof of Lemma 1.8.

There are 4 mutually exclusive ways of choosing such pairs, and these 4 cases exhaust all possibilities.



Case 1. $\sigma$ and $\tau$ belong to a simplex of X and this simplex does not contain $Q_i$ as a vertex $N+1 \le i \le N+s$. We clearly have

$$\Delta_{\sigma\tau}(f) \ge \lambda_2 > 0 \qquad \text{for some constant } \lambda_2$$

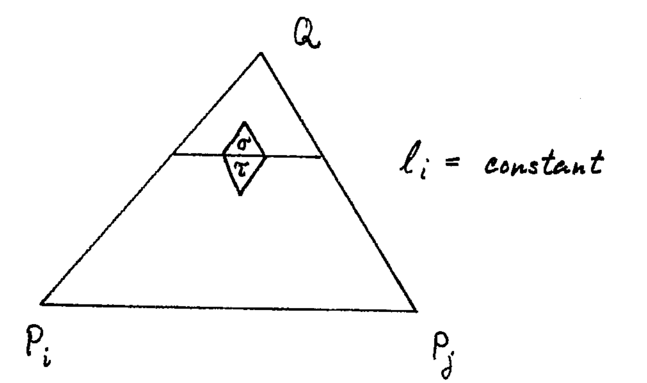
$$\Delta_{\sigma\tau}(g) \ge 0$$

$$\Delta_{\sigma\tau}(h) \ge 0.$$

Hence

$$\Delta_{\sigma\tau}(F_{\epsilon_1\epsilon_2}) \ge \epsilon_2\lambda_2 \quad .$$

Case 2. $\sigma$ and $\tau$ lie in a simplex which has $Q_i$ as a vertex, and the hyperplane that separates $\sigma$ and $\tau$ is of the form $\ell_i$ = constant



Clearly

$$|\Delta_{\sigma\tau}(f)| < K_2 \quad \text{for some constant } K_2$$

$$|\Delta_{\sigma\tau}(g)| < K_1 \quad \text{for some constant } K_1$$

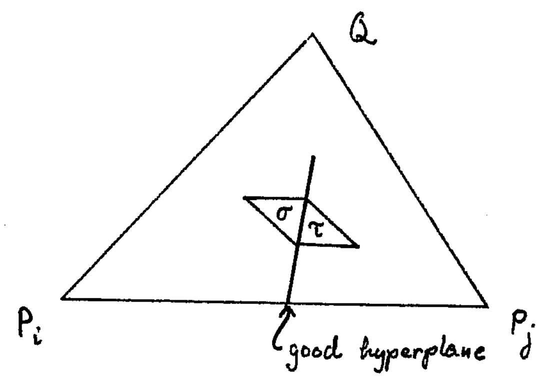
$$\Delta_{\sigma\tau}(h) \geq \lambda_o > 0 \quad \text{for some constant } \lambda_o.$$

Hence we have

$$\Delta_{\sigma\tau}(F_{\epsilon_1\epsilon_2}) \geq \lambda_o - \epsilon_1 K_1 - \epsilon_2 K_2$$

Case 3. $\sigma$ and $\tau$ lie in a simplex which has $Q_i$ as a vertex, and moreover the hyperplane that separates $\sigma$ and $\tau$ intersects the bottom face in a good hyperplane,

i.e.,




Now g "breaks" along all good hyperplanes so that



$$|\Delta_{\sigma\tau}(f)| < K_2$$

$$\Delta_{\sigma\tau}(g) \geq \lambda_1 > 0 \quad \text{for some constant } \lambda_1$$

$$\Delta_{\sigma\tau}(h) = 0$$

and we have

$$\Delta_{\sigma\tau}(F_{\epsilon_1\epsilon_2}) \geq \epsilon_1\lambda_1 - \epsilon_2 K_2$$

Case 4. Again $\sigma$ and $\tau$ lie in a simplex of X which has $Q_i$ as a vertex but this time the hyperplane that separates $\sigma$ and $\tau$ intersects the botto face in a bad hyperplane. We have

$\Delta_{\sigma\tau}(f) \geq \lambda_2 > 0$ by Lemma 2.20, a)

$\Delta_{\sigma\tau}(g) = 0$ this because g is linear except across good hyperplanes or $\ell_i$ = const. hyperplan

$\Delta_{\sigma\tau}(h) = 0.$

Hence

$$\Delta_{\sigma\tau}(F_{\epsilon_1\epsilon_2}) \geq \epsilon_2\lambda_2$$

By choosing $\epsilon_1$ and $\epsilon_2$ such that $0 < \epsilon_1 K_1 < \lambda_o/2$ and $0 < \epsilon_2 K_2 < \min\{\lambda_o/2, \epsilon_1\lambda_1\}$ which is clearly possible, we see that

$$\Delta_{\sigma,\tau}(F_{\epsilon_1\epsilon_2}) > 0$$

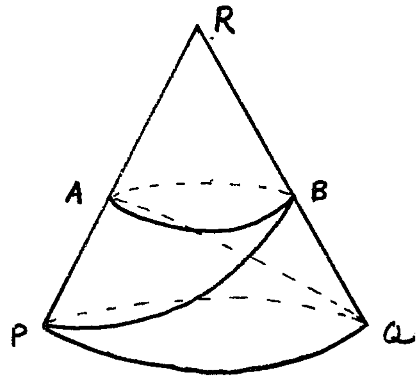
in all the cases. The theorem now follows from Lemma 1.8.

q.e.d.

## §2E

We conclude this section by giving two rather simple examples of non-projective subdivisions.

Example 1 (Hironaka).




This is two triangles which are glued together along the edges PR and QR. (Hence as a complex not embeddable in $\mathbb{R}^n$). Suppose f is a good function for this subdivision. Adding to f a function linear on each triangle we may suppose that f(P) = f(Q) = f(R) = 0. If f is good on the front triangle we necessarily get f(B) > f(A). But looking at the back we must have f(A) > f(B). Hence the subdivision is non-projective. This example corresponds to the following "blow-up":

PR and QR "correspond" to two lines $\ell$ and m in $\mathbb{P}^3$

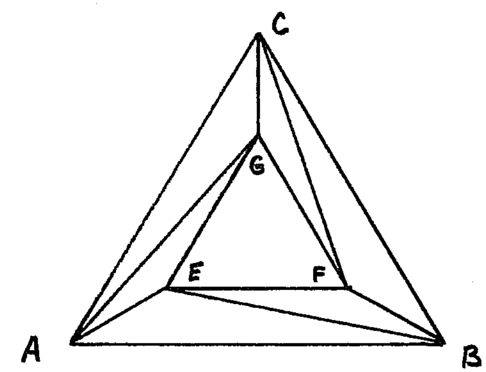

intersecting in two points S and T "corresponding" to the faces PQR front



and PQR back resp.

Now the subdivision corresponds to blowing up the two lines outside S and T. However at S we first blow up the line m and then blow up the proper transform of $\ell$ and at T we reverse the order. Of course we can glue these things together algebraically. The result yields a non-singular, non-projective complete algebraic threefold first discovered by Hironaka (Annals of Math., 75 (1962), p.190).

Example 2. (F. Commoner)




Again if f is a good function, we can assume f(A) = f(B) = f(C) = 0; but then we must have

$$f(E) > f(F) > f(G) > f(E) .$$

The geometric analogue in this case yields a proper non-projective birational morphism

$$X \xrightarrow{\pi} \mathbb{A}^3$$

such that $\pi$ is an isomorphism on $X - \pi^{-1}(o)$.



## §3. Waterman points

In this section we fix the following:

$\Delta$ is a simplex in $\mathbb{R}^n$ with an ordered set of integral vertices $P_o, \ldots, P_n$ and we denote the vector $\overrightarrow{P_oP_i}$ by $e_i$. The $e_i$'s generate a lattice which we call L.

Considering $\Delta$ as a simplicial complex we get a rational structure on $\Delta$ by restricting all integral combinations of 1 and the coordinate functions to $\Delta$. We note this rational structure by M. Moreover we suppose that the multiplicity of $\Delta$ with respect to M and the integer 1 is $k > 1$, i.e.,

$$m(\Delta, M, 1) = k .$$

The quotient group $\mathbb{Z}^n/L$ we denote by $W(\Delta)$. Clearly we have

$$\#(W(\Delta)) = k.$$

If $M_o$ denotes the set of linear plus constant functions which take integral values on the vertices of $\Delta$ we have $M \subset M_o$ and the evaluation map gives us a pairing

$$M_o/M \times W(\Delta) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$f(e_i) \longmapsto f(P_i) - f(P_o).$$

By definition $M_o/M \simeq \widehat{W(\Delta)}$ and therefore $\#(M_o/M) = k$ also.

If $[\omega]$ is a nonzero element of $W(\Delta)$, $[\omega]$ has a unique representation as a vector

$$\omega = \sum_{i=1}^{n} \alpha_i e_i \qquad \text{where} \quad 0 \leq \alpha_i < 1 \ .$$

We define $\nu([\omega]) = \{\sum \alpha_i\}$ where

$$\{\beta\} = \min\{n \in \mathbb{Z},\ n \geq \beta\},$$

i.e., $\{\beta\} = -[-\beta]$.

The point $P([\omega]) = \omega/\nu([\omega]) + P_o$ clearly lies in $\Delta$ so P is a mapping

$$P\colon W(\Delta) - (0) \longrightarrow \Delta,$$

but not necessarily 1-1.

Definition 3.1. The points in the image of P will be called the waterman points of $\Delta$.

For any waterman point $P([\omega])$ in $\Delta$, let $[\omega],[\omega_1],[\omega_2],\cdots,[\omega_s]$ be the full inverse image of $P([\omega])$. The integer

$$\nu = \min\{\nu([\omega]),\nu([\omega_1]),\cdots,\nu([\omega_s])\}$$

will be called the value of the waterman point.

Note that for a $\nu$-valued waterman point $P = P([\omega])$, $\nu\cdot P$ is an integral point.

Let P be a $\nu$-valued waterman point in $\Delta$. For each $i \in \{0,\cdots,n\}$ such that P does not belong to the face $\{P_0,P_1,\cdots,\hat{P}_i,\cdots,P_n\}$ we subdivide the simplex $\{P_0,\cdots\hat{P}_i\cdots P_n\}$ regularly with respect to this ordering and the integer $\nu$, and take the cones over the



simplices of this subdivision with apex P. This gives us a rational subdivision $\Delta'$ of $\Delta$. The following lemma is due to Alan Waterman:

Lemma 3.2. With the above notations we have $(\Delta',M)$ integral over $\frac{1}{\nu}\mathbb{Z}$ and

$$m(\Delta',M,\nu) < k.$$

Proof. We have $\nu\cdot(P-P_o) = \alpha_1 e_1+\alpha_2 e_2+\cdots+\alpha_n e_n$ where for all i, $0 \leq \alpha_i < 1$.

If $\Gamma$ is a simplex of $\Delta$ and a face of $\Gamma$ lies in the i-th face of $\Delta$ for $1 \leq i \leq n$ we have

$$\mathrm{vol}(\Gamma) = \frac{1}{\nu^n\cdot n!}\left|\det(e_1\cdots,\alpha_i e_i,\cdots e_n)\right|$$

and hence

$$m(\Gamma,M,\nu) = \alpha_i\cdot k < k.$$

If a face of $\Gamma$ lies in the face opposite $P_o$ we get

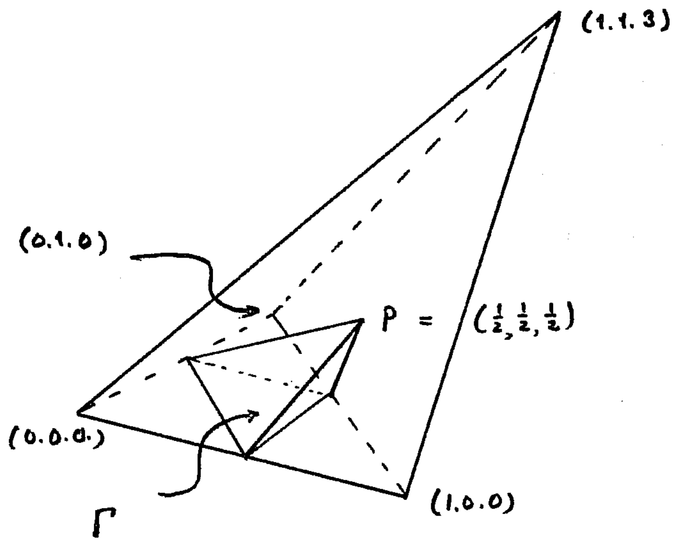
$$m(\Gamma,M,\nu) = (\nu - \sum\alpha_i)\cdot k$$

and $\nu - \sum\alpha_i < 1$ by the very definition of $\nu$. q.e.d.

We can illustrate this lemma by a simple example: let $\Delta$ be the simplex in $\mathbb{R}^3$ given by the coordinates $\{(0,0,0)(100)(010)(1,1,3)\}$. Clearly

$$m(\Delta,M,1) = 3$$

and we have two 2-valued waterman points $P = (\frac{1}{2} \frac{1}{2} \frac{1}{2})$, $Q = \{\frac{1}{2},\frac{1}{2},1\}$




If $\Delta'$ is the above subdivision of $\Delta$ with respect to P we have

$$m(\Delta',M,2) = 2.$$

In particular if $\Gamma$ is as in the picture above we have

$$m(\Gamma,M,2) = 1 .$$

Remark. Note that we can always find a waterman point P of value $\nu \leq \{n/2\}$.



Lemma 3.3. Let $\Delta_\alpha$ be any face of $\Delta$, then $\Delta_\alpha$ and $\Delta$ have the same multiplicity if and only if all the waterman points of $\Delta$ lie in the face $\Delta_\alpha$.

Proof. It is clear that for any $\alpha$ we have injections

$$W(\Delta_\alpha) \longrightarrow W(\Delta).$$

To say that all the waterman points actually lie in $\Delta_\alpha$ means that the above injection is an isomorphism and so the two groups have the same cardinality, i.e., $\Delta_\alpha$ and $\Delta$ have some multiplicity. The converse is also clear.

Corollary 3.4. Let $\Delta'$ and $\Delta''$ be two faces of $\Delta$ such that $m(\Delta') = m(\Delta'') = m(\Delta) = k$ then $m(\Delta' \cap \Delta'') = k$.

This last corollary gives us a remarkable decomposition theorem.

Proposition 3.5. Let X be a simplicial complex with a rational structure L and $\mu$ an integer such that the functions in L take values in $1/\mu\mathbb{Z}$ on the vertices of X.

Suppose $m(X,L,\mu) = k > 1$ and let

$$U = \bigcup_{m(\sigma_\alpha,L,\mu)=k} \operatorname{int}(\sigma_\alpha) .$$

Then U breaks up into connected components $U_i$ each being the open star neighborhood of a simplex of multiplicity k, i.e.,

$$U = \bigcup_{\text{disjoint}} \operatorname{star}(\sigma_i).$$

We end this section with a discussion of the existence of waterman points in the interior of a simplex.

Lemma 3.6. Let $\Delta$ be as before and suppose that

i) $W(\Delta)$ is a cyclic group.

ii) For all proper faces $\Delta_\alpha$ of $\Delta$ $\quad W(\Delta_\alpha) < k$.

Then if $[\omega]$ is a generator of $W(\Delta)$

$$P([\omega]) \in \text{int } \Delta.$$

Proof. If $P([\omega])$ belonged to some face, say $\Delta_\alpha$, we would have $[\omega] \in W(\Delta_\alpha)$.

Since $[\omega]$ is a generator of $W(\Delta)$ and $W(\Delta_\alpha)$ injects into $W(\Delta)$ this would imply that $W(\Delta_\alpha) \cong W(\Delta)$ contradicting ii). q.e.d.

Corollary 3.7. Let $X$ be as in Proposition 3.5 and let $\sigma_1 \cdots \sigma_s$ be the simplexes such that

$$U = \bigcup_{\text{disjoint}} \text{star}(\sigma_i) .$$

Then if $k$ is a prime number each of the simplices $\sigma_i$ have interior waterman points.



## §4. Statement and proof of the main theorem.

Theorem 4.1 (Main Theorem) Let $X$ be a polyhedral complex, $L$ a rational structure, and $\mu$ an integer such that $(X,L)$ is integral over $\frac{1}{\mu}\mathbb{Z}$.

Then there exists an integer $\nu$ and a rational projective subdivision $X'$ of $X$ such that $(X',L)$ is integral over $\frac{1}{\mu\cdot\nu}\mathbb{Z}$ and

$$m(X',L,\mu\cdot\nu) = 1 \quad .$$

Proof. By the transitivity of projective subdivisions we may as well suppose that $X$ is simplicial since the barycentric subdivision is rational and projective. We will prove the theorem with induction on the number $k = m(X,L,\mu)$. So suppose the theorem is true for all simplicial complexes $X''$ such that $m(X'',L'',\mu'') < k$. For the inductive step we divide into two cases.

Case 1. $k$ is a composite number.

For each simplex $\sigma_i$ of multiplicity $k$ we pick out a waterman point $P_i \in \sigma_i$, say of value $\nu_i$ and of order $p_i < k$, $p_i$ a prime number and $1 \leq i \leq s$. Then we define a decreasing sequence of rational structures $M_o \supset M_1 \supset \cdots \supset M_s \supset L$ as follows:

$$M_o = \left\{ \begin{array}{l} \text{all linear functions which take values} \\ \text{in } \frac{1}{\mu}\mathbb{Z} \text{ on the vertices of } X \end{array} \right\}$$

$$M_1 = \left\{ \begin{array}{l} \text{all linear functions } f \in M_o \text{ with the} \\ \text{extra condition } \quad f(P_1) \in \frac{1}{\mu\cdot\nu_1}\mathbb{Z} \end{array} \right\}$$

$$M_i = \begin{Bmatrix} \text{all linear functions } f \in M_{i-1} \text{ with the} \\ \text{extra condition } f(P_i) \in \frac{1}{\mu \cdot \nu_i}\mathbb{Z} \end{Bmatrix}$$

Since $M_i$ is obtained from $M_{i-1}$ by one extra condition it follows by duality that for any subdivision X" of X and any simplex $\sigma_\alpha \in |X"|$ we have

$$\#(M_{i-1,\alpha}/M_{i,\alpha}) = p_i \quad \text{or} \quad 1 \quad .$$

Note also that for each i

$$p_i \,\big|\, m(\sigma_i, M_i, \mu)$$

hence

$$m(\sigma_i, M_s, \mu) > 1 \quad .$$

Now if X" is a projective subdivision of X such that $(X", M_{i-1})$ is integral over $\frac{1}{\mu\nu"}\mathbb{Z}$ and

$$m(X", M_{i-1}, \mu\nu") = 1$$

we will have $(X", M_i)$ integral over $\frac{1}{\mu\nu"}\mathbb{Z}$ and

$$m(X", M_i, \mu\nu") \le p_i < k \quad .$$

So by induction we can find a projective subdivision X"' of X" and an integer $\nu'''$ such that $(x''', M_i)$ is integral over $\frac{1}{\mu}\nu"\nu'''\mathbb{Z}$ and



$$m(X''', M_i \cdot \mu\nu"\nu''') = 1 \quad .$$

By induction and transitivity of projective subdivisions we may proceed with this process all the way to s, i.e., there is a projective subdivision X" of X and a number $\nu"$ such that $(X", M_s)$ is integral over $\frac{1}{\mu}\nu"\mathbb{Z}$ and

$$m(X", M_s, \mu\nu") = 1 \quad .$$

Since $M_s \supset L$, (X',L) is integral over $\mu\nu"$ as well and moreover all simplices of multiplicity k with respect to L and $\mu$ have been decomposed into simplices with lower multiplicity with respect to L and $\mu\nu"$, cf. Observation 1.2.

Hence

$$m(X", L, \mu\nu") < k$$

and the theorem follows by induction.

<u>Case</u> 2. <u>k = p = prime number</u>.

Let U be the union of all the interiors of simplices of multiplicity p. Then as in Corollary 3.7, U splits up into a disjoint union:

$$U = \bigcup_{\text{disjoint}} \text{star}(\sigma_i) \quad .$$

And each $\sigma_i$ has an interior waterman point $Q_i$, say of value $\nu_i$. Let $\mu_i$ be integers such that $\nu_1\mu_1 = \nu_2\mu_2 = \nu_3\mu_3 = \cdots = \nu_i\mu_i = \cdots = \nu$, $1 \leq i \leq s$, and let X' be the mixed $(\nu_i,\mu_i)$ subdivision of X with respect to the points $Q_i$, $1 \leq i \leq s$.

Then X' is projective and all the simplices of multiplicity p have been refined, hence

$$m(X',L,\mu\cdot\nu) < p,$$

and the theorem follows by induction. q.e.d.