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A Note on Convex Decompositions of a Set of Points in the Plane*

Víctor Neumann-Lara[1], Eduardo Rivera-Campo[2], and Jorge Urrutia[1]

[1] Instituto de Matemáticas,Universidad Nacional Autónoma de México, Ciudad Universitaria, México, D.F. 04510
[2] Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Av. San Rafael Atlixco 186, México D.F. 09340

Abstract. For any set P of n points in general position in the plane there is a convex decomposition of P with at most $\frac{10n-18}{7}$ elements. Moreover, any minimal convex decomposition of such a set P has at most $\frac{3n-2k}{2}$ elements, where k is the number of points in the boundary of the convex hull of P.

Key words. Minimal convex decomposition

1. Introduction

Let P be a set of points in general position in the plane. A set Π of convex polygons with vertices in P and with pairwise disjoint interiors is a *convex decomposition* of P if their union is the convex hull $CH(P)$ of P and no point of P lies in the interior of any polygon in Π. A convex decomposition Π of P is *minimal* if the union of any two polygons in Π is not a convex polygon.

J. Urrutia [3] conjectured that for any set P of $n \geq 3$ points in general position in the plane, there is a convex decomposition of P with at most $n+1$ elements. Later, O. Aichholzer and H. Krasser [1] gave a set P_n with n points, for each $n \geq 13$, such that any convex decomposition of P_n has at least $n+2$ elements.

In this article we prove that for any set P of $n \geq 3$ points in general position in the plane, there is a convex decomposition of P with at most $\frac{10n-18}{7}$ elements. Moreover, we prove that if Π is a minimal convex decomposition of P, then Π has at most $\frac{3n-2k}{2}$ elements, where k is the number of points in the boundary of $CH(P)$.

2. Convex Decompositions

Let Π be a convex decomposition of a set P of points in general position in the plane. An edge e of Π is *essential* in Π if either e is contained in the boundary of

* Partially supported by Conacyt, México

$CH(P)$ or $\alpha \cup \beta$ is not convex, where α and β are the two polygons in Π that share the edge e. If e is not essential in Π, then $(\Pi \backslash \{\alpha, \beta\}) \cup \{\alpha \cup \beta\}$ is a convex decomposition of P which we denote by $\Pi - e$.

Let u, v and w be points in P. We say that the triangle $\triangle uvw$ is empty (with respect to P) if there are no vertices of P in the interior of $\triangle uvw$.

Theorem 1. *For each set P of $n \geq 3$ points in general position in the plane, there is a convex decomposition Π of P with at most $\frac{10n-18}{7}$ elements.*

Proof. If $n = 3$, then the boundary of $CH(P)$ is a convex decomposition of P with 1 element. We proceed by induction assuming $n \geq 4$ and that the result holds for every proper subset of P with at least 3 points.

If possible, let x and y be two non consecutive points in the boundary of $CH(P)$ and let L and R be the closed halfplanes defined by the line joining x and y. Let $P_1 = P \cap L$ and $P_2 = P \cap R$. By induction, there is a convex decomposition Π_1 of P_1 with at most $\frac{10n_1-18}{7}$ elements and a convex decomposition Π_2 of P_2 with at most $\frac{10n_2-18}{7}$ elements where n_1 and n_2 are the number of points in P_1 and P_2 respectively.

Clearly $\Pi_1 \cup \Pi_2$ is a convex decomposition of P. Let α and β be the unique polygons in Π_1 and Π_2, respectively, that contain the edge $e = xy$. Since $\alpha \cup \beta$ is a convex polygon, e is not essential in $\Pi_1 \cup \Pi_2$ and therefore $\Pi = (\Pi_1 \cup \Pi_2) - e$ is a convex decomposition of P with at most $\frac{10n_1-18}{7} + \frac{10n_2-18}{7} - 1$ elements. Since $n_1 + n_2 = n + 2$, Π has at most $\frac{10n-23}{7}$ elements.

We may now assume that the boundary of $CH(P)$ has exactly 3 points which we denote by a, b and c.

Case 1. There is an internal point x of P such that none of $\triangle axb$, $\triangle axc$ and $\triangle bxc$ is an empty triangle.

Let $P_1 = P \cap \triangle axb$, $P_2 = P \cap \triangle axc$ and $P_3 = P \cap \triangle bxc$. By induction, for $i = 1, 2, 3$, there is a convex decomposition Π_i of P_i with at most $\frac{10n_i-18}{7}$ elements, where n_i is the number of points in P_i. Clearly $\Pi_1 \cup \Pi_2 \cup \Pi_3$ is a convex decomposition of P.

Since $\triangle axb$ is not empty, there is a point u in the interior of $\triangle axb$ which is adjacent to x in Π_1. This implies that at least one of the edges ax or bx is not essential in $\Pi_1 \cup \Pi_2 \cup \Pi_3$ (see Fig. 1).

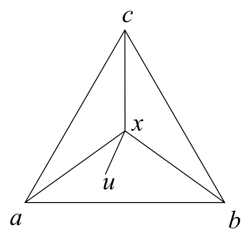

Fig. 1. Edge ax is not essential in $\Pi_1 \cup \Pi_2 \cup \Pi_3$

Analogously, at least one of the edges cx or bx and at least one of the edges ax or cx are not essential in $\Pi_1 \cup \Pi_2 \cup \Pi_3$. We claim that there are two edges $e_1, e_2 \in \{ax, bx, cx\}$ such that $\Pi = (\Pi_1 \cup \Pi_2 \cup \Pi_3) - \{e_1, e_2\}$ is a convex decomposition of P.

Since $n = n_1 + n_2 + n_3 - 5$, the number of elements in Π is

$$\begin{aligned}
|\Pi| &= |\Pi_1 \cup \Pi_2 \cup \Pi_3| - 2 \\
&= |\Pi_1| + |\Pi_2| + |\Pi_3| - 2 \\
&\leq \frac{10n_1 - 18}{7} + \frac{10n_2 - 18}{7} + \frac{10n_3 - 18}{7} - 2 \\
&= \frac{10(n_1 + n_2 + n_3) - 68}{7} \\
&= \frac{10(n + 5) - 68}{7} \\
&= \frac{10n - 18}{7}
\end{aligned}$$

Case 2. There is an internal point x of P such that two of $\triangle axb$, $\triangle axc$ and $\triangle bxc$ are empty triangles.

Without loss of generality we assume that $\triangle axb$, $\triangle axc$ contain no points of P in their interiors.

Subcase 2.1. $\triangle bxc$ is not an empty triangle.

By induction there is a convex decomposition Π_1 of $P \backslash \{a\}$ with at most $\frac{10(n-1)-18}{7}$ elements. Clearly $\Pi_1 \cup \{\Delta axb, \Delta axc\}$ is a convex decomposition of P. Since Δbxc is not empty, there is a point in the interior of Δbxc which is adjacent to x in Π_1 . This implies that there is an edge $e \in \{xb, xc\}$ which is not essential in $\Pi_1 \cup \{\Delta axb, \Delta axc\}$ and therefore $\Pi = (\Pi_1 \cup \{\Delta axb, \Delta axc\}) - e$ is a convex decomposition of P.

In this case the number of elements of Π is

$$\begin{aligned}
|\Pi| &= |\Pi_1 \cup \{\Delta axb, \Delta axc\}| - 1 \\
&= |\Pi_1| + |\{\Delta axb, \Delta axc\}| - 1 \\
&\leq \frac{10(n-1) - 18}{7} + 2 - 1 \\
&= \frac{10n - 21}{7}
\end{aligned}$$

Subcase 2.2. $\triangle bxc$ is an empty triangle.

In this case $n = 4$ and $\Pi = \{\Delta axb, \Delta axc, \Delta bxc\}$ is a convex decomposition of P with 3 elements.

Case 3. For each interior point u of P, exactly one of Δaub, Δauc and Δbuc is an empty triangle.

Let z be an interior point of P. Without loss of generality we assume that Δazb is an empty triangle.

Subcase 3.1. There is an interior point x of P such that either $\square axzb$ or $\square azxb$ is a convex quadrilateral that contains no points of P in its interior. Without loss of generality we assume the former.

Let $P_1 = P \cap \Delta axc$, $P_2 = P \cap \Delta xcz$ and $P_3 = \Delta bzc$. By induction, for $i = 1, 2, 3$, there is a convex decomposition Π_i of P_i with at most $\frac{10n_i-18}{7}$ elements, where n_i is the number of points in P_i. Clearly $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \{\square axyb\}$ is a convex decomposition of P.

Since Δaxb is empty, Δaxc cannot be empty. Therefore there is a point in the interior of Δaxc which is adjacent to x in Π_1. This implies that there is an edge $e_1 \in \{xa, xc\}$ which is not essential in $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \{\square axzb\}$. Analogously there is an edge $e_2 \in \{zc, zb\}$ which is not essential in $\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \{\square axzb\}$. We claim that $\Pi = (\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \{\square axzb\}) - \{e_1, e_2\}$ is a convex decomposition of P.

Since $n = n_1 + n_2 + n_3 - 4$, the number of elements in Π is

$$\begin{aligned}
|\Pi| &= |\Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \{\square axzb\}| - 2 \\
&= |\Pi_1| + |\Pi_2| + |\Pi_3| + |\{\square axzb\}| - 2 \\
&\leq \frac{10n_1 - 18}{7} + \frac{10n_2 - 18}{7} + \frac{10n_3 - 18}{7} + 1 - 2 \\
&= \frac{10(n_1 + n_2 + n_3) - 61}{7} \\
&= \frac{10(n + 4) - 61}{7} \\
&= \frac{10n - 21}{7}
\end{aligned}$$

Subcase 3.2. For each other interior point u of P, z is an interior point of Δaub.

Since Δazb is an empty triangle, Δazc must contain at least one point of P in its interior. Let x be a point of P in the interior of Δazc such that Δxza is an empty triangle. Analogously, there is a point y of P in the interior of Δczb such that Δyzb is an empty triangle.

Subcase 3.2.1. Δcxb is an empty triangle.

Let $P_1 = P \cap \Delta axc$ and $P_2 = P \cap \Delta xzb$. By induction, there is a convex decomposition Π_1 of P_1 with at most $\frac{10n_1-18}{7}$ elements and a convex decomposition Π_2 of P_2 with at most $\frac{10n_2-18}{7}$ elements where n_1 and n_2 are the number of points in P_1 and P_2, respectively. Clearly $\Pi_1 \cup \Pi_2 \cup \{\Delta azb, \Delta axz, \Delta cxb\}$ is a convex decomposition of P.

Since Δcxb is an empty triangle, Δaxc cannot be empty, therefore there is a point in the interior of Δaxc which is adjacent to x in Π_1. This implies that there is an edge $e_1 \in \{xa, xc\}$ which is not essential in $\Pi_1 \cup \Pi_2 \cup \{\Delta azb, \Delta axz, \Delta cxb\}$.

Since Δczb is not empty and Δcxb is an empty triangle, Δxzb is not empty. Therefore there is a point in the interior of Δxzb which is adjacent to z in Π_2. This implies that there is an edge $e_2 \in \{zx, zb\}$ which is not essential in $\Pi_1 \cup \Pi_2 \cup \{\Delta azb, \Delta axz, \Delta cxb\}$.

We claim that $\Pi = (\Pi_1 \cup \Pi_2 \cup \{\Delta azb, \Delta axz, \Delta cxb\}) - \{e_1, e_2\}$ is a convex decomposition of P.

Since $n = n_1 + n_2 - 1$, the number of elements in Π is

$$\begin{aligned}
|\Pi| &= |\Pi_1 \cup \Pi_2 \cup \{\Delta azb, \Delta ayz, \Delta cyb\}| - 2 \\
&= |\Pi_1| + |\Pi_2| + |\{\Delta azb, \Delta ayz, \Delta cyb\}| - 2 \\
&\leq \frac{10n_1 - 18}{7} + \frac{10n_2 - 18}{7} + 3 - 2 \\
&= \frac{10(n_1 + n_2) - 29}{7} \\
&= \frac{10(n + 1) - 29}{7} \\
&= \frac{10n - 19}{7}
\end{aligned}$$

Subcase 3.2.2. Δcya is an empty triangle.

Interchange y and x and a and b in Case 3.2.1.

Subcase 3.2.3. Both Δcxb and Δcya contain at least one point of P in their interiors.

Since z lies in the interior of Δaxb, both triangles Δcxb and Δaxb are not empty and therefore Δaxc is empty. Analogously Δbyc is also empty.

Subcase 3.2.3.1. The quadrilateral $\square cyzx$ contains at least one point u of P in its interior.

Without loss of generality we assume that u lies in the interior of Δcxz. Let $P_1 = P \cap \Delta cxz$ and $P_2 = P \cap \Delta czb$. By induction there is a convex decomposition Π_1 of P_1 with at most $\frac{10n_1-18}{7}$ elements and a convex decomposition Π_2 of P_2 with at most $\frac{10n_2-18}{7}$ elements, where n_1 and n_2 are the number of points in P_1 and P_2, respectively. Clearly $\Pi_1 \cup \Pi_2 \cup \{\Delta axc, \Delta axz, \Delta azb\}$ is a convex decomposition of P.

Since Δcxz is not empty then there is a point in the interior of Δcxz which is adjacent to x in Π_1. This implies that there is an edge $e_1 \in \{xc, xz\}$ which is not essential in $\Pi_1 \cup \Pi_2 \cup \{\Delta axc, \Delta axz, \Delta azb\}$. Analogously, there is an edge $e_2 \in \{zc, zb\}$ which is no essential in $\Pi_1 \cup \Pi_2 \cup \{\Delta axc, \Delta axz, \Delta azb\}$. We claim that $\Pi = (\Pi_1 \cup \Pi_2 \cup \{\Delta ayc, \Delta ayz, \Delta azb\}) - \{e_1, e_2\}$ is a convex decomposition of P.

Since $n = n_1 + n_2 - 1$ as in Subcase 3.2.1, the number of elements in Π is at most $\frac{10n-19}{7}$.

Subcase 3.2.3.2. The quadrilateral $\square cyzx$ contains no points of P in its interior.

In this case $P = \{a, b, c, z, y, x\}$ and $\{\Delta ayc, \Delta ayz, \Delta azb, \Delta bzx, \Delta bxc, \square cyzx\}$ is a convex decomposition of P with $6 = \frac{10(6)-18}{7}$ elements. Since cases 1, 2 and 3 cover all possibilities, the result follows.

3. Minimal Convex Decompositions

Let P be a set of n points in general position in the plane and T be a triangulation of P. An edge e of T is flippable if e is contained in the boundary of two triangles r and s of T such that $r \cup s$ is a convex quadrilateral. F. Hurtado *et al* proved in [2] that T has at least $\frac{n-4}{2}$ flippable edges.

In this section we use similar technics to show that T has a set $\{e_1, e_2, \ldots, e_t\}$ with at least $\frac{n-4}{2}$ edges such that the faces of $T - \{e_1, e_2, \ldots, e_t\}$ form a convex decomposition of P.

For every convex decomposition Π of P let $G(\Pi)$ denote the *skeleton graph* of Π. That is the plane geometric graph with vertex set P in which the edges are the sides of all polygons in Π.

If Π is a minimal convex decomposition of P, then for every internal edge e of $G(\Pi)$, the graph $G(\Pi) - e$ has an internal face Q_e which is not convex and at least one end of e is a reflex vertex of Q_e.

Therefore we can orient the edges of $G(\Pi)$ as follows: The edges lying in the boundary of $CH(P)$ are oriented clockwise, and every internal edge e is oriented towards a reflex vertex of Q_e. If both ends of e are reflex vertices of Q_e, the orientation of e is arbitrary. Let $\overrightarrow{G(\Pi)}$ denote the corresponding oriented geometric graph.

The following lemma is presented here without proof.

Lemma 1. *If Π is a minimal convex decomposition of P then:*

a) The indegree $d^-(u)$ of every vertex u of $\overrightarrow{G(\Pi)}$ is at most 3.
b) If $\overrightarrow{uz}, \overrightarrow{vz}$ are arcs of $\overrightarrow{G(\Pi)}$, then uz and vz lie in a common face of $G(\Pi)$.
c) If $\overrightarrow{uz}$, $\overrightarrow{vz}$ and $\overrightarrow{wz}$ are arcs of $\overrightarrow{G(\Pi)}$, then z has degree 3 in $G(\Pi)$ and lies in the interior of the triangle uvw.

Lemma 2. *Let Π be a minimal convex decomposition of P. If k is the number of vertices in the boundary of $CH(P)$, then $|V_3| \leq 2|V_0| + 2|V_1| + |V_2| - (k+2)$, where V_i denotes the set of vertices of $\overrightarrow{G(\Pi)}$ with indegree i.*

Proof. By Lemma 1b, the graph $G(\Pi)$ can be extended to plane geometric graph F_1 in which all internal faces are triangles such that if $\overrightarrow{uz}$ and $\overrightarrow{vz}$ are arcs of $\overrightarrow{G(\Pi)}$, then F_1 contains the edge uv. For each vertex $z \in V_2$, let $T(z)$ denote the triangular face of F_1 bounded by the edges uz, vz and uv, where $\overrightarrow{uz}$ and $\overrightarrow{vz}$ are the two arcs of $\overrightarrow{G(\Pi)}$ with head in z.

Let F_2 be the plane geometric graph with vertex set $V_0 \cup V_1 \cup V_2$, obtained from $G(\Pi)$ by deleting all vertices in V_3. Notice that each internal face of F_2 is a triangle and that $T(z)$ is a face of F_2 for each $z \in V_2$. By Euler's formula, the number of internal faces of F_2 is $2(|P_0| + |P_1| + |P_2|) - (k+2)$. Since each vertex $u \in V_3$ must lie in the interior of a face of F_2 which is not a face of F_1, there are at most as many vertices in V_3 as faces in F_2 which are not faces of F_1. That is $|V_3| \leq (2(|V_0| + |V_1| + |V_2|) - (k+2)) - |V_2| = 2|V_0| + 2|V_1| + |V_2| - (k+2)$.

Theorem 2. *If Π is a minimal convex decomposition of P, then Π has at most $\frac{3n-2k}{2}$ elements, where k is the number of points in the boundary of $CH(P)$.*

Proof. Let $G(\Pi)$ be the skeleton graph of Π and $\overrightarrow{G(\Pi)}$ be the corresponding oriented graph. By Lemma 2, $d^-(u) \leq 3$ for each $u \in V\left(\overrightarrow{G(\Pi)}\right)$ and therefore $\left|E\left(\overrightarrow{G(\Pi)}\right)\right| = |V_1| + 2|V_2| + 3|V_3|$, where V_i is the set of vertices of $\overrightarrow{G(\Pi)}$ with indegree i. It follows that

$$\begin{aligned} 2\left|E\left(\overrightarrow{G(\Pi)}\right)\right| &= 2|V_1| + 4|V_2| + 6|V_3| \\ &= 5(|V_0| + |V_1| + |V_2| + |V_3|) - 5|V_0| - 3|V_1| - |V_2| + |V_3| \end{aligned}$$

Since $n = \left|V\left(\overrightarrow{G(\Pi)}\right)\right| = |V_0| + |V_1| + |V_2| + |V_3|$ and, by Lemma 2, $|V_3| \leq 2|V_0| + 2|V_1| + |V_2| - (k+2)$,

$$\begin{aligned} 2\left|E\left(\overrightarrow{G(\Pi)}\right)\right| &\leq 5n - 5|V_0| - 3|V_1| - |V_2| + (2|V_0| + 2|V_1| + |V_2| - (k+2)) \\ &= 5n - 3|V_0| - |V_1| - k - 2 \end{aligned}$$

Since all vertices in the boundary of $CH(P)$ have indegree 1 in $\overrightarrow{G(\Pi)}$, $|V_1| \geq k$ and therefore

$$\begin{aligned} 2\left|E\left(\overrightarrow{G(\Pi)}\right)\right| &\leq 5n - 3|V_0| - 2k - 2 \\ &\leq 5n - 2k - 2 \end{aligned}$$

By Euler's formula, the number of internal faces of $\overrightarrow{G(\Pi)}$ is

$$1 - \left|V\left(\overrightarrow{G(\Pi)}\right)\right| + \left|E\left(\overrightarrow{G(\Pi)}\right)\right| \leq 1 - n + \frac{5n - 2k - 2}{2} = \frac{3n - 2k}{2}$$

The result follows since the elements of Π correspond to the internal faces of $\overrightarrow{G(\Pi)}$.

Let G_1 be the geometric graph in Fig. 2, and for $i \geq 1$ let G_{i+1} be geometric graph obtained from G_i as in Fig. 3, where $\overline{G_i}$ is a copy of G_i with the 3 convex hull edges removed and placed upside down.

For $i \geq 1$, G_i is the skeleton graph of a minimal convex decomposition Π_i of a set P_i with $n_i = 6i - 2$ points and. Since Π_i has $r_i = 9i - 6 = \frac{3n_i - 6}{2}$ elements, this shows that the bound in Theorem 2 is tight for $k = 3$. An analogous family of convex decompositions can be constructed for any $k \geq 3$.

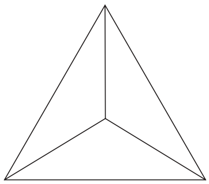

Fig. 2. The geometric graph G_1

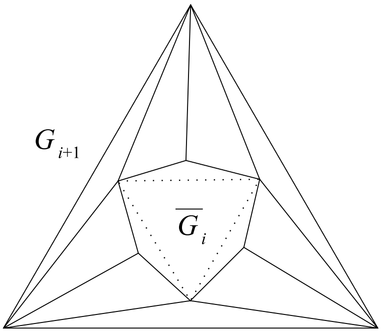

Fig. 3. The geometric graph G_{i+1}

Corollary 1. *If T is a triangulation of a set P of n points in convex position in the plane, then T* it contains a set $\{e_1, e_2, \dots, e_t\}$ *with at least $\frac{n-4}{2}$ flippable edges such that the faces of $T - \{e_1, e_2, \dots, e_t\}$ form a convex decomposition of P.*

Proof. Let Π be a minimal convex decomposition of P such that all edges of $G(\Pi)$ are edges of T. By the proof of Theorem 2, the graph $G(\Pi)$ has at most $\frac{5n-2k-2}{2}$ edges, where k is the number of points in the boundary of $CH(P)$. Since T has $3n - k - 3$ edges, there are at least $3n - k - 3 - \frac{5n-2k-2}{2} = \frac{n-4}{2}$ edges in T which are not edges of $G(\Pi)$. Clearly each of these edges is flippable in T. □

4. Final Remark

It remains as an unsolved problem to decide whether there exists a constant c such that for any set P of n points in general position in the plane, there is a convex decomposition of P with at most $n + c$ elements.

References

1. Aichholzer, O., Krasser, H.: The point set order type data base, A collection of applications and results. Proc. 13th Canadian Conference on Computational Geometry, pp. 17–20, University of Waterloo: Waterloo 2001

2. Hurtado, F., Noy, M., Urrutia, J.: Flipping edges in triangulations. Discrete Comput. Geom. **22,** 333–346 (1999)
3. Urrutia, J.: Open problem session, 10th Canadian Conference on Computational Geometry. Montreal: McGill University 1998

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