# On Binary Differential Equations, Quadratic Differentials, and Minimal Surfaces

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## 1 Introduction

This paper stems from our desire to understand the configuration of asymptotic and principal curves on a minimal surface in $R^3$ and their degeneration at flat umbilics. More precisely, at a general point of a minimal surface there are two asymptotic and two principal directions, and the integral lines of, say, the asymptotic directions form a family of orthogonal lines in a neighbourhood of this point. However, because the surface is minimal, umbilic points are isolated and *flat*, that is all sectional curvatures are 0. They are of interest because all directions through such points can be viewed as both asymptotic and principal.

Any family of pairs of lines can be described locally by a binary differential equation. It turns out that the family of principal curves (resp. asymptotic curves) on a minimal surface is given by a very restricted type of binary differential equation, namely one of the form $\Re\{f(z)dz^2\} = 0$ (resp., $\Im\{f(z)dz^2\} = 0$) where $f(z)$ is some holomorphic function in the complex plane and, as usual, we write $z = x + iy$ and $dz = dx + idy$. Binary differential equations of this form arise in Teichmuller theory and the theory of quasiconformal maps under the name of *quadratic differentials*. In that language, we show that one can associate to any minimal surface in $R^3$ a natural quadratic differential (the horizontal and vertical trajectories of which are the principal lines of curvature). This observation may provide a way to approach the study of the global topology of minimal surfaces. We

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hope to return to this in a future paper. For now our interests are purely local.

The flat umbilic points of a minimal surface are precisely the points where $f(z)$ has a zero. The order of the zero provides a natural measure of the 'degree of flatness' of the umbilic. We want not merely to describe the configurations of principal and asymptotic curves at these umbilics up to conformal equivalence, but also to understand the ways in which these configurations split when we deform a degenerate flat point. To do this we produce a classification of zeroes of BDE's which are quadratic differentials, and a description of their *versal unfoldings*, that is a finite-dimensional family of deformations which contains (in some precise sense) all possible deformations of the BDE. The question of relating the deformation theory of the BDE's above to those arising on the minimal surfaces is quite delicate. As a start we show that any flat umbilic can be deformed, via a family of minimal surfaces to one with only non-degenerate flat umbilics (whose degree of flatness is one). Needless to say, BDE's which are quadratic differentials are of interest in their own right.

The outline of the paper is as follows. In Section 2 we introduce some basic facts about minimal surfaces, and show how quadratic differentials arise. We also prove the deformation result mentioned above. In Section 3 we make a classification of the singularities of these quadratic differentials, corresponding to zero's of $f$, and discuss their geometry. This is more or less classical [?] and has been redone without the geometry in a much more general context by [?]. We need the geometry, and the methods of proofs we use extend to give deformation results. In Section 4 we discuss unfoldings, and prove the versality result (every singularity corresponding to a zero of $f$ has a versal unfolding). As a by-product there is an easy to apply infinitesimal criterion for versality. The versality result was obtained by J. Hubbard and H. Masur [?] in 1979. In 1984, the versality result and the infinitesimal versality criterion were obtained by V.P. Kostov [?] for forms of any order. We give proofs of both results which differ from both the above, and which are of interest because they are constructive and can easily be made effective. In Section 5 we show that the BDE's under consideration split as two ODE's, both of which are integrable in a natural way. In Section 6 we investigate the simplest degenerate singularity and its unfolding. In Section 7 we make a start on describing the configuration of integral curves for a generic deformation of the singularities. In [?] Hubbard and Masur also explore the geometry of unfoldings; however, their interests are in quadratic differentials with compact singular trajectories, their analysis is more global than ours, and they are primarily interested in one stratum of the unfolding space which is, from a purely local viewpoint, very degenerate and highly non-generic. Our analysis is purely local. In a later paper we shall continue the study these configurations.

We are grateful to Kit Nair and Phil Rippon for helping out with some problems concerning power series used in the proof of the existence of versal unfoldings. We thank Vlodya Zakalyukin for referring us to Kostov's paper. We thank Richard Morris for helping us understand the problem better by drawing many examples of solution curves to these BDE's using his Liverpool Surface Modeller Package. In particular, he produced Figures 1-5 of this paper. Finally, we thank the National Science Foundation and Esprit for partial support of this work.

## 2 Minimal Surfaces

Our basic reference for minimal surfaces will be Osserman's beautiful book [?]. As described in the introduction we wish to determine the local nature of the asymptotic and principal curves on such a surface. It turns out that we can make a useful classification using the group of conformal mappings. First we recall some basic facts from differential geometry.

Let $X$ be a surface in Euclidean 3-space $R^3$ with $p$ a point of $X$. The Gauss map associates to each such point a (coherently chosen) normal $N(p)$ in the unit sphere. The derivative of the map $N$ is the Weingarten map $L$ which is an endomorphism of the tangent space $T_pX$. If $<, >$ denotes the Euclidean inner product then a vector $v$ in $T_pX$ is asymptotic if $< Lv, v >= 0$. Clearly any multiple of $v$ is also asymptotic, so we can refer to asymptotic directions. The principal directions are the eigenspaces of $L$; since $L$ is self-adjoint the eigenvalues or principal curvatures are real, and when they are distinct the principal directions are orthogonal. When the product of the eigenvalues, the Gauss curvature, is negative there are two asymptotic directions, bisected by the principal directions. A curve is principal if its tangent vector at each point is principal, it is asymptotic if its tangent vector at each point is asymptotic.

Now minimal surfaces are those for which the sum of the principal curvatures are zero, so the Gauss curvature is non positive. They are also characterised as those surfaces whose asymptotic directions are orthogonal. When studying the configuration of principal curves on a surface [?], [?], the intersecting points are those for which the eigenvalues of $L$ coincide, the umbilics, for at such points every direction is principal. When studying the configuration of asymptotic curves [?], [?], the interesting points are those at which the Gauss curvature vanishes, for there are two asymptotic directions when the Gauss curvature is negative and none when it is positive. We see that for minimal surfaces both types of point coincide, at such *flat umbilics* all sectional curvatures vanish, and these are our principal objects of study.

The main result we shall need is the following classical result of Enneper and Weierstrass.

**Theorem 2.1** *(See [?], pp. 63, 64.) Let $D$ be a domain in the complex $z$-plane, $g(z)$ an arbitrary meromorphic function in $D$, $f(z)$ an analytic function in $D$ having the property that at each point where $g(z)$ has a pole of order $m$, $f(z)$ has a zero of order $2m$. Then if we set*

$$\phi_1 = \frac{1}{2}f(1-g^2),\ \phi_2 = \frac{i}{2}f(1+g^2),\ \phi_3 = fg$$

*the parametrisation*

$$a_k(z) = \Re\Big\{ \int_0^z \phi_k(z)dz \Big\} + constant$$

*is isothermal and determines a simply connected miminal surface. Conversely any such surface can be represented in this form.*

What we actually need is something considerably weaker than this. Namely

**Theorem 2.2** *Let $X$ be a minimal surface, $p \in X$. Then there is a neighbourhood $U$ of $p$ in $X$ having a representation as above.*

This follows directly from the proof of Theorem 2.1 above given in [?].

We now turn to the determination of the principal and asymptotic directions. Using the given parametrisation a direction corresponding to $e^{i\theta}$ gives rise to a normal curvature

$$\Big[\frac{2}{\mid f \mid (1+ \mid g \mid^2)}\Big]^2 Re\{-fg'e^{2i\theta}\}.$$

This is maximised (resp. minimised) when $2\theta = \arg(-fg')$ (resp. $\arg(fg')$). Writing $(fg') = u + iv$ the equations for the principal directions are

$$v(dy)^2 - 2udydx - v(dx)^2 = 0 \quad (a).$$

Given that the asymptotic directions bisect the principal directions and the parametrisation is conformal the equation for the asymptotic directions is

$$u(dy)^2 + 2vdydx - u(dx)^2 = 0 \quad (b).$$

Alternatively considering directions in which curvature vanishes one arrives at the same result.

Clearly we have special points when $u = v = 0$; at such points every direction is principal and asymptotic; these are the flat umbilics. They correspond to zeros of $fg'$; in other words to zeros of $f$ and of $g'$. If $g$ has a pole of order $m$ then $fg'$ has a zero of order $m - 1$.

We are interested in the zeros of $fg'$, that is the flat umbilics. In particular we wish to understand how especially flat umbilics corresponding to higher order zeros of $fg'$ will break up under deformation. Naturally the deformation should take place in the space of minimal surfaces. Our first result ensures that we can deform any flat umbilic locally so that it breaks up into simple flat umbilics, in other words so that $fg' = 0$ has only simple zeros.

**Theorem 2.3** *Let $p$ be a flat umbilic point on a minimal surface $X$, with $U$ a neighbourhood of $p$. Then we can find a smooth deformation $X_t$ of $X$ with the $X_t$ all minimal surfaces, $X_0 = X$ and $X_t$ having only simple flat umbilics.*

**Proof** We shall supose tha the point $p$ corresponds to $0 \in D$ and that $f(z) = z^{2m}F(z)$ while $g(z) = z^{-m}G(z)$ where $F$ and $G$ are holomorphic and $F(0)G(0) \neq 0$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$. We shall consider the following deformations

$$f_\alpha(z) = \prod(z - \alpha_i)^2 F(z), \quad g_{\alpha,\beta,\gamma}(z) = (\prod(z - \alpha_i)^{-1})(G(z) + \beta z + \gamma).$$

This has the right properties from Theorem 2.1 to yield, for each all $(\alpha, \beta, \gamma)$ sufficiently near $(0, 0, 0)$ a minimal surface. We now need to find $f_\alpha g'_{\alpha,\beta,\gamma}$. Well

$$g'_{\alpha,\beta,\gamma} = (\prod(z-\alpha_i)^{-1})(G'(z)+\beta) - \sum(\{(z-\alpha_j)\prod(z-\alpha_i)\}^{-1})(G(z)+\beta z+\gamma)$$

and the product

$$f_\alpha g'_{\alpha,\beta,\gamma} = (\prod(z-\alpha_i))F(z)(G'(z)+\beta) - \sum(\{\prod(z-\alpha_i)\}/(z-\alpha_j))F(z)(G(z)+\beta z+\gamma).$$

We claim that for generic choices of $\alpha, \beta, \gamma$ this product will have $m-1$ distinct simple zeros. Indeed we shall show that this is so for all $(\alpha, \beta, \gamma)$ off an analytic subset, and the result then follows from the Curve Selection Lemma, [?].

The key is the fact that we want 0 to be a regular value of $f_\alpha g'_{\alpha,\beta,\gamma}$. So we consider the map $\Lambda : D \times C^{m+2} \to C$ given by $(z, \alpha, \beta, \gamma) \mapsto f_\alpha(z) g'_{\alpha,\beta,\gamma}(z)$. Let $L_{ij}$ denote the subset of $C^{m+2}$ given by $\{(\alpha, \beta, \gamma) : \alpha_i = \alpha_j\}$. What we shall show is that 0 is a regular value of the restriction of $\Lambda$ (also denoted by $\Lambda$) to $D \times (C^{m+2} \setminus \cup L_{ij})$. For then $\Lambda^{-1}(0)$ is a smooth manifold, and if we consider the projection $\pi : \Lambda^{-1}(0) \to C^{m+2}$ then 0 is a regular value of $f_\alpha g'_{\alpha,\beta,\gamma}$ if and only if $(\alpha, \beta, \gamma)$ is not a critical value of $\pi$. But the set of critical values will be a proper analytic subset of the image space by Sard's Theorem. To establish that $\Lambda$ is indeed a submersion on the fibre over 0 we compute the partial derivatives

$$\partial\Lambda/\partial\beta = \prod(z-\alpha_i)F(z) - \sum_j\{\prod_i(z-\alpha_i)(z-\alpha_j)^{-1}\}zF(z),$$

$$\partial\Lambda/\partial\gamma = -\sum_j\{\prod_i(z-\alpha_i)(z-\alpha_j)^{-1}\}F(z).$$

Now we only have problems if $\Lambda = \partial\Lambda/\partial\beta = \partial\Lambda/\partial\gamma = 0$. It is clear that $\partial\Lambda/\partial\beta = \partial\Lambda/\partial\gamma = 0$ imply that $\prod(z-\alpha_i)F(z) = 0$. On the other hand we may suppose that $F$ does not vanish on $D$ so $z = \alpha_r$ for some $r$. But then the condition that $\partial\Lambda/\partial\gamma = 0$ shows that $\prod_{i\neq r}(\alpha_i - \alpha_r) = 0$. But the $\alpha_i$ are all distinct, and the result is proved.

We now turn to studying the pair of equations $(a)$ and $(b)$ above and their deformations, where $u + iv$ is a holomorphic function. We believe that these equations are of interest in their own right.

## 3 Binary Differential Equations

Let $U$ be an open subset of $C$ and $f : U \to C$ a holomorphic function (not to be confused with the $f$ above). Writing $f = u + iv$ we consider the corresponding equations $(a), (b)$ in Section 2. We aim to describe the corresponding integral curves, especially in a neighbourhood of zeros of $f$. If we write vectors at $z$ as $\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{dy}$ equations $(a)$ and $(b)$ can be rewritten

$$v(x,y)(\beta^2 - \alpha^2) - 2u(x,y)\alpha\beta = 0$$

$$u(x,y)(\beta^2 - \alpha^2) + 2v(x,y)\alpha\beta = 0.$$

In turn these can be rewritten respectively as $\Im\{f(z)c^2\} = 0$ and $\Re\{f(z)c^2\} = 0$ where $c = \alpha + i\beta$, or better writing $z = x + iy$ as usual, as

$$\Im\{f(z)dz^2\} = 0 \text{ and } \Re\{f(z)dz^2\} = 0.$$

We now attempt to simplify these equations by making holomorphic changes of co-ordinates. So suppose that we change co-ordinates using a holomorphic function $h : V \to U$. If $h(w) = z$ and $h'(w)dw = dz$ then the equations reduce to

$$\Im\{f(h(w))h'(w)^2dw^2\} = 0, \Re\{f(h(w))h'(w)^2dw^2\} = 0.$$

This allows us to replace $f$ by $(f \circ h)(h')^2$. Moreover since $h$ is holomorphic we still have the same picture up to conformal transformation. In fact we shall look at a more general case, namely the D.E.

$$\Im\{f(z)dz^r\} = 0 \text{ and } \Re\{f(z)dz^r\} = 0,$$

since this is really no more difficult to consider. Now we need to replace $f(z)$ by $(f \circ h)(h')^r$. The first results yields a normal local form for such BDE's.

**Definition 3.1** *We shall say that the holomorphic germs $f,\ g : C, 0 \to C, 0$ are RDE-equivalent if $g(z) = (f \circ h)(h')^r$ for some biholomorphic $h : C, 0 \to C, 0$.*

It is not difficult to check that this gives a group action of the set of biholomorphic germs on the space of function germs. In the case $r = 2$ this is the equivalence relation traditionally associated with quadratic differentials – in fact, a quadratic differential is usually defined as an expression of the form $f(z)\,dz^2$ modulo precisely this equivalence relation (see, for example, [?]). The result that follows is classical in the case $r = 2$, has been established for any non-zero complex number $r$ by Kostov in the paper [?] and has been generalised by Lando, Varchenko and others to the case of several variables; see [?] for example.

**Proposition 3.2** *Let $f : C, 0 \to C, 0$ be a holomorphic function with $f$ having a zero of order $k \geq 0$ at 0. Then we can find a holomorphic function $h : C, 0 \to C, 0$ with $h'(0) \neq 0$, such that $(f \circ h)(h')^r = z^k$ (when $k = 0$ the RHS is 1). In other words $f$ is RDE-equivalent to $z^k$.*

**Proof** Write $f(z) = g(z)^k$ with $g$ holomorphic, $g(0) = 0, g'(0) \neq 0$, so we need to solve $(g \circ h)^k (h')^r = z^k$, that is $g^k$ is equivalent to $z^k$. Since we have a group action (in particular an equivalence relation) this is the same as asking that $z^k$ is equivalent to $g^k$, in other words $g^k = h^k (h')^r$. We can now reduce to the case $g'(0) = 1$ by multiplying $h$ by some constant. Clearly now $h'(0) \neq 0$, indeed it must be one of the $(k+r)$th roots of unity. In what follows we take it to be 1.

So writing $g(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $h(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, $g_m(z) = z + a_2 z^2 \cdots + a_m z^m$, $h_m(z) = z + b_2 z^2 + \cdots + b_r z^m$ we need to solve

$$h(z)(h'(z))^{\frac{r}{k}} = g(z)$$

for $h$. If we write $\equiv_m$ for equality modulo terms of degree $m$ we first show that there is a formal power series solution. Suppose that we have found $b_2, \cdots, b_m$ with

$$h(z)(h'(z))^{\frac{2}{k}} \equiv_m g(z). \quad (*)_r$$

We seek $b_{m+1}$ such that $(*)_{m+1}$ holds. But this reduces to

$$h_m(z)(h_m'(z))^{\frac{r}{k}} + b_{m+1} z^{m+1} + \frac{2(m+1)}{k} b_{r+1} z^{m+1} \equiv_{m+1} g(z).$$

Writing $c_{m+1}$ for the coefficient of $z^{m+1}$ in $h_r(z)(h_m'(z))^{\frac{2}{k}}$ we find that

$$b_{m+1} = \{a_{m+1} - c_{m+1}\}\{k/(k + rm + 2)\}.$$

This shows that the $b_m$ exist and are unique. So a formal power series solution exists, and given $h'(0) = 1$ is unique. We could now proceed to prove that it is convergent. Instead we give an alternative argument.

Let $U$ be a neighbourhood of 0 on which $g$ is holomorphic with $g^{-1}(0) \cap U = \{0\}$ and let $w_0 \in U$. Suppose that we want to solve $g^k = (h^k)(h')^r$ at $w_0$. We write $h = y$, and note that the above equation reduces formally to

$$y^{\frac{k}{r}} \frac{dy}{dz} = g^{\frac{k}{r}}(z).$$

Proceeding formally and integrating both sides we find

$$\frac{r}{(k+r)} y^{\frac{k+r}{r}} = \int g^{\frac{k}{r}}(z) dz + c$$

so that

$$y = \left\{ \frac{(k+r)}{r} \int g^{\frac{k}{r}}(z) dz \right\}^{\frac{r}{(k+r)}}.$$

If $k$ is a multiple of $r$ there is no problem with the formal integral $h(z) = \int g^{\frac{k}{r}}(z)dz$, and one can check that it has the required properties. If $k$ is odd we write $g^{\frac{k}{r}}(z) = \pm z^{\frac{1}{2}}H(z)$ for some convergent $H$ and integrate term by term to obtain $\pm z^{\frac{k}{r}}H_1(z)$ for some $H_1$. (We set the constant equal to zero.) Now $\{\frac{(k+r)}{r}\int g^{\frac{k}{r}}(z)dz\}^r$ in both ($\pm$) cases yields a well defined holomorphic function on $U$. Moreover this function has a zero of order $k+r$ at the origin, so can be written $z^{k+r}H(z)$ with $H(0) \neq 0$. So we get $h(z) = z(H(z))^{\frac{1}{(k+r)}}$, with $h(0) = 1$, and this has the required properties.

So we are reduced (in the case of major interest to us $r = 2$) to studying the binary differential equations

$$\Re(z^k dz^2) = 0, \Im(z^k dz^2) = 0.$$

Note that since replacing $f(z)$ by $if(z)$ interchanges real and imaginary parts, we can concentrate on one of the cases (the $\Re$ case in what follows). The case where the BDE's are equivalent to $\Re/\Im\{z^k dz^2\} = 0$ is referred to as type $A_k$.

Before proceeding further note that there is an action of the positive multiplicative reals $R^+$ via $z \mapsto tz$, since $\Re\{z^k dz^2\} = 0$ if and only if $\Re\{t^{k+2}z^k dz^2\} = 0$. So the corresponding mappings preserve the integral curves. In particular the local picture of the integral curves near $(0,0)$ is the same as that inside a disk of any positive radius.

**Example 3.3** *Cases* $f(z) = z^k$.

In the case $k = 0$ we obtain repectively the curves $y = \pm x + c$; $y = c$, $x = c$.

More generally writing $z = x + iy$ we can rewrite $z^k = A(x,y) + iB(x,y)$ where $A$ and $B$ are homogeneous of degree $k$ in $x$ and $y$. We show how one can solve the resulting BDE's by quadrature; the fact that one can do so is, as is usual, because of the existence of the 1-parameter family of automorphisms mentioned above.

For the pair then becomes

$$B(x,y)(dy)^2 - 2A(x,y)dydx - B(x,y)(dx)^2 = 0,$$

(*)

$$A(x,y)(dy)^2 + 2B(x,y)dydx - A(x,y)(dx)^2 = 0.$$

These are homogeneous equations and can be solved by integration. So write $u = \frac{y}{x}$, $v = x$, so that $dx = dv, dy = udv + vdu$, and the second BDE (for example) becomes

$$A(v,uv)((dv)^2 - (udv + vdu)^2) - 2B(v,uv)dv(udv + vdu) = 0.$$

Factoring out the $v^k$ terms this is

$$\{A(1,u) - 2uB(1,u) - u^2A(1,u)\}(dv)^2 +$$

$$\{-2uvA(1,u) - 2vB(1,u)\}dvdu + \{-v^2A(1,u)\}(du)^2 = 0$$

i.e.

$$Q(u)(\frac{dv}{du})^2 + R(u)v(\frac{dv}{du}) + S(u)v^2 = 0$$

where

$Q(u) = A(1,u) - 2uB(1,u) - u^2A(1,u), R(u) = -2(A(1,u) + B(1,u)), S(u) = -A(1,u).$

So

$$\frac{dv}{du} = \frac{-vR(u) \pm \sqrt{v^2R(u)^2 - 4v^2Q(u)S(u)}}{2Q(u)}.$$

Note that $Q(0) = A(1,0) = 1$ so the RHS makes sense near $u = 0$, and $R^2(0) - 4Q(0)S(0) = 8$. We can now separate variables to obtain

$$\frac{dv}{v} = \frac{-R(u) \pm \sqrt{R^2(u) - 4Q(u)S(u)}}{2Q(u)}du$$

which can now be integrated.

(Working with polar co-ordinates one can also replace $z$ by $re^{i\theta}$ and $c$ by $se^{i\varphi}$, so the BDE's become

$$\Re \text{ or } \Im(e^{i(k\theta+2\varphi)}) = 0,$$

so that

$$k\theta + 2\varphi = m\pi + \frac{\pi}{2}, k\theta + 2\varphi = m\pi$$

for some integer $m$.)

Returning to the equations (*) and taking $k = 1$ we obtain

$$y((dy)^2 - (dx)^2) - 2xdydx = 0 \quad (a)$$

$$x((dy)^2 - (dx)^2) + 2ydydx = 0 \quad (b).$$

Using the terminology of, for example [?], these are both stars. We seek the singular directions: in $(a)$ we set $y = \alpha x$ and deduce that $\alpha(\alpha^2 - 3) = 0$ so the singular directions, in polar co-ordinates, are given by $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$; for $(b)$ the singular directions are $\theta = \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and the pictures for the integral

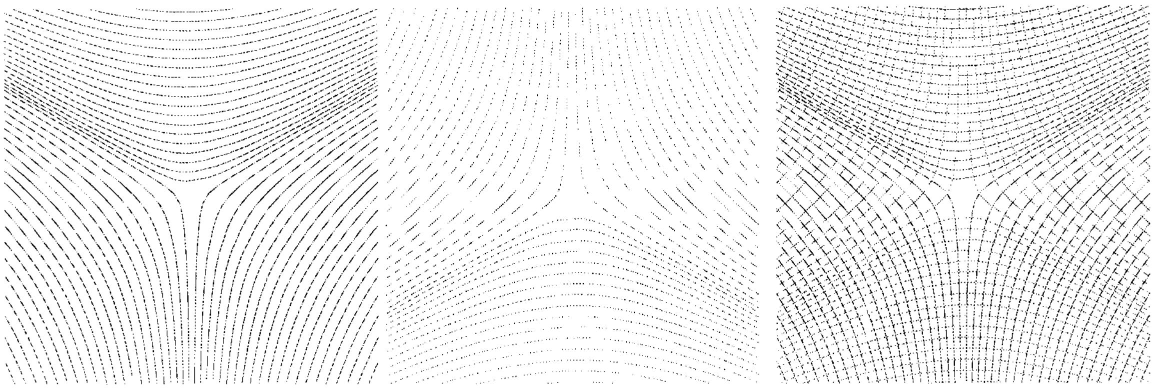

Figure 1: Non-degenerate singular point

curves of $(a)$ and $(b)$ is simply that of Figure 1 together with the same figure rotated through $\frac{\pi}{2}$ and superimposed on itself.

Indeed returning to the general case (*) above note that replacing $z$ by $\omega z$ for a constant $\omega$ replaces the original DE by

$$\Re \text{ or } \Im(\omega^{k+2}z^k dz^2) = 0.$$

If $\omega^{k+2} = \pm i$ this means that the original equations are interchanged; if $\omega^{k+2} = \pm 1$ we have an automorphism. So in general we expect a $(k+2)$-fold symmetry of our DE's (rotation through an angle of $\pi/(k+2)$) and the picture of the one DE can be obtained from the other by rotation through $\pi/2(k+2)$. So taking $k = 2$ we obtain Figure 2.

Note that in the polar form of the BDE the singular directions are given by $\theta = \varphi$ so $(k+2)\theta = m\pi + \frac{\pi}{2}$, (resp. $m\pi$).

Note that since our normal forms were up to a conformal transformation the angles between the singular directions at any singular point are always equal.

**Remark 3.4** *It is interesting to also consider the case of BDE's of the form* $\Re/\Im\{f(z)dz^2\} = 0$ *where the function* $f(z)$ *could have poles. This case is not needed for minimal surfaces, but it is important for Riemann surfaces, where one often needs to consider meromorphic quadratic differentials. One seeks local normal forms in a neighbourhood of the poles. It is not difficult to see that things will not be as simple as in the zeros case: one obtains a single modulus when the order of the pole is even, and none when the order of the modulus is odd. Although this can be proved by the methods above, we*

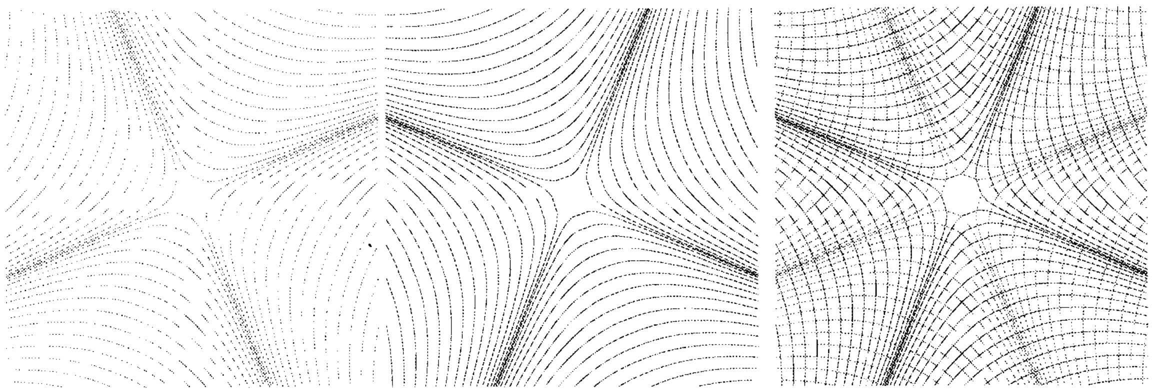

Figure 2: Case k=2

*do not give details. For a different proof see [?]. We remark that one can deduce deformation results using [?] together with the observation that one can replace $gdz^2$ with $g^{-1}dz^{-2}$ in the case that $g$ has a pole (since one can always replace $f^\alpha dz$ with $fdz^{(1/\alpha)}$).*

# 4 Unfoldings

When studying the binary differential equation $\Re\{f(z)dz^2\} = 0$ we have seen that there are a (countably) infinite family of local models $f(z) = z^k$, one for each non-negative integer $k$. The higher the value of $k$ the more degenerate (in some sense) the equations. On the minimal surfaces these correspond to flatter and flatter umbilics. It is natural to ask how these degenerate umbilics break up under deformation, and what the corresponding deformed configuration of lines looks like. This leads to the following definitions.

**Definition 4.1** *Let $f : C, 0 \to C, 0$ be a (holomorphic) function. A function $F : C \times C^r, 0 \to C, 0$ with $F(z, 0) = f(z)$ is an **unfolding** of $f$. Let $a : C \times C^s, (0, 0) \to C, 0, b : C^s, 0 \to C^r, 0$ be holomorphic with $a(z, 0) = z$. Then $G : C \times C^s, 0 \to C, 0$ defined by $G(z, w) = F(a(z, w), b(w))\{\frac{\partial a}{\partial z}(z, w)\}^2$ is said to be **induced** from $F$. If $F$ is such that every unfolding of $f$ can be induced from $F$ then $F$ is called a **BDE-versal unfolding** of $f$.*

**Remark 4.2** *These are just the standard definitions for unfoldings under right equivalence adapted to our new equivalence relation. See for example [?].*

The basic fact that we need is the following.

**Proposition 4.3** *For small $v$ the maps $a_v(z) = a(z,v)$ map the integral curves of the family $\Re/\Im\{F_{b(v)}(z)dz^2\} = 0$ to those of $\Re/\Im\{G_v(z)dz^2\} = 0$.*

In particular if $F$ is a versal unfolding of $f$ then the solution curves of the families $\Re/\Im\{F_u(z)dz^2\} = 0$ contain pictures, up to conformal equivalence, of all deformations of $\Re/\Im\{f(z)dz^2\} = 0$. The main result we wish to prove is the following, which can be found in [?], where it is proved using the inverse function theorem for Banach spaces. More general results have been established by [?], by Lando [?] and by Kostov and Lando in [?], using singularity-theoretic methods. The proof given here is an adaptation of the one given in [?] for the case of right-equivalence of functions of a single variable and has the virtue of being constructive.

**Theorem 4.4** *If $f(z) = z^k$ then $G : C \times C^{k-1}, 0 \to C, 0$ defined by $G(z,u) = z^k + u_1 z^{k-2} + u_1 z^{k-3} + \cdots + u_{k-1}$ is BDE-versal.*

**Proof** Let $F : C \times C^s, 0 \to C, 0$ be an unfolding of $z^k$. We first note that it is enough to produce

$$a : C \times C^s, 0 \to C, 0, b : C^s, 0 \to C^{k-1}, 0$$

with $a(z,0) = z$ and

$$G(z, b(w)) = F(a(z,w), w)\{\frac{\partial a}{\partial z}(z,w)\}^2 \quad (**).$$

This just uses the implicit function theorem. We first show that we can find formal power series $a, b$ which satisfy (**). Indeed we shall show that $a$ and $b$ are unique.

Because of further applications we shall actually establish the existence of $a$ and $b$ as above satisfying

$$G(z, b(w)) = F(a(z,w)w)\{\frac{\partial a}{\partial z}(z,w)\}^r$$

for any integer $r \geq 1$, so our theorem is just the special case $r = 2$. So by hypothesis we have

$$F(z,0) = z^k \text{ and } G(z,u) = z^k + u_1 z^{k-2} + \cdots + u_{k-1}.$$

We seek $a(z,w), b(w)$ with

$$(1)\quad F(a(z,w),w)\{\frac{\partial a}{\partial z}(z,w)\}^r = G(z,b(w)),$$

$$(2)\quad a(z,0)=z.$$

Write

$$a(z,w) = a_0(z,w) + a_1(z,w) + a_2(z,w) + \cdots +$$
$$b(w) = b_0(w) + b_1(w) + b_2(w) + \cdots +$$
$$a^p(z,w) = (a_0 + a_1 + \cdots + a_p)(z,w)$$
$$b^p(z) = (b_0 + b_1 + \cdots + b_p)(w)$$

where the $a_i$ and $b_i$ are homogeneous of degree $i$ in $w$. To solve (1), our proof is by induction on $p$. We need to show that we can find $a_0, \cdots, a_p, b_0, \cdots, b_p$ with

$$F(a^p(z,w),w)\{\frac{\partial a^p}{\partial z}(z,w)\}^r = G(z,b^p(w))$$

modulo terms (in $w$) of degree $p+1$; we write this as $\equiv_{p+1}$. We get the induction started for $p=0$ by taking $a_0(z,w)=z, b_0(w)=0$. Suppose we have done the case $p$. We need to find $a_{p+1}, b_{p+1}$ with

$$F((a^p + a_{p+1})_g(z,w),w)\{(\frac{\partial a^p}{\partial z} + \frac{\partial a_{p+1}}{\partial z})(z,w)\}^r \equiv_{p+2} G(z, b^p(w) + b_{p+1}(w)).$$

Using Taylor's theorem this is equivalent to

$$(a)\qquad F(a^p(z,w),w)\{\frac{\partial a^p}{\partial z}(z,w)\}^r$$

$$(b)\qquad + rF(a^p(z,w),w)\{\frac{\partial a_{p+1}}{\partial z}(z,w)\}$$

$$(c)\qquad +a_{p+1}(z,w)\frac{\partial F}{\partial z}(a^p(z,w),w)\{\frac{\partial a^p}{\partial z}(z,w)\}^r \equiv_{p+1} G(z,b^p(w)+b_{p+1}(w)).$$

Since we are only interested in terms of degree $\leq p+1$ in (b) the only relevant term is $rz^k\frac{\partial a_{p+1}}{\partial z}(z,w)$ while for (c) we need only consider $a_{p+1}(z,w)kz^{k-1}$. So since

$$G(z,b^p(w)+b_{p+1}(w)) = G(z,b^p(w)) + \sum_{j=0}^{k-2}(b_{p+1})_j z^j$$

this reduces to

$$-F(a^p(z,w),w)\{\frac{\partial a^p}{\partial z}(z,w)\}^r + G(z,b^p(w)) \equiv_{p+2}$$

$$rz^k\frac{\partial a_{p+1}}{\partial z}(z,w) + kz^{k-1}a_{p+1}(z,w) - \sum_{j=0}^{k-2}(b_{p+1})_j z^j.$$

By induction the expression on the left hand side of the equivalence contains no terms of degree $\leq p$. So we are reduced (comparing coefficients of terms of degree $p+1$) to writing any power series $\theta(z)$ in the form

$$\theta(z) = z^{k-1}\{rz\alpha'(z) + k\alpha(z)\} + \sum_{j=0}^{k-2}\beta_j z^j \quad (***)$$

for some power series $\alpha$.

Write $\alpha(z) = \alpha_0 + \alpha_1 z + \alpha_2 + \cdots$; then

$$rz\alpha'(z) + k\alpha(z) = r(\alpha_1 z + 2\alpha_2 z^2 + 3\alpha_3 z^3 + \cdots)$$

$$+k(\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots) = k\alpha_0 + (r+k)\alpha_1 z + (2r+k)\alpha_2 z^2 + \cdots.$$

Since any $\theta(z)$ can be written (uniquely) as

$$\theta(z) = z^{k-1}\Lambda(z) + \sum_{j=0}^{k-2}\beta_j z^j$$

for some power series $\Lambda = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots$ the result follows: take $\alpha_j = \lambda_j/(jr+k)$. (Note we have shown that $a, b$ are unique.) This completes the proof of the existence of $a$ and $b$ at the formal power series level.

Now for the proof of convergence. This clearly involves some estimate on the size of the terms $\alpha$ and $\beta_j$ which arise in the solution of (***). Since $\theta$ already depends on $a^p$ and $b^p$ any such estimate will necessarily be obtained by induction on $p$. We need the following notation and results. All power series will be expanded about the origin in some complex space $C^m$.

**Notation**

(1) For $\epsilon > 0$ let $D_\epsilon$ denote the closed disc in the complex plane $D_\epsilon = \{z \in C : |z| \leq \epsilon\}$. For $s \geq 2$ the product $\{(x_1, \ldots, x_s) \in C^s : |x_i| \leq \epsilon, 1 \leq i \leq s\}$ is denoted by $B_\epsilon$.

(2) Suppose that $h(z,x)$ (resp. $h(x)$) is analytic on some neighbourhood of the origin in $C \times C^s$ (resp. $C^s$), and let $\nu = (i_1, \ldots, i_s)$ be a multi-index.

Then $h_\nu(z)$ (resp. $h_\nu$) will denote the coefficient of $x_1^{i_1} x_2^{i_2} \dots x_s^{i_s}$ in the power series expansion of $h$. We also write $|\nu|$ for $i_1 + \dots + i_s$.

(3) Let $h$ be analytic in some neighbourhood of $D_\epsilon$ in $C$, so we can expand it as a power series $\sum a_k z^k$. We write $||h||$ for $\sum |a_k| \epsilon^k$, this sum converges since $h$ is absolutely convergent on some neighbourhood of $D_\epsilon$. Clearly $|h(z)| \leq ||h||$ for all $z \in D_\epsilon$

(4) Let $h$ be analytic in some neighbourhood of $D_\epsilon \times 0$ in $C \times C^r$ (resp. of $0 \in C^r$) and let $g$ be analytic on some neighbourhood of 0 in $C^r$. Then we write $h \ll g$ to mean that $||h_\nu|| \leq |g_\nu|$ (resp. $|h_\nu| \leq |g_\nu|$) for all $\nu$.

(5) For a vector-valued function $h$ consisting of $k$ analytic functions, $h = (h_1, \dots, h_k)$ we write $h \ll g$ when each $h_i \ll g$.

**Lemma 4.5** *(See [?], p 244.) Suppose that $H$, $G$, $h_i$, $g_i$, $1 \leq i \leq m$, are analytic functions, with*

$$H(x_1, \dots, x_m) \ll G(x_1, \dots, x_m),$$

$$h_i(y_1, \dots, y_n) \ll g_i(y_1, \dots, y_n), \quad 1 \leq i \leq m,$$

*and suppose that the coefficients in the power series expansion of $G$ and the $g_i$ are real and nonnegative. Then*

$$H(h_1(y), \dots, h_m(y)) \ll G(g_1(y), \dots, g_m(y)).$$

Next we have a key result.

**Lemma 4.6** *Given any $\epsilon$ with $0 < \epsilon < 1$ there is a constant $L > 0$ (depending only on $\epsilon$) so that every function $\theta(z)$, analytic on $D_\epsilon$ can be written uniquely in the form*

$$\theta(z) = z^{k-1}\{rz\alpha'(z) + k\alpha(z)\} + \sum_{j=0}^{k-2} \beta_j z^j$$

*with $\alpha(z)$ analytic on $D_\epsilon$ and $||\alpha|| \leq L||\theta||$, $||\alpha'|| \leq L||\theta||$ and $|\beta_j| \leq ||\theta||$.*

We first need the following result.

**Lemma 4.7** *Let $f(z) = \sum_{j \geq 0} a_j z^j$ be a function analytic on $D_\epsilon$, $r$, $k \geq 1$ and $g(z) = \sum_{j \geq 0} \frac{a_j z^j}{jr+k}$, so that $f(z) = rzg'(z) + kg(z)$ where this makes sense. Then $g$ is also analytic on $D_\epsilon$. Moreover $||g|| \leq ||f||$, and $||g'|| \leq ||f||$.*

**Proof** The first assertion is easy. Since $f$ is analytic on $D_\epsilon$ its radius of convergence is larger than $\epsilon$. So the series is absolutely convergent on some closed disk $D_{\epsilon'}$ with $\epsilon' > \epsilon$. So $\sum |a_j| z^j$ converges for $|z| \leq \epsilon$, hence so does $\sum |a_j||z|^j$ and consequently $\sum \frac{|a_j|}{jr+k} z^j$. So $g$ is absolutely convergent on $D_\epsilon$. The inequalities are trivial.

We can now return to the proof of the preceding result.

**Proof** Any $\theta(z)$ can be written as a sum

$$\theta(z) = z^{k-1}\{\gamma(z)\} + \sum_{j=0}^{k-2} \beta_j z^j$$

with $\gamma(z)$ analytic on $D_\epsilon$. Clearly $||\gamma|| \leq ||\theta||\epsilon^{1-k}$ and $|\beta_j| \leq ||\theta||$. Now apply the previous result to represent $\gamma(z)$ in the form $\{rz\alpha'(z) + k\alpha(z)\}$ for some function $\alpha(z)$ with $||\alpha||$ and $||\alpha'|| \leq ||\gamma|| \leq ||\theta||\epsilon^{1-k}$.

The final ingredients of the proof are two convergent power series which will be used in the comparison process. Let $e, d, E, D$ be real and positive constants and set

$$A(w) = \frac{e}{d}\sum_{j=1}^{\infty} \frac{d^j}{j^2}(w_1 + \ldots + w_s)^j,$$

$$B(z, w) = \frac{E}{D}\sum_{j=1}^{\infty} D^j(z + w_1 + \ldots + w_s)^j.$$

These power series have the following important properties.

**Lemma 4.8** *(i) $A$ (resp. $B$) is convergent in some neighbourhood of $0 \in C^s$ (resp. $C \times C^s$).*

*(ii) $A(w)^p \ll (3e/d)^{p-1} A(w)$ for $p \geq 2$.*

For a proof see [?] page 246.

We now come to the proof of convergence. First the initial case, then the inductive step.

In what follows we shall suppose that $F(z, w)$ is analytic on a neighbourhood of $D_{2\epsilon} \times B_\epsilon \subset C \times C^s$, and the $a(z, w)$ and $b(w)$ are the power series constructed above.

**Proposition 4.9** *We may choose $D$, $E$ and $e > 0$ so that*

*(i) $a^1(z, w), b^1(w), w_i$ are all $\ll A(w)$, $1 \leq i \leq r$.*

*(ii) $F(z + t, w) - F(z, 0) \ll B(t, w)$.*

**Proof** We follow that given in [?] page 247.

(i) So writing $a^1(z,w) - z = \sum_{j=1}^{s}(a_1(z))_j w_j$, $b^1(w) = \sum_{j=1}^{s}(b_1)_j w_j$ we choose $e$ so that $e \geq \max\{||(a_1(z))_j||,\ |(b_1)_j|,\ 1\}$; note that this choice is independent of $d$.

(ii) Writing $F(z+t,w) - F(z,0) = \sum A_{ij\nu} z^i t^j w^\nu = \sum g_{j\nu}(z) t^j w^\nu$ we set $M = \sum |A_{ij\nu}| \epsilon^{i+j+|\nu|}$ which converges, and clearly $||g_{j\nu}|| \leq M\epsilon^{-(j+|\nu|)}$. Writing $c_{j\nu}$ for the binomial coefficient of $t^j w^\nu$ in $(t + w_1 + \ldots + w_s)^{j+|\nu|}$, the coefficient of $t^j w^\nu$ in $B(t,w)$ is $ED^{j+|\nu|-1}c_{j\nu}$. Choosing $D \geq 1/\epsilon$, $E \geq M/\epsilon$ and using $c_{j\nu} \geq 1$ we have

$$|ED^{j+|\nu|-1}c_{j\nu}| \geq ED^{j+|\nu|-1} \geq M\epsilon^{-(j+|\nu|)} \geq ||g_{j\nu}||$$

as required.

Now for the inductive step.

**Proposition 4.10** *For a suitable choice of $d$ we have*

$$a_\nu(z)w^\nu,\ a'_\nu(z)w^\nu,\ b_\nu w^\nu \text{ all} \ll A(w)$$

*for all $\nu$ with $|\nu| \geq 1$. In particular the $a^p(z,w) - z$, $b^p(w)$ are all $\ll A(w)$ for all $p$.*

**Proof** The case $|\nu| = 1$ is dealt with by the proof of the previous result. Before starting the induction we remark that, denoting the linear part in $t$ and $w$ of $F(z+t,w) - F(z,0)$ by $\mathcal{L}(z,t,w)$, we have by the previous result

$$F(z+t,w) - F(z,0) - \mathcal{L}(z,t,w) \ll (E/D)\sum_{j=2}^{\infty} D^j(t + w_1 + \ldots + w_s)^j.$$

Now let $p \geq 1$ and suppose that the proposition is proved for $|\nu| \leq p$. In what follows the subscript $\nu$ denotes the coefficient of $w^\nu$ in the indicated expression as before. Now with $|\nu| = p+1$,

$$F(a^p(z,w),w)_\nu = (F(z + (a^p(z,w) - z), w) - F(z,0) - \mathcal{L}(z, a^p(z,w) - z, w))_\nu.$$

By Lemma 5.5, Proposition 5.9 and the inductive hypothesis $a^p(z,w) - z \ll A(w)$, $w_i \ll A(w)$ we find that

$$F(a^p(z,w),w)_\nu \ll (E/D)(\sum_{j=2}^{\infty} D^j((s+1)A(w))^j.$$

Using $(A(w))^j \ll (3e/d)^{j-1}A(w)$ this implies that

$$F(a^p(z,w),w)_\nu \ll (E/D)(\sum_{j=2}^{\infty} D^j(s+1)^j(3e/d)^{j-1})A(w).$$

Provided that $0 < 3D(s+1)e/d < 1$, we can sum this geometric series and obtain $3ED(s+1)^2eA(w)/(d-3D(s+1)e)$. On the other hand since $G(z,b^p(w)) = z^{k+1} + \sum_{j=1=0}^{k-1}(b^p(w))_j z^j$ we see that $G(z,b^p(w))_\nu = 0$. Hence

$$(F(a^p(z,w),w)\{\frac{\partial a}{\partial z}(z,w)\}^r + G(z,b^p(w)))_\nu \ll 3ED(s+1)^2eA(w)^{r+1}/(d-3D(s+1)e)$$

$$\ll 3ED(s+1)^2e(3e/d)^rA(w)/(d-3D(s+1)e).$$

We may now invoke Lemma 5.6 to deduce that the terms $a_\nu(z)$, $b_\nu$ obtained in solving the equation ?? above satisfy

$$||a_\nu||,\ ||\partial a_\nu/\partial z||,\ |b_\nu|\ \textit{all} \ll 3LED(s+1)^2 3^r e^{r+1}A_\nu/(d^r(d-3D(s+1)e))$$

where $L$ is independent of $\nu$.

Now choose $d$ so large that $0 < 3D(s+1)e/d < 1$ and

$$0 < 3LED(s+1)^2 3^r e^{r+1}/(d^r(d-3D(s+1)e)) < 1$$

this choice being independent of $\nu$. We now deduce that

$$||a_\nu||,\ ||\partial a_\nu/\partial z||,\ b_\nu \ll A_\nu,$$

and in particular $a^p(z,w) - z$, $b^p(z) \ll A(w)$ for all $p$.

**Completion of Proof of Theorem 4.4**

So we claim that the formal power series $a$ and $b$ constructed above converge on some neighbourhood of the origin. In particular $G(z,u)dz^2 = (z^{k+1} + \sum_{i=0}^{k-1} u_i z^i)$ is a versal unfolding of $z^{k+1}$. The power series $a$, $b$ obtained above converge because

$$a(z,w) - z = \lim_{p\to\infty}(a^p(z,w) - z) \ll A(w)$$

$$b(w) = \lim_{p\to\infty} b^p(w) \ll A(w).$$

Since $A(w)$ is convergent in some neighbourhood of the origin it easily follows from the comparison test that $a$ and $b$ are also analytic in some neighbourhood of the origin.

**Corollary 4.11** *An unfolding $F$ of a germ $f$ is BDE-versal if and only if the partial derivatives $\partial F(z,0)/\partial u_i$ span the quotient $\mathcal{O}_1/J(f)$, where $\mathcal{O}_1$ is the ring of holomorphic functions $C,0 \to C$, and $J(f)$ is the ideal of functions spanned by $f'(z)$.*

**Proof** A deduction of the same kind for $\mathcal{R}$-equivalence is given in [?] on page 141. The proof required here is virtually the same.

# 5 Integrability

In this section we show that our BDE's have first integrals.

First recall that given an analytic function $h : C \to C$ we can look at the level curves of the real and imaginary parts of $h$. So if $h = U + iV$ we consider the families of curves $U = constant$ and $V = constant$. Since holomorphic maps are conformal it is not difficult to see that at any point $z$ in the plane where the derivative $h'(z)$ is non-zero the corresponding pair of level curves are orthogonal. Indeed we have the following result.

**Proposition 5.1** *Let $h = U + iV$ be as above. Then the level curves $U = constant$ and $V = constant$ are the solution curves of the BDE*

$$U_x U_y (dy^2 - dx^2) + (U_x^2 - U_y^2) dx dy = 0.$$

**Proof** Since $U = constant$ we have $U_x dx + U_y dy = 0$ and similarly $V_x dx + V_y dy = 0$. Multiplying these together and using the Cauchy-Riemann equations gives the result.

These level curves are of some interest in themselves, but we shall see that generic BDE's of the type we are studying do not yield integral curves corresponding to the level sets of the real and imaginary parts of generic functions $h$. Nevertheless we do have the following important result.

**Theorem 5.2** *(i) Given an analytic function $h$ we can find an analytic function $f$ such that the integral curves of $\Re\{f(z)dz^2\} = 0$ coincide with the level curves of the real and imaginary parts of $h$.*

*(ii) Away from the zeros of $f$ the integral curves of the BDE given by $\Re\{f(z)dz^2\} = 0$ locally coincide with the level curves of the real and imaginary parts of some analytic function $h$.*

**Proof** (i) Writing $f = u + iv$ we know that $\Re\{f(z)dz^2\} = 0$ reduces to $u(dy^2 - dx^2) + 2v dx dy = 0$. So this would coincide with a BDE corresponding to the level curves of the real and imaginary parts of an analytic function $h = U + iV$ if $u = U_x U_y$ and $v = (U_x^2 - U_y^2)/2$. The conditions that the resulting $f$ is analytic is that the Cauchy-Riemann equations hold. But

$$u_x = U_{xx}U_y + U_x U_{xy}, u_y = U_{xy}U_y + U_x U_{yy} = U_{xy}U_y - U_x U_{xx}.$$

$$v_x = U_x U_{xx} - U_y U_{yx}, v_y = U_x U_{xy} - U_y U_{yy} = U_x U_{xy} + U_y U_{xx}.$$

Clearly $u_x = v_y$ and $v_x = -u_y$ as required.

(ii) Now for the converse assertion. Suppose given $f = u + iv$ a holomorphic function. We now seek a function $h = U + iV$ such that the level curves of $U$ and $V$ are the integral curves of the BDE $\Re\{f(z)dz^2\} = 0$. By the above we seek $U$ and $V$ with

$$u = U_x U_y = -U_x V_x, v = (U_x^2 - U_y^2)/2 = (U_x^2 - V_x^2)/2.$$

But now we note that

$$(U_x + iV_x)^2 = 2v - 2iu = -2i(u + iv) = -2if.$$

Since $h_x = h_z = h'$ we deduce that $(h')^2 = -2if$, so that formally $h = \pm\sqrt{-2i}\int\sqrt{f}dz + c$. Of course the function $h$ is generally not holomorphic at the zeros of $f$, has two branches away from the zeros, and is well defined only up to a constant. Nevertheless the level curves of the real and imaginary parts of $h$ *are* clearly well defined away from the zeros of $f$. (Of course we know that the zeros of $f$ play a key role in the geometry of the solution curves of the BDE, so their appearance above is expected.)

# 6 Geometry of the Unfolding

We have seen that there is a holomorphic function $h$ properly defined on a cover of the complement of the zeros of $f$ with the property that the level curves of the real or imaginary parts of $h$ project down (unambiguously) to the integral curves of the BDE. We wish to understand the way in which these integral curves will change as we move around the unfolding space. We have already seen that there is a codimension-2 catastrophe, namely the coming together of zeros of $f$, and hence singular points of the solution curves. Another low(er) co-dimensional change occurs when a singular solution passes through two singularities. We identify the corresponding stratum in the versal unfolding of the $A_1$ singularity.

**Example 6.1** $A_1$ *singularity.*

Consider the function $f(z) = z^2$ with associated BDE $\Re\{z^2 dz^2\} = 0$ and its versal unfolding $\Re\{F_w(z)dz^2\} = 0$ where $F(z,w) = F_w(z) = z^2 + w$. We know that the $A_1$ stratum is simply given by $w = 0$. We now consider the values of $w$ for which there is a singular trajectory passing through two singular points. What we shall do is construct, more or less explicitly, the relevant function $h$ whose real and imaginary parts have level curves which give the solutions of the original BDE. Indeed we need to find a family $H(z,w) = H_w(z)$ which does the job for $F_w$ for each $w$. We then seek points $(z_1, w)$, $(z_2, w)$ with $H'_w(z_i) = 0$ for $i = 1,\ 2$ and $\Re\{H_w(z_1)\} = \Re\{H_w(z_2)\}$ (or we have equality of the imaginary parts instead).

Now we have seen that $H(z,w) = \pm\sqrt{-2i}\int\sqrt{z^2+w}dz$ and it is not difficult to see that this is an odd function of $z$ (note that $\sqrt{-2i} = \pm(-1+i)$). However

$$(w+z^2)^{\frac{1}{2}} = \pm w^{\frac{1}{2}}\sqrt{(1+z^2/w)}.$$

This can then be expanded as a function $z$, indeed can be written in the form $\pm w^{\frac{1}{2}} zG(z^2, w)$ for some analytic $G$. Now the zeros of $H'$ are the zeros of $F$ namely $\pm i\nu$, where $\nu^2 = w$. Substituting these in for $z$ the condition that the (say real) parts of the $H_u$ coincide is $\Re\{(-1+i)i\nu G(-w,w)\} = \Re\{-(-1+i)i\nu G(-w,w)\}$. However by expanding and substituting one can check that the function $G(z^2, w)$ when evaluated at $z^2 = -w$ is a non-zero real multiple of $i\nu$. Indeed

$$(1+z^2/w)^{1/2} = \sum(1/2)(-1/2)(-3/2)\ldots((-2k+3)/2)z^{2k}/(k!w^k).$$

Integrating we obtain

$$\sum(1/2)(-1/2)(-3/2)\ldots((-k+1)/2)z^{2k+1}/((2k+1)k!w^k).$$

Setting $z^2 = -w$ we obtain

$$i\nu\sum(1/2)(-1/2)(-3/2)\ldots((-2k+3)/2)(-1)^k/((2k+1)k!) =$$

$$-i\nu\sum(1/2)(1/2)(3/2)\ldots((2k-3)/2)/((2k+1)k!) \neq 0$$

as required. So the above condition comes to $\Re(w) + \Im(w) = 0$. Similarly the condition for this to happen with the other level curves is $\Re(w) = \Im(w)$. (Compare the example in §III.2 of [?], where there is a slight error.) So in the unfolding space we have two strata. Figure 3 show the change in the integral curves as the parameter moves across these strata. We label this the $YY$ stratum for reasons which are clear from Figure 3.

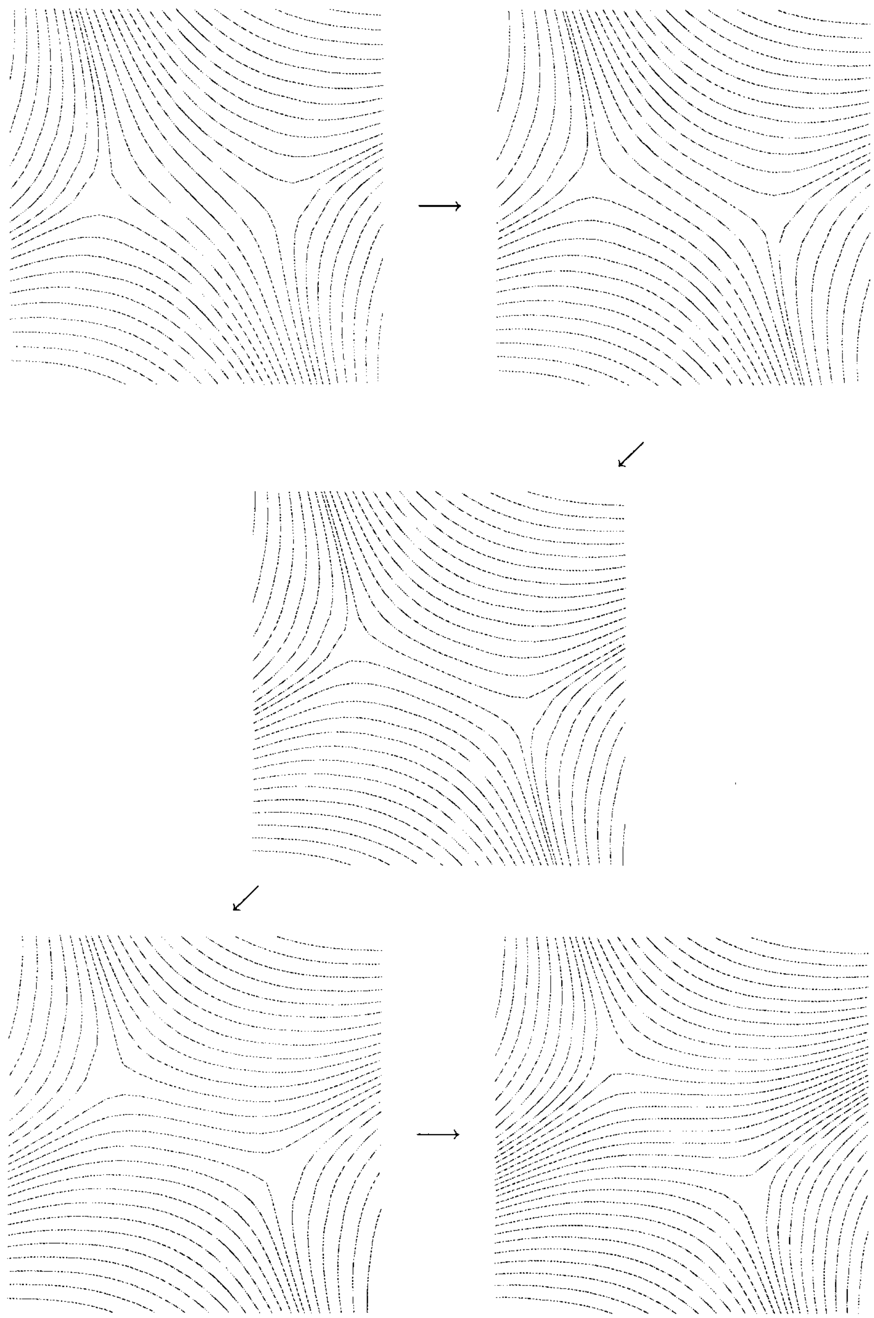

Figure 3: YY catastrophe

## 7 Generic Configurations for an Unfolding of $z^k$

We now wish to understand the topology of the integral curves of a generic deformation of the BDE given by $\Re\{z^k dz^2\} = 0$. So in particular these deformations will only have simple singular points and no integral curves joining two (or more) singular points. One can view these a being obtained by selecting a generic point in the versal unfolding space. These configurations will change as the unfolding parameter moves across the $YY$ strata, and we saw in the previous section the way this occurs. (The degenerate singular point stratum is of codimension 2 and so not relevant.) In [?], Hubbard and Masur study the case, almost diametrically oppsed to what we study, where all singular points are connected to one another by singular trajectories.

Note that our discussion above shows that the BDE really does factorise; that is the integral curves can be split into two (singular) foliations, namely that given by $\Re\{h(z)\} = constant$ and $\Im\{h(z)\} = constant$. Because of this we shall start by considering each of these foliations separately, looking later at the question of how the two mesh together. First recall the configuration of the solution curves of the undeformed BDE. There is a single singular point and $k+2$ integral curves through that point. If we split the BDE each resulting ODE has $k+2$ half branch solutions emanating from the singular point. See Figure 4.

Out at the boundary of the unit disc other solutions get closer to these singular half branch solutions. More importantly there are $k+2$ points where the integral curves are tangent to the bounding circle of the disc. It is easy to check that this tangency is simple, with the tangent curve meeting the disc only at the point of tangency.

We now need a few definitions. In what follows we have a singular foliation of the unit disc.

**Definition 7.1** *(i) A singular point of the boundary circle is a point of intersection with a singular solution of the foliation.*

*(ii) We say that a pair of points on the boundary are adjacent if they are not separated by a point of tangency.*

*(iii) We say that a pair of arms are adjacent if they meet the boundary circle in adjacent singular points. We say that a pair of Y's are adjacent if each has a pair of arms which are adjacent.*

There are three basic facts needed to understand these configurations.

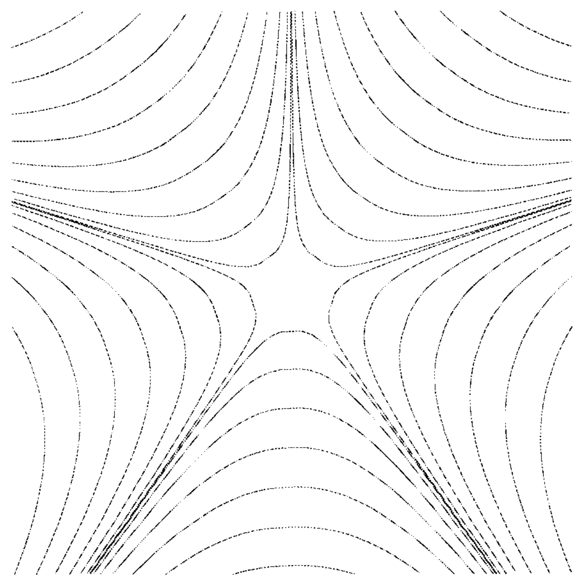

Figure 4: Solution curves for ODE

**Theorem 7.2** *(i) The singular foliation is determined up to topological equivalence by the configuration (up to topological equivalence) of the singular solutions in the unit disk.*

*(ii) In a generic unfolding of an $A_{k-1}$ singular point there are $k$ simple singular points of the BDE. (We shall refer to simple singular points hereafter as $Y$'s.)*

*(iii) There are $k+2$ tangency points in any small deformation of the BDE. Moreover no two of the three singular solutions through any simple singular points are adjacent.*

**Proof** (i) See below.

(ii) The first result is immediate from the fact that an $A_{k-1}$ singularity splits generically into $k$ $A_1$s.

(iii) Suppose given a family $F(z,t)$ with $F(z,0) = z^k$ and so that for $t > 0$ (and sufficiently small) the configuration of solution curves of $\Re\{F(z,t)dz^2\} = 0$ is that to be studied. Recall that there are $k+2$ tangency points of each of the foliations corresponding to the unperturbed BDE $\Re\{z^k dz^2\} = 0$ and these remain under small perturbation. Moreover the tangency is simple with the integral curve lying outside the open unit disc, and this property will also be preserved on perturbation. Since in the deformation the simple singular points are created close to the origin each of the associated three arms exits the circle between two points of tangency.

Suppose now that two of the arms associated with a simple singular point are adjacent. The integral curves of the foliation then are all transverse to the boundary circle between the points of contact of the two singular solutions. In the enclosed region there may of course be further simple singularities whose singular solutions meet the boundary between the two points already mentioned. There is certainly one for which two of the points of intersections are adjacent. But now integral curves leaving the boundary between the two points return there at a different point because there ae no tangency points. But the integral curves are the level curves of the real part of some function $h$, and are all smooth 1-manifolds with boundary where they meet the bounding circle. If we assign to each point on this interval the other point on the boundary where the integral curve meets it again we obtain a smooth map of an interval with no fixed points: a contradiction.

(i) It is now not difficult to see that the singular solutions cut the disc into two types of region. Up to homeomorphism these are a rectangular strip whose two ends are parts of the boundary circle, and whose sides are each two arms of a $Y$. There are no points of tangency on the boundary. The other is a sector of a circle, the circular part being the boundary circle, the two bounding radii arms of a $Y$, with a single tangency on the boundary. The integral curves in these regions are as indicated in Figure 5. The region types are completely determined by the configuration of the singular solutions, and the result now follows.

**Remark 7.3** *(1) If one assigns an index of $-1/2$ to each $Y$ and $1/2$ for each boundary tangency (tangent integral curve lying outside interior of disc) then on any surface with boundary the sum of the indices is the Euler characteristic.*

*(2) It is not difficult to see (using a little pasting and Euler characteristics) that in a deformation of $z^k$ there are $k-1$ rectangular regions and $k+2$ sectors.*

**Corollary 7.4** *Each $Y$ has two arms adjacent to two arms of at least one other $Y$.*

**Proof** Suppose that we start on the unit circle at some fixed singular point and draw in the corresponding $Y$. We then move counterclockwise say and insert each new $Y$ as we meet its singular points on the boundary. Now we

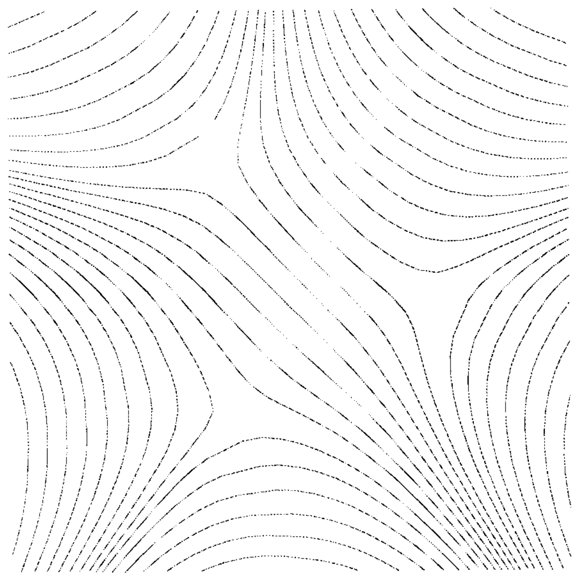

Figure 5: The two types of region

count the number of tangency points on the boundary, using all the while the fact that any two of the three singular points on the boundary of the $Y$ are separated by a point of tangency. So after the first insertion we have register 3 corresponding tangency points. The claim is that thereafter each new insertion of a $Y$ requires at least one new tangency point. This follows from the fact that any new insertion can have at most two of its arms already separated by tangency point. So after we have inserted all $k$ singular points we have registered at least $3 + (k - 1) = k + 2$ tangency points. But this *is* in fact the total number of tangency points, and so with each new insertion exactly two of the arms are adjacent to singular points on the boundary which are already registered.

We have seen that for each $Y$ there is at least one *pair* of arms adjacent to a pair of arms of another $Y$. This fact allows us to associate a graph with each foliation.

**Definition 7.5** *The graph associated to the ODE of the BDE has for its vertices the $Y$ singularities of the deformation, and edges correspond to pairs of singular points which have pairs of arms adjacent.*

**Proposition 7.6** *(i) The resulting graph is a tree in other words is con-*

*nected with no loops.*

*(ii) If the the number of vertices of valency $i$ is $\alpha(i)$ then $\alpha(i) = 0$ for $i \geq 4$ and $\alpha(1) + \alpha(2) + \alpha(3) = k$ while $\alpha(1) + 2\alpha(2) + 3\alpha(3) = 2(k-1)$.*

*(ii) This tree does not determine the configuration of the singular solutions.*

**Proof** (i) It is not difficult to see that there is at least one $Y$ whose 3 associated boundary singular points are separated by no others. If $k \geq 2$ there will be a (unique) second $Y$ with two sides adjacent to the outer arms of the first. Continuing in this way (see the proof of the previous Corollary) we obtain a connected graph.

We now need to show that it contains no loops. But if there was a loop then choosing a vertex or $Y$ it is clear that all subsequent $Y$'s occur in one of the regions partitioned off by two of the arms, and so it is not possible for the loop to close up.

(ii) This is now clear from (i).

(iii) The configurations given in Figure 6 are clearly distinct but give rise to the same graph.

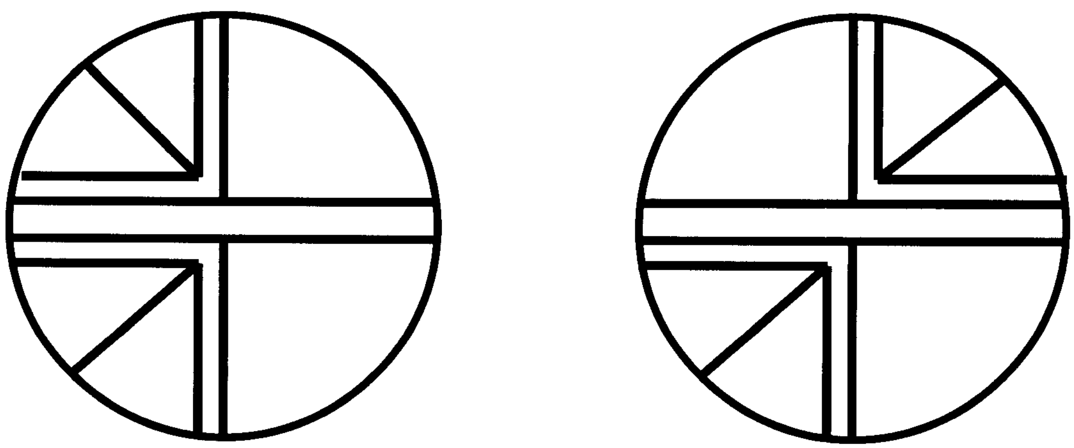

Figure 6: Distinct configurations yielding the same graph

As mentioned above we would like to enumerate all possible generic configurations of singular solutions in the split ODE. We now describe what one might term the fundamental move repeated application of which allows one to change from one possible configuration of the foliation to any other.

Basically this is the local $YY$ catastrophe discussed above which occurs across a codimension 1 subset of the space of BDE's of the type discussed here. The basic change is indicated in Figure 7, where at the point of transition a pair of $Y$'s share a common arm.

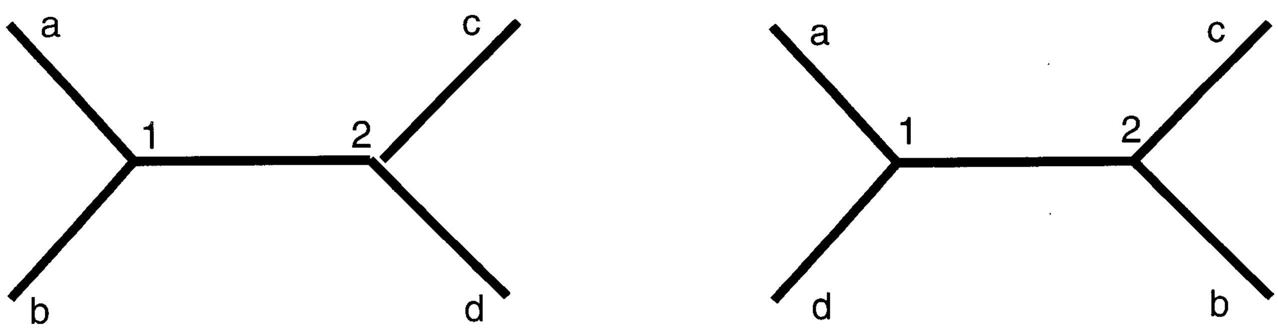


Figure 7: Effect of YY change on graph

The effect on the graph of the configuration is indicated in the same diagram. The idea here is that each vertex has valency at most 3, that is there is at most 3 adjacent edges; the effect of the move is to swap pairs of edges, or in the case where one of the $Y$ has valency 1 to transfer one of the edges from the other $Y$. The effect on the three possible configurations in the case $k = 4$ is illustrated in Figure 8.

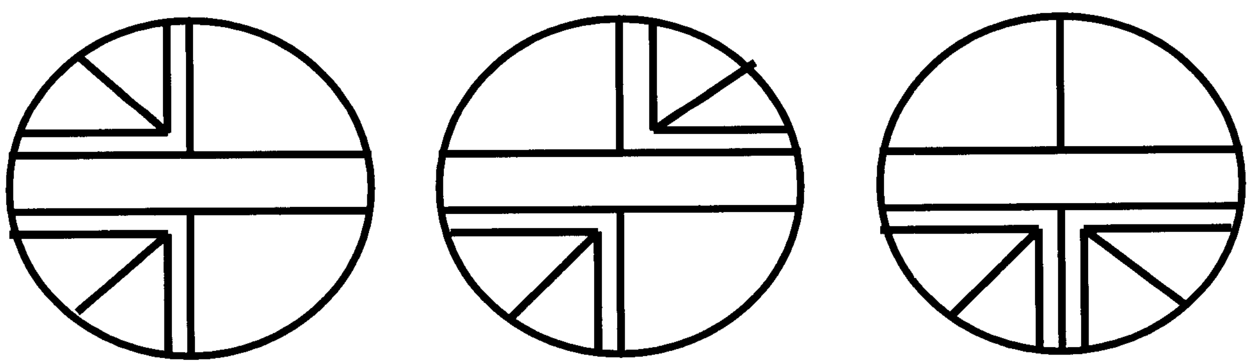

Figure 8: The three configurations for k=4

Another way of describing the singular solutions is by first labelling the

$Y$'s, say 1 to $k$, and then starting at some point on the circumference writing down the numbers associated to each singular point. In this way we arrive at a string of $3k$ numbers, with each of the numbers 1 to $k$ occuring 3 times. It would be of interest to characterise the strings that can occur in this way.

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