(TR # 050) 07/08/96

TECHNICAL REPORT

Combinatorics of Visibility and Illumination: 37 Open Problems

Joseph O'Rourke

Department of Computer Science McConnell Hall Smith College
Northampton, MA 01063 Tel: (413) 585-3804

Combinatorics of Visibility and Illumination: 37 Open Problems *(Draft)*

Joseph O'Rourke*

July 8, 1996

Abstract

A compendium of thirty-seven open problems is presented: on point visibility, floodlight illumination, and visibility graphs.

In general, a point x can *see*, is *visible* to, or *illuminates* a point y if the line segment xy is not obstructed. (The metaphors of "visibility" and "illumination" are used interchangeably.) Often this means that xy cannot contain any point of a set of obstacles, but sometimes xy is permitted to contain boundary points of the obstacles (line-of-sight grazing contact). Other nuances to the definition of visibility are appropriate for certain problems, and will be detailed below.

Throughout γ will be used to denote the constant of proportionality for quantities of the form $\gamma n + c$ for some constant c.

1 Point Visibility Problems

1.1 Art Gallery Theorems

All these problems concern simple polygons. Grazing contact is permitted, so a guard g sees an interior point x if gx is nowhere exterior, and sees an exterior point y if gy is nowhere interior. A *vertex guard* is a point located at a vertex of the polygon. An *edge guard* is an edge e of the polygon, which sees a point x if there is a point $y \in e$ such that xy is nowhere exterior. Edge guard problems can be viewed as illumination by fluorescent lightbulbs.

*Department of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu. Supported by NSF grant CCR-9421670.

1. Orthogonal Prison Yard. How many vertex guards are needed to cover both the interior and the exterior of a simple orthogonal polygon? An *orthogonal polygon* is one whose edges meet at right angles. The best lower and upper bounds are $\lceil (5n-10)/16 \rceil$ and $\lfloor 5n/12 \rfloor + 2$ [HK93]. ($\gamma \in [0.31, 0.42]$.)

2. Edge guard problem. How many edge guards are needed to cover the interior of a simple polygon? The best lower and upper bounds (for $n > 11$) are $\lfloor n/4 \rfloor$ and $\lfloor 3n/10 \rfloor$ [She94]. ($\gamma \in [0.25, 0.30]$.)

3. How many edge guards are needed to cover the interior of a star-shaped polygon? Toussaint showed that $\lfloor n/5 \rfloor$ are necessary [O'R87]. ($\gamma \in [0.2, 0.3]$.)

4. Prove or disprove that $\lfloor n/4 \rfloor$ edge guards suffice to cover a star-shaped polygon of n vertices [O'R94a].

5. The problem of finding the minimum number of guards needed to cover a given simple polygon is NP-complete [O'R87]. Is there a polynomial approximation algorithm with some guarantee of performance?

1.2 Exterior Illumination

These problems ask for the fewest exterior point light sources to illuminate the boundary of a collection of objects. Rectangles are *isothetic* if their sides are parallel to coordinate axes.

6. How many points are need to illuminate n disjoint isothetic rectangles in the plane? Fejes Tóth proved that $4n - 7$ points suffice to illuminate $n \geq 4$ disjoint compact convex sets in the plane [Fej77], but restricted cases often require fewer lights. For rectangles, examples are known that require $n - 1$ lights, and $\lfloor (4n+4)/3 \rfloor$ lights suffice [CRCU93]. ($\gamma \in [1.00, 1.33]$.) Here visibility excludes grazing contact.

7. How many points are need to illuminate n disjoint triangles in the plane? The bounds known here are exactly the same as for rectangles, for different reasons [CRCU93].

8. How many points are needed to illuminate n disjoint segments in the plane? Examples are known that require $\lceil 4n/9 \rceil$ lights [Zak96], and for $n \geq 11$, it has been proven that $\lceil 2n/3 \rceil - 3$ suffice [CRCUZ90]. ($\gamma \in [0.44, 0.67]$; it has been conjectured that $\frac{1}{2}$ is the right value.)

9. Hadwiger-Levi Covering. What is the fewest number of exterior points sufficient to illuminate any closed convex body in three dimensions? The conjecture is that $2^d = 8$ point suffice, but this is only proven for polytopes with some affine symmetry [Bez93]. Illumination here requires the lines of sight to penetrate to the interior.

2 Floodlights

A *floodlight* of angle α is a light that projects in a (contiguous) cone of angle at most α. A floodlight placed in a simple polygon with its apex at a vertex is called a *vertex floodlight*; if it has angle α, I will call it a *vertex α-light.*

10. Stage Illumination. Given a line segment (the stage) and a set of floodlights of fixed apexes and apertures, find an algorithm to decide if the floodlights can be rotated to illuminate the stage [BGL+93].

11. Do four vertex $\pi/4$-floodlights suffice to illuminate every convex polygon of $n > 4$ vertices? Only known to hold for $n \leq 4$ [OSS95].

12. Do five vertex $\pi/5$-floodlights suffice to illuminate every convex pentagon [OSS95]?

13. For what value of α is it the case that a set of vertex floodlights, whose total angle sum is at least α, suffices to cover any convex polygon of n vertices? It is known that $\alpha = 2\pi$ does suffice, as such floodlights can cover the entire plane [BGL+93], and that $\alpha = \pi$ does not suffice [OSS95].

14. Find a subquadratic algorithm for finding an optimal pair of floodlights to illuminate a convex polygon, i.e., floodlights whose angle sum is as small as possible. An $O(n^2)$ algorithm is available [ECU95].

15. Are the apexes of an optimal set of k floodlights that illuminate a convex polygon always at vertices of the polygon? Open for $k \geq 3$ [ECU95].

16. How many vertex π-floodlights are sufficient to cover any polygon of n vertices? (It is known that no angle $\alpha < \pi$ suffices, even with one light per vertex [ECOUX95].) Larman proved $\lfloor (4n+1)/9 \rfloor$ suffice. Perhaps $\lfloor n/3 \rfloor$ suffice.

3 Visibility Graphs

3.1 Polygon Vertex Visibility Graphs

Here visibility permits boundary contact: x sees y if the segment xy is nowhere exterior to the polygon. The *polygon vertex visibility graph* of a simple polygon P has a node for each vertex of P, with an arc between two nodes if the corresponding vertices are visible to one another. Note the definition of visibility implies that every edge of P is included as an arc of the visibility graph (and therefore the graph is Hamiltonian).

17. How many different vertex visibility graphs are there of polygons of n vertices? The number is in $\Omega(n^{(2-\epsilon)n})$ and $O(n^{3n})$ [HN95].

18. What is the minimum possible sum of the number of (strictly) internal and the number of (strictly) external visibility edges of a simple polygon of n vertices. Perhaps $\sim 3n/2$ is right [Bag95].

19. Develop an algorithm to recognize whether a polygon vertex visibility graph is planar. Necessary and sufficient conditions are known [LC94], but they seem not to lead to a polynomial algorithm.

20. Is the class of vertex visibility graphs realizable by polygons whose vertices are in general position, a proper subset of the class realized by all polygons? In other words, does some graph require collinearities? There are polygon vertex-edge visibility graphs that do require collinearities [OS96b].

3.2 Computational Complexity

21. Is recognition of polygon vertex visibility graphs in NP? It is only known to be in $PSPACE$ [Eve90].

22. Can any polygon vertex visibility graph be realized by a polygon drawn on an exponential-size grid? If so, visibility graph recognition is in P [LS92].

23. Is recognition of polygon vertex pseudo-visibility graphs in P? It is only known to be in NP [OS96a].

24. What is the computational complexity of computing the chromatic number of a polygon vertex visibility graph G? Finding a maximum independent set or a minimum vertex cover are both NP-complete problems when restricted to visibility graphs [LS92]. Of course $\chi(G) \geq 3$ because G can be triangulated.

3.3 Segment Visibility Graphs

There are two varieties of segment visibility graphs. The most-studied I call the "endpoint visibility graph." An *endpoint visibility graph* G of a set S of closed, disjoint line segments has a node for each segment endpoint, and an arc between two nodes x and y can see each other in the sense that $[x, y] \cap S = \{x, y\}$, or $[x, y]$: the intersection is either just the two endpoints, or the entire closed segment. Note that here visibility is blocked by grazing contact with a segment, but that G contains an arc corresponding to each segment in S.

The second version, the "segment visibility graph," will be defined in Problem 28.

25. Does the endpoint visibility graph for a set of disjoint non-collinear segments in the plane always contain a Hamiltonian cycle [Mir92]?

26. Does the endpoint visibility graph for a set of shellable disjoint non-collinear segments in the plane always contain a Hamiltonian cycle [OR94b]? A set of segments is *shellable* when it is possible to order the segments $s_1, s_2, \ldots$ such that s_i lies in the exterior of the convex hull of $\{s_1, s_2, \ldots, s_{i-1}\}$.

27. What is the complexity of recognizing whether a set of disjoint non-collinear segments in the plane admits a simple circuit? A *simple circuit* is a simple (non-crossing) Hamiltonian circuit in the segment endpoint visibility graph that includes every segment. This is known to be an NP-complete problem when the segments are non necessarily disjoint, and therefore may form polygonal chains [Rap87].

28. Prove or disprove the conjecture that every segment visibility graph for an even number of segments has a perfect matching [OR92]. A *segment visibility graph* has an arc between two segment nodes if some point of one sees some point of the other.

3.4 Rectangle Visibility Graphs

There are several varieties of "rectangle visibility graphs," sometimes known as RVG's. G is a *rectangle visibility graph* if it can be realized by closed isothetic rectangles in the plane, with pairwise disjoint interiors, with nodes corresponding to rectangles, and arcs corresponding to visibility, under the following definition of visibility: two rectangles see one another iff there is either a horizontal or a vertical unobstructed "beam" of visibility of finite width connecting them. The "beam"-visibility requirement is often called "ϵ-visibility" in the literature. Variations are obtained by permitting this beam to be a segment, or demanding that the rectangles be entirely disjoint, or that they have no collinear sides.

29. Is every rectangle visibility graph a doubly-linear graph [HSV96]? A graph is *doubly-linear* if it can be realized in the plane as the union of two plane graphs drawn with straight lines. (All rectangle visibility graphs are the union of two planar graphs, for the horizontal and for the vertical visibility edges.)

30. Is every graph with aboricity two a rectangle visibility graph [BDHS96]? The *aboricity* of a graph G is the minimum k such that G can be decomposed into k forests.

31. Is every maximum-degree 4 graph a collinear rectangle visibility graph? [BDHS96] A rectangle representation is called *noncollinear* if no line contains two distinct rectangle sides. It is *collinear* if collinearities are permitted. $K_{4,4}$ is not a noncollinear rectangle visibility graph, although it is a collinear one [DH95].

3.5 Visibility Graphs in $2\frac{1}{2}$D and 3D

Here it is reasonable to extend the ϵ-visibility definition to require a finite-volume beam of visibility for an arc of the graph.

32. What is the largest complete graph realizable by disjoint isothetic rectangles in $2\frac{1}{2}$D with vertical visibility (all rectangles are parallel to the xy-plane)? It is known that K_{20} is realizable [BEF$^+$94] and K_{55} is not [SPF96].

33. What is the largest complete graph that is a box visibility graph? Such a graph is realized by disjoint isothetic boxes in three-dimensions with lines of sight parallel to a coordinate axis. It is known that K_{42} is realizable and $K_{10^{75}}$ is not [BJMO94].

34. Is it true that in every collection of isothetic boxes in three dimensions, there is one whose removal reduces the number of edges in the associated box visibility graph? This property holds for rectangles in the plane [JMO94].

3.6 Point Visibility Graphs

A *point visibility graph* [She92] for a simple polygon P is a "continuous" graph with an uncountable number of nodes, one for each point in P, with an arc between nodes x and y iff the segment xy is nowhere exterior.

35. Find an algorithm to decide whether or not two polygons have isomorphic point visibility graphs. An $O(n^2)$ algorithm to detect such isomorphism between spiral polygons is known [MS96].

36. Is it true that if two polygons have isomorphic point visibility graphs, then there is an isomorphism between the graphs that is also a homeomorphism? This is a conjecture in [MS96].

37. Can a node corresponding to a reflex vertex be identified in a given polygon point visibility graph [MS96]?

References

[Bag95] J. Bagga. Personal communication. 1995.

[BDHS96] P. Bose, A. Dean, J. Hutchinson, and T. Shermer. On rectangle visibility graphs I: k-trees and caterpillar forests. Technical report, DIMACS and Simon Fraser University, May 1996.

[BEF+94] P. Bose, H. Everett, S. Fekete, A. Lubiw, H. Meijer, K. Romanik, T. Shermer, and S. Whitesides. On a visibility representation for graphs in three dimensions. Submitted to *J. Graph Alg. Appl*, 1994.

[Bez93] K. Bezdek. Hadwiger-Levi's covering problem revisited. In J. Pach, editor, *New Trends in Discrete and Computational Geometry*, volume 10 of *Algorithms and Combinatorics*, pages 199–233. Springer-Verlag, 1993.

[BGL+93] P. Bose, L. Guibas, A. Lubiw, M. Overmars, D. Souvaine, and J. Urrutia. The floodlight problem. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 399–404, Waterloo, Canada, 1993.

[BJMO94] P. Bose, A. Josefczyk, J. Miller, and J. O'Rourke. K_{42} is a box visibility graph. In *Snapshots of Computational and Discrete Geometry*, volume 3, pages 88–91. School Comput. Sci., McGill Univ., Montreal, PQ, July 1994. Technical Report SOCS-94.50.

[CRCU93] J. Czyzowicz, E. Rivera-Campo, and J. Urrutia. Illuminating rectangles and triangles in the plane. *J. Combin. Theory Ser. B*, 57(1):1–17, 1993.

[CRCUZ90] J. Czyzowicz, E. Rivera-Campo, J. Urrutia, and J. Zaks. Illuminating and guarding segments in the plane. Technical Report RR 90/09-13, Dept. d'informatique, Univ. du Québec à Hull, 1990. to appear in *Geometriae Dedicata.*

[DH95] A. M. Dean and J. P. Hutchinson. Rectangle-visibility representations of bipartite graphs. In R. Tamassia and I. G. Tollis, editors, *Graph Drawing (Proc. GD '94)*, volume 894 of *Lecture Notes in Computer Science*, pages 159–166. Springer-Verlag, 1995.

[ECOUX95] V. Estivill-Castro, J. O'Rourke, J. Urrutia, and D. Xu. Illumination of polygons with vertex floodlights. *Inform. Process. Lett.*, 56:9–13, 1995.

[ECU95] V. Estivill-Castro and J. Urrutia. Two-floodlight illumination of convex polygons. In *Proc. 4th Workshop Algorithms Data Struct.*, volume 955 of *Lecture Notes in Computer Science*, pages 62–73. Springer-Verlag, 1995.

[Eve90] H. Everett. Visibility graph recognition. Report 231/90, Dept. Comput. Sci., Univ. Toronto, Toronto, ON, 1990. Ph.D. Thesis.

[Fej77] L. Fejes Tóth. Illumination of convex discs. *Acta Math. Acad. Sci. Hungar.*, 29(3–4):355–360, 1977.

[HK93] F. Hoffmann and K. Kriegel. A graph coloring result and its consequences for some guarding problems. In *Proc. 4th Annu. Internat. Sympos. Algorithms Comput. (ISAAC 93)*, volume 762 of *Lecture Notes in Computer Science*, pages 78–87. Springer-Verlag, 1993.

[HN95] F. Hurtado and M. Noy. On the number of visibility graphs of simple polygons. 1995.

[HSV96] J. P. Hutchinson, T. Shermer, and A. Vince. On representations of some thickness-two graphs. In F. Brandenburg, editor, *Graph Drawing '95*, volume 1027 of *Lecture Notes in Computer Science*, pages ??–?? Springer-Verlag, 1996.

[JMO94] A. Josefczyk, J. Miller, and J. O'Rourke. Arkin's conjecture for rectangle and box visibility graphs. Technical Report 036, Dept. Comput. Sci., Smith College, July 1994.

[LC94] S.-Y. Lin and C. Chen. Planar visibility graphs. In *Proc. 6th Canad. Conf. Comput. Geom.*, pages 30–35, 1994.

[LS92] Y. Lin and S. Skiena. Complexity aspects of visibility graphs. Technical Report 92/08, SUNY Stony Brook, 1992.

[Mir92] A. Mirzaian. Hamiltonian triangulations and circumscribing polygons of disjo int line segments. *Comput. Geom. Theory Appl.*, 2(1):15–30, 1992.

[MS96] G. MacDonald and T. Shermer. Isomophism of spiral polygons. *Disc. Comput. Geom.*, 16(2):??–??, 1996.

[O'R87] J. O'Rourke. *Art Gallery Theorems and Algorithms.* Oxford University Press, New York, NY, 1987.

[OR92] J. O'Rourke and J. Rippel. Segment visibility graphs: several results. In *Proc. 4th Canad. Conf. Comput. Geom.*, pages 35–38, 1992.

[O'R94a] J. O'Rourke. Guarding corpulent stars. In *Snapshots of Computational and Discrete Geometry*, volume 3, pages 63–71. School Comput. Sci., McGill Univ., Montreal, PQ, July 1994. Technical Report SOCS-94.50.

[OR94b] J. O'Rourke and J. Rippel. Two segment classes with Hamiltonian visibility graphs. *Comput. Geom. Theory Appl.*, 4:209–218, 1994.

[OS96a] J. O'Rourke and I. Streinu. Pseudo-visibility graphs in pseudo-polygons: Part II. Technical Report 042, Dept. Comput. Sci., Smith College, Northampton, MA, April 1996.

[OS96b] J. O'Rourke and I. Streinu. The vertex-edge visibility graph of a polygon. Technical Report 047, Dept. Comput. Sci., Smith College, Northampton, MA, June 1996. Submitted for publication.

[OSS95] J. O'Rourke, T. Shermer, and I. Streinu. Illuminating convex polygons with vertex floodlights. In *Proc. 7th Canad. Conf. Comput. Geom.*, pages 151–156, 1995.

[Rap87] D. Rappaport. Computing simple circuits from a set of line segments is NP-c omplete. In *Proc. 3rd Annu. ACM Sympos. Comput. Geom.*, pages 322–330, 1987.

[She92] T. C. Shermer. Recent results in art galleries. *Proc. IEEE*, 80(9):1384–1399, September 1992.

[She94] T. Shermer. A tight bound on the combinatorial edge guarding problem. In *Snapshots in Comput. Geom.*, pages 191–223. Univ. Saskatchewan, August 1994.

[SPF96] S. Whitesides S. P. Fekete, M. E. Houle. New results on a visibility representation of graphs in 3-d. In F. Brandenburg, editor, *Graph Drawing '95*, volume 1027 of *Lecture Notes in Computer Science*, pages 234–241. Springer-Verlag, 1996.

[Zak96] J. Zaks. On illuminating segments in the plane. *Ars Combinatoria*, ??:??–??, 1996. To appear.