compensate for somewhat inadequate qualifications in Applied Mathematics. Such a deficiency could be helped by the provision of a tutorial class in the second year of study.

In addition, an effort should be made to help students who decide rather late in their courses, say at the end of the Part I year, that they wish to qualify as mathematical physicists. They might be admitted to the necessary courses, provided that they have shown suitable ability, and if they rectify the omissions in their earlier courses by private reading.

ON INDECOMPOSABLE POLYHEDRA

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1. Introduction. We shall prove the following

THEOREM. *If n is an integer not less than* 6, *then there exists a polyhedron,* π_n, *with n vertices and the following properties:*

(I) π_n *is simple, and every one of its faces is a triangle.*

(II) *If* τ *is a tetrahedron, each of whose vertices is a vertex of* π_n, *then not every interior point of* τ *is an interior point of* π_n.

(III) *Every open segment whose endpoints are vertices of* π_n, *but which is not an edge of* π_n, *lies wholly exterior to* π_n.

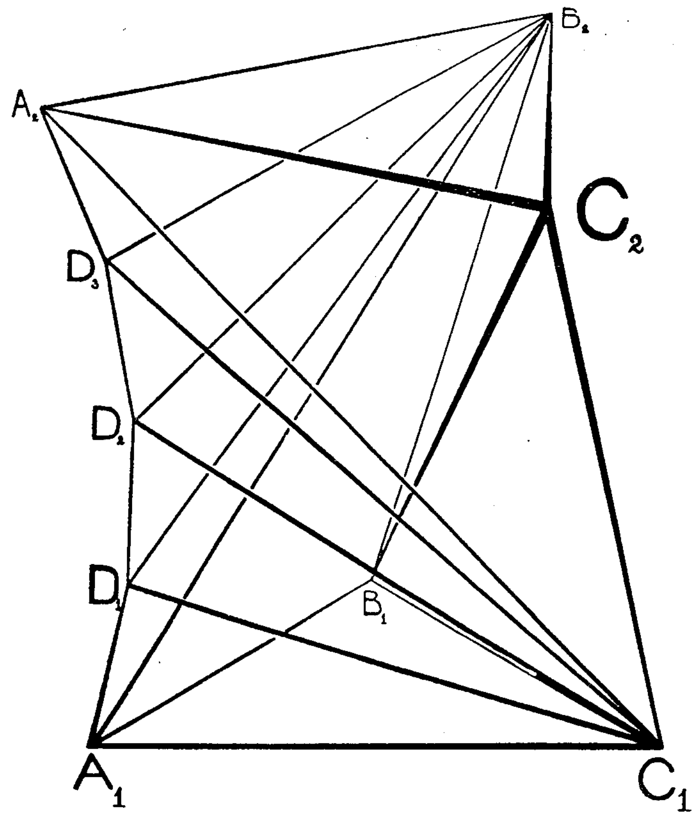
It is well known that every *convex* polyhedron can be decomposed into a set of tetrahedra whose vertices are all vertices of the given polyhedron [1, p. 280 or 2, p. 57]; and *every* simple polygon can be decomposed into a set of triangles whose vertices are all vertices of the given polygon [1, p. 246 or 2, p. 46]. Lennes, however, proved [2, p. 55] the existence of indecomposable polyhedra by constructing a polyhedron which has properties (I) and (II). His polyhedron, which possesses seven vertices, does not satisfy (III). Schönhardt [3] subsequently gave an example of a polyhedron having six vertices and all three of the above properties. He showed, moreover, that there is no indecomposable polyhedron with less than six vertices.

We shall take π_6 to be a Schönhardt polyhedron, and base our construction of π_n, for $n>6$, on π_6. It will be apparent first that, for every $n\geqq 6$, π_n satisfies (I) and the following condition:

(IV) *Every triangle whose sides are edges of* π_n *is a face of* π_n.

Then we shall demonstrate that (III) holds. Finally, (II) follows from (I), (III), and (IV). For suppose that, on the contrary, there existed a tetrahedron, τ, with its vertices and interior points consisting exclusively of vertices and interior points, respectively, of π_n. Then, because of (III), every edge of τ would be an edge of π_n, otherwise some interior point of τ would be exterior to π_n; by (IV), every face of τ would be a face of π_n; and, according to (I), ... is simple, so that π_n would have to be identical with τ. But this would contrad... our assumption that $n \geqq 6$.

2. **Description of π_6.** Let A_1, B_1, C_1 be the vertices of an equilateral triangl... each of whose sides has length 1; and let A_2, B_2, C_2 be the vertices of the tri... angle obtainable from triangle $A_1B_1C_1$ by first rotating the latter about its cente... and in its plane, through 30° in the direction $A_1B_1C_1$, and then translating it one unit in the direction perpendicular to the plane $A_1B_1C_1$, as in the figure, wher...



π_9

A_2, B_2, C_2 correspond, respectively, to A_1, B_1, C_1 under this transformation. Then π_6 consists of

6 vertices: A_1, B_1, C_1; A_2, B_2, C_2;

12 edges: A_1B_1, B_1C_1, C_1A_1; A_2B_2, B_2C_2, C_2A_2; A_1A_2, B_1B_2, C_1C_2; A_1B_2, B_1C_2, C_1A_2;

8 triangular faces: $A_1B_1C_1$, $A_2B_2C_2$; $A_1A_2C_1$, $B_1B_2A_1$, $C_1C_2B_1$; $A_1A_2B_2$, $B_1B_2C_2$, $C_1C_2A_2$.

It is evident that π_6 possesses properties (I) and (IV); and (III) also holds, because the only (open) segments whose endpoints are vertices of π_6, but which are not edges of π_6, are A_1C_2, B_1A_2, C_1B_2, and these are obviously wholly exterior to π_6.

3. **Construction of** π_n, $n>6$. Let $n=6+k$. In the interior of π_6, take an open circular arc $\overset{\frown}{A_1A_2}$, with endpoints A_1, A_2, and with a radius so large, that every point of $\overset{\frown}{A_1A_2}$ is on the same side of the plane $C_1A_2C_2$ as A_1, and on the same side of the plane $B_1A_1B_2$ as A_2. (See the figure.) On $\overset{\frown}{A_1A_2}$, choose k distinct points, $D_1, D_2, \cdots, D_k$, in the order $A_1D_1D_2 \cdots D_{k-1}D_kA_2$. Then π_n shall consist of

n vertices: A_1, B_1, C_1; A_2, B_2, C_2; $D_1, D_2, \cdots, D_k$;

$3k+12$ edges: A_1B_1, B_1C_1, C_1A_1; A_2B_2, B_2C_2, C_2A_2; B_1B_2, C_1C_2; A_1B_2, B_1C_2, C_1A_2; A_1D_1, D_1D_2, $D_2D_3, \cdots, D_{k-1}D_k$, D_kA_2; C_1D_1, $C_1D_2, \cdots, C_1D_{k-1}$, C_1D_k; B_2D_1, $B_2D_2, \cdots, B_2D_{k-1}$, B_2D_k;

$2k+8$ triangular faces: $A_1B_1C_1$, $A_2B_2C_2$; $B_1C_2B_2$, $C_1B_1C_2$, $A_1B_2B_1$, $A_2C_1C_2$; $C_1A_1D_1$, $C_1D_1D_2$, $C_1D_2D_3, \cdots, C_1D_{k-1}D_k$, $C_1D_kA_2$; $B_2A_1D_1$, $B_2D_1D_2$, $B_2D_2D_3, \cdots, B_2D_{k-1}D_k$, $B_2D_kA_2$.

Obviously π_n satisfies (I); and it is easy to see that (IV) also holds, by simply comparing the various triangles in question, with the list of faces of π_n.

The open segments whose endpoints are vertices of π_n, but which are not edges of π_n, are

A_1C_2, B_1A_2, C_1B_2; D_1C_2, $D_2C_2, \cdots, D_kC_2$; D_1B_1, $D_2B_1, \cdots, D_kB_1$; and all segments connecting pairs of nonadjacent vertices of π_n on $\overset{\frown}{A_1A_2}$, its endpoints included. It is clear that every segment of this last class is wholly outside π_n, because $\overset{\frown}{A_1A_2}$ is a circular arc. A_1C_2, B_1A_2, C_1B_2 were seen to be completely exterior to π_6; and since $\overset{\frown}{A_1A_2}$ was chosen to lie *within* π_6, these segments are also entirely exterior to π_n. Every one of the segments D_1C_2, $D_2C_2, \cdots, D_kC_2$ is wholly outside π_n. For, each intersects the (open) triangle $A_1A_2C_1$, because $\overset{\frown}{A_1A_2}$ was taken to lie inside π_6 and, at the same time, on the same side of the plane $C_1A_2C_2$ as A_1. Consequently, it is obvious that the only faces of π_n which any of these segments could possibly intersect, are the faces $C_1A_1D_1$, $C_1D_1D_2, \cdots, C_1D_{k-1}D_k$, $C_1D_kA_2$. But, since $\overset{\frown}{A_1A_2}$ is a *circular* arc, any open segment joining a point of this arc to a point of the open triangle $A_1A_2C_1$ cannot contain a point of any one of the faces in question. An analogous argument shows that every one of the segments D_1B_1, $D_2B_1, \cdots, D_{k-1}B_1$, D_kB_1 is completely exterior to π_n. Hence, π_n satisfies (III).

References

1. H. G. Forder, The Foundations of Euclidean Geometry, Cambridge, 1927.
2. N. J. Lennes, Theorems on the Simple Finite Polygon and Polyhedron, American Journal of Mathematics, vol. 33, 1911.
3. E. Schönhardt, Über die Zerlegung von Dreieckspolyedern in Tetraeder, Mathematische Annalen, vol. 98, 1928, pp. 309–312.