Geometric And Functional Analysis
Vol. 5, No. 1 (1995)
1016-443X/95/010076-16$1.50+0.20/0

Part of it (only appendix is relevant to me)

EINSTEIN STRUCTURES: EXISTENCE VERSUS UNIQUENESS

A. NABUTOVSKY

Abstract

Two well-known questions in differential geometry are "Does every compact manifold of dimension greater than four admit an Einstein metric?" and "Does an Einstein metric of a negative scalar curvature exist on a sphere?" We demonstrate that these questions are related: For every $n \geq 5$ the existence of metrics for which the deviation from being Einstein is arbitrarily small on every compact manifold of dimension n (or even on every smooth homology sphere of dimension n) implies the existence of metrics of negative Ricci curvature on the sphere S^n for which the deviation from being Einstein is arbitrarily small. Furthermore, assuming either a version of the Palais-Smale condition or the plausible looking existence of an algorithm deciding when a given metric on a compact manifold is close to an Einstein metric, we show for any $n \geq 5$ that: 1) If every n-dimensional smooth homology sphere admits an Einstein metric then S^n admits infinitely many Einstein structures of volume one and of negative scalar curvature; 2) If every compact n-dimensional manifold admits an Einstein metric then every compact n-dimensional manifold admits infinitely many distinct Einstein structures of volume one and of negative scalar curvature.

Einstein metrics are metrics of constant Ricci curvature, i.e. solutions of the Einstein equation $Ric_g = \lambda g$, where λ is a constant. Isometry classes of Einstein metrics are called Einstein structures. Many facts about Einstein structures can be found in [Be] and [S]. A study of Einstein structures is motivated by possible applications to General Relativity and also, according to [Be], by the following question of R. Thom: "Are there any best (or nicest) Riemannian structures on a given compact manifold M?" (Einstein structures are natural candidates to be considered as nice structures.) One of our results (Theorem 3) is that there is no quite satisfactory positive answer to Thom's question. More precisely, assume that one defines nice structures in such a manner that: 1) (Existence) A nice structure exists on every compact manifold of a fixed dimension $n \geq 5$; 2) (Scale-invariance) If a metric is nice, then metrics obtained by the multiplication of this metric by a positive constant are nice; and 3) (Recognizability) It is possible to recognize when a given Riemannian structure is close to a nice structure. Then the set of nice structures of volume one for *every* compact manifold

of the dimension n is infinite. This result follows from S. Novikov's theorem stating the non-existence of an algorithm recognizing S^n in the class of n-dimensional manifolds (or even n-dimensional homology spheres). (In the Appendix, we present the proof of a smooth version of this theorem which we use in this paper.) The main idea of this paper is that a sufficiently good positive answer for Thom's question would imply the existence of an algorithm recognizing S^n, which is impossible. The same idea will be used to prove other results of the paper. Informally, Theorem 1 below states that the existence of almost Einstein metrics on homology spheres with "large" fundamental groups implies the existence of almost Einstein metrics with negative scalar curvature on S^n, $(n \geq 5)$. Note that in view of results of [L] the existence of such metrics seems quite plausible. Theorem 2 states that the existence of Einstein metrics on homology spheres implies the existence of infinitely many distinct Einstein structures of volume one and of negative scalar curvature on S^n if a plausible conjecture about algorithmic recognizability of Einstein metrics (namely, statement (D) below) is true. Formulating these theorems the author was, in particular, motivated by the hope that more complicated topology of the space of Riemannian structures on homology spheres with "large" fundamental groups will make proving the existence of Einstein metrics of negative scalar curvature on such homology spheres easier than proving the existence of such metrics on S^n (see also the discussion before the proof of Theorem 1 below).

Let us say that a compact manifold M *almost admits* an Einstein metric of negative scalar curvature if there exists an infinite sequence of metrics $\{\mu_i\}$ on M such that the (C^0) norm of the tensor $Ric_{\mu_i} + \mu_i$ tends to zero. (It is clear that if M admits an Einstein metric of negative scalar curvature, then multiplying this metric by a positive constant one gets an Einstein metric g of scalar curvature $-n$ (hence, $Ric_g + g = 0$).) Note that this definition of the intuitive concept of "almost Einstein metrics" can be modified in various ways, if necessary. It can be replaced, for example, by the following definition (and all results below will remain true): Fix arbitrary positive integer k_1. We could demand the existence of a sequence of Riemannian metrics on M such that $\exp(\|R\|_{C^{k_1}} Diam^2 \max\{1,\ 1/vol\})\|Ric + g\| \to 0$ for this sequence. (R is the curvature tensor of a metric, vol is its volume, and $Diam$ is its diameter). Or, alternatively, we could demand the existence of a sequence of Riemannian metrics on M such that $Diam^2\|Ric + g\| \to 0$ for this sequence. More generally, we could put before $\|Ric + g\|$ any computable function of volume, diameter and $\|R\|_{C^{k_1}}$.

THEOREM 1. *Assume that for some $n \geq 5$ every n-dimensional smooth homology sphere with fundamental group of exponential growth almost admits an Einstein metric of negative scalar curvature. Then S^n almost admits an Einstein metric of negative scalar curvature.*

Riemannian structure on a compact manifold can be approximated to any accuracy by isometry classes of such Nash submanifolds.) We used Nash submanifolds of Euclidean space only as a way to approximately represent a Riemannian manifold by a finite set of data. Alternatively, one can use other ways to represent Riemannian manifolds by a finite set of rational or algebraic numbers in this definition. For example, one can represent Riemannian manifolds as was done in the proof of Theorem 1.

THEOREM 3. *Let $n \geq 5$ be fixed. Assume that for every compact manifold of dimension n $Nice(M)$ is a non-empty set of Riemannian structures of volume one on M, and $\bigcup_M Nice(M)$ is recognizable. Then for every compact n-dimensional manifold M $Nice(M)$ is an infinite set.*

Assume that for some $n \geq 5$ every compact n-dimensional manifold M admits an Einstein metric. Then we can regard the set of Einstein structures of volume one on M as $Nice(M)$. Theorem 3 immediately implies that:

THEOREM 4. *For any $n \geq 5$ at least one of the following three statements is false:*

(1) *Every compact n-dimensional manifold admits an Einstein metric;*
(2) *There exists a compact n-dimensional manifold admitting only a finite number of Einstein structures of volume one;*
(3) *The set of Einstein structures on compact n-dimensional manifolds is recognizable.*

Similarly to the proof of Theorem 2, one can alternatively replace statement (3) in the text of Theorem 4 by the analogue of statement (D) formulated above, but where g is allowed to be a Riemannian metric on any compact manifold of dimension n and not just only on S^n.

Appendix. A Smooth Version of S. Novikov's Theorem

Here we are going to give the proof of a smooth version of S. Novikov's theorem on the algorithmic unrecognizability of S^n, $n \geq 5$, in the class of non-singular algebraic hypersurfaces in $\mathbf{R}^{n+1}$. Our exposition is based on Novikov's proof sketched in [VKuF, ch. 10] and uses some technicalities from [BooHPo] and some ideas from semialgebraic geometry (cf. [Co], [BoCoR]).

THEOREM . *For any $n \geq 5$ there is no algorithm which for a given d and a vector of coefficients of a polynomial $p : \mathbf{R}^{n+1} \to \mathbf{R}$ with rational coefficients such that the zero set $Z(p)$ of p is non-empty, compact and non-singular, decides whether or not $Z(p)$ is diffeomorphic to the sphere S^n.*

Proof: The general idea of Novikov is to prove the algorithmic unsolvability of the triviality problem in the class of finitely presented groups with trivial first and second homology groups. Such groups can be realized as fundamental groups of homology spheres of any dimension $n \geq 5$ (see [K]). The problem of recognizing S^n among such homology spheres will be obviously equivalent to the triviality problem for their fundamental groups and, thus, unsolvable.

One starts from a finite presentation of a group G with an unsolvable word problem. Using this presentation one can effectively construct a sequence of finite presentations of groups $\{G_i\}$ such that there is no algorithm deciding which of them are finite presentations of the trivial group. (In other words the set of indices i for which groups G_i are trivial is not recursive.) (This construction was originally found independently by M. Rabin and S. Adyan.) An explicit construction of such a sequence from a given finite presentation of a group with an unsolvable word problem can be found in [M, pp. 13-14]. One can easily see from formulae on pp. 13-14 that groups with these finite presentations have trivial first homology groups. Now Novikov explicitly constructs for every group G_i a finite presentation of another group $\tilde{G}_i$, which turns out to be the universal central extension of G_i. This finite presentation can be described as follows. Let F be the free group generated by all generators $f_1, \ldots, f_k$ of G_i and R be the normal subgroup of F generated by all relators $g_1, \ldots, g_m$. Since $H_1(G_i) = G_i/[G_i, G_i]$ is trivial, every generator f_j of G_i can be written as the product $\tilde{g}_j c_j$, where $\tilde{g}_j$ is a product of some powers of the relators of G_i and $c_j \in [F, F]$. Define $\tilde{G}_i$ as the group generated by $f_1, \ldots, f_k$ with relations $\tilde{g}_j = e, j = 1, \ldots, k$ and $f_l g_j f_l^{-1} g_j^{-1} = e$, $l = 1, \ldots k$, $j = 1, \ldots, m$. To show that the universal central extension of G_i is, indeed, isomorphic to $\tilde{G}_i$, note that the universal central extension of G_i is isomorphic to $[F, F]/[R, F]$ (cf. [Mi2, Corollary 5.8]). Consider the homomorphism ψ of $[F, F]/[R, F]$ onto $\tilde{G}_i$ which is the composition of the inclusion of $[F, F]/[R, F]$ into $F/[R, F]$ and the natural surjection of $F/[R, F]$ onto $\tilde{G}_i$. Obviously, this homomorphism is surjective. (The image of c_j in $[F, F]/[R, F]$ goes to the image of f_j in $\tilde{G}_i$.) To show that ψ is injective note that $\psi(u) = e$ in $\tilde{G}_i$ implies that u lifts to an element $\bar{u}$ in F which is equal to a product of powers of $\tilde{g}$'s modulo $[R, F]$. Permuting $\tilde{g}$'s we see that $\bar{u} = \prod_{j=1}^{k} \tilde{g}_j^{n_j} mod[R, F]$. But $\bar{u} = e\, mod[F, F]$. Since the images of $\tilde{g}_j$ in $F/[F, F]$ form a system of linearly independent generators of $F/[F, F]$, all exponents n_j are equal to zero, and $\bar{u}$ is in $[R, F]$. Therefore, $u = e$ in $[F, F]/[R, F]$, and ψ is an isomorphism.

Groups $\tilde{G}_i$ have the following properties: (i) The first and the second homology groups of $\tilde{G}_i$ are trivial; (ii) $\tilde{G}_i$ is trivial if and only if G_i is trivial. Property (ii) is obvious. It implies that the triviality problem for the sequence $\{\tilde{G}_i\}$ is unsolvable. The triviality of $H_1(\tilde{G}_i)$ follows from Theorem 5.3 in [Mi2] or, alternatively, can be easily proven using the finite presentation of $\tilde{G}_i$ given above. Now Novikov notes that using homological algebra one can show that $H_2(\tilde{G}_i)$ is trivial. This can be done for example as follows (and the author wants to thank Vladimir Hinich who helped him to reconstruct this part of the proof): To show the triviality of $H_2(\tilde{G}_i)$ note that $\tilde{G}_i$ and the identical homomorphism constitute the universal central extension of $\tilde{G}_i$. (This immediately follows from Theorem 5.3 in [Mi2].) But the second homology group of a group H, such that $H_1(H)$ is trivial, is the kernel of the homomorphism of the universal central extension of H onto H (cf. Corollary 5.8 in [Mi2]). Therefore $H_2(\tilde{G}_i)$ is trivial. Thus, we have an effectively constructed sequence $\{\tilde{G}_i\}$ of finite presentations of groups with trivial first and second homology groups such that the triviality problem for this sequence is unsolvable. Now we are going to construct a sequence of compact non-singular algebraic hypersurfaces S_i in $\mathbf{R}^{n+1}$ in such a manner that S_i are homology spheres and $\pi_1(S_i) = \tilde{G}_i$. Observe that because S_i are *hypersurfaces* in $\mathbf{R}^{n+1}$, S_i will be diffeomorphic to S^n if and only if $\tilde{G}_i$ is trivial (cf. [Mi1, ch. 9, Theorem A]). Thus, the construction of such a sequence will complete the proof of the theorem.

First, for every i we use the Dehn construction and then smooth out the corners in order to build a smooth hypersurface Q_i in $\mathbf{R}^{n+1}$ such that $\pi_1(Q_i) = \tilde{G}_i$ and all homology groups of Q_i but the second and the $(n-2)$th are trivial (and the second homology group is the direct sum of several copies of $\mathbf{Z}$). The details can be found in [BooHPo]. Q_i is the smoothed out boundary of a sufficiently small neighborhood of a two-dimensional simplicial complex K_i, embedded in $\mathbf{R}^{n+1}$, with fundamental group $\tilde{G}_i$. Now we use the fact that $H_2(\tilde{G}_i)$ is trivial. By virtue of H. Hopf's theorem this fact implies that the Hurewicz homomorphism $\pi_2(Q_i) \to H_2(Q_i)$ is surjective. Thus, it is possible to realize the generators of $H_2(Q_i)$ by spheroids. Moreover, using a general position argument these generators can be realized by pairwise non-intersecting embedded spheres. These spheres will have trivial normal bundles. (This fact easily follows from Lemma 3.5 in [KMi]. These spheres can be effectively found by a trial and error algorithm which will be described below.) Now we are going to kill one by one these generators by surgeries, and we would like to perform these surgeries inside $\mathbf{R}^{n+1}$. At the beginning the unbounded connected component U_i of the complement of Q_i in $\mathbf{R}^{n+1}$ is homotopy equivalent to the complement of the two-dimensional complex K_i in $\mathbf{R}^{n+1}$ and, thus, is 2-connected. So, any embedded sphere σ in Q_i, realizing a generator of $H_2(Q_i)$, will be null homotopic in U_i. If $n+1 \geq 7$, then the standard general position argument implies that we can always realize this homotopy by a 3-disc embedded in U_i, meeting Q_i transversally along σ. If $n+1 = 6$, then one must also apply the Whitney

trick to get the embedded disk. Thus, we can perform the first surgery inside $\mathbf{R}^{n+1}$. After several surgeries we get a smooth hypersurface Q_{ij}. To show that the next surgery can be done inside $\mathbf{R}^{n+1}$ we need to demonstrate that the first and the second homotopy groups of the outer connected component U_{ij} of the complement of Q_{ij} are trivial. Q_{ij} is the boundary of a tubular neighborhood of a 3-dimensional complex K_{ij} embedded in $\mathbf{R}^{n+1}$. Thus, U_{ij} is homotopy equivalent to the complement of K_{ij}. If $n+1 \leq 7$ this implies that U_{ij} is 2-connected. If $n+1 = 6$, this immediately implies that U_{ij} is simply connected. In order to show that $\pi_2(U_{ij}) = H_2(U_{ij})$ is trivial, note that $H^3(K_{ij})$ is trivial (this follows from the fact that $H_2(K_i)$ was free abelian and K_{ij} was obtained from K_i by adding several 3-cells killing several linearly independent generators of $H_2(K_i)$), and apply the Alexander duality theorem.

When all generators of the second homology group of Q_i will be killed, we must smooth out the corners. The result will be a compact hypersurface S_i which is a boundary of a small neighborhood of a finite 3-dimensional acyclic complex $\bar{K}_i$ such that $\pi_1(\bar{K}_i) = \tilde{G}_i$. It is easy to see that the fundamental group of the constructed hypersurface is isomorphic to $\pi_1(\bar{K}_i)$ and, thus, to $\tilde{G}_i$. Using the Mayer-Vietoris exact sequence and the Alexander duality theorem one can easily see that the constructed hypersurface is a homology sphere.

All steps of the construction described above are, in fact, effective. But the shortest way to show the existence of an algorithm constructing homology spheres S_i is to use a semialgebraic trial and error algorithm (somewhat similar to the algorithm described in the proof of Theorem 1). For example, to perform the smoothing of the corners on the last stage, we look for a polynomial $p \in \mathbf{Q}[x_1, \ldots, x_{n+1}]$ such that its gradient does not vanish at any point of its zero set $Z(p)$, and such that its zero set approximates the piecewise smooth (semialgebraic) hypersurface Q obtained at the previous stage in the following sense. Let $r(Z(p))$ denote the injectivity radius of the normal exponentional map for $Z(p)$. (Informally, $r(Z(p))$ is the maximal radius of the non-selfintersecting open tube around $Z(p)$.) We require that on the normal to every point $x \in Z(p)$ there exists a single point $y(x) \in Q$ such that $|x - y(x)| \leq r(Z(p))/2$, and the map $x \to y(x)$ is a homeomorphism. It is not difficult to check that this condition is a first order formula in the theory of real closed fields. (Here we regard the coefficients of p as real variables.) By virtue of the Tarski-Seidenberg theorem (cf. [Co], [BoCoR]) this condition can be verified for every fixed vector of coefficients of a polynomial of $n+1$ variables. Thus, we can find p checking one-by-one all polynomials of degree d with rational coefficients with the numerator and denominator bounded by M (and gradually raising d and M).

Also, to find the disjoint embedded spheres realizing generators of $H_2(Q_i)$

(represented by simplicial chains in a triangulation of Q_i) we should look for a collection of disjoint polynomial (over $\mathbf{Q}$) embeddings of S^2 not to the manifold Q_i but to its neighborhood $N(Q_i)$ of a sufficiently small radius (say, half the injectivity radius of the normal exponential map for Q_i). For every such embedding we check whether or not its projection to Q_i is an embedded sphere (this is a semialgebraic condition, as well as the disjointness with the other embeddings) and whether or not this sphere represents the required element of $H_2(Q_i)$. The fact that we will find such a collection of embeddings in a recursive time follows from the fact that the algorithm must stop for every i by virtue of the already proven existence part of the construction. □

Remark: Observe that we could choose the original group G with unsolvable word problem as we like. In particular, we can assume that G has the exponential growth. Now Lemma 3.6 from [M], which we used to construct the groups G_i, implies that if G_i is not trivial, then the original group G embeds into G_i. So, in this case G_i is of exponential growth and has unsolvable word problem. One can easily see, that $\tilde{G}_i$ is also a group of exponential growth and has an unsolvable generalized word problem.

Acknowledgements. I would like to thank Professors Michael Gromov and Gang Tian for stimulating discussions.

References

[A] M.T. Anderson, Ricci curvature bounds and Einstein metrics on compact manifolds, J. Amer. Math. Soc. 2 (1989), 455-490.

[BT] E. Ballico, A. Tognoli, Algebraic models defined over $\mathbf{Q}$ of differentiable manifolds, Geom. Dedicata, 42 (1992), 155-162.

[Be] A. Besse, Einstein Manifolds, Springer, 1987.

[BoCoR] J. Bochnak, M. Coste, M.-F. Roy, Geometrie Algebrique Reelle, Springer, 1987.

[BooHPo] W. Boone, W. Haken, V. Poenaru, On recursively unsolvable problems in topology and their classification, in "Contributions to Mathematical Logic" (H. Arnold Schmidt, K. Scutte, H.-J. Thiele, eds.) North-Holland, 1968.

[C] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math. 92 (1970), 61-74.

[Co] M. Coste, Ensembles semi-algebriques, in "Geometrie Algebrique Reelle et Formes Quadratiques", Journees S.M.F. Universite de Rennes (J.-L. Colliot-Thelene, M. Coste, L. Mahe, M.-F. Roy, eds.) Springer, LN in Math. 959 (1982), 109-138.

[G1] M. Gromov, Hyperbolic manifolds, groups and actions, in: "Riemannian Surfaces and Related Topics" (I. Kra, B. Maskit, eds.) Princeton Univ. Press, Ann. of Math. Studies 97 (1981), 183-215.

[G2] M. Gromov, Asymptotic invariants of infinite groups, preprint IHES/M/92/8.

[K] M. Kervaire, Smooth homology spheres and their fundamental groups, Trans. Amer. Math. Soc. 144 (1969), 67-72.

[KMi] M. Kervaire, J. Milnor, Groups of homotopy spheres I, Ann. of Math. 77 (1963), 504-537.

[L] J. Lochkamp, Negatively curved Ricci manifolds, Bull. Amer. Math. Soc. 27 (1992), 288-291.

[M] C.F. Miller, Decision problems for groups – survey and reflections, in "Combinatorial Group Theory" (G. Baumslag, C.F. Miller, eds.), Springer, 1989.

[Mi1] J. Milnor, Lectures on the h-cobordism Theorem, Princeton Univ. Press, 1965.

[Mi2] J. Milnor, Introduction to Algebraic K-theory, Princeton Univ. Press, Ann. of Math. Studies 72 (1971).

[N1] A. Nabutovsky, Isotopies and non-recursive functions in real algebraic geometry, in "Real Algebraic Geometry" (M. Galbiati, A. Tognoli, eds.), Springer, LN in Math., 1420 (1990), 194-205.

[N2] A. Nabutovsky, Geometry of spaces of objects with complexity: algebraic hypersurfaces, knots "with thick ropes" and semi-linear elliptic boundary value problems, Ph.D. Thesis, The Weizmann Institute of Science, 1992.

[N3] A. Nabutovsky, Non-recursive functions, knots "with thick ropes", and self-clenching "thick" hypersurfaces", to appear in Comm. on Pure and Appl. Math.

[P] S. Peters, Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds, J. fur die reigne und angew. Math. 349 (1984), 77-82.

[Pe] P. Petersen V, Gromov-Hausdorff convergence of metric spaces, Proc. of Symp. in Pure Math. 54:3 (1993), 489-504.

[S] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Springer, LN in Math., 1365 (1989), 120-154.

[VKuF] I.A. Volodin, V.E. Kuznetzov, A.T. Fomenko, The problem of discriminating algorithmically the standard three-dimensional sphere, Russian Math. Surveys 29:5 (1974), 71-172.

Alexander Nabutovky

Courant Inst. of Math. Sci.	and	Dept. of Math.
New York Univ.		Univ. of Toronto
251 Mercer Street		Toronto
New York, NY 10012		Ontario, M5S 1A1,
USA		Canada
e-mail: nabutovk@cims.nyu.edu		e-mail: alex@math.toronto.edu

Submitted: April 1994