ARRANGEMENTS OF HYPERPLANES

P. Orlik H. Terao

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Preface

An arrangement of hyperplanes is a finite collection of codimension one affine subspaces in a finite dimensional vector space. In this book we study arrangements with methods from combinatorics, algebra, algebraic geometry, topology, and group actions. These first sentences illustrate the two aspects of our subject that attract us most. Arrangements are easily defined and may be enjoyed at levels ranging from the recreational to the expert, yet these simple objects lead to deep and beautiful results. Their study combines methods from many areas of mathematics and reveals unexpected connections.

The idea to write a book on arrangements followed three semesters of lectures on the topic at the University of Wisconsin, Madison. Louis Solomon lectured on the combinatorial aspects in the fall of 1981. Peter Orlik continued the course with the topological properties of arrangements in the spring of 1982. Hiroaki Terao visited Madison for the academic year 1982–83 and gave a course on free arrangements in the fall of 1982. The original project was to enlarge the lecture notes into a book and a long but incomplete manuscript was typed up in 1984. The book was revived in 1988 when P. Orlik was invited to give a CBMS lecture series on arrangements. The resulting lecture notes provided a core for the present work. Also in 1988, H. Terao gave a course on arrangements at Ohio State University. Parts of his lectures are used in this book. The fresh start allowed us to take advantage of the recent TEXnical innovations of word processing and computer aided typesetting.

Time constraints forced L. Solomon to leave the completion to us. Much of the material in the book is his joint work with one or both of us, and a large part of chapter II was written by him. We are grateful for permission to use his work, and for his support and friendship.

We thank M. Falk and R. Randell for help on many technical points. W. Arvola obtained a presentation of the fundamental group of the complement in his PhD thesis. We thank him for permission to include a modified version of his work. In addition, we owe thanks to C. Greene, T. Zaslavsky, and S. Yuzvinsky for valuable suggestions, and to V. I. Arnold for references on the M–property. P. Orlik had the opportunity to lecture on arrangements at the Swiss Seminar in Bern. He wishes to thank the mathematicians in Geneva for their hospitality and the participants of the seminar for many helpful comments. H. Terao gave a graduate course on arrangements at International Christian University in Tokyo in the fall of 1989. He wishes to thank all the participants of the course.

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P. Orlik and H. Terao

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Chapter I

1 Introduction

Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

Problem E 554. *American Mathematical Monthly* **50** (1943), p. 59.
Proposed by J. L. Woodbridge, Philadelphia

Solution by Free Jamison, Pittsburgh. [*ibid* pp. 564–5] Since n straight lines can divide a plane into $(n^2+n+2)/2$ areas, the $(n+1)$st plane can be divided by the first n planes into that number of areas. For each of these areas the $(n+1)$st plane divides a piece of cheese already formed into two, and increases the total number of pieces by $(n^2+n+2)/2$. Since $(n^3+5n+6)/6$ gives the number of pieces for $n = 1$ or 2, and since

$$\frac{n^3+5n+6}{6}+\frac{n^2+n+2}{2}=\frac{(n+1)^3+5(n+1)+6}{6},$$

the expression $(n^3+5n+6)/6$ holds for every n.

It is interesting to note that

(1) n points can divide a line into $1+n$ parts,

(2) n lines can divide a plane into $1+n+\binom{n}{2}$ parts,

(3) n planes can divide space into $1+n+\binom{n}{2}+\binom{n}{3}$ parts.

Editorial Note. The general formula

$$1+n+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{m},$$

for the case of an m-dimensional cheese, was obtained by L. Schläfli on page 39 of his great posthumous work, *Theorie der vielfachen Kontinuität* (Denkschriften der Schweizerischen naturforschenden Gesellschaft, vol. 38, 1901).

History

Such are the humble origins of our subject. In order to maximize the number of pieces, the arrangement of planes in the problem must be in "general position." This means that any two planes have a common line, and these lines are distinct, and that any three planes have a common point, and these points are distinct. Allowing degeneracy makes the problem of counting parts much harder. S. Roberts [159] gave a formula in 1889 for the number of regions formed by an arbitrary arrangement of n lines in the plane. It is "the number of regions formed by n lines in general position" minus "the number of regions lost because of multiple points" minus "the number of regions lost because of parallels." See J. Wetzel's article [205] for a modern treatment. There is an extensive literature on partition problems in Euclidean space and projective spaces. B. Grünbaum summarized much of what was known in 1971 in [83, 84]. We quote from the introduction of his paper [83], whose title we borrowed for this book.

> ...I would like to survey the somewhat related field of *arrangements of hyperplanes*, which I expect to become increasingly popular during the next few years ... the theory of arrangements may be developed, much like topology, in rectilinear or curved versions as well as in discrete and continuous variants, and that in these developments it impinges upon many aspects of convexity, topology, and geometry which seemed to be quite unrelated.

The complement of certain hyperplanes in complex space had been studied by E. Fadell, R. Fox, and L. Neuwirth [58, 71] in connection with the braid space. The braid arrangement consists of the hyperplanes $H_{i,j} = \ker(z_i - z_j)$. Let $M = \{z \in \mathbb{C}^\ell \mid z_i \neq z_j \text{ for } i \neq j\}$ be the complement of these hyperplanes, called the pure braid space. They proved that M is a $K(\pi, 1)$ space. Let $\mathrm{Poin}(M,t) = \sum_{k\geq 0} \mathrm{rank} H^k(M)t^k$ be its Poincaré polynomial. In 1969 V. I. Arnold [6] proved the beautiful formula

$$(1) \qquad \mathrm{Poin}(M,t) = (1+t)(1+2t)\cdots(1+(\ell-1)t)$$

in connection with his work on Hilbert's 13th problem. He also constructed a graded algebra A as the quotient of an exterior algebra by a homogeneous ideal, and showed that there is an isomorphism of graded algebras

$$(2) \qquad H^*(M) \simeq A.$$

This gives a presentation of the cohomology ring of the pure braid space in terms of generators and relations. The study of the topological properties of the complement of an arbitrary arrangement over the complex numbers was launched by Arnold with the following remark at the end of his paper.

> Let M be the manifold obtained from $\mathbb{C}^n$ by discarding an arbitrary number of hyperplanes
>
> $$M = \{z \in \mathbb{C}^n \mid \alpha_k(z) \neq 0,\ k = 1,\ldots,N\}.$$
>
> Probably, the ring $H^*(M,\mathbb{Z})$ is torsion free and is generated by the one-dimensional classes $\omega_k = (1/2\pi i)(d\alpha_k/\alpha_k)$, a polynomial in ω_k being cohomologous to 0 in H^* only when it is zero.

E. Brieskorn [33] proved these conjectures in a 1971 Bourbaki Seminar talk. One of his results captured an essential topological feature of arrangements.

> ...une famille finie quelconque d'hyperplans affines complexes V_i, $i \in I$, dans un espace affine complexe V. Pour calculer le p-iéme groupe de cohomologie, $0 \leq p \leq n$, on considére les sousensembles maximaux $I_{p,1},\ldots,I_{p,k_p}$ de I pour lequels on ait la propriété:
>
> $$\mathrm{codim} \bigcap_{i\in I_{p,k}} V_i = p.$$

Lemme 3. *Pour les complémentaires d'union d'hyperplans $Y = V - \cup_{i\in I} V_i$ et $Y_{p,k} = V - \cup_{i\in I_{p,k}} V_i$ les inclusions $i_k : Y \to Y_{p,k}$ induisent un isomorphisme:*

$$H^p(Y,\mathbb{Z}) = \bigoplus_{k=1}^{k_p} H^p(Y_{p,k},\mathbb{Z}).$$

Brieskorn also generalized Arnold's results in another direction. He replaced the symmetric group and the braid arrangement by a finite Coxeter group W and its reflection representation in a real vector space $V_{\mathbb{R}}$ of dimension ℓ. Let V be the complexification of $V_{\mathbb{R}}$. Then W acts as a reflection group in V. Let $M_W \subset V$ be the complement of the reflecting hyperplanes of W. He proved that the analog of (1) involves the exponents $m_1,\ldots m_\ell$ of W.

$$(3) \qquad \mathrm{Poin}(M_W,t) = (1+m_1t)(1+m_2t)\cdots(1+m_\ell t).$$

Brieskorn conjectured that M_W is a $K(\pi,1)$ space for all Coxeter groups W. He proved this for some of the groups by representing M_W as the total space of a sequence of fibrations.

In the 1971 paper quoted above, Grünbaum [83] reported the

> ...finding of a rock (or rather, unpolished gem) discovered thirty years ago by one of the lone wanderers in the wilderness of specialization. The "simplicial arrangements" which will be discussed below were first discovered by Melchior [127]; though they are a very natural notion and appear in the solutions of many problems about arrangements, they remained unnoticed.

Grünbaum classified all simplicial arrangements in the affine and projective planes with ≤ 38 lines. It seems like poetic justice that these "unpolished gems" became the central objects in the solution of Brieskorn's conjecture by P. Deligne [49] in 1972.

> **Théorème.** *Soit V un espace vectoriel réel de dimension finie, $\mathcal{M}$ un ensemble fini d'hyperplans homogénes de V, $V_{\mathbb{C}}$ le complexifié de V et $Y = V_{\mathbb{C}} - \cup_{M\in\mathcal{M}} M_{\mathbb{C}}$. On suppose que les composants connexes de $V - \cup_{M\in\mathcal{M}} M$ sont des cônes simpliciaux ouverts. Alors, Y est un $K(\pi,1)$.*

The next significant advance was made by T. Zaslavsky [211] in 1975. The title of his AMS Memoir tells it all: "*Facing up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes.*" He introduced the method of deletion and restriction to obtain recursion formulas for counting problems. Let $\mathcal{A}$ be an arrangement and let $H \in \mathcal{A}$ be a hyperplane. Then $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is called the deleted arrangement. The arrangement in H defined by $\mathcal{A}'' = \{K \cap H \mid K \in \mathcal{A}'\}$ is called the restricted arrangement. The triple $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ may be used to solve the problem of counting the parts of the complement of the hyperplanes of an arbitrary real arrangement. The parts are called **chambers** in modern terminology. Let $C(\mathcal{A})$ be the set of chambers of $\mathcal{A}$. Let $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ be a triple of real arrangements. Then

$$|C(\mathcal{A})| = |C(\mathcal{A}')| + |C(\mathcal{A}'')|.$$

To prove this recursion, let P be the set of those chambers in $\mathcal{C}(\mathcal{A}')$ which intersect the distinguished hyperplane H. Let Q be the set of those chambers in $\mathcal{C}(\mathcal{A}')$ which do not intersect H. Evidently $|\mathcal{C}(\mathcal{A}')| = |P|+|Q|$. The hyperplane H divides each chamber of P into two chambers of $\mathcal{C}(\mathcal{A})$ and leaves the chambers of Q unchanged. Thus $|\mathcal{C}(\mathcal{A})| = 2|P| + |Q|$. Finally, there is a bijection between P and $\mathcal{C}(\mathcal{A}'')$ given by $C \mapsto C \cap H$. Thus $|\mathcal{C}(\mathcal{A}'')| = |P|$. Figure 1 illustrates this in the plane. Let H be the broken line. Then $|P| = 4$ and $|Q| = 10$, so we get $|\mathcal{C}(\mathcal{A})| = 18$.

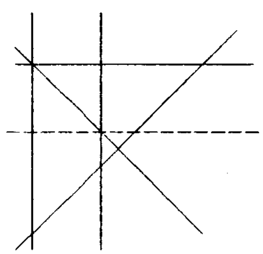


Figure 1: An illustration of chamber counting

Zaslavsky defined the set $L(\mathcal{A})$ of intersections of elements of $\mathcal{A}$, and partially ordered $L(\mathcal{A})$ by reverse inclusion. He used the Möbius function of $L(\mathcal{A})$ to define the characteristic polynomial of $L(\mathcal{A})$. There is a closely related polynomial $\pi(\mathcal{A}, t)$, defined on $L(\mathcal{A})$, which we call the Poincaré polynomial. It follows from the definition that for the empty arrangement $\pi(\mathcal{A}, t) = 1$. He proved a result about the characteristic polynomial, which amounts to the following recursion for the Poincaré polynomial: $\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t)$. Since $|\mathcal{C}(\mathcal{A})|$ and $\pi(\mathcal{A}, 1)$ agree on the empty arrangement and satisfy the same recursion for deletion and restriction, this proves Zaslavsky's beautiful result:

(4) $$|\mathcal{C}(\mathcal{A})| = \pi(\mathcal{A}, 1).$$

Analysis led to development in a different direction. The classical hypergeometric function $F(a, b; c; z)$ is defined by the series

$$F(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a, m)(b, m)}{(c, m)(1, m)} z^m,$$

where (a, m) denotes the factorial function

$$a(a+1)\cdots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)}.$$

The hypergeometric function satisfies the differential equation

$$z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,$$

and it has the Euler integral representation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{a-c}(t-1)^{c-b-1}(t-z)^{-a} dt.$$

The function is normalized to depend on the arrangement of the three points $0, 1, z$ in the complex line. Hypergeometric functions have been defined in several complex variables, and they have analogous integral representations. These generalizations naturally depend on arrangements of hyperplanes in affine space, see [2, 92]. Much work has been done in studying these integrals over various domains. This was the motivation for the investigation by A. Hattori [91] in 1975 of the homotopy type of the complement of an arrangement of complex hyperplanes in general position.

> We denote by $\mathbf{k}$ the set $\{1, 2, \ldots, k\}$. If I is a subset of $\mathbf{k}$, we denote by $|I|$ the cardinal number of I. We define the subtorus T_I of T^k by
>
> $$T_I = \{z \mid z = (z_1, \ldots, z_k) \in T^k,\ z_j = 1 \text{ for } j \notin I\}.$$
>
> The dimension of T_I is equal to $|I|$.
>
> THEOREM 1. *Let $L_1, \ldots, L_k$ be affine hyperplanes in $\mathbb{C}^n$ in general position, where $n + 1 \leq k$. Then the space $X = \mathbb{C}^n - L_1 \cup \cdots \cup L_k$ has the same homotopy type as the space*
>
> $$X_0 = \bigcup_{\substack{I \subset \mathbf{k} \\ |I| = n}} T_I.$$

This is the complex analog of the cheese cutting problem. There the number of parts in the complement of a real arrangement in general position depends only on the number of hyperplanes, but not on their location. Here the homotopy type of the complement of a complex arrangement in general position depends only on the number of hyperplanes, but not on their location.

More tools were added to the study of arrangements in 1980. P. Orlik and L. Solomon [142] used combinatorial methods to study the complement $M(\mathcal{A})$ of a complex hyperplane arrangement $\mathcal{A}$. They used Brieskorn's results to compute the Poincaré polynomial of the complement of an arbitrary complex arrangement:

(5) $$\text{Poin}(M(\mathcal{A}), t) = \pi(\mathcal{A}, t).$$

Thus the Betti numbers of the complement depend only on the lattice of intersections of the hyperplanes. I. Petrowsky [152] defined the M-property for real algebraic curves.

> As early as 1876 Harnack [90] showed that the maximal number of components (maximal connected subsets) of a real algebraic curve of order n in the projective plane is precisely $\frac{1}{2}(n-1)(n-2)+1$. At the same time Harnack proposed a process for the construction of curves with this maximal number of components. Such curves we shall call in the sequel, M-curves. ...

> In 1891 D. Hilbert [94] proposed a new method of constructing M-curves. ...In his report to the International Mathematical Congress in 1900 on modern problems of mathematics Hilbert considers the investigation of the topology of M-curves and of the corresponding algebraic surfaces as most timely.

Work on Hilbert's 16th problem continued. For recent progress see [8, 86, 201]. Suppose $X_{\mathbf{R}}$ is a real algebraic variety and $X_{\mathbf{C}}$ is its complexification. Complex conjugation induces an involution on $X_{\mathbf{C}}$ whose fixed point set is $X_{\mathbf{R}}$. Let $b_i(X_{\mathbf{R}})$ and $b_i(X_{\mathbf{C}})$ be their respective Betti numbers with $\mathbb{Z}/2$ coefficients. An application of Smith theory, see [69, Theorem 4.4], provides the following inequality:

(6) $$\sum_{i\geq 0} b_i(X_{\mathbf{R}}) \leq \sum_{i\geq 0} b_i(X_{\mathbf{C}}).$$

The natural generalization of an M-curve is to say that the real algebraic variety $X_{\mathbf{R}}$ has the M-property if equality holds in (6).

Let $\mathcal{A}_{\mathbf{R}}$ be a real arrangement and let $\mathcal{A}_{\mathbf{C}}$ be its complexification. Let $M_{\mathbf{R}}$ and $M_{\mathbf{C}}$ be the real and complex complements. Let $Q \in \mathbf{R}[x_1, \dots, x_\ell]$ be a product of linear polynomials whose zero set is the union of the hyperplanes in $\mathcal{A}_{\mathbf{R}}$. Note that the complement is also an algebraic variety, $M_{\mathbf{R}} \approx \{v \in \mathbf{R}^{\ell+1} \mid x_0 Q(v) = 1\}$. It follows that $M_{\mathbf{R}}$ has the M-property:

$$\sum_{i\geq 0} b_i(M_{\mathbf{R}}) = b_0(M_{\mathbf{R}}) = \pi(\mathcal{A}_{\mathbf{R}}, 1) = \pi(\mathcal{A}_{\mathbf{C}}, 1) = \sum_{i\geq 0} b_i(M_{\mathbf{C}}).$$

Here we used (4), (5), the fact that $M_{\mathbf{C}}$ has no torsion in homology, and the fact that $L(\mathcal{A}_{\mathbf{R}}) = L(\mathcal{A}_{\mathbf{C}})$, so their Poincaré polynomials are equal.

Orlik and Solomon [142] also defined a graded algebra $A(\mathcal{A})$, which is constructed using only $L(\mathcal{A})$. It is the quotient of the exterior algebra $E(\mathcal{A})$ based on $\mathcal{A}$ by a homogeneous ideal $I(\mathcal{A})$, $A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$. They showed that there is an isomorphism of graded algebras

(7) $$H^*(M(\mathcal{A})) \simeq A(\mathcal{A}).$$

This generalizes (2) by giving a presentation of the cohomology ring of the complement of a complex arrangement in terms of generators and relations. In [143] they considered complex reflection arrangements. If G is a complex reflection group acting in a complex vector space V of dimension ℓ then it has exponents $m_1, \dots m_\ell$. However, if $M_G \subset V$ is the complement of the reflecting hyperplanes of G then formula (3) does not hold for M_G. Orlik and Solomon [143] defined coexponents $n_1, \dots n_\ell$ for G and showed that for real groups $m_i = n_i$. They proved the following generalization of (3) for complex reflection groups:

(8) $$\mathrm{Poin}(M_G, t) = (1 + n_1 t)(1 + n_2 t) \cdots (1 + n_\ell t).$$

A different line of investigation was inspired by singularity theory. In the present context the study of logarithmic vector fields and logarithmic differential forms on a hypersurface

was initiated by K. Saito [168]. He defined free hypersurfaces in the analytic category. Arrangements represent a special case. Here the hypersurface is the union of the hyperplanes of $\mathcal{A}$. Its singular set consists of linear subspaces. This special case was studied by H. Terao [186]. He showed that we can pass from analytic to algebraic considerations.

Let S denote the polynomial algebra of V, and let Q be a product of defining linear forms for $\mathcal{A}$. Suppose $\theta : S \to S$ is a derivation. It is called an $\mathcal{A}$-derivation if $\theta(Q) \in QS$. The set of $\mathcal{A}$-derivations is an S-module, $D_S(\mathcal{A})$. The arrangement $\mathcal{A}$ is called free if $D_S(\mathcal{A})$ is a free S-module. It is an enduring mystery of the subject just what makes an arrangement free, but it is known that the property is not generic. If $\mathcal{A}$ is free then Terao [186] associated to $\mathcal{A}$ a collection of non-negative integers called its exponents, $\exp \mathcal{A} = \{b_1, \dots, b_\ell\}$. These integers are unique up to order, but they are not necessarily distinct. Terao [186] proved several results concerning a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. These results may be combined to assert that any two of the following statements imply the third:

$$\begin{aligned}
\mathcal{A} \text{ is free with } \exp\mathcal{A} &= \{b_1, \dots, b_{\ell-1}, b_\ell\}, \\
\mathcal{A}' \text{ is free with } \exp\mathcal{A}' &= \{b_1, \dots, b_{\ell-1}, b_\ell - 1\}, \\
\mathcal{A}'' \text{ is free with } \exp\mathcal{A}'' &= \{b_1, \dots, b_{\ell-1}\}.
\end{aligned}$$

The following year Terao [188] proved that if $\mathcal{A}$ is a free ℓ-arrangement with $\exp \mathcal{A} = \{b_1, \dots, b_\ell\}$ then

(9) $$\pi(\mathcal{A}, t) = (1 + b_1 t) \cdots (1 + b_\ell t).$$

He also proved [187] that complex reflection arrangements are free. Thus (8) is a consequence of (9). These early developments were reviewed by P. Cartier [38] in a Bourbaki Seminar talk in November 1980. Much exciting work by many authors followed. We can only describe a fraction of it here.

Recent Advances

The combinatorial vein is carried on by Zaslavsky [212, 213, 216] and others. There are several papers on various counting problems by A. Björner, P. Edelman and G. Ziegler [24, 28, 219, 220]. There are close connections with matroid theory, see [26]. Note also related work by N. E. Mnëv [130] and by A. V. Zelevinsky [217]. Some of this combinatorial research is closely related to computer science. See articles in *Discrete and Combinatorial Geometry*, particularly the special issue on "*The Complexity of Arrangements*," Volume 5, Number 2, 1990. There are also connections with coding theory [41, Chapter 21]. For connections with the theory of box splines, see the work of C. de Boor.

Much of the topological work is focused on the homotopy type of the complement of a complex arrangement. The rational homotopy type of the complement was studied by T. Kohno [107, 108, 110, 114], M. Falk [61] and R. Randell [65]. This work is reviewed in the survey article by Falk and Randell [66]. A presentation of the fundamental group of the complement of a complexified real arrangement was obtained by Randell [157] and M. Salvetti [170]. The problem was solved for arbitrary complex arrangements by W. Arvola [11]. In their work on stratified Morse theory, M. Goresky and R. MacPherson [76] generalized the notion of hyperplane arrangements to arrangements of affine subspaces, without restriction on their dimensions. They computed the Betti numbers of the complement and showed that the complement of a real subspace arrangement has the M-property. Salvetti [170] constructed a finite cell complex of the homotopy type of the complement of a complexified real arrangement. Orlik [140] constructed a finite cell complex of the homotopy type of the complement of an arbitrary arrangement of affine subspaces. Despite these constructions, there is still no good criterion to detect nonzero elements in the homotopy groups of the complement or to determine whether the complement is a $K(\pi,1)$ space. Even the special case of complex reflection arrangements is open. Orlik and Solomon [149] defined a subclass of complex reflection groups, called Shephard groups, and proved that the complements of their reflection arrangements are $K(\pi,1)$ spaces. Using the classification of irreducible complex reflection groups [176], this leaves an infinite family of groups where a fibration argument may be used, some exeptional groups of rank two where the complement is always $K(\pi,1)$, and six exceptional groups of rank ≥ 3 where the problem is still open.

The study of algebraic geometry over the complex numbers leads to different questions. The general plane cubic curve has nine inflection points. They have the property that a line through any two contains a third, see Figure 40. In 1893 J. J. Sylvester conjectured that it was impossible to have a non-linear finite set of points in real space with this property. This was proved by T. Gallai.

> Let A_n be the projective n-space ($n \geq 2$) over some field $\mathbb{K}$. A finite subset X of A_n is called a Sylvester-Gallai (S-G) configuration if it verifies the following condition:
>
> (*) If $P, Q \in X$, with $P \neq Q$, the line joining P and Q contains at least one more point of X. (Equivalently: no line intersects X in exactly two points.)
>
> An S-G configuration is called linear (planar) if it is contained in a line (plane). If $\mathbb{K}$ is the field of real numbers, it is known that any S-G configuration is linear. Over the field of complex numbers there are well-known examples of nonlinear S-G configurations (e. g. the nine inflection points of a nonsingular cubic.)
>
> Is there a nonplanar Sylvester-Gallai configuration over the field of complex numbers?
>
> Problem 5359. *American Mathematical Monthly* **73** (1966), p. 89.
> *Proposed by Jean-Pierre Serre, Paris, France*

In 1977 Y. Miyaoka and Sh. -T. Yau proved the inequality $c_1^2 \leq 3c_2$ for the Chern numbers of an algebraic surface of general type. Equality occurs if and only if the univeral cover of the surface is the complex ball. The corresponding surfaces are called ball quotients. Several explicit examples of ball quotients were constructed by F. Hirzebruch [96]. Related results were obtained by A. Sommese [178] and B. Hunt [97]. We quote from the introduction of [19], where there is an excellent description of this work:

> Die hier untersuchten Flächen erhalten wir hauptsächlich als "Kummer–Überlagerungen" zu Geradenkonfigurationen in der komplex–projektiven Ebene. Diese Überlagerungen der projektiven Ebene sind entlang einer Menge von Geraden lokal mit der Ordnung $n \geq 2$ verzweigt. Die Chernschen Zahlen (der minimalen Desingularisierungen) dieser Flächen sind dann durch die kombinatorischen Invarianten der Geradenkonfiguration bestimmt. Wenn wir nun die Miyaoka–Yau–Ungleichung $c_1^2 \leq 3c_2$ anwenden, so erhalten wir wiederum Ungleichungen für diese kombinatorischen Invarianten, aus denen sich überraschenderweise Sätze über Geradenkonfigurationen in der Ebene (und durch Dualisierung auch über Punktkonfigurationen) ergeben, die sich bisher nicht auf elementarem Wege beweisen lassen.

Let $\mathcal{A}$ be an arrangement of lines in the plane. Let t_j be the number of points which lie on exactly j lines. For a real arrangement, Melchior [127] showed that if $\mathcal{A}$ has at least three noncollinear points then $t_2 \geq 3 + t_4 + 2t_5 + 3t_6 + \cdots$. As a consequence of his work on ball quotients, Hirzebruch [95] proved the following inequality for a complex arrangement of k lines with $t_k = t_{k-1} = 0$:

$$t_2 + t_3 \geq k + t_5 + 2t_6 + 3t_7 + \cdots$$

This enabled L. M. Kelly [106] to show that every S-G configuration over the field of complex numbers is planar.

The work on hypergeometric functions in several variables has become a major new area of research. In this generalization there are linear functions $f = \{f_i\}$, complex exponents $\alpha = \{\alpha_i\}$, and a real polytope $\Delta \subset \mathbb{R}^n$. The integral

$$I(\Delta, f, \alpha) = \int_\Delta f_1^{\alpha_1} \cdots f_N^{\alpha_N} dx_1 \cdots dx_n$$

is a function of these variables. The linear functions $\{f_i\}$ determine an arrangement $\mathcal{A}$ and the algebra $A(\mathcal{A})$ enters this work. There are unexpected connections with algebraic K-theory and conformal field theory. Among the principal practitioners are K. Aomoto [3, 4], I. M. Gelfand [72, 75], V. Schechtman and A. Varchenko [174]. For a recent survey see [199].

In order to study the structure of the algebra $A(\mathcal{A})$, it is sometimes useful to have a standard way to choose a basis. Such a basis, called the broken circuit basis, was constructed independently by Björner [24], Gelfand and Zelevinsky [75], and Jambu and Terao [104].

Most of the recent work on free arrangements is attempting to solve one of three outstanding conjectures: (i) the complement of a complex free arrangement is a $K(\pi,1)$ space, (ii) the restriction of a free arrangement to one of its hyperplanes is free, (iii) the property that $\mathcal{A}$ is free depends only on $L(\mathcal{A})$. Ziegler [221] gave an example of a free arrangement and a non-free arrangement such that their lattices are isomorphic. One is defined over a field of characteristic three, while the other is over characteristic not equal to three. Conjecture (iii) is still open for arrangements defined over the same field. S. Yuzvinsky [209, 210] gave interesting necessary conditions in terms of lattice homology for an arrangement to be free.

A general formula for $\pi(\mathcal{A}, t)$ in terms of derivations was proved by Solomon and Terao [177]. It yields the factorization theorem when it is applied to free arrangements. The holonomy Lie algebra associated with an arrangement was studied by Kohno [109], who also studied the Hecke algebra representations of braid groups in [112]. Other applications include the work of T. tom Dieck and T. Petrie [52, 53]. Arrangements and group representations are combined in [142], in work of H. Barcelo [19, 17], G. Lehrer [118, 119], and Lehrer and Solomon [120].

2 Definitions and Examples

The following special symbols are used throughout this book: natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, the field of q elements $\mathbb{F}_q$, and arbitrary fields $\mathbb{K}$, $\mathbb{L}$.

Definition 2.1 *Let $\mathbb{K}$ be a field and let $V_{\mathbb{K}}$ be a vector space of dimension ℓ. A* **hyperplane** *H in $V_{\mathbb{K}}$ is an affine subspace of dimension $(\ell - 1)$. A* **hyperplane arrangement** *$\mathcal{A}_{\mathbb{K}} = (\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}})$ is a finite set of hyperplanes in $V_{\mathbb{K}}$.*

More generally, a **subspace arrangement** is a finite set of affine subspaces of V with no dimension restrictions. Since this book is mostly about hyperplane arrangements, we agree to use "arrangement" in place of "hyperplane arrangement."

The subscript $\mathbb{K}$ will be used only when we want to call attention to the field. We call $\mathcal{A}$ an ℓ**-arrangement** when we want to emphasize the dimension of V. Let Φ_ℓ denote the empty ℓ-arrangement. Let V^* be the dual space of V, the space of linear forms on V. Let $S = S(V^*)$ be the symmetric algebra of V^*. Choose a basis $\{e_1, \ldots, e_\ell\}$ in V and let $\{x_1, \ldots, x_\ell\}$ be the dual basis in V^* so $x_i(e_j) = \delta_{i,j}$. We may identify $S(V^*)$ with the polynomial algebra $S = \mathbb{K}[x_1, \ldots, x_\ell]$. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree 1 defined up to a constant.

Definition 2.2 *The product*

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called a **defining polynomial** *of $\mathcal{A}$. It is defined up to a nonzero constant multiple. We agree that $Q(\Phi_\ell) = 1$ is the defining polynomial of the empty arrangement.*

Definition 2.3 *We call $\mathcal{A}$* **centered** *if $\cap_{H \in \mathcal{A}} H \neq \emptyset$ and* **centerless** *otherwise. If $\mathcal{A}$ is centered then coordinates may be chosen so that each hyperplane contains the origin. In this case we call $\mathcal{A}$* **central**. *If $\mathcal{A}$ is central then each α_H is a linear form and $Q(\mathcal{A})$ is a homogeneous polynomial whose degree is the cardinality of $\mathcal{A}$. We agree that the empty arrangement Φ_ℓ is central. When we want to emphasize that an arrangement can be either centered or centerless we call it* **affine**.

Definition 2.4 *A* **projective arrangement** *is a finite set of projective hyperplanes in projective space.*

Since the complement of a hyperplane in projective space is affine space, the complement of a nonempty projective arrangement may be viewed as the complement of an affine arrangement. We shall not discuss projective arrangements separately.

Examples

First consider some real arrangements. The only central 1-arrangement consists of the hyperplane $\{0\}$. An affine 1-arrangement consists of a finite set of points. For $\ell = 2, 3$ we agree to use x, y, z in place of x_1, x_2, x_3. A central real 2-arrangement is a finite set of lines through the origin. An affine 2-arrangement is a finite set of lines in the plane.

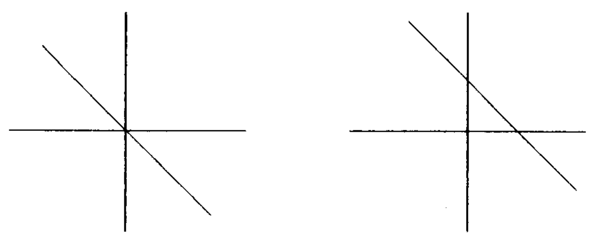


Figure 2: $Q(\mathcal{A}) = xy(x+y)$ and $Q(\mathcal{A}) = xy(x+y-1)$

Example 2.5 *Define $\mathcal{A}$ by $Q(\mathcal{A}) = xy(x+y)$. It consists of three lines through the origin,* see Figure 2.

Example 2.6 *Define $\mathcal{A}$ by $Q(\mathcal{A}) = xy(x+y-1)$. Then $\mathcal{A}$ is centerless. It consists of three affine lines,* see Figure 2.

Real 3-arrangements are examples which display some of the intricacies of the general case.

Example 2.7 *Let $\mathbb{R}^3$ have its usual basis. Consider the cube with vertices $(\pm 1, \pm 1, \pm 1)$. Its nine planes of symmetry form a central 3-arrangement defined by*

$$Q(\mathcal{A}) = xyz(x-y)(x+y)(x-z)(x+z)(y-z)(y+z).$$

These nine planes intersect in lines which are axes of rotational symmetry for the cube. The group of symmetries of the cube is the Coxeter group of type B_3. We shall refer to this arrangement as the B_3-arrangement.

We can visualize central real 3-arrangements by the deconing construction of Definition 2.15. As usual, we think of the projective plane $P^2_{\mathbb{R}}$ as the disk with identification of diametrically opposite points on the boundary. The picture we draw is therefore the intersection of the arrangement with the upper hemisphere. The same idea may be conveyed by a slightly different picture which is easier to draw. Since we assume that the line at infinity is in $\mathcal{A}$, we may identify its complement in $P^2_{\mathbb{R}}$ with $\mathbb{R}^2$ and draw the corresponding affine arrangement. **Here we must remember that parallel lines meet at infinity.** In Example 2.7 we let the plane $z = 0$ go to the line at infinity. If we substitute $z = 1$ in the remaining linear forms we get an affine 2-arrangement. In order to remember that the line at infinity is in our arrangement we draw a frame at "infinity" in Figure 3. It is easy to find and label the 13 lines of intersection in the 3-arrangement of Example 2.7 by finding the 13 points of intersection in Figure 3.

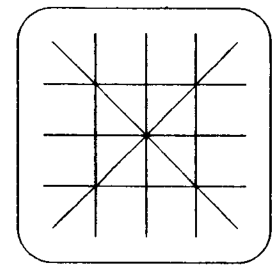


Figure 3: The B_3-arrangement

Example 2.8 *Let $\mathcal{A}_{\mathbb{R}}$ be the* **Boolean** *arrangement defined by*

$$Q(\mathcal{A}) = x_1 x_2 \cdots x_\ell.$$

This is the arrangement of the coordinate hyperplanes in $\mathbb{R}^\ell$.

Example 2.9 *For $1 \le i < j \le \ell$ let $H_{i,j} = \ker(x_i - x_j)$. Let $\mathcal{A}_{\mathbb{R}}$ be the* **braid** *arrangement defined by*

$$Q(\mathcal{A}) = \prod_{1 \le i < j \le \ell} (x_i - x_j).$$

While the examples above also make sense over finite fields, the next example can only be stated in that setting.

Example 2.10 *Let V be an ℓ-dimensional vector space over the finite field of q elements, $\mathbb{F}_q$. Let $\mathcal{A}$ be the arrangement of all hyperplanes in V.*

Basic Constructions

Definition 2.11 *Let $|\mathcal{A}|$ denote the* **cardinality** *of $\mathcal{A}$.*

In Example 2.8 we have $|\mathcal{A}| = \ell$. In Example 2.9 we have $|\mathcal{A}| = \ell(\ell-1)/2$. In Example 2.10 we have
$$|\mathcal{A}| = 1 + q + q^2 + \cdots + q^{\ell-1}.$$
The last assertion may be seen in two ways. Using induction we count the number of distinct linear forms $\alpha = c_1x_1 + \cdots + c_\ell x_\ell$. If $c_\ell \neq 0$ then there are $q^{\ell-1}$ such forms. If $c_\ell = 0$ then by induction there are $1 + q + \cdots + q^{\ell-2}$ such forms. If we introduce an inner product in V then we may count the number of lines instead. Since V has $q^\ell - 1$ nonzero elements and each line contains $q-1$ nonzero elements, there are $(q^\ell - 1)/(q-1)$ lines in V.

It is clear from these examples that some of the complexity of $\mathcal{A}$ may be captured by knowledge of the intersection pattern of its hyperplanes.

Definition 2.12 *Let $L(\mathcal{A})$ be the set of all nonempty intersections of elements of $\mathcal{A}$. We agree that $L(\mathcal{A})$ includes V as the intersection of the empty collection of hyperplanes.*

We should remember that if $X \in L(\mathcal{A})$ then $X \subseteq V$. Strictly speaking these objects should have different names, but it is always clear from the context which one is in consideration.

Definition 2.13 *Let $(\mathcal{A}, V)$ be an arrangement. If $\mathcal{B} \subseteq \mathcal{A}$ is a subset then $(\mathcal{B}, V)$ is called a* **subarrangement**. *For $X \in L(\mathcal{A})$ define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by*
$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$
Note that $\mathcal{A}_X$ has center X in any arrangement. Define an arrangement $(\mathcal{A}^X, X)$ in X by
$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$
We call $\mathcal{A}^X$ the **restriction** *of $\mathcal{A}$ to X.*

The method of **deletion and restriction** is a basic construction in this book. It allows for induction on the cardinality of $\mathcal{A}$.

Definition 2.14 *Let $\mathcal{A}$ be a nonempty arrangement and let $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ and let $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a* **triple** *of arrangements and H_0 the* **distinguished** *hyperplane.*

The method of **coning** is another basic construction. It allows for comparing affine and central arrangements.

Definition 2.15 *An affine ℓ-arrangement $\mathcal{A}$ defined by $Q(\mathcal{A}) \in S$ gives rise to a central $(\ell+1)$-arrangement $\mathbf{c}\mathcal{A}$, called the* **cone** *over $\mathcal{A}$. Let $Q' \in \mathbb{K}[x_0, x_1, \ldots, x_\ell]$ be the polynomial $Q(\mathcal{A})$ homogenized and define $Q(\mathbf{c}\mathcal{A}) = x_0Q'$. Note that $|\mathbf{c}\mathcal{A}| = |\mathcal{A}|+1$. We call $K_0 = \ker(x_0)$ the* **additional** *hyperplane.*

Note that in the coning construction the arrangement $\mathcal{A}$ is embedded in $\mathbf{c}\mathcal{A}$ by identifying its total space with the affine subspace $\ker(x_0 - 1)$ in the total space of $\mathbf{c}\mathcal{A}$. There is an inverse operation. A nonempty central $(\ell+1)$-arrangement $\mathcal{A}$ gives rise to an ℓ-arrangement $\mathbf{d}\mathcal{A}$, which is in general not centered, by the following **deconing** construction. Choose a hyperplane $K_0 \in \mathcal{A}$. Choose coordinates so that $K_0 = \ker(x_0)$. Let $Q(\mathcal{A}) \in \mathbb{K}[x_0, x_1, \ldots, x_\ell]$ be a defining polynomial for $\mathcal{A}$. The defining polynomial $Q(\mathbf{d}\mathcal{A})$ is obtained by substituting 1 for x_0 in $Q(\mathcal{A})$. The deconing construction may be viewed as first projectivizing the central arrangement $\mathcal{A}$ then removing the image of K_0, the hyperplane at infinity, and identifying its complement with affine space.

There are two sets of fundamental interest in the study of arrangements: the variety of $\mathcal{A}$ and the complement of $\mathcal{A}$.

Definition 2.16 *Define the* **variety** *of $\mathcal{A}$ by*
$$N(\mathcal{A}) = \bigcup_{H \in \mathcal{A}} H = \{v \in V \mid Q(\mathcal{A})(v) = 0\}.$$

Definition 2.17 *Define the* **complement** *of $\mathcal{A}$ by*
$$M(\mathcal{A}) = V \setminus N(\mathcal{A}).$$

It is sometimes convenient to suppress dependence on $\mathcal{A}$ and write $Q = Q(\mathcal{A})$, $L = L(\mathcal{A})$, $N = N(\mathcal{A})$, $M = M(\mathcal{A})$, etc.

The Module of $\mathcal{A}$–Derivations

The variety $N(\mathcal{A})$ is a hypersurface with a very complicated singular set. In the present context the study of logarithmic vector fields and logarithmic differential forms on a hypersurface was initiated by Saito [168]. It was shown in [192] that in the case of an arrangement we can pass from analytic to algebraic considerations.

Definition 2.18 *A $\mathbb{K}$-linear map $\theta : S \to S$ is a* **derivation** *if for $f, g \in S$*
$$\theta(fg) = f\theta(g) + g\theta(f).$$
Let $\mathrm{Der}_{\mathbb{K}}(S)$ be the S-module of derivations of S.

Definition 2.19 *Define an S-submodule of $\mathrm{Der}_{\mathbb{K}}(S)$, called the* **module of $\mathcal{A}$–derivations,** *by*
$$D_S(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q) \in QS\}.$$

Definition 2.20 *The arrangement $\mathcal{A}$ is called* **free** *if the module $D_S(\mathcal{A})$ is a free S-module.*

The Complement of a Complex Arrangement

Next we consider field extensions. Let $\mathbb{L}$ be an extension of $\mathbb{K}$. Every arrangement $(\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}})$ gives rise to an arrangement over $\mathbb{L}$.

Definition 2.21 *Let $(\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}})$ be an arrangement with defining polynomial $Q(\mathcal{A}_{\mathbb{K}})$. The $\mathbb{L}$-extended arrangement is in $V = V_{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}$. It consists of the hyperplanes: $\mathcal{A}_{\mathbb{L}} = \{H \otimes_{\mathbb{K}} \mathbb{L} \mid H \in \mathcal{A}_{\mathbb{K}}\}$. Thus $Q(\mathcal{A}_{\mathbb{L}}) = Q(\mathcal{A}_{\mathbb{K}})$.*

One example of this is a **complexified** real arrangement. It is already quite difficult to visualize the complexification of Example 2.5. In real dimensions we have three 2-planes in 4-space which meet only at the origin. The complexification of the Boolean arrangement is the arrangement of the coordinate hyperplanes in $\mathbb{C}^\ell$.

The complexified braid arrangement occurs in the theory of configuration spaces and braids. Recall that a **braid** on ℓ strands may be viewed as the graph of the motion of ℓ distinct points in the complex line between times $t = 0$ and $t = 1$, subject to the condition that the points remain distinct throughout the motion. Thus we have a map $f : [0, 1] \to \mathbb{C}^\ell$ such that for each t the image point $(f_1(t), \ldots, f_\ell(t))$ satisfies the condition $f_i(t) \neq f_j(t)$. The braid is **pure** if $f(0) = f(1)$. Thus a pure braid is the image of a circle in the complement of the hyperplanes $H_{i,j}$. The variety $N(\mathcal{A})$ is called the **superdiagonal** and its complement $M(\mathcal{A}) = V \setminus N(\mathcal{A})$ is the **pure braid space.**

Reflection Arrangements

Next we define a collection of arrangements with particularly nice properties. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $GL(V)$ denote the general linear group of V.

Definition 2.22 *An element $s \in GL(V)$ is a* **reflection** *if it has finite order and its fixed point set is a hyperplane H_s. We call H_s the* **reflecting hyperplane** *of s. A finite subgroup $G \subset GL(V)$ is called a* **reflection group** *if it is generated by reflections.*

Definition 2.23 *Let $G \subset GL(V)$ be a finite reflection group. The set $\mathcal{A} = \mathcal{A}(G)$ of reflecting hyperplanes of G is called the* **reflection arrangement** *of G.*

3 Outline

Combinatorics

The **intersection poset** $L(\mathcal{A})$ is an important combinatorial invariant of the arrangement $\mathcal{A}$. We study its properties in chapter II. In section 4 we give $L(\mathcal{A})$ a partial order by reverse inclusion and show that it is a geometric lattice in case $\mathcal{A}$ is a central arrangement. We construct the face poset of a real arrangement and show its connection with oriented matroids. We also define supersolvable arrangements here and a generalization called arrangements with a nice partition. In section 5 we define the Möbius function and study its properties. We also present notes on the interesting history of this function dating back to Euler. In section 6 we define the Poincaré polynomial of $\mathcal{A}$, $\pi(\mathcal{A}, t)$. It is related to another combinatorial function called the characteristic polynomial. We show that if $\mathbf{c}\mathcal{A}$ is the cone over the affine arrangement $\mathcal{A}$ then

$$\pi(\mathbf{c}\mathcal{A}, t) = (1 + t)\pi(\mathcal{A}, t). \tag{1}$$

A fundamental technical tool in this book is the method of **deletion and restriction** which allows induction on the number of hyperplanes in the arrangement. It uses the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ of Definition 2.14. We prove a theorem of Brylawski [35] about the Poincaré polynomial under deletion and restriction:

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t). \tag{2}$$

In section 6 we also prove a theorem of Stanley [181] which asserts that if $L(\mathcal{A})$ is supersolvable then

$$\pi(\mathcal{A}, t) = (1 + b_1 t) \cdots (1 + b_\ell t), \tag{3}$$

where the b_i are non-negative integers. We close the chapter with a section on the connections between arrangements and graph theory. This includes the chromatic polynomial, a precursor of the characteristic and Poincaré polynomials.

Algebras

In chapter III let $\mathcal{K}$ be a commutative ring. We construct certain algebras over $\mathcal{K}$ associated with $\mathcal{A}$. We construct the graded algebra $A(\mathcal{A})$ for a central arrangement $\mathcal{A}$ in section 8. This construction is generalized to affine arrangements in section 9. The algebra $A(\mathcal{A})$ is the quotient of the exterior algebra $E(\mathcal{A})$ based on $\mathcal{A}$ by a homogeneous ideal $I(\mathcal{A})$, $A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$. This algebra is constructed using only $L(\mathcal{A})$. In the literature $A(\mathcal{A})$ is sometimes called the Orlik-Solomon algebra. It will reappear in chapter V with a topological significance. We prove that the $\mathcal{K}$-algebra $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module and that its Poincaré polynomial is equal to $\pi(\mathcal{A}, t)$. This gives an interpretation of the coefficients of $\pi(\mathcal{A}, t)$. We construct a $\mathcal{K}$-basis for $A(\mathcal{A})$ as a free graded $\mathcal{K}$-module using broken circuits. We also show that given a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ there is an exact sequence of $\mathcal{K}$ modules

$$(4) \qquad 0 \to A(\mathcal{A}') \to A(\mathcal{A}) \to A(\mathcal{A}'') \to 0.$$

Inspection of the maps shows that (2) is a consequence of (4). We prove some algebra factorization theorems in section 10. If $\mathbf{c}\mathcal{A}$ is the cone over $\mathcal{A}$ then

$$(5) \qquad A(\mathbf{c}\mathcal{A}) \simeq (\mathcal{K} + \mathcal{K}a_0) \otimes A(\mathcal{A}).$$

In fact (1) is a consequence of (5). If $\mathcal{A}$ is supersolvable then $A(\mathcal{A})$ has a tensor product decomposition

$$(6) \qquad (\mathcal{K} + B_1) \otimes \cdots \otimes (\mathcal{K} + B_\ell)$$

as graded $\mathcal{K}$-module with $b_i = \operatorname{rank} B_i$. In fact (3) is a consequence of (6). This decomposition of $A(\mathcal{A})$ is generalized to arrangements with a nice partition. In section 11 we define another graded algebra $B(\mathcal{A})$ whose multiplication is a shuffle product. We prove that $B(\mathcal{A})$ is algebra isomorphic to $A(\mathcal{A})$. In section 12 we assume that $\mathcal{K}$ is a subring of $\mathbf{K}$. We associate to the arrangement $\mathcal{A}$ the $\mathcal{K}$-algebra $R(\mathcal{A})$ generated by the differential forms $\omega_H = d\alpha_H/\alpha_H$. Note that this algebra is not a purely combinatorial object since the defining polynomials α_H enter the definition. The main result of section 12 is that there is an isomorphism of algebras $A(\mathcal{A}) \simeq R(\mathcal{A})$. This shows that $R(\mathcal{A})$ depends only on $L(\mathcal{A})$. The argument uses the fact that there is a short exact sequence of $\mathcal{K}$-modules

$$0 \to R(\mathcal{A}') \to R(\mathcal{A}) \to R(\mathcal{A}'') \to 0.$$

The Module of $\mathcal{A}$-Derivations

The most important algebraic geometric invariant of $\mathcal{A}$ is the module $D_S(\mathcal{A})$. In chapter IV we assume that $\mathcal{A}$ is a central arrangement and study the algebraic properties of $D_S(\mathcal{A})$. Section 13 contains the basic definitions. In section 14 we define free arrangements and establish their fundamental properties. If $\mathcal{A}$ is free then we can associate with it a collection of non-negative integers, called its exponents, $\exp \mathcal{A} = \{b_1, \ldots, b_\ell\}$. These integers are unique up to order, but they are not necessarily distinct. In section 15 we prove the Addition–Deletion Theorem. It asserts that if $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple then any two of the following statements imply the third:

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp \mathcal{A} &= \{b_1, \ldots, b_{\ell-1}, b_\ell\}, \\ \mathcal{A}' \text{ is free with } \exp \mathcal{A}' &= \{b_1, \ldots, b_{\ell-1}, b_\ell - 1\}, \\ \mathcal{A}'' \text{ is free with } \exp \mathcal{A}'' &= \{b_1, \ldots, b_{\ell-1}\}. \end{aligned}$$

This result leads to the definition of inductively free arrangements. We give several examples and prove that a supersolvable arrangement is inductively free. In section 16 we define the

module $\Omega^p(\mathcal{A})$ of logarithmic p-forms with poles on the hypersurface $N(\mathcal{A})$. We show that the complex $\Omega^\cdot(\mathcal{A})$ is closed under exterior product and that $\Omega^1(\mathcal{A})$ is the dual of $D(\mathcal{A})$. In section 17 we construct a simplicial complex $\mathsf{F}(\mathcal{A})$ associated to $L(\mathcal{A})$ by Folkman [70]. We compute its homology groups and show that $\mathsf{F}(\mathcal{A})$ has the homotopy type of a wedge of spheres. We also construct another chain complex whose homology is naturally isomorphic to $B(\mathcal{A})$, and show how these constructions are related. In section 18 we generalize these two constructions to order complexes with arbitrary functor coefficient. This allows proof of an important technical result due to Yuzvinsky [209]. It is used in the proof of the Factorization Theorem which asserts that if $\mathcal{A}$ is a free ℓ-arrangement with $\exp \mathcal{A} = \{b_1, \ldots, b_\ell\}$ then

$$\pi(\mathcal{A}, t) = (1 + b_1 t) \cdots (1 + b_\ell t).$$

In the first four chapters the field $\mathbf{K}$ is arbitrary and the development of the material is essentially self-contained. In the last two chapters we work mostly over the complex numbers and we make more use of results from the literature.

Topology

In chapter V we return to the convention that an arrangement is not necessarily central. The subject of this chapter is the topology of the complement of a complex arrangement, $M(\mathcal{A})$. In section 19 we prove some elementary facts about $M = M(\mathcal{A})$ and discuss a few examples. In particular, if $\mathbf{c}\mathcal{A}$ is the cone over $\mathcal{A}$ then

$$(7) \qquad M(\mathbf{c}\mathcal{A}) \approx M(\mathcal{A}) \times \mathbf{C}^*,$$

where $\mathbf{C}^*$ denotes the nonzero complex numbers and $\approx$ denotes homeomorphism. We also give a review of fundamental work of Arnold, Brieskorn, Deligne and Hattori. The rest of the chapter does not follow the chronology of discovery. In section 20 we construct a finite simplicial complex $\mathbf{M}$ of the homotopy type of M. The construction uses an embedding in V of the order complex of the face poset of a real arrangement. In principle, $\mathbf{M}$ contains all information about the homotopy type of M. In the special case of a complexified real arrangement, Salvetti [170] constructed a smaller complex $\mathbf{W}$ of the homotopy type of M. Arvola [13] constructed a simplicial map $\mathbf{M} \to \mathbf{W}$ which is a homotopy equivalence. In practice, $\mathbf{M}$ and $\mathbf{W}$ are very large and unsuited for explicit calculations. It is therefore desirable to find simple algorithms to compute topological invariants of M.

Arvola's presentation of the fundamental group of M is in section 21. It generalizes Randell's presentation of the fundamental group of the complexification of a real arrangement. In section 22 we consider the cohomology groups of $M(\mathcal{A})$ with integer coefficients. We use our results on $R(\mathcal{A})$ from section 12 to prove that given a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, there are split short exact sequences for all $k \geq 0$

$$(8) \qquad 0 \to H^k(M(\mathcal{A}')) \to H^k(M(\mathcal{A})) \to H^{k-1}(M(\mathcal{A}'')) \to 0.$$

It follows that the map $R(\mathcal{A}) \to H^\cdot(M(\mathcal{A}))$ induced by $\omega_H \mapsto [(1/2\pi i)\omega_H]$ is an algebra isomorphism. Together with the algebra isomorphism $R(\mathcal{A}) \simeq A(\mathcal{A})$ established in section

12, this provides a presentation of the cohomology algebra in terms of generators and relations. This is the topological interpretation of $A(\mathcal{A})$. Thus the cohomology algebra of $M(\mathcal{A})$ depends only on $L(\mathcal{A})$. These results have several consequences. They provide an elementary proof of Brieskorn's Lemma [33]. They show that the Poincaré polynomial of the complement is

$$\mathrm{Poin}(M(\mathcal{A}),t) = \pi(\mathcal{A},t).$$

Thus the coefficients of $\pi(\mathcal{A},t)$ are also the Betti numbers of the complement. They show the common origin of formulas (2), (4), (8), and of (1), (5), (7). They also show that if $\mathcal{A}$ is a real arrangement then $M(\mathcal{A})$ has the M-property. In section 23 we prove that the complement of a supersolvable arrangement admits a strictly linear fibration. This is the topological interpretation of (6). In section 24 we describe some related recent results: work of Falk and Kohno on minimal models, Manin and Schechtman's work on discriminantal arrangements, Falk's geometric linking, the cohomology of the Milnor fiber of a generic arrangement, and the results of Goresky and MacPherson on arrangements of subspaces of arbitrary codimension.

Reflection Arrangements

In chapter VI we assume the presence of a symmetry group. Suppose $(\mathcal{A}, V)$ is an arrangement and $G \subseteq GL(V)$ is a finite group such that $G(\mathcal{A}) \subseteq \mathcal{A}$. All our constructions may be done equivariantly. We obtain particularly nice results for the arrangements which arise as reflecting hyperplanes of complex reflection groups. In particular every reflection arrangement is free. It follows from work of Brieskorn and Deligne that the complement of a complexified real reflection arrangement is a $K(\pi,1)$ space. We conclude with an outline of the proof that the complements of the reflection arrangements of certain complex reflection groups, called Shephard groups, are also $K(\pi,1)$ spaces.

The book concludes with an appendix. In section 26 we collect certain facts from commutative algebra which are needed in the text.

Chapter II

The intersection poset $L(\mathcal{A})$ is an important combinatorial invariant of the arrangement $\mathcal{A}$. We study its properties in this chapter. In section 4 we give $L(\mathcal{A})$ a partial order by reverse inclusion and show that it is a geometric lattice in case $\mathcal{A}$ is a central arrangement. We construct the face poset of a real arrangement and show its connection with oriented matroids. We also define supersolvable arrangements here and a generalization called arrangements with a nice partition. In section 5 we define the Möbius function and study its properties. We also present notes on the interesting history of this function dating back to Euler. In section 6 we define the Poincaré polynomial $\pi(\mathcal{A},t)$. It is related to another combinatorial function called the characteristic polynomial. A fundamental technical tool in this book is the method of **deletion and restriction**, which allows induction on the number of hyperplanes in the arrangement. It uses the triple $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ of Definition 2.14. The Deletion-Restriction Theorem states:

$$\pi(\mathcal{A},t) = \pi(\mathcal{A}',t) + t\pi(\mathcal{A}'',t).$$

We prove a theorem of Stanley [181] which asserts that if $L(\mathcal{A})$ is supersolvable then

$$\pi(\mathcal{A},t) = (1+b_1t)\cdots(1+b_\ell t),$$

where the b_i are non-negative integers. We close the chapter with a section on the connections between arrangements and graph theory. This includes the chromatic polynomial, a precursor of the characteristic polynomial and the Poincaré polynomial. Many of the definitions and results may be extended to a larger class of objects. We use Aigner's book [1] as a general reference for undefined terms.

4 The Poset $L(\mathcal{A})$

Definitions

Definition 4.1 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. Define a* **partial order** *on L by:*

$$X \le Y \Longleftrightarrow Y \subseteq X.$$

Note that this is **reverse** inclusion. Thus V is the unique minimal element. Ordinary inclusion also gives a partial order and it is preferred by many authors. Our reason for this convention is that with it the intersection poset of a central arrangement has all the properties of a **geometric lattice** shown in Lemma 4.3.

Definition 4.2 *Define a rank function on L by $r(X) = \mathrm{codim}X$. Thus $r(V) = 0$ and $r(H) = 1$ for $H \in \mathcal{A}$. Call H an* **atom** *of L. Let $X, Y \in L$. Define their* **meet** *by $X \wedge Y = \cap\{Z \in L \mid X \cup Y \subseteq Z\}$. If $X \cap Y \neq \emptyset$ we define their* **join** *by $X \vee Y = X \cap Y$.*

Lemma 4.3 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. Then:*
(1) Every element of $L \setminus \{V\}$ is a join of atoms.
(2) For every $X \in L$ all maximal linearly ordered subsets

$$V = X_0 < X_1 < \ldots < X_p = X$$

have the same cardinality. Thus $L(\mathcal{A})$ is a geometric poset.
(3) If $\mathcal{A}$ is central then all joins exist so L is a lattice. For all $X, Y \in L$ the rank function satisfies

$$r(X \wedge Y) + r(X \vee Y) \leq r(X) + r(Y).$$

Thus for a central arrangement $L(\mathcal{A})$ is a geometric lattice.

Proof. Assertion (1) follows from the definition. Assertion (2) is a consequence of the fact that the maximal number of linearly independent hyperplanes which can contain a subspace is its codimension. To see (3) recall that

$$\dim(X + Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$$

and $\dim(X + Y) \leq \dim(X \wedge Y)$. □

Lemma 4.4 *Maximal elements of $L(\mathcal{A})$ have the same rank.*

Proof. This is clear if $\mathcal{A}$ is a central arrangement since $L(\mathcal{A})$ has a unique maximal element. If $\mathcal{A}$ is centerless then it may have several maximal elements. Observe that $T \in L(\mathcal{A})$ is a maximal element if and only if for every $H \in \mathcal{A}$ either $T \subset H$ or $T \cap H = \emptyset$. Since this condition is invariant under affine linear transformations, maximal elements are affine linear images of each other, hence of the same dimension. □

Definition 4.5 *The* **rank** *of $\mathcal{A}$, $r(\mathcal{A})$ is the rank of a maximal element of $L(\mathcal{A})$. Call the ℓ-arrangement $\mathcal{A}$* **essential** *if $r(\mathcal{A}) = \ell$. If $\mathcal{A}$ is a central arrangement let $T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H$ be the* **unique maximal element** *of $L(\mathcal{A})$.*

Definition 4.6 *Call the arrangements $\mathcal{A} = (\mathcal{A}, V)$ and $\mathcal{B} = (\mathcal{B}, W)$* **lattice equivalent**, *or L-equivalent, if there is an order preserving bijection $\pi : L(\mathcal{A}) \to L(\mathcal{B})$.*

Definition 4.7 *Let $L_p(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid r(X) = p\}$. The* **Hasse diagram** *of $L(\mathcal{A})$ has vertices labeled by the elements of $L(\mathcal{A})$ and arranged on levels L_p for $p \geq 0$. Suppose $X \in L_p$ and $Y \in L_{p+1}$. An edge in the Hasse diagram connects X with Y if $X < Y$.*

If $\mathcal{A}$ is defined by a polynomial $Q(\mathcal{A})$ it is sometimes convenient to label elements of $L(\mathcal{A})$ by the equations they satisfy. The Hasse diagrams of Examples 2.5, 2.6, 2.7 appear in Figures 4, 5, 6.

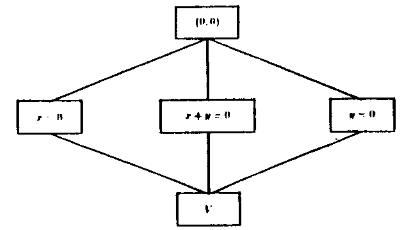


Figure 4: The Hasse diagram of $Q(\mathcal{A}) = xy(x + y)$

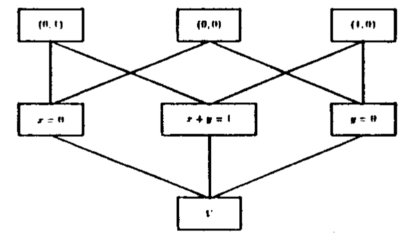


Figure 5: The Hasse diagram of $Q(\mathcal{A}) = xy(x + y - 1)$

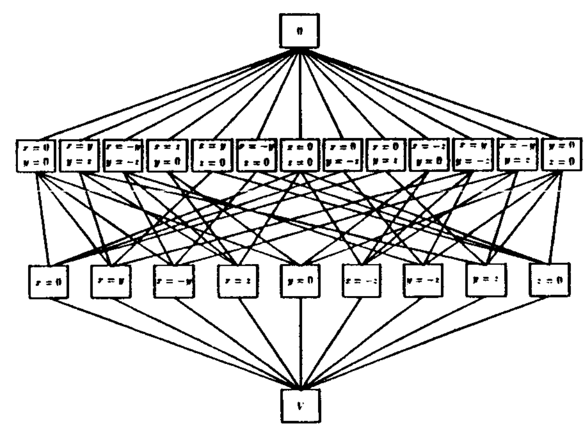


Figure 6: The Hasse diagram of the B_3-arrangement

Examples

Example 4.8 *The lattice $L(\mathcal{A})$ of the Boolean arrangement.*

Let $H_i = \ker(x_i)$. Let $I = \{i_1, \ldots, i_p\}$ where $1 \le i_1 < \cdots < i_p \le \ell$. Let $H_I = H_{i_1} \cap \cdots \cap H_{i_p}$. The lattice $L(\mathcal{A})$ consists of the 2^ℓ subspaces H_I for all subsets I.

Proposition 4.9 *The lattice $L(\mathcal{A})$ of the braid arrangement is isomorphic to the* **partition lattice.**

Proof. Let $I = \{1, \ldots, \ell\}$. Let $\mathcal{P}(\ell)$ be the set of partitions of I. An element of $\mathcal{P}(\ell)$ is a collection $\Lambda = \{\Lambda_1, \ldots, \Lambda_r\}$ of nonempty pairwise disjoint subsets of I, called the blocks of Λ, whose union is I. There is a natural partial order on $\mathcal{P}(\ell)$ given by $\Lambda \le \Gamma$ if Λ is finer than Γ. Thus blocks of Γ are unions of blocks of Λ. In order to find a lattice isomorphism from the braid lattice $L(\mathcal{A})$ to $\mathcal{P}(\ell)$ it is convenient to define $H_{i,i} = V$ for all i. Let $X \in L(\mathcal{A})$. Define a relation $\sim_X$ on I by $i \sim_X j$ if and only if $X \subseteq H_{i,j}$. Since $H_{i,i} = V$, $H_{i,j} = H_{j,i}$ and $H_{i,j} \cap H_{j,k} \subseteq H_{i,k}$, this is an equivalence relation. Let Λ_X be the partition of I defined by $\sim_X$. The map $\pi : L(\mathcal{A}) \to \mathcal{P}(\ell)$ given by $\pi(X) = \Lambda_X$ is a lattice isomorphism. It is injective because

$$X = \bigcap_{k=1}^{r}\Big(\bigcap_{i,j \in \Lambda_k} H_{i,j}\Big)$$

is determined by the blocks of Λ. It is surjective because given any partition $\Lambda = \{\Lambda_1, \ldots, \Lambda_r\}$ we may define X by the intersection above and we get $\Lambda_X = \Lambda$. Note also that $X \le Y$ if and only if every block of Λ_Y is a union of blocks of Λ_X. □

Definition 4.10 *Given a poset L and $X, Y \in L$ with $X < Y$, define the following subposets and segments*

$$L_X = \{Z \in L \mid Z \le X\}, \quad L^X = \{Z \in L \mid Z \ge X\}.$$
$$[X,Y] = \{Z \in L \mid X \le Z \le Y\}, \quad [X,Y) = \{Z \in L \mid X \le Z < Y\}.$$

Lemma 4.11 *Let $\mathcal{A}$ be an arrangement and let $X \in L(\mathcal{A})$. Then*
(1) $L(\mathcal{A})_X = L(\mathcal{A}_X)$,
(2) $L(\mathcal{A})^X = L(\mathcal{A}^X)$,
(3) if $Y \in L$ and $X \le Y$ then $L((\mathcal{A}_Y)^X) = L(\mathcal{A}_Y)^X = [X, Y]$. □

Example 4.12 *The lattice $L(\mathcal{A})$ of the arrangement of* Example 2.10 .

The lattice $L(\mathcal{A})$ consists of all subspaces of V. If $X \in L(\mathcal{A})$ is p-dimensional then $L(\mathcal{A}^X)$ is the lattice of all hyperplanes in X and $L(\mathcal{A}_X)$ is isomorphic to the lattice of all hyperplanes in the $(\ell - p)$-dimensional space V/X. If $X < Y$ then $[X, Y]$ is isomorphic to the lattice of all subspaces of X/Y.

Recall that $\mathcal{A}$ is essential if and only if it contains ℓ linearly independent hyperplanes. For a central arrangement this is equivalent to the condition $T(\mathcal{A}) = \{0\}$. The braid arrangement is not essential, $T(\mathcal{A})$ is the line $x_1 = x_2 = \ldots = x_\ell$. All the other arrangements considered so far are essential.

Definition 4.13 *Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be arrangements and let $V = V_1 \oplus V_2$. Define the* **product** *arrangement $(\mathcal{A}_1 \times \mathcal{A}_2, V)$ by*

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}.$$

Remark 4.14 *Let $\mathcal{A}_1, \mathcal{A}_2$ be arrangements. Define a partial order on the set $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$ of pairs (X_1, X_2) with $X_i \in L(\mathcal{A}_i)$ by*

$$(X_1, X_2) \le (Y_1, Y_2) \iff X_1 \le Y_1 \text{ and } X_2 \le Y_2.$$

There is a natural isomorphism of lattices

$$\pi : L(\mathcal{A}_1) \times L(\mathcal{A}_2) \to L(\mathcal{A}_1 \times \mathcal{A}_2).$$

The map $\pi(X_1, X_2) = X_1 \oplus X_2$ provides the required isomorphism.

Definition 4.15 *Call the arrangement $(\mathcal{A}, V)$* **reducible** *if after a change of coordinates $(\mathcal{A}, V) = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$. Otherwise call $(\mathcal{A}, V)$* **irreducible.**

Example 4.16 *The B_3-arrangement is irreducible. The Boolean arrangement is the product of ℓ copies of the 1-arrangement $(\{0\}, \mathbb{K})$. The braid arrangement is the product of the empty 1-arrangement with an irreducible arrangement.*

Let $\mathcal{A}$ be an affine ℓ-arrangement. Recall the coning construction from Definition 2.15. Let $H \in \mathcal{A}$ be the kernel of the degree 1 polynomial $\alpha_H \in \mathbb{K}[x_1, \ldots, x_\ell]$. It corresponds to $\mathbf{c}H \in \mathbf{c}\mathcal{A}$, the kernel of the linear form obtained by homogenizing α_H in $\mathbb{K}[x_0, x_1, \ldots, x_\ell]$. Note that $\dim H = \ell - 1$ and $\dim \mathbf{c}H = \ell$. Recall that $\mathbf{c}\mathcal{A}$ also contains the additional hyperplane $K_0 = \ker(x_0)$.

Proposition 4.17 *Let $\mathcal{A}$ be an affine arrangement with cone $\mathbf{c}\mathcal{A}$. Let $\mathcal{B} = (\mathbb{K}, \{0\})$ be the nonempty central 1-arrangement. Define the bijection $\phi : \mathcal{A} \times \mathcal{B} \to \mathbf{c}\mathcal{A}$ by $\phi(H \times \mathbb{K}) = \mathbf{c}H$ and $\phi(V \times \{0\}) = K_0$. Then ϕ induces a rank preserving surjective map of posets $\phi : L(\mathcal{A} \times \mathcal{B}) \to L(\mathbf{c}\mathcal{A})$.*

Proof. Let $X \in L(\mathcal{A})$. Write $X = H_1 \cap \cdots \cap H_p$. Define $\mathbf{c}X = \mathbf{c}H_1 \cap \cdots \cap \mathbf{c}H_p$. A direct argument shows that $\mathbf{c}X$ is independent of the representation of X as an intersection of hyperplanes. We define $\phi(X \times \mathbb{K}) = \mathbf{c}X$ and $\phi(X \times \{0\}) = \mathbf{c}X \cap K_0$. It is easy to see that the map is rank preserving and surjective. In general it is not injective. □

Oriented Matroids

Next we consider the special case of a real arrangement $\mathcal{A}$. Recall that $C(\mathcal{A})$ is the set of chambers of $\mathcal{A}$. Thus $M(\mathcal{A}) = \cup_{C \in \mathcal{C}(\mathcal{A})} C$.

Definition 4.18 *Let $\mathcal{A}$ be a real arrangement. Let*

$$\mathcal{L}(\mathcal{A}) = \bigcup_{X \in L(\mathcal{A})} \mathcal{C}(\mathcal{A}^X).$$

View $\mathcal{L}(\mathcal{A})$ as a collection of subsets of V. An element $P \in \mathcal{L}(\mathcal{A})$ is a **face**. *The support $|P|$ of a face P is its affine linear span. Each face is open in its support. Let $\bar{P}$ denote the closure of P in V. The set $\mathcal{L}(\mathcal{A})$ is partially ordered by reverse inclusion: $P \le Q$ if $Q \subseteq \bar{P}$. We call $\mathcal{L}(\mathcal{A})$ the* **face poset** *of $\mathcal{A}$.*

Definition 4.19 *Let $\mathcal{A}$ be a real arrangement. The poset map $\zeta : \mathcal{L}(\mathcal{A}) \to L(\mathcal{A})$ defined by $\zeta(P) = |P|$ is order preserving.*

There is a particularly efficient way to store the information in the face poset by using the associated oriented matroid, see [26]. Choose linear polynomials α_H so that $H = \ker \alpha_H$. Let $J = \{+, -, 0\}$. We may view each face $P \in \mathcal{L}(\mathcal{A})$ as a map $P : \mathcal{A} \to J$ defined by $P(H) = \alpha_H(p)$ for any $p \in P$. Note that $P(H) = 0$ if and only if $P \subseteq H$, and if $P(H) \neq 0$ then the sign indicates whether P is in the positive or negative half-space determined by H. If we choose a linear order in $\mathcal{A}$ then we may write $\mathcal{A} = \{H_1, \ldots, H_n\}$ and let $H_k = \ker \alpha_k$. Let $W = J^n$, and let $\pi_k : W \to J$ be projection onto the k-th coordinate. Define a map $\sigma : V \to W$ by

$$\pi_k\sigma(v) = \begin{cases} + & \text{if } \alpha_k(v) > 0, \\ 0 & \text{if } \alpha_k(v) = 0, \\ - & \text{if } \alpha_k(v) < 0. \end{cases}$$

We illustrate this concept in Example 2.6. In Figure 7 we labeled the chambers only. The reader is invited to label the remaining faces.

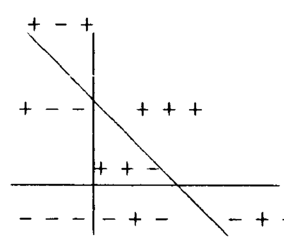


Figure 7: The chambers of $Q(\mathcal{A}) = xy(x + y - 1)$

Definition 4.20 *Let $\mathcal{G}(\mathcal{A}) = \sigma(V) \subseteq W$. Define a partial order in J by $+ < 0$, $- < 0$, while $+$ and $-$ are incomparable. This induces a partial order in W and in $\mathcal{G}(\mathcal{A})$. The poset $\mathcal{G}(\mathcal{A})$ is called the* **oriented matroid** *of $\mathcal{A}$.*

Thus an element $P \in \mathcal{G}(\mathcal{A})$ is an ordered n-tuple $P = (P(1), \ldots, P(n))$ where each $P(k) \in J$ is one of $+, -, 0$. Equivalently, we may view it as a map $P : \mathcal{A} \to J$ where $P(H_k) = P(k)$. Let G_0 be the set of all subsets of $\{1, 2, \ldots, n\}$ partially ordered by inclusion. Define $\rho : \mathcal{G}(\mathcal{A}) \to G_0$ by $\rho(P) = \{k \mid P(k) = 0\}$. Let $G(\mathcal{A}) = \rho(\mathcal{G}(\mathcal{A}))$. Define $\tau : L(\mathcal{A}) \to G_0$ by $\tau(X) = \{k \mid H_k \le X\}$. Then the following diagram of posets is commutative, and the vertical maps are poset isomorphisms:

$$\begin{array}{ccc} \mathcal{L}(\mathcal{A}) & \xrightarrow{\zeta} & L(\mathcal{A}) \\ \sigma\downarrow & & \downarrow\tau \\ \mathcal{G}(\mathcal{A}) & \xrightarrow{\rho} & G(\mathcal{A}) \end{array}$$

Definition 4.21 *Identify $\mathcal{G}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$. Define the* **vector product** *$\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ by*

$$(PQ)(i) = \begin{cases} P(i) & \text{if } P(i) \neq 0, \\ Q(i) & \text{if } P(i) = 0. \end{cases}$$

Proposition 4.22 *The vector product is associative but not commutative. For every $Q \in \mathcal{L}$ we have $P \ge PQ$. For fixed $P \in \mathcal{L}$ the map $\mathcal{L} \to \mathcal{L}$ defined by $Q \mapsto PQ$ is order preserving. In particular it carries chambers to chambers.* □

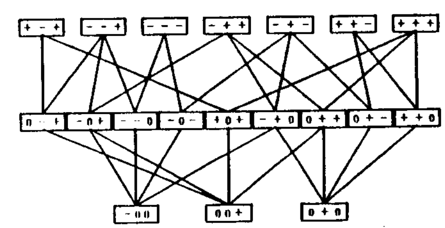


Figure 8: The face poset of $Q(\mathcal{A}) = xy(x + y - 1)$

The face poset of Example 2.6 is in Figure 8. It is identified with the oriented matroid. The face poset is a sharper invariant then the intersection poset. Arrangements $\mathcal{A}$ and $\mathcal{B}$ in Figure 9 are L-equivalent but they have different face posets.

Supersolvable Arrangements

The next definitions are standard for lattices in general, see [1, 21, 180]. We are only going to use them for central arrangements. It simpifies notation to assume that $L = L(\mathcal{A})$ and $\mathcal{A}$ is central and essential. Let $r(\mathcal{A}) = \ell$ and write $T = T(\mathcal{A})$.

Definition 4.23 *A pair $(X, Y) \in L \times L$ is called a* **modular pair** *if for all Z with $Z \le Y$*

$$Z \vee (X \wedge Y) = (Z \vee X) \wedge Y.$$

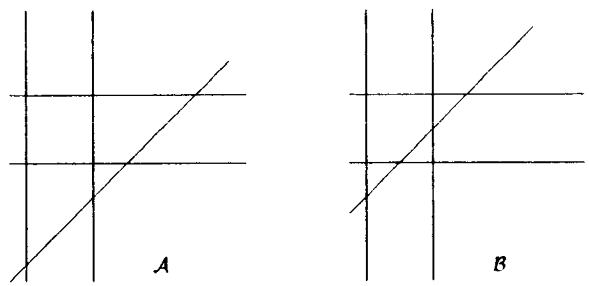


Figure 9: Different face posets

Definition 4.24 *An element $X \in L$ is called* **modular** *if (X,Y) is a modular pair for all $Y \in L$.*

Lemma 4.25 *Let $X \in L$ be a modular element. For $Y \in L$ the map $\sigma_X : [Y, X \vee Y] \to [X \wedge Y, X]$ defined by $\sigma_X(Z) = X \wedge Z$ is an isomorphism with inverse $\tau_Y(Z) = Y \vee Z$.*

Proof. Clearly both maps are order preserving. Since X is modular, if $Z \in [X \wedge Y, X]$ then $\sigma_X \tau_Y(Z) = X \wedge (Y \vee Z) = (X \wedge Y) \vee Z = Z$. Similarly, if $Z \in [Y, X \vee Y]$ then $\tau_Y \sigma_X(Z) = Y \vee (X \wedge Z) = (Y \vee X) \wedge Z = Z$. □

Lemma 4.26 *The following statements are equivalent:*
(1) the pair $(X,Y) \in L \times L$ is modular,
(2) $r(X) + r(Y) = r(X \vee Y) + r(X \wedge Y)$,
(3) $X \wedge Y = X + Y$,
(4) $X + Y \in L$.

Proof. The conditions (3) and (4) are obviously equivalent.
(1) ⇒ (2) If (X,Y) is a modular pair then by Lemma 4.25 the segments $[X \wedge Y, X]$ and $[Y, X \vee Y]$ are isomorphic. In particular

$$r(X) - r(X \wedge Y) = r(X \vee Y) - r(Y).$$

(2) ⇒ (3) Recall that $X + Y \subseteq X \wedge Y$. If (2) holds then $\mathrm{codim}(X \wedge Y) = \mathrm{codim}(X + Y)$ so the spaces are equal. Finally, (3) ⇒ (1) follows from the definition. □

Corollary 4.27 *An element $X \in L$ is modular if and only if $X + Y \in L$ for all $Y \in L$.* □

Example 4.28 *For any central arrangement $\mathcal{A}$ the elements V, $T = T(\mathcal{A})$, and all atoms are modular.*

Example 4.29 *Consider $Q(\mathcal{A}) = xyz(x + y - z)$. This is the cone over the 2-arrangement of* Example 2.6. *In $L(\mathcal{A})$ every $H \in \mathcal{A}$ is modular, but no element of rank 2 is modular.*

Lemma 4.30 *An element $X \in L$ is modular if and only if (X,Y) is a modular pair for every $Y \in L$ such that $X \wedge Y = V$.*

Proof. Fix $X \in L$ and assume that (X,Y) is a modular pair for every $Y \in L$ such that $X \wedge Y = V$. We want to show that (X,Z) is a modular pair for every $Z \in L$. Set $a = r(Z) - r(X \wedge Z)$. There exist linearly independent hyperplanes $H_1, \ldots, H_a \in \mathcal{A}$ such that

$$Z = (X \wedge Z) \vee H_1 \vee \cdots \vee H_a.$$

Let $Y = H_1 \vee \cdots \vee H_a$. Then $r(Y) = a$ and $Z = (X \wedge Z) \vee Y$. We have

$$r(X \wedge Y) = r(X \wedge Z \wedge Y) \le r(X \wedge Z) + r(Y) - r((X \wedge Z) \vee Y) = r(X \wedge Z) + a - r(Z) = 0.$$

Therefore $X \wedge Y = V$. By assumption (X,Y) is a modular pair and hence $X + Y = V$. Then

$$X + Z = X + ((X \wedge Z) \vee Y) = X + ((X \wedge Z) \cap Y) = (X \wedge Z) \cap (X + Y) = X \wedge Z.$$

This shows that (X,Z) is a modular pair. □

Lemma 4.31 *If Y is a modular element in L and X is a modular element in L_Y then X is a modular element in L.*

Proof. Let $Z \in L$. By Lemma 4.26 it is sufficient to show that $X + Z = X \wedge Z$. We have

$$X + Z = (X+Y) + Z = X + (Y + Z) = X + (Y \wedge Z) = X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z = X \wedge Z. \quad □$$

Definition 4.32 *Let $\mathcal{A}$ be an arrangement with $r(\mathcal{A}) = \ell$. We call $\mathcal{A}$* **supersolvable** *if $L(\mathcal{A})$ has a maximal chain of modular elements*

$$V = X_0 < X_1 < \ldots < X_\ell = T.$$

Example 4.33 *The arrangement in* Example 4.29 *is not supersolvable because it has no modular element of rank 2. The B_3-arrangement, the braid arrangement, and the Boolean arrangement are supersolvable.*

In the Boolean arrangement every element is modular, so we may take any maximal chain. In the B_3-arrangement the only modular elements of rank 2 are $x = y = 0$, $x = z = 0$, and $y = z = 0$, see Figure 6. Thus a maximal chain of modular elements is given by

$$V < \{x = 0\} < \{x = y = 0\} < \{0\}.$$

Not all elements of the braid arrangement are modular, but

$$V < \{x_1 = x_2\} < \{x_1 = x_2 = x_3\} < \ldots < \{x_1 = x_2 = \ldots = x_\ell\} = T$$

is a maximal chain of modular elements.

5 The Möbius Function

The Möbius Function

Definition 5.1 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. Define the* **Möbius function** *$\mu_{\mathcal{A}} = \mu : L \times L \to \mathbb{Z}$ as follows:*

$$\begin{array}{ll} \mu(X,X) = 1 & \text{if } X \in L, \\ \sum_{X \leq Z \leq Y} \mu(X,Z) = 0 & \text{if } X,Y,Z \in L \text{ and } X < Y, \\ \mu(X,Y) = 0 & \text{otherwise.} \end{array}$$

Note that for fixed X the values of $\mu(X,Y)$ may be computed recursively. It follows that if ν is any other function which satisfies the defining properties of μ then $\nu = \mu$. There is a useful reformulation of $\mu(X,Y)$.

Lemma 5.2 *Let $\mathcal{A}$ be an arrangement. For $X,Y \in L$ with $X \leq Y$, let $S(X,Y)$ be the set of central subarrangements $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{A}_X \subseteq \mathcal{B}$ and $T(\mathcal{B}) = Y$. Then*

$$\mu(X,Y) = \sum_{\mathcal{B} \in S(X,Y)} (-1)^{|\mathcal{B} \setminus \mathcal{A}_X|}.$$

Proof. Let $\nu(X,Y)$ denote the right side of the expression. Note that

$$\bigcup_{X \leq Z \leq Y} S(X,Z) = \{\mathcal{B} \subseteq \mathcal{A} \mid \mathcal{A}_X \subseteq \mathcal{B} \subseteq \mathcal{A}_Y\}$$

where the union is disjoint. Thus

$$\sum_{X \leq Z \leq Y} \nu(X,Z) = \sum_{\mathcal{A}_X \subseteq \mathcal{B} \subseteq \mathcal{A}_Y} (-1)^{|\mathcal{B} \setminus \mathcal{A}_X|} = \sum_{\mathcal{C}} (-1)^{|\mathcal{C}|}.$$

The last sum is over all subsets $\mathcal{C}$ of $\mathcal{A}_Y \setminus \mathcal{A}_X$. If $X = Y$ the sum is 1. If $X < Y$ then $\mathcal{A}_X$ is a proper subset of $\mathcal{A}_Y$ so the sum is zero. □

There is another interesting formula for $\mu(X,Y)$.

Definition 5.3 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. Let $ch(L)$ be the set of all chains in L:*

$$ch(L) = \{(X_1, \ldots, X_p) \mid X_1 < \cdots < X_p\}.$$

Denote the first element of $c \in ch(L)$ by $\underline{c}$, the last element of c by $\bar{c}$, and the cardinality of c by $|c|$. Let $ch[X,Y] = \{c \in ch(L) \mid \underline{c} = X,\ \bar{c} = Y\}$ and $ch[X,Y) = \{c \in ch(L) \mid \underline{c} = X,\ \bar{c} < Y\}$

Proposition 5.4 *For all $X,Y \in L$*

$$\mu(X,Y) = \sum_{c \in ch[X,Y]} (-1)^{|c|-1}.$$

Proof. We prove that the right hand side satisfies the defining properties of μ. This is clear for $X = Y$ and when X,Y are incomparable. Suppose $X < Y$. Then

$$\begin{aligned} \sum_{Z \in [X,Y]} \sum_{c \in ch[X,Z]} (-1)^{|c|-1} &= \sum_{Z \in [X,Y)} \sum_{c \in ch[X,Z]} (-1)^{|c|-1} + \sum_{c \in ch[X,Y]} (-1)^{|c|-1} \\ &= \sum_{c \in ch[X,Y)} (-1)^{|c|-1} + \sum_{c \in ch[X,Y]} (-1)^{|c|-1} \\ &= \sum_{c \in ch[X,Y]} (-1)^{|c|-2} + \sum_{c \in ch[X,Y]} (-1)^{|c|-1} = 0, \end{aligned}$$

because the map $(X, X_2, \ldots, X_p) \mapsto (X, X_2, \ldots, X_p, Y)$ from $ch[X,Y)$ to $ch[X,Y]$ is a bijection. □

Möbius Inversion

Lemma 5.5 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. Then*

$$\begin{array}{ll} \mu(X,X) = 1 & \text{if } X \in L, \\ \sum_{X \leq Z \leq Y} \mu(Z,Y) = 0 & \text{if } X,Y \in L \text{ and } X < Y. \end{array}$$

Proof. Write $L = \{X_1, \ldots, X_r\}$ where the numbering is chosen so that $X_i \leq X_j$ implies $i \leq j$. Let A be the $r \times r$ matrix with (i,j) entry $\mu(X_i, X_j)$. Let B be the $r \times r$ matrix with (i,j) entry 1 if $X_i \leq X_j$ and 0 otherwise. Both A and B are upper unitriangular. It follows from the definition of μ that $\mathsf{AB} = \mathsf{I}_r$ is the identity matrix. Thus $\mathsf{BA} = \mathsf{I}_r$, which implies the assertions. □

The next result is the Möbius inversion formula.

Proposition 5.6 *Let f,g be functions on $L(\mathcal{A})$ with values in an abelian group. Then*

$$g(Y) = \sum_{X \in L_Y} f(X) \iff f(Y) = \sum_{X \in L_Y} \mu(X,Y) g(X)$$

$$g(X) = \sum_{Y \in L^X} f(Y) \iff f(X) = \sum_{Y \in L^X} \mu(X,Y) g(Y).$$

Proof. Each of the four implications is based on an interchange of summation and the properties of μ given in the definition and in Lemma 5.5. We prove left to right implication in the first formula.

$$\begin{aligned} \sum_{Z \in L_Y} \mu(Z,Y) g(Z) &= \sum_{Z \in L_Y} \mu(Z,Y) \sum_{X \in L_Z} f(X) \\ &= \sum_{X \in L_Z} \Big(\sum_{X \leq Z \leq Y} \mu(Z,Y) \Big) f(X) \\ &= f(Y). \quad \square \end{aligned}$$

The next result is due to Weisner [204].

Lemma 5.7 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$.*
(1) Suppose $Y \in L$ and $Y \neq V$. Then for all $Z \in L$

$$\sum_{X \vee Y = Z} \mu(V, X) = 0.$$

(2) Suppose $Y \in L$ and $T \in L$ is a maximal element such that $Y < T$. Then for all $Z \in L$

$$\sum_{X \wedge Y = Z} \mu(X, T) = 0.$$

Proof. We prove (1). The proof of (2) is similar. Note that $X \vee Y = Z$ implies $X \leq Z$, $Y \leq Z$, and $r(Z) \geq r(Y)$. We argue by induction on $r(Z)$. If $Z = Y$ then the sum to be computed is $\sum_{X \leq Y} \mu(V, X) = 0$ since $Y \neq V$. If $Z > Y$ then

$$\begin{aligned}\sum_{X \vee Y = Z} \mu(V, X) &= \sum_{X \vee Y \leq Z} \mu(V, X) - \sum_{X \vee Y < Z} \mu(V, X) \\ &= \sum_{X \leq Z} \mu(V, X) - \sum_{W < Z} \Big(\sum_{X \vee Y = W} \mu(V, X) \Big).\end{aligned}$$

The first term is zero by the definition of μ. The second term is zero by induction. □

Lemma 5.8 *Let $(\mathcal{A}, V) = (\mathcal{A}_1, V_1) \times (\mathcal{A}_2, V_2)$ be the direct product of two arrangements. Let $\mu_i = \mu_{\mathcal{A}_i}$ and let $\mu = \mu_{\mathcal{A}}$. Let $X, Y \in L(\mathcal{A})$ where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ with $X_i, Y_i \in L(\mathcal{A}_i)$. Then*

$$\mu(X, Y) = \mu_1(X_1, Y_1)\mu_2(X_2, Y_2).$$

Proof. Define ν on $L(\mathcal{A}_1 \times \mathcal{A}_2)$ by $\nu(X, Y) = \mu_1(X_1, Y_1)\mu_2(X_2, Y_2)$. Then ν satisfies the defining conditions of μ, hence they are equal. □

The Function $\mu(X)$

Definition 5.9 *For $X \in L$ define $\mu(X) = \mu(V, X)$.*

Clearly $\mu(V) = 1$, $\mu(H) = -1$, for all $H \in L$ and if $r(X) = 2$ then $\mu(X) = |\mathcal{A}_X| - 1$. In general it is not possible to give a formula for $\mu(X)$. Recall the map $\phi : L(\mathcal{A} \times \mathcal{B}) \to L(\mathbf{c}\mathcal{A})$ of Proposition 4.17.

Proposition 5.10 *Let $\phi : L(\mathcal{A} \times \mathcal{B}) \to L(\mathbf{c}\mathcal{A})$. Let μ be the Möbius function of $L(\mathcal{A} \times \mathcal{B})$ and let μ_c be the Möbius function of $L(\mathbf{c}\mathcal{A})$. For all $Z \in L(\mathbf{c}\mathcal{A})$ we have*

$$\mu_c(Z) = \sum_{Y \in \phi^{-1}(Z)} \mu(Y).$$

Proof. It follows from Lemma 5.8 that $\mu(X \times \mathbb{K}) = -\mu(X \times \{0\})$ for $X \in L(\mathcal{A})$. If $K_0 \not\leq Z$ then there is a unique $Y \in L(\mathcal{A})$ such that $Z = \mathbf{c}Y$. Thus $\phi^{-1}(Z) = \{Y \times \mathbb{K}\}$ and $\mu(Y \times \mathbb{K}) = \mu_c(\mathbf{c}Y)$. If $K_0 \leq Z$ we argue by induction on $r(Z)$. If $r(Z) = 1$ then $Z = K_0$. Since $\phi^{-1}(K_0) = \{V \times \{0\}\}$ we have $\mu_c(K_0) = -1 = \mu(V \times \{0\})$. Now suppose $r(Z) \geq 2$.

$$\begin{aligned}\mu_c(Z) &= -\sum_{\substack{Y \in L(\mathbf{c}\mathcal{A}) \\ Y < Z}} \mu_c(Y) \\ &= -\sum_{\substack{Y \in L(\mathbf{c}\mathcal{A}) \\ Y < Z}} \sum_{X \in \phi^{-1}(Y)} \mu(X) \\ &= -\sum_{\substack{X \in L(\mathcal{A} \times \mathcal{B}) \\ \phi(X) < Z}} \mu(X) \\ &= -\sum_{\substack{X \in L(\mathcal{A}) \\ \mathbf{c}X < Z}} \mu(X \times \mathbb{K}) - \sum_{\substack{X \in L(\mathcal{A}) \\ \mathbf{c}X \cap K_0 < Z}} \mu(X \times \{0\}) \\ &= \sum_{\substack{X \in L(\mathcal{A}) \\ \mathbf{c}X \cap K_0 \leq Z}} \mu(X \times \{0\}) - \sum_{\substack{X \in L(\mathcal{A}) \\ \mathbf{c}X \cap K_0 < Z}} \mu(X \times \{0\}) \\ &= \sum_{\substack{X \in L(\mathcal{A}) \\ \mathbf{c}X \cap K_0 = Z}} \mu(X \times \{0\}) \\ &= \sum_{\substack{Y \in L(\mathcal{A} \times \mathcal{B}) \\ \phi(Y) = Z}} \mu(Y).\end{aligned}$$

The second equality is by the induction assumption. The fifth is by the fact that if $K_0 \leq Z$ then $\{X \in L(\mathcal{A}) \mid \mathbf{c}X < Z\} = \{X \in L(\mathcal{A}) \mid \mathbf{c}X \cap K_0 \leq Z\}$. □

Proposition 5.11 *Define $\mathcal{A}$ by $Q(\mathcal{A}) = x_1 x_2 \cdots x_\ell$. Then for $X \in L$*

$$\mu(X) = (-1)^{r(X)}.$$

Proof. Define $\nu(X) = (-1)^{r(X)}$. It suffices to show that ν satisfies the defining properties of μ. Clearly $\nu(V) = (-1)^0 = 1$. If $X \neq V$ then $X = H_I$ for some I with $|I| = p = r(X) > 0$. If $V \leq Y \leq X$ then $Y = H_J$ where $J \subseteq I$ and

$$q = r(Y) = |J| \leq |I| = r(X) = p.$$

Thus

$$\sum_{Z \leq X} \nu(Z) = \sum_{J \subseteq I} (-1)^q = \sum_{q=0}^{p} (-1)^q \binom{p}{q} = 0. \quad \square$$

Definition 5.12 *For a central arrangement $\mathcal{A}$ let $\mu(\mathcal{A}) = \mu(T(\mathcal{A}))$.*

It follows from Lemma 5.8 that when $\mathcal{A}_1$ and $\mathcal{A}_2$ are central arrangements we have $\mu(\mathcal{A}_1 \times \mathcal{A}_2) = \mu_1(\mathcal{A}_1)\mu_2(\mathcal{A}_2)$.

Proposition 5.13 *Let $\mathcal{A}$ be the arrangement of* Example 2.10. *If $X \leq Y$ then $\mu(X,Y) = (-1)^k q^{k(k-1)/2}$ where $k = \dim(X/Y)$. In particular $\mu(\mathcal{A}) = (-1)^\ell q^{\ell(\ell-1)/2}$.*

Proof. Since the segment $[X,Y]$ is isomorphic to the lattice of all subspaces of X/Y, it suffices to prove the second formula. We argue by induction on ℓ. If $\ell = 1$ then $L = \{V, T\}$ so $\mu(\mathcal{A}) = -1$. Suppose $\ell \geq 2$ and recall that the number of lines in V is $1 + q + \cdots + q^{\ell-1}$. Apply Lemma 5.7.1 with $Z = T = \{0\}$ and $Y \in \mathcal{A}$. Then $X = T$ or $X \in R$, where R is the set of lines not contained in the hyperplane Y. Thus we have

$$0 = \mu(T) + \sum_{X \in R} \mu(X) = \mu(\mathcal{A}) + \sum_{X \in R} \mu(\mathcal{A}_X)$$

Clearly

$$|R| = (1 + q + \cdots + q^{\ell-1}) - (1 + q + \cdots + q^{\ell-2}) = q^{\ell-1}.$$

Since each $\mathcal{A}_X$ is isomorphic to the arrangement of all hyperplanes in V/X, we may assume by induction that $\mu(\mathcal{A}_X) = (-1)^{\ell-1} q^{(\ell-1)(\ell-2)/2}$. Thus

$$\mu(\mathcal{A}) = -q^{\ell-1}(-1)^{\ell-1} q^{(\ell-1)(\ell-2)/2}$$

as required. □

Theorem 5.14 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. If $X, Y \in L$ and $X \leq Y$ then $\mu(X,Y) \neq 0$ and $\mathrm{sign}\mu(X,Y) = (-1)^{r(X)-r(Y)}$.*

Proof. Since the the Möbius function of the segment $[X,Y]$ is the restriction of the Möbius function $\mu_{\mathcal{A}}$, we have $\mu(X,Y) = \mu((\mathcal{A}_Y)^X)$. Thus it suffices to show the assertion for a central arrangement $\mathcal{A}$, and there it suffices to prove that $\mu(\mathcal{A}) \neq 0$ and $\mathrm{sign}\mu(\mathcal{A}) = (-1)^{r(\mathcal{A})}$. We argue by induction on $r(\mathcal{A})$. The assertion is clear if $r(\mathcal{A}) = 0$. Suppose $r(\mathcal{A}) \geq 1$. Choose $H \in \mathcal{A}$ and apply Lemma 5.7.1 with $Y = H$ and $Z = T(\mathcal{A})$. We get

$$\begin{aligned} 0 &= \mu(\mathcal{A}) + \sum_{X \in M} \mu(X) \\ &= \mu(\mathcal{A}) + \sum_{X \in M} \mu(\mathcal{A}_X) \end{aligned}$$

where M is the set of all $X \in L$ such that $X \not= T(\mathcal{A})$ but $X \vee H = T(\mathcal{A})$. If $X \in M$ then

$$\begin{aligned} r(\mathcal{A}) &= r(X \vee H) \\ &\leq r(X \vee H) + r(X \wedge H) \\ &\leq r(X) + r(H) \\ &= r(X) + 1. \end{aligned}$$

Thus $r(X) = r(\mathcal{A}) - 1$ and therefore $r(\mathcal{A}_X) = r(\mathcal{A}) - 1$. By induction $\mu(\mathcal{A}_X) \neq 0$ and $\mathrm{sign}\mu(\mathcal{A}_X) = (-1)^{r(\mathcal{A}_X)} = (-1)^{r(\mathcal{A})-1}$. The assertion follows from the equation $\mu(\mathcal{A}) = -\sum_{X \in M} \mu(\mathcal{A}_X)$. □

We close with the results of a hand calculation. Figure 10 gives the values of $\mu(X)$ for all $X \in L$ in Example 2.7.

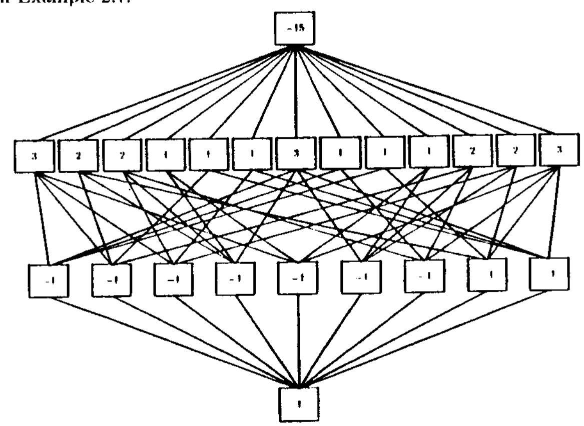


Figure 10: The values of $\mu(X)$ for the B_3-arrangement

Historical Notes

Next we describe some of the history of the Möbius function. It started with inversion problems and continued in number theory. The extension to partially ordered sets dates from around 1935. A revival started in 1964 with Rota's paper [163]. There is an excellent survey of the modern theory by C. Greene [79]. Recall the Möbius function of elementary number theory. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the natural numbers. Define $\mu : \mathbb{N} \to \{0, \pm 1\}$ by

$$\begin{aligned} \mu(1) &= 1 \\ \mu(p_1 \ldots p_r) &= (-1)^r \quad \text{if } p_1, \ldots, p_r \text{ are distinct primes} \\ \mu(n) &= 0 \quad \text{otherwise.} \end{aligned}$$

Suppose $n > 1$ and write $n = p_1^{a_1} \ldots p_r^{a_r}$, where $p_1, \ldots, p_r$ are distinct primes. Since $r > 0$ we have

$$\sum_{d|n} \mu(d) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k = 0.$$

The condition $\mu(1) = 1$ and this formula determine μ recursively.

The first known result which may be viewed as a precursor of the Möbius function appeared in the work of Euler. He started with the formula

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\cdots$$

and formally inverted this infinite series to obtain

$$\frac{6}{\pi^2} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2}\cdots$$

In modern notation Euler's formula reads

$$\frac{6}{\pi^2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}.$$

In 1832 Möbius [132] considered the following problem. Given a function

$$f(x) = \sum_{j=1}^{\infty} a_j x^j$$

find coefficients b_k such that

$$x = \sum_{k=1}^{\infty} b_k f(x^k).$$

Since $f(x^k) = \sum_{j=1}^{\infty} a_j x^{jk}$ we have

$$x = \sum_{k=1}^{\infty} b_k \sum_{j=1}^{\infty} a_j x^{jk} = \sum_{n=1}^{\infty} (\sum_{jk=n} a_j b_k) x^n.$$

Thus the solution is given recursively by

$$\begin{aligned} a_1 b_1 &= 1 \\ \sum_{jk=n} a_j b_k &= 0. \end{aligned}$$

Möbius solved the problem explicitly for several functions. For

$$f(x) = \frac{x}{1-x} = x + x^2 + x^3 + \ldots$$

the $a_j = 1$ for all j. Thus $b_1 = 1$ and for $n > 1$ we have $\sum_{d|n} b_d = 0$. He proved that, in modern notation, $b_n = \mu(n)$. As a second example let

$$f(x) = -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

so $a_j = 1/j$. He showed that $b_k = \mu(k)/k$. To see this note that $b_1 = 1$ and for $n > 1$ we have

$$\sum_{jk=n} a_j b_k = \sum_{jk=n} \frac{1}{j}\frac{\mu(k)}{k} = \frac{1}{n}\sum_{k|n} \mu(k) = 0.$$

This gives

$$x = \sum_{k=1}^{\infty} \frac{\mu(k)}{k}(-\ln(1-x^k)).$$

Formal exponentiation gives the remarkable formula:

$$e^x = \prod_{k=1}^{\infty} (1-x^k)^{\frac{-\mu(k)}{k}}.$$

Work of Dedekind and Liouville is also relevant to these developments. Their problem concerned a function f defined on $\mathbb{N}$, and a second function g defined by the formula

$$g(n) = \sum_{d|n} f(d).$$

Can $f(n)$ be expressed in terms of g? Their formula

$$f(n) = g(n) - \sum g(\frac{n}{a}) + \sum g(\frac{n}{ab}) + \ldots$$

is written today as

$$f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d}). \tag{1}$$

For an application recall Euler's function $\phi(n)$ which counts the number of integers k relatively prime to n such that $1 \le k \le n$. Let $f(n) = \phi(n)$. It is well known that $n = \sum_{d|n} \phi(d)$, thus $g(n) = n$. This gives the expression

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}.$$

Let R be a commutative ring and let A denote the set of all functions from $\mathbb{N} \to R$. Dirichlet convolution

$$f * g(n) = \sum_{d|n} f(d) g(\frac{n}{d})$$

makes A an R-algebra. It is commutative and associative with identity function δ defined by $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. Let $U(R)$ denote the group of units in R, and let $U(A)$

be the group of units in A. If $f(1) \in U(R)$ then $f \in U(A)$ since we can define its inverse, g recursively by

$$\begin{aligned} f(1)g(1) &= 1, \\ f(1)g(n) + \sum_{d>1} f(d)g(\frac{n}{d}) &= 0. \end{aligned}$$

Define $\zeta \in A$ by $\zeta(n) = 1$ for all $n \in \mathbf{N}$. Clearly $\zeta \in U(A)$. Let μ be its inverse. This is again the Möbius function. If

$$g(n) = \sum_{d|n} f(d)$$

then $g = f * \zeta$. It follows that $f = g * \mu$, which is formula (1).

In case $R = \mathbf{C}$ we may associate with $f \in A$ the **formal Dirichlet series**

$$\hat{f}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

where $s \in \mathbf{C}$ and there is no assumption of convergence. Let $\hat{A}$ denote the set of formal Dirichlet series. We may define addition and multiplication in $\hat{A}$ pointwise. Note that

$$\begin{aligned} \hat{f}(s)\hat{g}(s) &= (\sum_{a=1}^{\infty} \frac{f(a)}{a^s})(\sum_{b=1}^{\infty} \frac{g(b)}{b^s}) \\ &= \sum_{n=1}^{\infty} (\sum_{ab=n} f(a)g(b)) \frac{1}{n^s} \\ &= (\widehat{f * g})(s). \end{aligned}$$

Thus the map $A \to \hat{A}$ given by $f \to \hat{f}$ preserves addition and multiplication. In particular the image of ζ is the Riemann zeta function.

$$\begin{aligned} \hat{\zeta}(s) &= \sum_{n=1}^{\infty} \frac{\zeta(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \\ \hat{\mu}(s) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \\ \hat{\delta}(s) &= 1. \end{aligned}$$

From $\zeta * \mu = \delta$ we get

$$\hat{\zeta}(s)(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}) = 1.$$

Work of P. Hall [87, 88], L. Weisner [204], and M. Ward [203] extended these considerations to locally finite partially ordered sets. Let L be a poset and let $A(L)$ be the set of

functions $f : L \times L \to R$ such that $f(x,y) = 0$ unless $x \leq y$. If we define addition pointwise and multiplication by

$$f * g(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$$

then $A(L)$ forms an associative algebra, called the **incidence algebra** of L. It has an identity

$$\delta(x,y) = \begin{cases} 1 & x = y \\ 0 & \textit{otherwise.} \end{cases}$$

If $f(x,x) \in U(R)$ for all $x \in L$ then f is a unit of $A(L)$ because we can define its inverse, g recursively by

$$\begin{aligned} f(x,x)g(x,x) &= 1, \\ f(x,x)g(x,y) + \sum_{x<z\leq y} f(x,z)g(z,y) &= 0. \end{aligned}$$

Define

$$\zeta(x,y) = \begin{cases} 1 & x \leq y \\ 0 & \textit{otherwise.} \end{cases}$$

Clearly ζ is a unit in $A(L)$. Let μ be its inverse so $\zeta * \mu = \delta = \mu * \zeta$. We call μ the **Möbius function** of L. Note that $\mu(x,x) = 1$ for all $x \in L$ and if $x < y$ then $\mu * \zeta = \delta$ implies

$$\begin{aligned} \mu * \zeta(x,y) &= \sum_{x \leq z \leq y} \mu(x,z)\zeta(z,y) \\ &= \sum_{x \leq z \leq y} \mu(x,z) \\ &= 0. \end{aligned}$$

In order to recover the number theoretic Möbius function from the Möbius function of a poset, recall that there is a natural partial order on the set $\mathbf{N}$ defined by $m \leq n \Leftrightarrow m$ divides n. Let $L(\mathbf{N})$ denote this poset. Its unique minimal element is 1. Let $\mu_L : L \times L \to \mathbb{Z}$ denote the Möbius function of this poset. Then $\mu(n) = \mu_L(1,n)$.

6 The Poincaré Polynomial

In this section we define one of the most important combinatorial invariants of an arrangement, its Poincaré polynomial, and study its properties.

Definition 6.1 *Let $\mathcal{A}$ be an arrangement with intersection poset L and Möbius function μ. Let t be an indeterminate. Define the* **Poincaré polynomial** *of $\mathcal{A}$ by*

$$\pi(\mathcal{A},t)=\sum_{X\in L}\mu(X)(-t)^{r(X)}.$$

It follows from Theorem 5.14 that $\pi(\mathcal{A},t)$ has non-negative coefficients. In some cases it is easy to compute the values of μ directly and obtain the Poincaré polynomial.

Examples

Example 6.2 *If $\mathcal{A}=\Phi_\ell$ is the empty ℓ-arrangement then $\pi(\mathcal{A},t)=1$.*

The Poincaré polynomial in Example 2.5 *is:*

$$\pi(\mathcal{A},t)=1+3t+2t^2=(1+t)(1+2t).$$

The Poincaré polynomial in Example 2.6 *is:*

$$\pi(\mathcal{A},t)=1+3t+3t^2.$$

The Poincaré polynomial in Example 2.7 *is:*

$$\pi(\mathcal{A},t)=1+9t+23t^2+15t^3=(1+t)(1+3t)(1+5t).$$

The Poincaré polynomial of the Boolean arrangement is:

$$\pi(\mathcal{A},t)=\sum_{p=0}^{\ell}\binom{\ell}{p}t^p=(1+t)^\ell.$$

These examples may give the false impression that the Poincaré polynomial of every central arrangement is a product of linear terms $(1+bt)\in\mathbb{Z}[t]$. The reader is invited to check that in Example 4.29 $\pi(\mathcal{A},t)=1+4t+6t^2+3t^3=(1+t)(1+3t+3t^2)$. Proposition 6.4 shows that $(1+t)$ divides the Poincaré polynomial of every central arrangement. More factors of the form $(1+bt)\in\mathbb{Z}[t]$ do not exist in general.

Lemma 6.3 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrangements, and let $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$. Then*

$$\pi(\mathcal{A},t)=\pi(\mathcal{A}_1,t)\pi(\mathcal{A}_2,t).$$

Proof. Let $V=V_1\oplus V_2$. Since $L(\mathcal{A})=L(\mathcal{A}_1)\times L(\mathcal{A}_2)$ we have

$$\begin{aligned}\pi(\mathcal{A},t) &= \sum_{X\in L(\mathcal{A})}\mu(X)(-t)^{r(X)}\\ &= \sum_{X_1\oplus X_2\in L(\mathcal{A}_1)\times L(\mathcal{A}_2)}\mu(X_1\oplus X_2)(-t)^{r(X_1\oplus X_2)}\\ &= \sum_{X_1\in L(\mathcal{A}_1),X_2\in L(\mathcal{A}_2)}\mu(X_1)\mu(X_2)(-t)^{r(X_1)}(-t)^{r(X_2)}\\ &= \pi(\mathcal{A}_1,t)\pi(\mathcal{A}_2,t).\ \square\end{aligned}$$

Proposition 6.4 *Let $\mathcal{A}$ be an affine arrangement with cone $\mathbf{c}\mathcal{A}$. Then*

$$\pi(\mathbf{c}\mathcal{A},t)=(1+t)\pi(\mathcal{A},t).$$

Proof. If $\mathcal{B}=(\mathbb{K},\{0\})$ then $\pi(\mathcal{B},t)=1+t$. It follows from Lemma 6.3 that $\pi(\mathcal{A}\times\mathcal{B},t)=(1+t)\pi(\mathcal{A},t)$. Since $\phi: L(\mathcal{A}\times\mathcal{B})\to L(\mathbf{c}\mathcal{A})$ is surjective and rank preserving, it follows from Proposition 5.10 that $\pi(\mathbf{c}\mathcal{A},t)=\pi(\mathcal{A}\times\mathcal{B},t)$. $\square$

Definition 6.5 *Define the* **characteristic polynomial** *of $\mathcal{A}$ by*

$$\chi(\mathcal{A},t)=t^\ell\pi(\mathcal{A},-t^{-1})=\sum_{X\in L}\mu(X)t^{\dim(X)}.$$

Note that $\chi(\mathcal{A},t)$ is a monic polynomial of degree ℓ. Our characteristic polynomial is slightly different from the usual definition of the characteristic polynomial of the lattice L. The definitions agree if $\mathcal{A}$ has rank ℓ. In some cases it is natural to compute the characteristic polynomial. Our first nontrivial computations obtain the characteristic polynomials of the arrangements of Examples 2.9 and 2.10. These computations have two interesting features. They use the combinatorial technique of proving an identity by expressing the cardinality of a set in two different ways, and they use Möbius inversion to find $\chi(\mathcal{A},t)$ without computing the individual values of $\mu(X)$.

Proposition 6.6 *Let $\mathcal{A}$ be the arrangement of* Example 2.10. *Then*

$$\pi(\mathcal{A},t)=(1+t)(1+qt)\cdots(1+q^{\ell-1}t).$$

Proof. We prove the equivalent formula:

$$\chi(\mathcal{A},t)=(t-1)(t-q)\cdots(t-q^{\ell-1}).$$

Let W be a vector space of finite dimension over $\mathbf{F}_q$. Let $w=|W|$ be the cardinality of W. Then $|\hom(W,V)|=w^\ell$. We classify elements of $\hom(W,V)$ according to their images. If $X\in L$ define subsets P_X, Q_X of $\hom(W,V)$ by

$$P_X=\{\phi\in\hom(W,V)\mid \operatorname{im}\phi=X\},\ Q_X=\{\phi\in\hom(W,V)\mid\operatorname{im}\phi\subseteq X\}.$$

If $\mathrm{im}\phi \subseteq X$ then $\mathrm{im}\phi = Y$ for some $Y \subseteq X$. Thus we have a disjoint union

$$Q_X = \bigcup_{Y \geq X} P_Y.$$

By Möbius inversion

$$|P_Y| = \sum_{X \geq Y} \mu(Y, X)|Q_X|.$$

Since $Q_X = \hom(W, X)$ we have $|Q_X| = w^{\dim X}$. Taking $Y = V$ we get

$$|P_V| = \sum_{X \in L} \mu(X) w^{\dim X}.$$

If $\phi \in \hom(W, V)$ let $\phi^* \in \hom(V^*, W^*)$ be the transpose map. Since $\ker \phi^* = (\mathrm{im}\phi)^\circ$ is the annihilator of $(\mathrm{im}\phi)$, ϕ is surjective if and only if ϕ^* is injective. Thus $|P_V|$ is the number of injective maps in $\hom(V^*, W^*)$. This number is zero if $\dim W < \dim V$. Suppose $\dim W \geq \dim V$. Let $x_1, \ldots, x_\ell$ be a basis for V^*. If ϕ is injective there are $w - 1$ possibilities for $\phi(x_1)$. Since $\phi(x_2)$ must lie outside of the one-dimensional subspace spanned by $\phi(x_1)$, there are $w - q$ possibilities for $\phi(x_2)$. Similarly there are $w - q^2$ possibilities for $\phi(x_3)$, and so on. Thus

$$|P_V| = (w-1)(w-q)\cdots(w-q^{\ell-1}).$$

Since the formulas hold for infinitely many integers w, the proposition follows by equating the two expressions for $|P_V|$. □

Proposition 6.7 *Let $\mathcal{A}$ be the braid arrangement. Then*

$$\pi(\mathcal{A}, t) = (1+t)(1+2t)\cdots(1+(\ell-1)t).$$

Proof. We prove the equivalent formula:

$$\chi(\mathcal{A}, t) = t(t-1)(t-2)\cdots(t-(\ell-1)).$$

Let $I = \{1, \ldots, \ell\}$. Let W be a set with cardinality $|W| = w$. Let $M = W^I$ denote the set of all maps from I into W, so $|M| = w^\ell$. Each $\phi \in M$ determines an equivalence relation $\sim_\phi$ on I by $i \sim_\phi j$ if and only if $\phi(i) = \phi(j)$. Let Λ_ϕ be the corresponding partition. We classify the elements of M using the partitions Λ_ϕ. Given $X \in L(\mathcal{A})$ define subsets P_X and Q_X of M by

$$P_X = \{\phi \in M \mid \Lambda_\phi = \Lambda_X\}, \qquad Q_X = \{\phi \in M \mid \Lambda_\phi \geq \Lambda_X\}.$$

If $\Lambda_\phi \geq \Lambda_X$ then $\Lambda_\phi = \Lambda_Y$ for some $Y \geq X$ so that

$$Q_X = \bigcup_{Y \geq X} P_Y.$$

Thus by Möbius inversion

$$|P_Y| = \sum_{X \geq Y} \mu(Y, X)|Q_X|.$$

Let $B(X)$ be the set of blocks of Λ_X and let $b(X) = |B(X)|$. If $\phi \in Q_X$ then ϕ is constant on the blocks of Λ_X. Thus there is a bijection from Q_X to $W^{B(X)}$. In particular $|Q_X| = w^{b(X)}$.

Next note that $b(X) = \dim X$. We see this by choosing a basis for X consisting of vectors v^k defined by

$$v_i^k = \begin{cases} 1 & \text{if } i \in \Lambda_k, \\ 0 & \text{otherwise.} \end{cases}$$

In case $Y = V$ this gives

$$|P_V| = \sum_{X \in L} \mu(X) w^{\dim X}.$$

Since Λ_V is the partition where each block is a singleton, P_V is the set of one-to-one maps from I into W. Therefore we have

$$|P_V| = w(w-1)\cdots(w-(\ell-1)).$$

Since these formulas hold for all positive integers w, we are done. □

Deletion–Restriction Theorem

The formula for $\mu(X, Y)$ obtained in Lemma 5.2 provides a useful expression for $\pi(\mathcal{A}, t)$.

Lemma 6.8 *Let $\mathcal{A}$ be an arrangement. Then*

$$\pi(\mathcal{A}, t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})},$$

where the sum is over all central subarrangements $\mathcal{B}$ of $\mathcal{A}$.

Proof. Let $S(X) = S(V, X)$. From Lemma 5.2 we get

$$\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{r(X)} = \sum_{X \in L}\Big(\sum_{\mathcal{B} \in S(X)} (-1)^{|\mathcal{B}|}(-t)^{r(X)}\Big).$$

If $\mathcal{B} \in S(X)$ then $T(\mathcal{B}) = X$ so $r(\mathcal{B}) = r(X)$. The result follows since every central subarrangement $\mathcal{B}$ of $\mathcal{A}$ occurs in a unique $S(X)$. □

We are now prepared for the main result of this section, the deletion–restriction theorem. A similar result was first proved by Brylawski [35] for central arrangements, and by Zaslavsky [211] in general.

Theorem 6.9 (Deletion–Restriction) *If $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple of arrangements then*

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t).$$

Proof. We use the formula in Lemma 6.8. It is convenient here to let $H = H_0$ be the distinguished hyperplane. Separate the sum over $\mathcal{B} \subseteq \mathcal{A}$ into two sums: R' and R''. Here R' is the sum over those $\mathcal{B}$ which do not contain H and R'' is the sum over those $\mathcal{B}$ which contain H. It follows from Lemma 6.8 with $\mathcal{A}'$ in place of $\mathcal{A}$ that

$$R' = \pi(\mathcal{A}', t).$$

In order to compute R'' recall the definition of $S(X,Y)$ from Lemma 5.2. Since $H \in \mathcal{B}$, $\mathcal{A}_H \subseteq \mathcal{B}$. Thus if $T(\mathcal{B}) = Y$ then $\mathcal{B} \in S(H,Y)$. Let $L'' = L(\mathcal{A}'')$. Then

$$\begin{aligned} R'' &= \sum_{H \in \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|}(-t)^{r(\mathcal{B})} \\ &= \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H,Y)} (-1)^{|\mathcal{B}|}(-t)^{r(Y)} \\ &= -\sum_{Y \in L''} \sum_{\mathcal{B} \in S(H,Y)} (-1)^{|\mathcal{B} \setminus \mathcal{A}_H|}(-t)^{r(Y)} \\ &= -\sum_{Y \in L''} \mu(H,Y)(-t)^{r(Y)} \\ &= t\pi(\mathcal{A}'', t). \end{aligned}$$

The last equality follows from Lemma 5.2, the fact that the Möbius function μ'' of L'' is the restriction of μ to L'' so $\mu''(Y) = \mu(H,Y)$, and that the rank function r'' of L'' satisfies $r(Y) = r''(Y) + 1$. □

Corollary 6.10 *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. Then*

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t). \quad \square$$

Definition 6.11 *Let $\mathcal{A}$ be a central arrangement and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple with respect to the hyperplane $H \in \mathcal{A}$. Call H a* **separator** *if $T(\mathcal{A}) \notin L(\mathcal{A}')$.*

Corollary 6.12 *Let $\mathcal{A}$ be a central arrangement and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple with respect to $H \in \mathcal{A}$.*

(1) If H is a separator then

$$\mu(\mathcal{A}) = -\mu(\mathcal{A}'')$$

and hence

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}'')|.$$

(2) If H is not a separator then

$$\mu(\mathcal{A}) = \mu(\mathcal{A}') - \mu(\mathcal{A}'')$$

and

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}')| + |\mu(\mathcal{A}'')|.$$

Proof. It follows from Theorem 5.14 that $\pi(\mathcal{A}, t)$ has leading term

$$(-1)^{r(\mathcal{A})}\mu(\mathcal{A})t^{r(\mathcal{A})}.$$

The conclusion follows by comparing coefficients of the leading terms on both sides of the equation in Theorem 6.9. If H is a separator then $r(\mathcal{A}') < r(\mathcal{A})$ and there is no contribution from $\pi(\mathcal{A}', t)$. □

Definition 6.13 *Call the arrangements $\mathcal{A} = (\mathcal{A}, V)$ and $\mathcal{B} = (\mathcal{B}, W)$* **$\pi$–equivalent** *if $\pi(\mathcal{A}, t) = \pi(\mathcal{B}, t)$.*

Example 6.14 *It is clear that L–equivalent arrangements are π–equivalent. The converse is false. Consider the arrangements:*

$$Q(\mathcal{A}) = xyz(x - z)(x + z)(y - z)(y + z).$$

$$Q(\mathcal{B}) = xyz(x + y + z)(x + y - z)(x - y + z)(x - y - z).$$

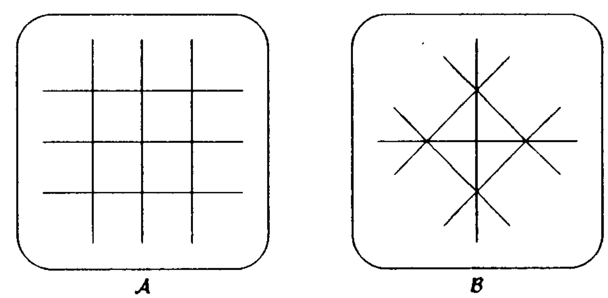


Figure 11: π–equivalent but not L–equivalent

The reader should check that

$$\pi(\mathcal{A}, t) = (1 + t)(1 + 3t)(1 + 3t) = \pi(\mathcal{B}, t)$$

but these arrangements are not L–equivalent. Figure 11 shows that $\mathcal{A}$ has two lines which are contained in four hyperplanes. These appear in the picture as the two common points on the line at infinity of the two sets of three parallel lines. Figure 11 also shows that $\mathcal{B}$ has no such lines. The factorization of these Poincaré polynomials is remarkable. Next we prove some general results to explain their factorization.

Supersolvable Arrangements

We noted that the braid arrangement is supersolvable. Our next aim is to prove a theorem due to Stanley [181]. It gives a factorization of $\pi(\mathcal{A},t)$ for supersolvable arrangements and serves as a model for several results in this book. Recall that a supersolvable arrangement is central.

Lemma 6.15 *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with a maximal chain of modular elements*

$$V = X_0 < X_1 < \ldots < X_\ell = T.$$

Let $H \in \mathcal{A}$ be a complement of $X_{\ell-1}$ and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the triple with respect to H. Then

(1) $\mathcal{A}_{X_{\ell-1}}$ is a supersolvable $(\ell-1)$-arrangement with a maximal chain of modular elements

$$V = X_0 < X_1 < \ldots < X_{\ell-1}.$$

The map $\tau_H : L(\mathcal{A}_{X_{\ell-1}}) \to L(\mathcal{A}'')$ defined by $\tau_H(Z) = H \vee Z$ is a lattice isomorphism. Thus $\mathcal{A}''$ is supersolvable.

(2) If H is not a separator then

$$V = X_0 < X_1 < \ldots < X_\ell = T$$

is a maximal chain of modular elements in $\mathcal{A}'$. If H is a separator then

$$V = X_0 < X_1 < \ldots < X_{\ell-1}$$

is a maximal chain of modular elements in $\mathcal{A}'$. Thus $\mathcal{A}'$ is supersolvable.

Proof. If X_p is modular in $L(\mathcal{A})$ then it is modular in $L(\mathcal{A}_{X_{\ell-1}})$. It follows from Lemma 4.25 that τ_H is an isomorphism. This implies (1).

To show that X_p is modular in $L(\mathcal{A}')$ note that $H \not\leq X_{\ell-1}$ and hence $H \not\leq X_p$. If we denote join and meet in $L(\mathcal{A}')$ by $\vee'$ and $\wedge'$ then for $Y \in L(\mathcal{A}')$ we have $Y \vee' X_p = Y \vee X_p$ and $Y \wedge' X_p = Y \wedge X_p$. Since $r(\mathcal{A}') = \ell$ if H is not a separator and $r(\mathcal{A}') = \ell - 1$ if H is a separator, this proves (2). □

Theorem 6.16 *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with a maximal chain of modular elements*

$$V = X_0 < X_1 < \ldots < X_\ell = T.$$

Let $b_i = |\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}|$. Then

$$\pi(\mathcal{A},t) = \prod_{i=1}^{\ell}(1 + b_i t).$$

Proof. We argue by induction on $|\mathcal{A}|$. The assertion is clear for $|\mathcal{A}| = 1$. For the induction step choose $H \in \mathcal{A}$ as in Lemma 6.15. It follows from Lemma 6.15.1 that $\mathcal{A}''$ is supersolvable. Since $|\mathcal{A}''| < |\mathcal{A}|$ the induction hypothesis applies to $\mathcal{A}''$. It follows from the maximal chain of modular elements given in Lemma 6.15.1 that

$$\pi(\mathcal{A}'',t) = \prod_{i=1}^{\ell-1}(1 + b_i t).$$

It follows from Lemma 6.15.2 that $\mathcal{A}'$ is supersolvable. Since $|\mathcal{A}'| < |\mathcal{A}|$ the induction hypothesis applies to $\mathcal{A}'$. If H is not a separator then it follows from the maximal chain of modular elements given in Lemma 6.15.2 that

$$\pi(\mathcal{A}',t) = \prod_{i=1}^{\ell-1}(1 + b_i t)(1 + (b_\ell - 1)t).$$

If H is a separator then $\mathcal{A}' = \mathcal{A}_{X_{\ell-1}}$. It follows from the maximal chain of modular elements given in Lemma 6.15.2 that

$$\pi(\mathcal{A}',t) = \prod_{i=1}^{\ell-1}(1 + b_i t).$$

Note that in this case $b_\ell = 1$. The conclusion follows from Theorem 6.9. □

The arrangement $\mathcal{A}$ defined by

$$Q(\mathcal{A}) = xyz(x - z)(x + z)(y - z)(y + z)$$

in Example 6.14 is supersolvable. The following is a maximal chain of modular elements: $V < \{x = 0\} < \{x = z = 0\} < \{0\}$. Thus Theorem 6.16 explains the factorization $\pi(\mathcal{A},t) = (1+t)(1+3t)(1+3t)$. The arrangement $\mathcal{B}$ in the same example is not supersolvable. We prove a result next which explains the factorization of its Poincaré polynomial.

Nice Partitions

In Theorem 6.16 we considered the sets $\pi_i = \mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}$. These sets provide a partition $(\pi_1, \ldots, \pi_\ell)$ of $\mathcal{A}$. Next we generalize supersolvable arrangements by defining the notion of a nice partition for a central arrangement. It was introduced in [195]. We show in Proposition 6.20 that the partition which arises from a maximal chain of modular elements is nice. Moreover, we show in Corollary 10.11 that the Poincaré polynomial of an arrangement with a nice partition has a factorization similar to Theorem 6.16. Nice partition of an affine arrangement is introduced in Definition 10.14.

Definition 6.17 *A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called* **independent** *if for every choice of hyperplanes $H_i \in \pi_i$ for $1 \leq i \leq s$, the resulting s hyperplanes are independent, $r(H_1 \vee \cdots \vee H_s) = s$.*

Definition 6.18 *Let $X \in L$. Let $\pi = (\pi_1, \ldots, \pi_s)$ be a partition of $\mathcal{A}$. Then the* **induced partition** π_X *is a partition of $\mathcal{A}_X$. Its blocks are the nonempty subsets $\pi_i \cap \mathcal{A}_X$.*

Definition 6.19 *A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called* **nice** *if:*
(1) π is independent and
(2) if $X \in L \setminus \{V\}$ then the induced partition π_X contains a block which is a singleton.

Proposition 6.20 *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with a maximal chain of modular elements $V = X_0 < X_1 < \ldots < X_\ell = T$. Let $\pi_i = \mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}$. Then the partition $(\pi_1, \ldots, \pi_\ell)$ is nice.*

Proof. Choose $H_i \in \pi_i$ for each i. First we use induction on i to prove that $r(H_1 \vee \cdots \vee H_i) = i$. This is clear when $i = 1$. Let $Y = H_1 \vee \cdots \vee H_{i-1}$. Then $Y \leq X_{i-1}$. Since $H_i \not\leq X_{i-1}$, we have $H_i \not\leq Y$. Thus $H_i \wedge Y = V$. By the inductive assumption, we have $r(Y) = i - 1$. Therefore we have

$$r(H_1 \vee \cdots \vee H_i) = r(Y \vee H_i) = r(Y) + r(H_i) - r(Y \wedge H_i) = (i-1) + 1 - r(V) = i.$$

This shows that the partition $(\pi_1, \ldots, \pi_\ell)$ is independent. Next let $X \in L \setminus \{V\}$. Let j be the largest integer such that $V = X \wedge X_j$. Then

$$\begin{aligned} 0 < r(X \wedge X_{j+1}) &= r(X) + r(X_{j+1}) - r(X \vee X_{j+1}) \\ &\leq r(X) + r(X_{j+1}) - r(X \vee X_j) \\ &= r(X) + r(X_{j+1}) - (r(X) + r(X_j) - r(X \wedge X_j)) = 1. \end{aligned}$$

This implies that $X \wedge X_{j-1}$ is a hyperplane belonging to $\mathcal{A}$. Thus $\mathcal{A}_X \cap \pi_j = \{X \wedge X_{j-1}\}$ is a singleton. □

We will prove in Corollary 10.11 that if $\mathcal{A}$ has a nice partition $\pi = (\pi_1, \ldots, \pi_s)$ and $b_i = |\pi_i|$, then

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{s}(1 + b_i t).$$

The arrangement $\mathcal{B}$ defined by

$$Q(\mathcal{B}) = xyz(x + y + z)(x + y - z)(x - y + z)(x - y - z)$$

in Example 6.14 has a nice partition $\pi = (\pi_1, \pi_2, \pi_3)$. We may take $\pi_1 = \{\ker(z)\}$, $\pi_2 = \{\ker(x), \ker(x - y + z), \ker(x - y - z)\}$, $\pi_3 = \{\ker(y), \ker(x + y - z), \ker(x + y + z)\}$. Thus Corollary 10.11 explains the factorization $\pi(\mathcal{B}, t) = (1 + t)(1 + 3t)(1 + 3t)$.

Counting Functions

The Poincaré polynomial of an arrangement will appear repeatedly in this book. It will be shown to equal the Poincaré polynomial of several graded algebras which we are going to associate with $\mathcal{A}$. It is also the Poincaré polynomial of the complement $M(\mathcal{A})$ of a complex arrangement. Here we prove that the Poincaré polynomial is also a counting function. First suppose that $\mathcal{A}$ is a real arrangement. Recall from the introduction that the complement $M(\mathcal{A})$ is a disjoint union of chambers

$$M(\mathcal{A}) = \bigcup_{C \in \mathcal{C}(\mathcal{A})} C.$$

Zaslavsky [211] showed that the number of chambers is determined by the Poincaré polynomial as follows.

Theorem 6.21 *Let $\mathcal{A}$ be a real arrangement. Then*

$$|\mathcal{C}(\mathcal{A})| = \pi(\mathcal{A}, 1).$$

Proof. If $\mathcal{A}$ is empty then $|\mathcal{C}(\mathcal{A})| = 1 = \pi(\mathcal{A}, 1)$. We showed in the introduction that $|\mathcal{C}(\mathcal{A})| = |\mathcal{C}(\mathcal{A}')| + |\mathcal{C}(\mathcal{A}'')|$. Thus the result is a consequence of Theorem 6.9. □

Next assume that $\mathcal{A}$ is an ℓ-arrangement over the finite field $\mathbf{F}_q$. Then the complement $M(\mathcal{A})$ is a finite set of points. Its cardinality is also determined by the Poincaré polynomial.

Theorem 6.22 *Let $\mathcal{A}$ be an ℓ-arrangement over $\mathbf{F}_q$. Let $|M(\mathcal{A})|$ denote the cardinality of the complement. Then*

$$|M(\mathcal{A})| = q^\ell \pi(\mathcal{A}, -q^{-1}) = \chi(\mathcal{A}, q).$$

Proof. If $\mathcal{A}$ is empty then $|M(\mathcal{A})| = q^\ell = \chi(\mathcal{A}, q)$. Suppose $\mathcal{A}$ is nonempty and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple. Evidently

$$|M(\mathcal{A})| = |M(\mathcal{A}')| - |M(\mathcal{A}'')|.$$

Thus the functions $|M(\mathcal{A})|$ and $\chi(\mathcal{A}, q)$ agree on Φ_ℓ and by Corollary 6.10 they satisfy the same recursion for deletion and restriction. It follows that they are equal. □

7 Graphic Arrangements

In this section we study certain special central arrangements obtained from finite simple non-oriented graphs. They are called **graphic arrangements**. Let G be a finite simple non-orieneted graph and let $\mathcal{A}(G)$ be the corresponding graphic arrangement. The correspondence $G \mapsto \mathcal{A}(G)$ gives a map from the set of finite simple non-oriented graphs into the set of arrangements. This map may be used to "pull back" results concerning arrangements to results concerning graphs. For example Zaslavsky's chamber counting theorem 6.21 for arrangements can be translated into Stanley's negative one color theorem 7.25 which determines the number of acyclic orientations for graphs. The characteristic polynomial of the arrangement $\mathcal{A}(G)$ corresponds to the chromatic polynomial of the graph G.

Definitions

Definition 7.1 *A finite simple non-oriented* **graph** $G = (\mathcal{V}, \mathcal{E})$ *is an ordered pair consisting of the set* $\mathcal{V}$ *of vertices and the set* $\mathcal{E}$ *of edges. They satisfy the following two conditions:*
(1) $\mathcal{V}$ *is a finite set,*
(2) $\mathcal{E}$ *is a collection of 2-element subsets of* $\mathcal{V}$.
When it is convenient we let $\mathcal{V} = \{1, 2, \ldots, \ell\}$.

Example 7.2 *If* $\mathcal{V} = \{1,2,3\}$ *and* $\mathcal{E} = \{\{1,2\},\{1,3\}\}$ *then* $G = (\mathcal{V}, \mathcal{E})$ *is a graph. This graph is shown in* Figure 12.

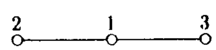


Figure 12: A graph with three vertices

Example 7.3 *When* $\mathcal{E}$ *is the set of all 2-element subsets of* $\mathcal{V}$, *the graph* $G = (\mathcal{V}, \mathcal{E})$ *is called a* **complete graph**. *For* $\mathcal{V} = \{1,2,3,4\}$ *it is shown in* Figure 13.

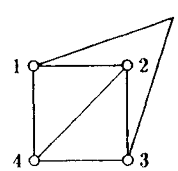


Figure 13: A complete graph

Definition 7.4 *Let* $\mathbb{K}$ *be a field and let* $V = \mathbb{K}^\ell$. *Let* $x_1, \ldots x_\ell$ *be a basis for the dual space* V^*. *Given the graph* $G = (\mathcal{V}, \mathcal{E})$, *define an arrangement* $\mathcal{A}(G)$ *by*

$$\mathcal{A}(G) = \{\ker(x_i - x_j) \mid \{i,j\} \in \mathcal{E}\}.$$

The arrangement $\mathcal{A}(G)$ *is called a* **graphic arrangement**.

Example 7.5 *If* G *is the graph in* Figure 12 *then a defining polynomial* Q *of the graphic arrangement is given by* $Q = (x_1 - x_2)(x_1 - x_3)$. *It consists of two planes in* $\mathbb{K}^3$.

Example 7.6 *Let* $\mathbb{K} = \mathbb{R}$. *For the complete graph with* ℓ *vertices* $\mathcal{A}(G)$ *has a defining polynomial*

$$Q(\mathcal{A}(G)) = \prod_{1 \le i < j \le \ell} (x_i - x_j).$$

Thus $\mathcal{A}(G)$ *is equal to the braid arrangement* $\mathcal{A}_{\mathbb{R}}$ *of* Definition 2.9.

It follows that an arrangement is graphic if and only if it is a subarrangement of the braid arrangement.

Definition 7.7 *Let* C *be a finite set of cardinality* n. *Let* $G = (\mathcal{V}, \mathcal{E})$ *be a graph. A* **coloring** *of* G *by* C *is a map* $\varphi : \mathcal{V} \to C$ *such that* $\varphi(i) \neq \varphi(j)$ *whenever* $\{i,j\} \in \mathcal{E}$.

Example 7.8 *If* G *is the graph in* Figure 12 *and* $C = \{a,b\}$ *then the map* $\varphi : \{1,2,3\} \to \{a,b\}$ *defined by* $\varphi(1) = a$, $\varphi(2) = b$, $\varphi(3) = a$ *is a coloring. The map* $\phi : \{1,2,3\} \to \{a,b\}$ *defined by* $\phi(1) = a$, $\phi(2) = a$, $\phi(3) = b$ *is not a coloring.*

The concept of coloring obviously originates from the "map coloring problem." Visualize a map of Canada, the United States, and Mexico. We assign a vertex to each country and connect two vertices only when the corresponding countries are adjacent. This map yields the graph in Figure 12. Here 1 = United States, 2 = Canada, 3 = Mexico. In this way we can assign a (finite simple non-oriented) graph to any map of countries. A graph coloring corresponds to a way of coloring the map so that two adjacent countries are colored differently.

Definition 7.9 *Let* $G = (\mathcal{V}, \mathcal{E})$ *be a graph. The* **chromatic function** $\chi(G,t)$ *is a function defined on the set of nonnegative integers by*

$$\chi(G,t) = \text{the number of colorings of } G \text{ with } t \text{ colors.}$$

The famous "four color theorem" is translated into the assertion that $\chi(G,4) > 0$ for any planar graph G, or any graph G obtained from an arbitrary map on the plane. The chromatic function $\chi(G,t)$ was introduced by G. Birkhoff in his study [22] of the four color problem.

Example 7.10 *If a graph G has ℓ vertices and no edges then there is no restriction for its coloring. Thus we have*

$$\chi(G,t) = t^{\ell}.$$

Example 7.11 *If G is the graph in* Figure 12 *then there are t ways to color the vertex 1 and there are $t-1$ ways to color the vertices 2 and 3 each. Thus we have*

$$\chi(G,t) = t(t-1)^2.$$

Example 7.12 *If G is the complete graph with ℓ vertices then there are t ways to color the first vertex, $t-1$ ways to color the second vertex, $t-2$ ways to color the third vertex, and so on. Thus we have*

$$\chi(G,t) = t(t-1)(t-2)\cdots(t-\ell+1).$$

It is easy to find the chromatic function directly in the examples above. It is not so easy to find the chromatic function of the graph in Figure 14 directly. To compute its chromatic function we use a deletion–contraction technique. It is closely related to the method of deletion and restriction of arrangements studied in section 6. We also prove that the chromatic function is a monic polynomial of degree ℓ.

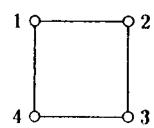


Figure 14: A graph with four vertices

Deletion–Contraction

Definition 7.13 *Let $G = (\mathcal{V},\mathcal{E})$, $\mathcal{V} = \{1,2,\ldots,\ell\}$, and $\mathcal{E} \neq \emptyset$. Fix an edge $e_0 = \{i,j\} \in \mathcal{E}$. The* **deletion** *$G' = (\mathcal{V}',\mathcal{E}')$ of G with respect to e_0 is defined by*

$$\mathcal{V}' = \mathcal{V}, \quad \mathcal{E}' = \mathcal{E} \setminus \{e_0\}.$$

The **contraction** *G'' of G with respect to e_0 is the graph $G'' = (\mathcal{V}'',\mathcal{E}'')$. Here $\mathcal{V}''$ is the vertex set with cardinality $\ell-1$ obtained by identifying i and j in $\mathcal{V}$. Write $\mathcal{V}'' = \{\bar{1},\bar{2},\ldots,\bar{\ell}\}$ where $\bar{p} = \bar{q}$ if and only if either $p = q$ or $\{p,q\} = \{i,j\}$. Define $\mathcal{E}''$ by*

$$\mathcal{E}'' = \{\{\bar{p},\bar{q}\} \mid \{p,q\} \in \mathcal{E}'\}.$$

Example 7.14 *Let G be the graph in* Figure 14 *and let $e_0 = \{2,3\}$.* Figure 15 *shows G together with the deletion G' and the contraction G'' with respect to e_0.*

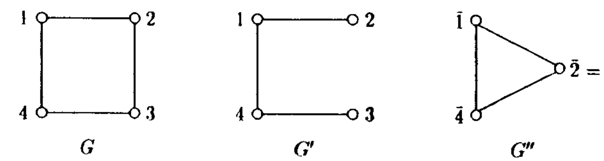


Figure 15: Deletion and contraction

Proposition 7.15 *Let $G = (\mathcal{V},\mathcal{E})$ be a graph with $\mathcal{E} \neq \emptyset$. Let G' and G'' be the deletion and contraction with respect to the edge e_0. Then*

$$\chi(G',t) = \chi(G,t) + \chi(G'',t).$$

Proof. Suppose $e_0 = \{1,2\}$. Every coloring of G induces a coloring of G'. Thus there is an injection from the set of all colorings of G to the set of all colorings of G'. The complement of the image is exactly equal to the set

$$\{\varphi \mid \text{colorings of } G' \text{ such that } \varphi(1) = \varphi(2)\}.$$

This set is in one-to-one correspondence with the set of all colorings of G''. □

Example 7.16 *This allows us to find the chromatic function of the graph in* Figure 14. *Let $e_0 = \{2,3\}$. The deletion G' and the contraction G'' are given in* Figure 15. Proposition 7.15 *gives*

$$\chi(G,t) = \chi(G',t) - \chi(G'',t) = t(t-1)^3 - t(t-1)(t-2) = t(t-1)(t^2-3t+3).$$

Corollary 7.17 *Let G be a graph with ℓ vertices. Then the chromatic function $\chi(G,t)$ is a monic polynomial in t of degree ℓ.*

Proof. We argue by induction on the number of edges. When there are no edges we have $\chi(G,t) = t^{\ell}$. Suppose that G has at least one edge. Fix an edge e_0. Consider the deletion G' and the contraction G''. By the induction assumption, $\chi(G',t)$ is a monic polynomial of degree ℓ and $\chi(G'',t)$ is a polynomial of degree less than ℓ. The result follows from Proposition 7.15. □

This shows that the chromatic function $\chi(G,t)$ is a polynomial. From now on we will call it the **chromatic polynomial**. As we have just seen, the construction of the deletion G' and the contraction G'' is very useful. Next we consider the corresponding graphic arrangements $\mathcal{A}(G')$ and $\mathcal{A}(G'')$.

Proposition 7.18 *Let G be a graph with edge e_0. Let G' and G'' be the deletion and contraction of G with respect to e_0. Let $H_0 \in \mathcal{A}(G)$ be the hyperplane corresponding e_0. Write $\mathcal{A} = \mathcal{A}(G)$. Let $\mathcal{A}'$ and $\mathcal{A}''$ denote the deletion and restriction of $\mathcal{A}$ with respect to H_0. Then $\mathcal{A}(G') = \mathcal{A}'$ and $\mathcal{A}(G'') = \mathcal{A}''$.*

Proof. Assume $e_0 = \{1,2\}$. Then $H_0 = \ker(x_1 - x_2)$. Clearly

$$\mathcal{A}(G') = \mathcal{A}(G) \setminus \{H_0\} = \mathcal{A}'.$$

Denote the set of vertices of G'' by $\{\bar{1} = \bar{2}, \bar{3}, \ldots, \bar{\ell}\}$. Write the corresponding basis for $\mathbf{K}^{\ell-1}$ as $\bar{x}_1 = \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_\ell$. It is naturally identified with a basis for the dual space H_0^* of H_0. We have

$$\begin{aligned}\mathcal{A}(G'') &= \{\ker(\bar{x}_i - \bar{x}_j) \mid \{\bar{i}, \bar{j}\} \in \mathcal{E}''\} = \{\ker(\bar{x}_i - \bar{x}_j) \mid \{i,j\} \in \mathcal{E}'\} \\ &= \{\ker(\bar{x}_i - \bar{x}_j) \mid \ker(x_i - x_j) \in \mathcal{A}'\} = \mathcal{A}''. \ \square\end{aligned}$$

Recall the characteristic polynomial of an arrangement from Definition 6.5:

$$\chi(\mathcal{A}, t) = t^\ell \pi(\mathcal{A}, -t^{-1}) = \sum_{X \in L} \mu(X) t^{\dim(X)}.$$

The Greek words for characteristic and chromatic begin with χ. We assume that this explains why the corresponding polynomials are called χ. It is a pleasant coincidence that these polynomials are equal.

Theorem 7.19 *Let G be a graph and let $\mathcal{A}(G)$ be the corresponding graphic arrangement. Then*

$$\chi(G, t) = \chi(\mathcal{A}(G), t).$$

Proof. We argue by induction on the number of edges in G. Equality holds when G has no edges: $\chi(G,t) = t^\ell = \chi(\mathcal{A}(G), t)$. The induction is completed by applying Proposition 7.15, Theorem 7.19, Proposition 7.18, and Corollary 6.10 to give

$$\chi(G,t) = \chi(G',t) - \chi(G'',t) = \chi(\mathcal{A}(G'),t) - \chi(\mathcal{A}(G''),t) = \chi(\mathcal{A}',t) - \chi(\mathcal{A}'',t) = \chi(\mathcal{A},t). \ \square$$

Theorem 6.21 on chamber counting gives

Corollary 7.20 *Let $\mathbf{K} = \mathbf{R}$. The number of chambers of the graphic arrangement $\mathcal{A}(G)$ is equal to $(-1)^\ell \chi(G, -1)$.*

Proof. By Theorem 6.21 and Theorem 7.19 we have

$$|\mathcal{C}(\mathcal{A}(G))| = \pi(\mathcal{A}(G), 1) = (-1)^\ell \chi(\mathcal{A}(G), -1) = (-1)^\ell \chi(G, -1). \ \square$$

Acyclic Orientations

Let $\mathcal{A}$ be an arrangement in $\mathbf{R}^\ell$. Denote the set of chambers of $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$ as before. Corollary 7.20 asserts that the value $\chi(G,-1)$ of the chromatic polynomial of the graph G has an interpretation as $(-1)^\ell |\mathcal{C}(\mathcal{A}(G))|$. If we give a graph theoretic meaning to the number $|\mathcal{C}(\mathcal{A}(G))|$ of chambers of the arrangement $\mathcal{A}(G)$, we will obtain a theorem in graph theory. Such a theorem was first proved by R. P. Stanley in [182]. It is called the negative one color theorem. The original proof is not directly related to arrangements. Here we will prove it via graphic arrangements.

Definition 7.21 *Let G be a graph. An* **orientation** *of G is an assignment of a direction to each edge $\{i,j\}$, denoted by $i \to j$ or $j \to i$. An orientation is called* **acyclic** *if it has no directed cycles.*

Example 7.22 *The orientation in* Figure 16 *is not acyclic because it contains the directed cycle $1 \to 2 \to 4 \to 1$.*

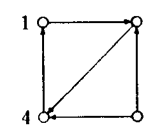


Figure 16: An oriented graph

Example 7.23 *The complete graph on three vertices has $8 = 2^3$ orientations. Six of these orientations are acyclic. The two shown in* Figure 17 *are not.*

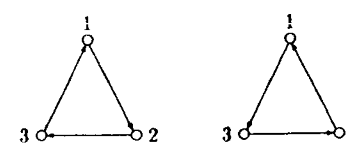


Figure 17: Not acyclic orientations

Lemma 7.24 *Let G be a graph. Denote the set of all acyclic orientations by $AO(G)$. Let $\mathcal{A} = \mathcal{A}(G)$. There exists a bijection from $AO(G)$ to $\mathcal{C}(\mathcal{A})$.*

Proof. Let $\omega \in AO(G)$. For $i \in \mathcal{V}$ let $p_i(\omega)$ denote the number of vertices which can be reached along the directions of the orientation from the vertex i. Consider a point $p(\omega) = (p_1(\omega), \ldots, p_\ell(\omega)) \in \mathbf{R}^\ell$. Let $\{i,j\} \in \mathcal{E}$. Define half spaces

$$H_{ij}^+ = \{(x_1, \ldots, x_\ell) \in \mathbf{R}^\ell \mid x_i > x_j\},$$

$$H_{ij}^- = \{(x_1, \ldots, x_\ell) \in \mathbf{R}^\ell \mid x_i < x_j\}.$$

If $i \to j$ then every vertex which can be reached from j can also be reached from i. Thus $p_i(\omega) \geq p_j(\omega)$. Since the orientation is acyclic, it is impossible to reach i from j. Thus we have

$$i \to j \text{ in } \omega \quad \Leftrightarrow \quad p_i(\omega) > p_j(\omega) \quad \Leftrightarrow \quad p(\omega) \in H_{ij}^+,$$
$$j \to i \text{ in } \omega \quad \Leftrightarrow \quad p_j(\omega) > p_i(\omega) \quad \Leftrightarrow \quad p(\omega) \in H_{ij}^-.$$

Therefore $p(\omega) \notin \ker(x_i - x_j)$. Thus $p(\omega) \in M(\mathcal{A}) = \mathbf{R}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ and there exists a unique chamber $C(\omega) \in \mathcal{C}(\mathcal{A})$ which contains $p(\omega)$. We have

(1) $$C(\omega) \subseteq H_{ij}^+ \quad \Leftrightarrow \quad i \to j \text{ in } \omega$$
(2) $$C(\omega) \subseteq H_{ij}^- \quad \Leftrightarrow \quad j \to i \text{ in } \omega\ .$$

We show next that the correspondence $\omega \mapsto C(\omega)$ gives a bijection from $AO(G)$ to $\mathcal{C}(\mathcal{A})$. It is obvious from (1) and (2) that this map is injective. In order to prove surjectivity let $C \in \mathcal{C}(\mathcal{A})$. Choose a point $p = (p_1, \ldots, p_\ell) \in C$. Define a direction on each edge $\{i,j\}$ by

(3) $$i \to j \quad \Leftrightarrow \quad p_i > p_j \quad \Leftrightarrow \quad p \in H_{ij}^+$$
(4) $$j \to i \quad \Leftrightarrow \quad p_i < p_j \quad \Leftrightarrow \quad p \in H_{ij}^-$$

Then by (1), (2), (3), and (4) we have

$$C(\omega) \subseteq H_{ij}^+ \quad \Leftrightarrow \quad p \in H_{ij}^+$$
$$C(\omega) \subseteq H_{ij}^- \quad \Leftrightarrow \quad p \in H_{ij}^-.$$

It follows that $p \in C(\omega)$ and thus $C = C(\omega)$. □

By combining Lemma 7.24 with Corollary 7.20, we obtain the following theorem due to R. P. Stanley [182]:

Theorem 7.25 *The number of acyclic orientations of G is $(-1)^\ell \chi(G, -1)$.* □

Chapter III

Let $\mathcal{K}$ be a commutative ring. We construct certain algebras over $\mathcal{K}$ associated with $\mathcal{A}$. We construct the graded algebra $A(\mathcal{A})$ for a central arrangement $\mathcal{A}$ in section 8. This construction is generalized to affine arrangements in section 9. The algebra $A(\mathcal{A})$ is the quotient of the exterior algebra $E(\mathcal{A})$ based on $\mathcal{A}$ by a homogeneous ideal $I(\mathcal{A})$, $A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$. This algebra is constructed using only $L(\mathcal{A})$. It will reappear in chapter V with a topological significance. We prove that the $\mathcal{K}$-algebra $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module and that its Poincaré polynomial is equal to $\pi(\mathcal{A}, t)$. This gives an interpretation of the coefficients of $\pi(\mathcal{A}, t)$. We construct a $\mathcal{K}$-basis for $A(\mathcal{A})$ as a free graded $\mathcal{K}$-module using broken circuits. We also show that given a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ there is an exact sequence of $\mathcal{K}$-modules

$$0 \to A(\mathcal{A}') \to A(\mathcal{A}) \to A(\mathcal{A}'') \to 0.$$

We prove some algebra factorization theorems in section 10. If $\mathbf{c}\mathcal{A}$ is the cone over $\mathcal{A}$ then

$$A(\mathbf{c}\mathcal{A}) \simeq (\mathcal{K} + \mathcal{K}a_0) \otimes A(\mathcal{A}).$$

Recall from Theorem 6.16 that if the central arrangement $\mathcal{A}$ is supersolvable then

(1) $$\pi(\mathcal{A}, t) = (1 + b_1 t) \cdots (1 + b_\ell t)$$

where the b_i are non-negative integers. In section 10 we prove that the algebra $A(\mathcal{A})$ has a tensor product decomposition

(2) $$(\mathcal{K} + B_1) \otimes \cdots \otimes (\mathcal{K} + B_\ell)$$

as graded $\mathcal{K}$-module with $b_i = \operatorname{rank} B_i$. It follows that (1) is a consequence of (2). This decomposition of $A(\mathcal{A})$ is generalized to arrangements with a nice partition. In particular the Poincaré polynomial of an arrangement with a nice partition has a factorization like (1). In section 11 we define another graded algebra $B(\mathcal{A})$ whose multiplication is a shuffle product. We prove that $B(\mathcal{A})$ is algebra isomorphic to $A(\mathcal{A})$. In section 12 we assume that $\mathcal{K}$ is a subring of $\mathbf{K}$. We associate to the arrangement $\mathcal{A}$ the $\mathcal{K}$-algebra $R(\mathcal{A})$ generated by the differential forms $\omega_H = d\alpha_H/\alpha_H$. Note that this algebra is not a purely combinatorial object since the defining polynomials α_H enter the definition. The main result of section 12 is that there is an isomorphism of algebras $A(\mathcal{A}) \simeq R(\mathcal{A})$. This shows that $R(\mathcal{A})$ depends only on $L(\mathcal{A})$. The argument uses the fact that there is a short exact sequence of $\mathcal{K}$-modules

$$0 \to R(\mathcal{A}') \to R(\mathcal{A}) \to R(\mathcal{A}'') \to 0.$$

8 The Algebra $A(\mathcal{A})$ for Central Arrangements

In this section we assume that $\mathcal{A}$ is a central arrangement. We associate to the arrangement $\mathcal{A}$ a graded anticommutative algebra $A(\mathcal{A})$ over $\mathcal{K}$. In the literature this algebra is sometimes called the **Orlik–Solomon algebra**. The algebra $A(\mathcal{A})$ was first defined in [142], where it was used to prove that for a complex arrangement $A(\mathcal{A})$ is isomorphic as graded algebra to the cohomology algebra of the complement $M(\mathcal{A})$. We show this in section 22. The algebra $A(\mathcal{A})$ has since been used by several authors in work on hypergeometric functions.

Construction of $A(\mathcal{A})$

Definition 8.1 *Let $\mathcal{A}$ be an arrangement over $\mathbf{K}$. Let $\mathcal{K}$ be a commutative ring. Let*

$$E_1 = \bigoplus_{H \in \mathcal{A}} \mathcal{K} e_H$$

and let

$$E = E(\mathcal{A}) = \Lambda(E_1)$$

be the exterior algebra of E_1.

Note that E_1 has a $\mathcal{K}$-basis consisting of elements e_H in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. Write $uv = u \wedge v$ and note that $e_H^2 = 0$, $e_H e_K = -e_K e_H$ for $H, K \in \mathcal{A}$. The algebra E is graded. If $|\mathcal{A}| = n$ then

$$E = \bigoplus_{p=0}^{n} E_p$$

where $E_0 = \mathcal{K}$, E_1 agrees with its earlier definition and E_p is spanned over $\mathcal{K}$ by all $e_{H_1} \cdots e_{H_p}$ with $H_k \in \mathcal{A}$.

Definition 8.2 *Define a $\mathcal{K}$-linear map $\partial_E = \partial : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1$ and for $p \geq 2$*

$$\partial(e_{H_1} \cdots e_{H_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \cdots \widehat{e_{H_k}} \cdots e_{H_p}$$

for all $H_1, \ldots, H_p \in \mathcal{A}$.

Definition 8.3 *Given a p-tuple of hyperplanes $S = (H_1, \ldots, H_p)$ write $|S| = p$,*

$$e_S = e_{H_1} \cdots e_{H_p} \in E, \qquad \cap S = H_1 \cap \cdots \cap H_p.$$

Since $\mathcal{A}$ is central, $\cap S \in L$ for all S.

If $p = 0$ we agree that $S = (\,)$ is the empty tuple, $e_S = 1$, and $\cap S = V$. Since the rank function on L is codimension, it is clear that $r(\cap S) \leq |S|$.

Definition 8.4 *Call S* **independent** *if $r(\cap S) = |S|$ and* **dependent** *if $r(\cap S) < |S|$.*

The terminology has geometric significance. The tuple S is independent if the corresponding linear forms $\alpha_1, \ldots, \alpha_p$ are linearly independent. Equivalently, the hyperplanes of S are in general position. Let $\mathbf{S}_p$ denote the set of all p-tuples $(H_1, \ldots, H_p)$ and let $\mathbf{S} = \cup_{p \geq 0} \mathbf{S}_p$.

Definition 8.5 *Let $\mathcal{A}$ be an arrangement. Let $I = I(\mathcal{A})$ be the ideal of E generated by ∂e_S for all dependent $S \in \mathbf{S}$.*

Since I is generated by homogeneous elements, it is a graded ideal. Let $I_p = I \cap E_p$. Then

$$I = \bigoplus_{p=0}^{n} I_p.$$

Definition 8.6 *Let $\mathcal{A}$ be an arrangement. Let $A = A(\mathcal{A}) = E/I$. Let $\varphi : E \to A$ be the natural homomorphism and let $A_p = \varphi(E_p)$. If $H \in \mathcal{A}$ let $a_H = \varphi(e_H)$ and if $S \in \mathbf{S}$ let $a_S = \varphi(e_S)$.*

Lemma 8.7 *If $S \in \mathbf{S}$ and $H \in S$ then $e_S = e_H \partial e_S$.*

Proof. Note that $H \in S$ implies that $e_H e_S = 0$. Thus $0 = \partial(e_H e_S) = e_S - e_H \partial e_S$. □

Since both E and I are graded, A is a graded anticommutative algebra. Since the elements of $\mathbf{S}_1$ are independent, we have $I_0 = 0$ and hence $A_0 = \mathcal{K}$. The only dependent elements of $\mathbf{S}_2$ are of the form $S = (H, H)$. Since $e_S = e_H^2 = 0$, we have $I_1 = 0$. Thus the elements a_H are linearly independent over $\mathcal{K}$ and $A_1 = \oplus_{H \in \mathcal{A}} \mathcal{K} a_H$. If $p > \ell$ then every element of $\mathbf{S}_p$ is dependent and it follows from Lemma 8.7 that $A_p = 0$. Thus

$$A = \bigoplus_{p=0}^{\ell} A_p.$$

Example 8.8 *Suppose $\ell = 2$ and $\mathcal{A} = \{H_1, \ldots, H_n\}$. Write $a_k = a_{H_k}$. Then*

$$A(\mathcal{A}) = \mathcal{K} \oplus \bigoplus_{p=1}^{n} \mathcal{K} a_p \oplus \bigoplus_{k=1}^{n-1} \mathcal{K} a_k a_n.$$

We have computed A_0, A_1 and we know that $A_p = 0$ for $p > 2$. It remains to compute A_2. Since $\dim V = 2$, (H_i, H_j, H_k) is dependent for all (i, j, k). Thus I_2 contains the element

$$\partial(e_i e_j e_k) = e_j e_k - e_i e_k + e_i e_j = e_i e_j + e_j e_k + e_k e_i.$$

It follows that A_2 is spanned by $a_p a_q$ subject to the relations

$$a_i a_j + a_j a_k + a_k a_i = 0$$

for all (i,j,k). This shows that A_2 is spanned by $a_k a_n$ for $1 \le k < n$. It remains to show that the sum is direct. Suppose $\sum_{k=1}^{n-1} c_k a_k a_n = 0$ with $c_k \in \mathcal{K}$. Then $\sum_{k=1}^{n-1} c_k e_k e_n \in I_2$. Recall that I_2 is spanned by the elements $\partial(e_i e_j e_k)$. Since $\partial\partial = 0$ we have $\partial I_2 = 0$ and hence

$$\partial\left(\sum_{k=1}^{n-1} c_k e_k e_n\right) = \sum_{k=1}^{n-1} c_k(e_n - e_k) = 0.$$

Since $e_1, \ldots, e_n$ are linearly independent over $\mathcal{K}$, we conclude that $c_k = 0$ for all k.

Example 8.9 *If $\mathcal{A}$ is the Boolean arrangement then $S = (H_1, \ldots, H_p)$ is independent if and only if $H_1, \ldots, H_p$ are distinct hyperplanes. Hence if S is dependent then $e_S = 0$. Thus $I = 0$ and $A = E$.*

An Acyclic Complex

It is convenient to introduce some more notation. If $S = (H_1, \ldots, H_p)$ we say that $H_i \in S$. If T is a subsequence of S we write $T \subseteq S$. If $T = (K_1, \ldots, K_q)$ we write $(S,T) = (H_1, \ldots, H_p, K_1, \ldots, K_q)$. Thus $e_{(S,T)} = e_S e_T$ and in particular for $H \in \mathcal{A}$ we have $e_{(H,S)} = e_H e_S$.

Lemma 8.10 *The map $\partial : E \to E$ satisfies:*
(1) $\partial^2 = 0$,
(2) If $u \in E_p$ and $v \in E$ then $\partial(uv) = (\partial u)v + (-1)^p u(\partial v)$.

Proof. Part (1) is the standard boundary formula. It suffices to check (2) for $u = e_S$ and $v = e_T$ for $S, T \in \mathbf{S}$, where it follows by direct computation. □

Note that this lemma has nothing to do with arrangements. It states two familiar properties of the exterior algebra. Since the map ∂ is homogeneous of degree -1, we see from (1) that (E, ∂) is a chain complex. Part (2) says that ∂ is a derivation of the exterior algebra. It may be characterized as the unique derivation of E with $\partial e_H = 1$.

Lemma 8.11 $\partial_E I \subseteq I$.

Proof. Recall that I is a $\mathcal{K}$-linear combination of elements of the form $e_T \partial e_S$ where $T, S \in \mathbf{S}$ and S is dependent. We have

$$\partial(e_T \partial e_S) = (\partial e_T)(\partial e_S) \pm e_T(\partial^2 e_S) = (\partial e_T)(\partial e_S) \in I. \ \square$$

Definition 8.12 *Since $\partial_E I \subseteq I$, we may define $\partial_A : A \to A$ by $\partial_A \varphi u = \varphi \partial_E u$ for $u \in E$.*

Lemma 8.13 *The map $\partial_A : A \to A$ satisfies*
(1) $\partial_A^2 = 0$,
(2) If $a \in A_p$ and $b \in A$ then $\partial_A(ab) = (\partial_A a)b + (-1)^p a(\partial_A b)$.
(3) If $\mathcal{A}$ is not empty then the chain complex (A, ∂_A) is acyclic.

Proof. Parts (1) and (2) follow from the corresponding facts for ∂_E. Since ∂_A is homogeneous of degree -1, (A, ∂_A) is a chain complex. It follows from (1) that $\mathrm{im}\partial_A \subseteq \ker\partial_A$. To prove that the complex is acyclic we must show the reverse inclusion. Since $\mathcal{A}$ is not empty, we may choose $H \in \mathcal{A}$. Let $v = e_H$. Then $\partial_E v = 1$. Let $b = \varphi v$ and let $a \in A$. Choose $u \in E$ with $\varphi u = a$. Then $\partial_E(vu) = (\partial_E v)u - v(\partial_E u) = u - v(\partial_E u)$. Since $\varphi\partial_E = \partial_A \varphi$ and φ is a $\mathcal{K}$-algebra homomorphism, applying φ to the first and last terms gives $a = \partial_A(ba) + b\partial_A a$ for all $a \in A$. Thus $\mathrm{im}\partial_A \supseteq \ker\partial_A$. □

Next we study the ideal I and return to the notation $\partial = \partial_E$.

Definition 8.14 *Let $J = J(\mathcal{A})$ be the submodule of E spanned over $\mathcal{K}$ by all e_S such that $S \in \mathbf{S}$ is dependent.*

Lemma 8.15 *J is an ideal of E and $I = J + \partial J$.*

Proof. If $T \in \mathbf{S}$ is dependent then (S,T) is dependent for all $S \in \mathbf{S}$. Thus $e_S e_T = e_{(S,T)} \in J$ and hence J is an ideal. The formula $e_S = e_H \partial e_S$ when $H \in S$ applied to a dependent S shows that $J \subseteq I$. The definitions of J and I imply that $\partial J \subseteq I$. Thus $J + \partial J \subseteq I$. For the reverse inclusion note that $J + \partial J$ contains the generators of I. It suffices to show that $J + \partial J$ is an ideal. Since J is an ideal, it is enough to show that $e_H \partial e_S \in J + \partial J$ when $H \in \mathcal{A}$ and $S \in \mathbf{S}$ is dependent. Since (H,S) is also dependent, this follows from the formula

$$e_H \partial e_S = e_S - \partial(e_H e_S) = e_S - \partial e_{(H,S)}. \ \square$$

The Structure of $A(\mathcal{A})$

We decompose the algebra E into a direct sum indexed by elements of L.

Definition 8.16 *For $X \in L$ let $\mathbf{S}_X = \{S \in \mathbf{S} \mid \cap S = X\}$ and let*

$$E_X = \sum_{S \in \mathbf{S}_X} \mathcal{K} e_S.$$

Note that $e_S \in E_{\cap S}$ for all $S \in \mathbf{S}$.

Lemma 8.17 *Since $\mathbf{S} = \bigcup_{X \in L} \mathbf{S}_X$ is a disjoint union, $E = \bigoplus_{X \in L} E_X$ is a direct sum.* □

Our next aim is to show that the algebra A has an analogous direct sum decomposition.

Definition 8.18 *Let $\pi_X : E \to E_X$ be the projection. Thus*

$$\pi_X e_S = \begin{cases} e_S & \text{if } \cap S = X \\ 0 & \text{otherwise.} \end{cases}$$

The next result follows from Lemma 8.17.

Lemma 8.19 *If F is a submodule of E write $F_X = F \cap E_X$. If $\pi_X(F) \subseteq F$ for all $X \in L$ then $\pi_X(F) = F_X$ and $F = \oplus_{X\in L} F_X$.* □

Lemma 8.20 $J = \bigoplus_{X\in L} J_X$.

Proof. Since J is spanned by elements e_S where $S \in \mathbf{S}$ is dependent, it follows from the definition of π_X that $\pi_X(J) \subseteq J$. The result follows from Lemma 8.19. □

Definition 8.21 *Let $J' = J'(\mathcal{A})$ be the submodule of E spanned by all e_S where $S \in \mathbf{S}$ is independent. Thus $E = J \oplus J'$. Let $\pi = \pi_{\mathcal{A}} : E \to J'$ be the projection which annihilates J. Let $K = K(\mathcal{A}) = \pi(\partial J)$.*

Lemma 8.22 $I = J \oplus K$.

Proof. The map $1 - \pi : E \to J$ is the projection which annihilates J'. Since $J \subseteq I$ we have $(1-\pi)I = J$. It follows from Lemma 8.15 that $\pi(I) = \pi(J + \partial J) = \pi(\partial J) = K$. From $(1-\pi)I = J \subseteq I$ we get $\pi I \subseteq I$ and hence $I = (1-\pi)I \oplus \pi I = J \oplus K$. □

Lemma 8.23 $K = \oplus_{X\in L} K_X$.

Proof. By Lemma 8.19 it suffices to show that $\pi_X K \subseteq K$ for all $X \in L$. By Lemma 8.20 we have $K = \pi(\partial J) = \sum_{Y\in L} \pi(\partial J_Y)$. The module ∂J_Y is spanned by elements ∂e_S where S is dependent and $\cap S = Y$. Let $S = (H_1, \ldots, H_p)$ and let $S_k = (H_1, \ldots, \hat{H}_k, \ldots, H_p)$. Then $\partial e_S = \sum_{k=1}^{p} (-1)^{k-1} e_{S_k}$. If S_k is dependent then $\pi e_{S_k} = 0$. If S_k is independent then $\cap S_k = \cap S = Y$ because S is dependent. This gives $\pi(\partial e_S) \in E_Y$ so $\pi(\partial J_Y) \subseteq E_Y$. For $Y \neq X$ we have $\pi_X(E_Y) = 0$. Thus $\pi_X(K) = \pi(\partial J_X) \subseteq \pi(\partial J) = K$. □

Proposition 8.24 $I = \bigoplus_{X\in L} I_X$.

Proof. Recall that $I = J \oplus K$. We showed $\pi_X(J) \subseteq J$ in Lemma 8.20 and $\pi_X(K) \subseteq K$ in Lemma 8.23. Thus $\pi_X I \subseteq I$. The conclusion follows from Lemma 8.19. □

Definition 8.25 *If $X \in L$ let $A_X = \varphi(E_X)$.*

Theorem 8.26 *Let $\mathcal{A}$ be a central arrangement and let $A = A(\mathcal{A})$. Then*

$$A = \bigoplus_{X\in L} A_X.$$

Proof. Since $E = \oplus E_X$, $A = \sum A_X$. Proposition 8.24 shows that the sum is direct. □

Corollary 8.27 *Let $\mathcal{A}$ be a central arrangement. Then $A_p = \bigoplus_{X\in L_p} A_X$.*

Proof. Suppose $a \in A_X$ where $X \in L_p$. Write $a = \varphi(u)$ where $u \in E_X$. Write $u = \sum_{S\in\mathbf{S}_X} c_S e_S$ where $c_S \in \mathbb{K}$. If $S \in \mathbf{S}_X$ is dependent then $e_S \in I$ and $\varphi(e_S) = 0$. If S is independent then $r(\cap S) = r(X) = p$ implies $e_S \in E_p$ and $\varphi(e_S) \in A_p$. Thus $a = \varphi(u) \in A_p$ and hence

$$\sum_{X\in L_p} A_X \subseteq A_p.$$

Conversely, suppose $a \in A_p$ and write $a = \varphi(u)$ where $u \in E_p$. Write $u = \sum_{S\in\mathbf{S}_p} c_S e_S$ where $c_S \in \mathbb{K}$. If $S \in \mathbf{S}_p$ is dependent then $\varphi(e_S) = 0$. If $S \in \mathbf{S}_p$ is independent let $X = \cap S$. Then $r(X) = p$ and $e_S \in E_X$ implies $\varphi(e_S) \in A_X$. Thus

$$A_p \subseteq \sum_{X\in L_p} A_X.$$

The sum is direct by Theorem 8.26. □

The Injective Map $A(\mathcal{A}_X) \to A(\mathcal{A})$

If $\mathcal{B}$ is a subarrangement of $\mathcal{A}$ then we view $E(\mathcal{B})$ as a subalgebra of $E(\mathcal{A})$ and $L(\mathcal{B})$ as a sublattice of $L(\mathcal{A})$. Note that $\mathbf{S}(\mathcal{B}) \subseteq \mathbf{S}(\mathcal{A})$ and an element $S \in \mathbf{S}(\mathcal{B})$ is dependent viewed in $\mathbf{S}(\mathcal{B})$ if and only if it is dependent in $\mathbf{S}(\mathcal{A})$. The map $\partial_{E(\mathcal{B})}$ is the restriction of $\partial_{E(\mathcal{A})}$ to $E(\mathcal{B})$. Since $J(\mathcal{B}) \subseteq J(\mathcal{A})$ and $J'(\mathcal{B}) \subseteq J'(\mathcal{A})$, the projection $\pi_{\mathcal{B}}$ of $E(\mathcal{B})$ onto $J'(\mathcal{B})$ is the restriction to $E(\mathcal{B})$ of the projection $\pi_{\mathcal{A}}$ of $E(\mathcal{A})$ onto $J'(\mathcal{A})$. For simplicity we let ∂ denote both $\partial_{E(\mathcal{A})}$ and $\partial_{E(\mathcal{B})}$ and we let π denote both $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$. Thus $K(\mathcal{B}) = \pi(\partial J(\mathcal{B})) \subseteq \pi(\partial J(\mathcal{A})) = K(\mathcal{A})$. It is convenient to agree that undefined modules are zero. It follows that $K_X(\mathcal{B}) \subseteq K_X(\mathcal{A})$ for all $X \in L(\mathcal{A})$, since by this convention $K_X(\mathcal{B}) = 0$ if $X \notin L(\mathcal{B})$.

Clearly $I(\mathcal{B}) \subseteq I(\mathcal{A}) \cap E(\mathcal{B})$ for any subarrangement $\mathcal{B}$ of $\mathcal{A}$. In [142, Lemma 2.14] it was asserted that

(1) $$I(\mathcal{B}) = I(\mathcal{A}) \cap E(\mathcal{B})$$

for any subarrangement $\mathcal{B}$ of $\mathcal{A}$. The proof given there is correct if $\mathcal{B} = \mathcal{A}_X$ for $X \in L(\mathcal{A})$ and this is the only case used in the rest of [142]. The mistake in [142, Lemma 2.14] lies in the claim that $E_X(\mathcal{A}) \subseteq E(\mathcal{B})$ if $X \in L(\mathcal{B})$. This was pointed out by George Glauberman, W. A. M. Janssen and Nguyễn Việt Dũng. Although this claim is false, equation (1) holds for all subarrangements $\mathcal{B}$ of $\mathcal{A}$. We will prove (1) in Proposition 9.22. Here we establish it in case $\mathcal{B} = \mathcal{A}_X$.

Lemma 8.28 *If $X \in L(\mathcal{A})$ then $I(\mathcal{A}_X) = I(\mathcal{A}) \cap E(\mathcal{A}_X)$.*

Proof. The inclusion $\subseteq$ is obvious. Suppose $Y \in L(\mathcal{A}_X)$. If $S \in \mathbf{S}_Y(\mathcal{A})$ then $S = (H_1, \ldots, H_p)$ with $\cap S = Y$. Thus $Y \subseteq H_k$ for $1 \le k \le p$. Since $Y \in L(\mathcal{A}_X)$ we have $X \subseteq Y$ and hence $X \subseteq H_k$ for $1 \le k \le p$. Thus $S \in \mathbf{S}_Y(\mathcal{A}_X)$. This gives $\mathbf{S}_Y(\mathcal{A}) \subseteq \mathbf{S}_Y(\mathcal{A}_X)$. Since $\mathcal{A}_X \subseteq \mathcal{A}$, we have $\mathbf{S}_Y(\mathcal{A}) = \mathbf{S}_Y(\mathcal{A}_X)$. Thus $E_Y(\mathcal{A}) = E_Y(\mathcal{A}_X)$ and $J_Y(\mathcal{A}) = J_Y(\mathcal{A}_X)$.

It follows from the argument of Lemma 8.23 that $K_Y(\mathcal{A}) = \pi(\partial J_Y(\mathcal{A})) = \pi(\partial J_Y(\mathcal{A}_X)) = K_Y(\mathcal{A}_X)$. Since $K_Y(\mathcal{A}_X) \subseteq E(\mathcal{A}_X)$, this shows

$$I(\mathcal{A}) \cap E(\mathcal{A}_X) \subseteq J(\mathcal{A}_X) \oplus (\bigoplus_{Y \in L(\mathcal{A}_X)} K_Y(\mathcal{A}_X)).$$

If we apply Lemmas 8.22 and 8.23 to $\mathcal{A}_X$ we see that the right side is $I(\mathcal{A}_X)$. □

Definition 8.29 *Let $\mathcal{B}$ be a subarrangement of $\mathcal{A}$. Since $I(\mathcal{B}) \subseteq I(\mathcal{A}) \cap E(\mathcal{B})$, the inclusion $E(\mathcal{B}) \subseteq E(\mathcal{A})$ induces a $\mathcal{K}$-algebra homomorphism $i : A(\mathcal{B}) \to A(\mathcal{A})$ such that for $H \in \mathcal{A}$*

$$i(e_H + I(\mathcal{B})) = e_H + I(\mathcal{A}).$$

Note that i is a monomorphism precisely when (1) holds.

The next result follows from Lemma 8.28.

Proposition 8.30 *The map i is a monomorphism for $\mathcal{B} = \mathcal{A}_X$.* □

Proposition 8.31 *Let $\mathcal{A}$ be a central arrangement. If $Y \leq X$ then $A_Y(\mathcal{A}_X) \simeq A_Y(\mathcal{A})$.*

Proof. Let $i : A(\mathcal{A}_X) \to A(\mathcal{A})$ be the homomorphism of Definition 8.29. It is a monomorphism by Lemma 8.28. The module $A_Y(\mathcal{A}) = \varphi(E_Y(\mathcal{A}))$ is spanned over $\mathcal{K}$ by all elements $e_S + I(\mathcal{A})$ with $S \in \mathbf{S}_Y(\mathcal{A})$. Similarly $A_Y(\mathcal{A}_X)$ is spanned over $\mathcal{K}$ by all elements $e_S + I(\mathcal{A}_X)$ with $S \in \mathbf{S}_Y(\mathcal{A}_X)$. Since $\mathbf{S}_Y(\mathcal{A}) = \mathbf{S}_Y(\mathcal{A}_X)$, we have $i(A_Y(\mathcal{A}_X)) = A_Y(\mathcal{A})$. Since i is a monomorphism, this completes the proof. □

The Broken Circuit Basis

Next we show that the $\mathcal{K}$-algebra $A(\mathcal{A})$ is a free $\mathcal{K}$-module by constructing a standard $\mathcal{K}$-basis for $A(\mathcal{A})$. These results can be extended to geometric lattices. For the more general results see [36] and [104].

We introduce an arbitrary linear order $\prec$ in $\mathcal{A}$. Call a p-tuple $S = (H_1, \ldots, H_p)$ **standard** if $H_1 \prec \ldots \prec H_p$. Note that $E = E(\mathcal{A})$ has a $\mathcal{K}$-basis consisting of all e_S with standard S.

Definition 8.32 *A p-tuple $S = (H_1, \ldots, H_p)$ is a* **circuit** *if it is minimally dependent. Thus $(H_1, \ldots, H_p)$ is dependent, but for $1 \leq k \leq p$ the $(p-1)$-tuple $(H_1, \ldots, \hat{H}_k, \ldots, H_p)$ is independent.*

Definition 8.33 *Given $S = (H_1, \ldots, H_p)$ let $\max S$ be the maximal element of S in the linear order $\prec$ in $\mathcal{A}$.*

Definition 8.34 *A standard p-tuple $S \in \mathbf{S}$ is a* **broken circuit** *if there exists $H \in \mathcal{A}$ such that $\max S \prec H$ and (S, H) is a circuit.*

Definition 8.35 *A standard p-tuple S is called* **χ-independent** *if it does not contain any broken circuit. Define $\mathbf{C}_p = \{S \in \mathbf{S}_p \mid S \text{ is standard and } \chi\text{-independent}\}$. Let $\mathbf{C} = \cup_{p \geq 0} \mathbf{C}_p$.*

Definition 8.36 *The* **broken circuit module** *$C = C(\mathcal{A})$ is defined as follows. Let $C_0 = \mathcal{K}$, and for $p \geq 1$ let C_p be the free $\mathcal{K}$-module with basis $\{e_S \in E \mid S \in \mathbf{C}_p\}$. Let $C = C(\mathcal{A}) = \oplus_{p \geq 0} C_p$. Then $C(\mathcal{A})$ is a free graded $\mathcal{K}$-module.*

It is clear that every broken circuit is obtained by deleting the maximal element in a standard circuit. Note that if S is χ-independent then S is independent. Thus every $S \in \mathbf{C}$ is independent. By definition $C(\mathcal{A})$ is a submodule of $E(\mathcal{A})$. In general $C(\mathcal{A})$ is not closed under multiplication in $E(\mathcal{A})$ so $C(\mathcal{A})$ is not a subalgebra. Recall the natural projection $\varphi : E(\mathcal{A}) \to A(\mathcal{A})$ and let $\psi : C(\mathcal{A}) \to A(\mathcal{A})$ be its restriction. Our aim is to show that ψ is an isomorphism of graded modules.

Example 8.37 *Define $\mathcal{A}$ by $Q(\mathcal{A}) = xyz(x+y)(x+y-z)$. Let $H_0 = \ker(x+y-z)$, $H_1 = \ker(x)$, $H_2 = \ker(y)$, $H_3 = \ker(z)$, and $H_4 = \ker(x+y)$. Define the linear order on $\mathcal{A}$ by $H_i \prec H_j \iff i < j$.*

The standard circuits are (H_0, H_1, H_2, H_3), (H_0, H_3, H_4), and (H_1, H_2, H_4). Thus the broken circuits are (H_0, H_1, H_2), (H_0, H_3), and (H_1, H_2). Writing $e_i = e_{H_i}$ we get the following basis for $C(\mathcal{A})$:

$$1$$
$$e_0,\ e_1,\ e_2,\ e_3,\ e_4$$
$$e_0e_1,\ e_0e_2,\ e_0e_4,\ e_1e_3,\ e_1e_4,\ e_2e_3,\ e_2e_4,\ e_3e_4$$
$$e_0e_1e_4,\ e_0e_2e_4,\ e_1e_3e_4,\ e_2e_3e_4$$

Definition 8.38 *Recall that for $S = (H_1, \ldots, H_p)$ we write $\cap S = H_1 \cap \cdots \cap H_p$ and that $E_X = \sum_{\cap S = X} \mathcal{K} e_S$. Let $C_X(\mathcal{A}) = C_X = C \cap E_X$. Then each C_X is a free $\mathcal{K}$-module.*

Lemma 8.39 *For $p \geq 0$ we have $C_p = \oplus_{X \in L_p} C_X$ and hence $C = \oplus_{X \in L} C_X$.*

Proof. If $S \in \mathbf{S}$ is χ-independent with $|S| = p$ then S is independent. If $Y = \cap S$ then $r(Y) = p$ and $e_S \in \oplus_{X \in L_p} C_X$. Conversely, if $e_S \in C_X$ and $r(X) = p$ then $|S| = p$ so $e_S \in C_p$. □

Lemma 8.40 *If $Y \leq X$ then $C_Y(\mathcal{A}_X) = C_Y(\mathcal{A})$.*

Proof. Let $S \in \mathbf{S}(\mathcal{A}_X) \subseteq \mathbf{S}(\mathcal{A})$. It suffices to show that S is a broken circuit of $\mathcal{A}_X$ if and only if S is a broken circuit of $\mathcal{A}$. First observe that S is dependent in $\mathcal{A}_X$ if and only if S is dependent in $\mathcal{A}$. Thus S is a circuit of $\mathcal{A}_X$ if and only if S is a circuit of $\mathcal{A}$. Suppose S is a broken circuit of $\mathcal{A}_X$. Then S is obtained by removing the maximal element of a standard circuit of $\mathcal{A}_X$. Since the latter is also a standard circuit of $\mathcal{A}$, S is a broken circuit of $\mathcal{A}$. Conversely, suppose S is a broken circuit of $\mathcal{A}$. Then there exists $H \in \mathcal{A}$ such that $\max S \prec H$ and (S, H) is a circuit. Since S is independent and (S, H) is dependent, we have $\cap S = \cap S \cap H$. Thus $X \geq \cap S \geq H$ and $(S, H) \in \mathbf{S}_X$. It follows that (S, H) is a circuit of $\mathcal{A}_X$ and S is a broken circuit of $\mathcal{A}_X$. □

Lemma 8.41 *Let H_n be the maximal element of $\mathcal{A}$ under $\prec$ and write $e_n = e_{H_n}$. Then $e_n C \subseteq C$ so C is closed under multiplication by e_n.*

Proof. Since a broken circuit is obtained from a standard circuit by deleting the maximal element, no broken circuit has the form (S, H_n). □

Lemma 8.42 *Suppose $\mathcal{A}$ is not empty. Let ∂_C denote the restriction of the map $\partial : E \to E$ to C. Then $\partial_C(C) \subseteq C$ and (C, ∂_C) is an acyclic complex.*

Proof. Deleting an element of a χ-independent p-tuple results in a χ-independent $(p-1)$-tuple. This shows that $\partial_C(C) \subseteq C$. Suppose $c \in C$ and $\partial_C c = 0$. By Lemma 8.41 $e_n c \in C$ and

$$c = c - e_n(\partial_C c) = \partial_C(e_n c) \in \partial_C C.$$

This shows that the complex is acyclic. □

Theorem 8.43 *For each $X \in L$ the restriction $\psi_X : C_X(\mathcal{A}) \to A_X(\mathcal{A})$ is an isomorphism. The map $\psi : C(\mathcal{A}) \to A(\mathcal{A})$ is an isomorphism of graded $\mathcal{K}$-modules. The set $\{e_S + I \in A(\mathcal{A}) \mid S$ is standard and χ-independent$\}$ is a basis for $A(\mathcal{A})$ as a graded $\mathcal{K}$-module.*

Proof. Clearly $\psi(C_X) \subseteq A_X$ so ψ induces a map $\psi_X : C_X \to A_X$. It suffices to show that this map is an isomorphism for all $X \in L(\mathcal{A})$. We use induction on $r = r(\mathcal{A})$. The assertion holds for the empty arrangement with $r = 0$ and $C(\mathcal{A}) = \mathcal{K} = A(\mathcal{A})$. Suppose $r > 0$. Let $X \in L(\mathcal{A})$ with $r(X) < r$. Then $r(\mathcal{A}_X) < r$ so by the induction hypothesis $\psi_X : C_X(\mathcal{A}_X) \to A_X(\mathcal{A}_X)$ is an isomorphism. We see from Proposition 8.31 that $A_X(\mathcal{A}_X) \simeq A_X(\mathcal{A})$ and from Lemma 8.40 that $C_X(\mathcal{A}_X) = C_X(\mathcal{A})$. It follows from the commutativity of the diagram

$$\begin{array}{ccc} C_X(\mathcal{A}_X) & \xrightarrow{\psi_X(\mathcal{A}_X)} & A_X(\mathcal{A}_X) \\ \downarrow & & \downarrow \\ C_X(\mathcal{A}) & \xrightarrow{\psi_X(\mathcal{A})} & A_X(\mathcal{A}) \end{array}$$

that $\psi_X(\mathcal{A})$ is an isomorphism for $X \in L$ with $r(X) < r$. Since $\mathcal{A}$ is central, it has a unique maximal element $T = T(\mathcal{A})$ of rank r. It remains to prove the isomorphism for $X = T$. In the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & C_r & \to & C_{r-1} & \to & \dots & \to & C_0 & \to & 0 \\ & & \psi_r \downarrow & & \psi_{r-1} \downarrow & & & & \psi_0 \downarrow & & \\ 0 & \to & A_r & \to & A_{r-1} & \to & \dots & \to & A_0 & \to & 0 \end{array}$$

the horizontal maps are the respective boundary operators in the two acyclic complexes, so the sequences are exact. Since $A_p = \oplus_{X \in L_p} A_X$ and $C_p = \oplus_{X \in L_p} C_X$, the first part of the argument shows that all vertical maps except ψ_r are isomorphisms. It follows from the diagram that ψ_r is an isomorphism. This completes the argument because $C_r = C_T(\mathcal{A})$, $A_r = A_T(\mathcal{A})$ and $\psi_r = \psi_T(\mathcal{A})$. The remaining assertions follow from Theorem 8.25 and Lemma 8.39. □

Corollary 8.44 *The algebra $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module. The $\mathcal{K}$-modules $A_X(\mathcal{A})$ for $X \in L$ and $A_p(\mathcal{A})$ for $p \geq 0$ are also free.*

Proof. The $\mathcal{K}$-modules $C_X(\mathcal{A})$ are free by definition. It follows from Theorem 8.43 that $C_X(\mathcal{A}) \simeq A_X(\mathcal{A})$. Thus $A_X(\mathcal{A})$ is a free $\mathcal{K}$-module. We showed in Corollary 8.27 that $A_p = \oplus_{X \in L_p} A_X$. Thus A_p is also free. □

9 The Algebra $A(\mathcal{A})$ for Affine Arrangements

In this section we generalize the constructions and results of the last section to affine arrangements. Our main tool is the interplay between the affine ℓ-arrangement $\mathcal{A}$ and the central $(\ell+1)$-arrangement $\mathbf{c}\mathcal{A}$. We recall the basic notation and properties of the coning construction from Definition 2.15 and Proposition 4.17. Let $Q(\mathcal{A}) \in S = \mathbb{K}[x_1,\dots,x_\ell]$ be a defining polynomial of $\mathcal{A}$ and let $Q' \in \mathbb{K}[x_0,x_1,\dots,x_\ell]$ be the polynomial $Q(\mathcal{A})$ homogenized. Then $\mathbf{c}\mathcal{A}$ is a central $(\ell+1)$-arrangement with defining polynomial $Q(\mathbf{c}\mathcal{A}) = x_0 Q'$. The cone $\mathbf{c}\mathcal{A}$ consists of the hyperplane $K_0 = \ker(x_0)$ together with $\{\mathbf{c}H \mid H \in \mathcal{A}\}$, where $\mathbf{c}H$ is the cone over the affine hyperplane H. If $H \in \mathcal{A}$ is the kernel of the degree 1 polynomial $\alpha_H \in \mathbb{K}[x_1,\dots,x_\ell]$ then $\mathbf{c}H \in \mathbf{c}\mathcal{A}$ is the kernel of the linear form $\alpha_{\mathbf{c}H}$ obtained by homogenizing α_H in $\mathbb{K}[x_0,x_1,\dots,x_\ell]$. For example, if $\alpha_H = x_1 + x_2 - 1$ then $\alpha_{\mathbf{c}H} = x_1 + x_2 - x_0$.

Constructon of $A(\mathcal{A})$

Let $\mathcal{K}$ be a commutative ring. The first definitions are the same as in the central case. Define a $\mathcal{K}$-module $E_1(\mathcal{A})$ which has a $\mathcal{K}$-basis consisting of elements e_H in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. Let

$$E(\mathcal{A}) = \Lambda(E_1(\mathcal{A}))$$

be the exterior algebra of E_1. Let $\mathbf{S}_p(\mathcal{A})$ denote the set of all p-tuples $(H_1,\dots,H_p)$ of hyperplanes in $\mathcal{A}$. Define $\mathbf{S}(\mathcal{A}) = \cup_{p\geq 0}\mathbf{S}_p(\mathcal{A})$. For $S = (H_1,\dots,H_p) \in \mathbf{S}(\mathcal{A})$, define $e_S = e_{H_1}\cdots e_{H_p} \in E(\mathcal{A})$. For $S \in \mathbf{S}(\mathcal{A})$ define the p-tuple $\mathbf{c}S \in \mathbf{S}(\mathbf{c}\mathcal{A})$ of hyperplanes in $\mathbf{c}\mathcal{A}$ by $\mathbf{c}S = (\mathbf{c}H_1,\dots,\mathbf{c}H_p)$. Write $e_0 = e_{K_0} \in E(\mathbf{c}\mathcal{A})$. Then $E(\mathbf{c}\mathcal{A})$ has a $\mathcal{K}$-basis

$$\{e_0 e_{\mathbf{c}S} \mid S \in \mathbf{S}(\mathcal{A})\} \cup \{e_{\mathbf{c}S} \mid S \in \mathbf{S}(\mathcal{A})\}.$$

Given $S = (H_1,\dots,H_p) \in \mathbf{S}(\mathcal{A})$, recall that $\cap S = H_1 \cap \cdots \cap H_p$. The crucial difference between central and affine arrangements is that here $\cap S$ may be empty. Since K_0 is sent to infinity in the deconing, $\cap S = \emptyset$ if and only if $\cap(\mathbf{c}S) \subseteq K_0$.

Definition 9.1 *Let $\mathcal{A}$ be an affine arrangement. We say that S is* **dependent** *if $\cap S \neq \emptyset$ and $r(\cap S) = \operatorname{codim} \cap S < |S|$. Let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ generated by*

$$\{e_S \mid \cap S = \emptyset\} \cup \{\partial e_S \mid S \text{ is dependent}\}.$$

Define the algebra $A(\mathcal{A})$ by

$$A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A}).$$

Example 9.2 *Recall the affine 2-arrangement $\mathcal{A}$ defined by $Q(\mathcal{A}) = xy(x+y-1)$ in Example 2.6. Let $H_1 = \ker(x)$, $H_2 = \ker(y)$, and $H_3 = \ker(x+y-1)$.*

Note that $H_1 \cap H_2 \cap H_3 = \emptyset$. Write $e_i = e_{H_i}$ and $a_i = e_i + I(\mathcal{A}) \in A(\mathcal{A})$. Then $e_1e_2e_3 \in I(\mathcal{A})$ and thus $a_1a_2a_3 = 0$. The ideal $I(\mathcal{A})$ is generated by $e_1e_2e_3$. We have

$$A(\mathcal{A}) = \mathcal{K} \oplus (\mathcal{K}a_1 \oplus \mathcal{K}a_2 \oplus \mathcal{K}a_3) \oplus (\mathcal{K}a_1a_2 \oplus \mathcal{K}a_2a_3 \oplus \mathcal{K}a_3a_1).$$

Lemma 9.3 *Let $S \in \mathbf{S}(\mathcal{A})$.*
(1) Assume $\cap S \neq \emptyset$. Then S is dependent if and only if $\mathbf{c}S$ is dependent.
(2) The $(p+1)$-tuple $(K_0, \mathbf{c}S)$ is dependent if and only if either $\cap S = \emptyset$ or S is dependent.

Proof. If $\cap S \neq \emptyset$ then $r(\cap S) = r(\cap(\mathbf{c}S))$. This proves (1). If $\cap S = \emptyset$ then $\cap(\mathbf{c}S) \subseteq K_0$. Thus $(K_0, \mathbf{c}S)$ is dependent. If S is dependent then $\mathbf{c}S$ is dependent and so is $(K_0, \mathbf{c}S)$. For the converse, suppose that $(K_0, \mathbf{c}S)$ is dependent. If we assume that $\cap S \neq \emptyset$ and that S is independent we derive a contradiction as follows. Since S is independent, $\mathbf{c}S$ is independent by (1). Since $(K_0, \mathbf{c}S)$ is dependent, $\cap(\mathbf{c}S) \subseteq K_0$ and hence $\cap S = \emptyset$. □

We define maps in both directions between $E(\mathcal{A})$ and $E(\mathbf{c}\mathcal{A})$.

Definition 9.4 *Let $S \in \mathbf{S}(\mathcal{A})$. Define a $\mathcal{K}$-algebra homomorphism*

$$s : E(\mathbf{c}\mathcal{A}) \to E(\mathcal{A}) \;\; by \;\; s(e_0 e_{\mathbf{c}S}) = 0, \;\; s(e_{\mathbf{c}S}) = e_S.$$

Define a $\mathcal{K}$-linear homomorphism

$$t : E(\mathcal{A}) \to E(\mathbf{c}\mathcal{A}) \;\; by \;\; t(e_S) = e_0 e_{\mathbf{c}S}.$$

Lemma 9.5 *We have $s(I(\mathbf{c}\mathcal{A})) \subseteq I(\mathcal{A})$. It follows that s induces a $\mathcal{K}$-algebra homomorphism $s : A(\mathbf{c}\mathcal{A}) \to A(\mathcal{A})$.*

Proof. Let $S = (H_1,\dots,H_p) \in \mathbf{S}(\mathcal{A})$. It follows from Definition 8.5 that the ideal $I(\mathbf{c}\mathcal{A})$ is generated by

$$\{\partial(e_0 e_{\mathbf{c}S}) \mid (K_0, \mathbf{c}S) \text{ is dependent}\} \cup \{\partial(e_{\mathbf{c}S}) \mid \mathbf{c}S \text{ is dependent}\}.$$

Case 1. If $(K_0, \mathbf{c}S)$ is dependent and $\cap S = \emptyset$ then we have $s(\partial(e_0 e_{\mathbf{c}S})) = s(e_{\mathbf{c}S}) = e_S \in I(\mathcal{A})$.

Case 2. If $(K_0, \mathbf{c}S)$ is dependent and $\cap S \neq \emptyset$ then S is dependent by Lemma 9.3.2. Thus we have $s(\partial(e_0 e_{\mathbf{c}S})) = e_S = e_{H_1}(\partial e_S) \in I(\mathcal{A})$.

Case 3. If $\mathbf{c}S$ is dependent and $\cap S \neq \emptyset$ then S is dependent by Lemma 9.3.1. Thus we have $s(\partial e_{\mathbf{c}S}) = \partial e_S \in I(\mathcal{A})$.

Case 4. Assume that $\mathbf{c}S$ is dependent and $\cap S = \emptyset$. Let $S_k = (H_1,\dots,\hat{H}_k,\dots,H_p)$ for $k = 1,\dots,p$. If $\cap S_k = \emptyset$ then $e_{S_k} \in I(\mathcal{A})$. If $\cap S_k \neq \emptyset$ then $\cap(\mathbf{c}S_k) \not\subseteq K_0 \supseteq \cap(\mathbf{c}S)$. Thus $\cap(\mathbf{c}S)$ is a proper subspace of $\cap(\mathbf{c}S_k)$. So $\mathbf{c}S_k$ is dependent and $e_{S_k} \in I(\mathcal{A})$ by Case 2. Then we have $s(\partial e_{\mathbf{c}S}) = \partial e_S = \sum_k (-1)^{k-1} e_{S_k} \in I(\mathcal{A})$. □

Lemma 9.6 *We have $t(I(\mathcal{A})) \subseteq I(\mathbf{c}\mathcal{A})$. It follows that t induces a $\mathcal{K}$-linear homomorphism $t : A(\mathcal{A}) \to A(\mathbf{c}\mathcal{A})$.*

Proof. *Case 1.* If S satisfies $\cap S = \emptyset$ then $(K_0, \mathbf{c}S)$ is dependent by Lemma 9.3.2. So we have $t(e_S) = e_0 e_{cS} \in I(\mathbf{c}\mathcal{A})$.

Case 2. If S is dependent then $\mathbf{c}S$ is dependent by Lemma 9.3.1. We have $t(\partial e_S) = e_0\partial(e_{cS}) \in I(\mathbf{c}\mathcal{A})$. □

Note that $st = 0$. Thus we have a complex

$$0 \to A(\mathcal{A}) \xrightarrow{t} A(\mathbf{c}\mathcal{A}) \xrightarrow{s} A(\mathcal{A}) \to 0.$$

We will prove that this is a short exact sequence by using the broken circuit basis.

The Broken Circuit Basis

We extend the construction of the broken circuit basis from central arrangements to affine arrangements. We introduce an arbitrary linear order $\prec$ in the affine arrangement $\mathcal{A}$. Call a p-tuple $S = (H_1, \ldots, H_p)$ **standard** if $H_1 \prec \ldots \prec H_p$. Note that $E = E(\mathcal{A})$ has a $\mathcal{K}$-basis consisting of all e_S with standard S. Recall that S is dependent if $\cap S \neq \emptyset$ and $r(\cap S) < |S|$. Next we generalize the notions of circuit and broken circuit of Definitions 8.32 and 8.34. A p-tuple is a **circuit** if it is minimally dependent. A standard p-tuple S is a **broken circuit** if there exists $H \in \mathcal{A}$ such that $\max S \prec H$ and (S, H) is a circuit.

Definition 9.7 *A standard p-tuple S is called* χ-**independent** *if $\cap S \neq \emptyset$ and it does not contain any broken circuit.*

Define a linear order $\prec$ in the central arrangement $\mathbf{c}\mathcal{A}$ by

(1) $\mathbf{c}H_1 \prec \mathbf{c}H_2$ if $H_1 \prec H_2$ for $H_1, H_2 \in \mathcal{A}$,

(2) K_0 is the maximal element in $\mathbf{c}\mathcal{A}$.

Lemma 9.8 *Let $S \in \mathbf{S}(\mathcal{A})$. The following three conditions are equivalent:*

(1) S is χ-independent,

(2) $\mathbf{c}S$ is χ-independent,

(3) $(\mathbf{c}S, K_0)$ is χ-independent.

Proof. Since K_0 is the maximal element in $\mathbf{c}\mathcal{A}$, (2) and (3) are equivalent. We show first that if (1) is false then either (2) or (3) must fail. Suppose that S is not χ-independent. Then either $\cap S = \emptyset$ or S contains a broken circuit. If $\cap S = \emptyset$ then $(\mathbf{c}S, K_0)$ is dependent by Lemma 9.3.2. This contradicts (3). If S contains a broken circuit then there exists $H \in \mathcal{A}$ with $\max S \prec H$ such that (S, H) is dependent. So $(\cap S) \cap H \neq \emptyset$. By Lemma 9.3.1 $(\mathbf{c}S, \mathbf{c}H)$ is dependent. Since $\max(\mathbf{c}S) \prec \mathbf{c}H$, $\mathbf{c}S$ is not χ-independent. This contradicts (2).

Next we show that (1) implies (2). Suppose that S is χ-independent. Then $\cap S \neq \emptyset$ and S is independent. It follows from Lemma 9.3.1 that $\mathbf{c}S$ is independent. Since $\cap S \neq \emptyset$, we have $\cap(\mathbf{c}S) \not\subseteq K_0$. If $\mathbf{c}S$ is not χ-independent then at least one of the following two conditions must be true:

(a) $(\mathbf{c}S, K_0)$ is dependent,

(b) $(\mathbf{c}S, \mathbf{c}H)$ is dependent for some $H \in \mathcal{A}$ with $\max S \prec H$.

Since $\mathbf{c}S$ is independent, (a) implies that $\cap(\mathbf{c}S) = \cap(\mathbf{c}S) \cap K_0 \subseteq K_0$. Thus $\cap S = \emptyset$, which is a contradiction. In case (b) we have $\cap(\mathbf{c}S) \cap \mathbf{c}H = \cap(\mathbf{c}S) \not\subseteq K_0$. Thus $(\cap S) \cap H \neq \emptyset$. It follows from Lemma 9.3.1 that (S, H) is dependent and thus S is not χ-independent. This is a contradiction. □

Definition 9.9 *The* **broken circuit module** *$C(\mathcal{A})$ is defined as follows. Let $C(\mathcal{A})$ be the free $\mathcal{K}$-module with basis $\{1\} \cup \{e_S \in E(\mathcal{A}) \mid S$ is χ-independent$\}$. Then $C(\mathcal{A})$ is a free graded $\mathcal{K}$-module.*

Proposition 9.10 *The following sequence is exact:*

$$0 \to C(\mathcal{A}) \xrightarrow{t} C(\mathbf{c}\mathcal{A}) \xrightarrow{s} C(\mathcal{A}) \to 0.$$

Proof. It is clear that t is injective. The implication (1) $\Rightarrow$ (3) in Lemma 9.8 shows that $t(C(\mathcal{A})) \subseteq C(\mathbf{c}\mathcal{A})$. The implication (2) $\Rightarrow$ (1) in Lemma 9.8 shows that $s(C(\mathbf{c}\mathcal{A})) \subseteq C(\mathcal{A})$. The implication (1) $\Rightarrow$ (2) in Lemma 9.8 shows the surjectivity of s. The implication (3) $\Rightarrow$ (1) in Lemma 9.8 shows $\ker(s) = t(C(\mathcal{A}))$. □

Theorem 9.11 *Let $\varphi : E(\mathcal{A}) \to A(\mathcal{A})$ be the natural homomorphism and $\psi : C(\mathcal{A}) \to A(\mathcal{A})$ be its restriction. The map $\psi : C(\mathcal{A}) \to A(\mathcal{A})$ is an isomorphism of graded $\mathcal{K}$-modules. The set $\{e_S + I \in A(\mathcal{A}) \mid S$ is standard and χ-independent$\}$ is a basis for $A(\mathcal{A})$ as a graded $\mathcal{K}$-module.*

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & C(\mathcal{A}) & \xrightarrow{t} & C(\mathbf{c}\mathcal{A}) & \xrightarrow{s} & C(\mathcal{A}) & \to & 0 \\ & & \psi\downarrow & & \psi\downarrow & & \psi\downarrow & & \\ 0 & \to & A(\mathcal{A}) & \xrightarrow{t} & A(\mathbf{c}\mathcal{A}) & \xrightarrow{s} & A(\mathcal{A}) & \to & 0. \end{array}$$

Note that $s : A(\mathbf{c}\mathcal{A}) \to A(\mathcal{A})$ is surjective and it follows from Theorem 8.43 that $\psi : C(\mathbf{c}\mathcal{A}) \to A(\mathbf{c}\mathcal{A})$ is an isomorphism. The top row is exact by Proposition 9.10. Diagram chasing shows that $\psi : C(\mathcal{A}) \to A(\mathcal{A})$ is an isomorphism. □

Corollary 9.12 *The algebra $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module.* □

Corollary 9.13 *The following sequence is exact:*

$$0 \to A(\mathcal{A}) \xrightarrow{t} A(\mathbf{c}\mathcal{A}) \xrightarrow{s} A(\mathcal{A}) \to 0. \quad \square$$

Let $\mathrm{Poin}(M, t)$ be the Poincaré polynomial of the free graded $\mathcal{K}$-module $M = \oplus_{p=0}^{\ell} M_p$:

$$\mathrm{Poin}(M, t) = \sum_{p=0}^{\ell} (\mathrm{rank}\, M_p) t^p.$$

In the commutative diagram above t is a homogeneous map of degree one and s is of degree zero in each row. This provides an analog of Proposition 6.4.

Corollary 9.14 $\mathrm{Poin}(A(\mathbf{c}\mathcal{A}), t) = (1 + t)\mathrm{Poin}(A(\mathcal{A}), t)$. □

Deletion and Restriction

Next we consider properties of the algebra A under deletion and restriction. Suppose that $\mathcal{A}$ is a nonempty affine arrangement. Let $H_0 \in \mathcal{A}$ be the distinguished hyperplane. Write $L = L(\mathcal{A})$, $L' = L(\mathcal{A}')$, $L'' = L(\mathcal{A}'')$ for the corresponding posets, and $A = A(\mathcal{A})$, $A' = A(\mathcal{A}')$, $A'' = A(\mathcal{A}'')$. We use similar notation like E, E', E'', I, I', I'', etc.

It is easy to see that $I' \subseteq I''$. Let $i : A' \to A$ be the $\mathcal{K}$-algebra homomorphism induced by the inclusion $E' \subseteq E$. If $H \in \mathcal{A}$ we write $a_H = e_H + I$. If $H \in \mathcal{A}'$ it is important to distinguish between a_H and $e_H + I'$. We cannot identify the two because we do not know that i is a monomorphism. If $S = (H_1, \ldots, H_p) \in \mathbf{S}$ write $a_S = a_{H_1} \ldots a_{H_p}$. If $S \in \mathbf{S}'$ then $a_S \in iA'$. The hyperplanes of $\mathcal{A}''$ have the form $H_0 \cap H$ where $H \in \mathcal{A}'$. We write the corresponding generators of E'' and A'' as $e_{H_0 \cap H}$ and $a_{H_0 \cap H}$. If $S = (H_1, \ldots, H_p) \in \mathbf{S}$ and σ is a permutation of $1, \ldots, p$ let $\sigma S = (H_{\sigma 1}, \ldots, H_{\sigma p})$. To define a $\mathcal{K}$-linear map θ from E to some module over $\mathcal{K}$ it suffices to prescribe the values $\theta(e_S)$ for $S \in \mathbf{S}$ and check that $\theta(e_{\sigma S}) = \text{sign}(\sigma)\theta(e_S)$. For convenience we agree that if $H_0 \in S$ then H_0 is the first element of the tuple S and write $S = (H_0, H_1, \ldots, H_p)$ where $H_1, \ldots, H_p \in \mathcal{A}'$.

Lemma 9.15 *There exists a surjective $\mathcal{K}$-linear map $\theta : E \to E''$ such that*

$$\theta(e_{H_1} \cdots e_{H_p}) = 0,$$
$$\theta(e_{H_0} e_{H_1} \cdots e_{H_p}) = e_{H_0 \cap H_1} \cdots e_{H_0 \cap H_p}.$$

for all $(H_1, \ldots, H_p) \in \mathbf{S}'$. This map satisfies $\theta(I) \subseteq I''$.

Proof. Since $E = E' \oplus e_{H_0} E'$ we may define θ by the formulas in the lemma. It is understood that $\theta(1) = 0$ and $\theta(e_{H_0}) = 1$. Define $\lambda : \mathcal{A}' \to \mathcal{A}''$ by $\lambda H = H_0 \cap H$ for $H \in \mathcal{A}'$. Extend this map to $\lambda : \mathbf{S}' \to \mathbf{S}''$ by $\lambda(H_1, \ldots, H_p) = (\lambda H_1, \ldots, \lambda H_p)$. In case $S = (\ )$ we agree that $\lambda S = (\ )$. In terms of this notation θ is defined by $\theta(e_S) = 0$ and $\theta(e_{H_0} e_S) = e_{\lambda S}$ for $S \in \mathbf{S}'$. Since θ is surjective, it suffices to show that $\theta(\partial e_T) \in I''$ for any dependent $T \in \mathbf{S}$. If $T \in \mathbf{S}'$ then $\theta(e_T) = 0$. Thus we may assume that $T = (H_0, S)$ is dependent. Note that $\cap(\lambda S) = H_0 \cap (\cap S) \neq \emptyset$. Thus λS is dependent and hence $\theta(\partial(e_{H_0} e_S)) = \theta(e_S - e_{H_0} \partial e_S) = -\partial(e_{\lambda T}) \in I''$. □

Corollary 9.16 *There exists a surjective $\mathcal{K}$-linear map $j : A \to A''$ such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E'' \\ \varphi \downarrow & & \downarrow \varphi'' \\ A & \xrightarrow{j} & A'' \end{array}$$

commutes. In particular for all $(H_1, \ldots, H_p) \in \mathbf{S}'$

$$j(a_{H_1} \cdots a_{H_p}) = 0,$$
$$j(a_{H_0} a_{H_1} \cdots a_{H_p}) = a_{H_0 \cap H_1} \cdots a_{H_0 \cap H_p}. \quad \Box$$

In order to prove that the sequence

$$0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$$

is exact we utilize broken circuits. Fix linear orders on $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ so that

(1) H_0 is the minimal element in $\mathcal{A}$,
(2) the linear order on $\mathcal{A}'$ is induced by the linear order of $\mathcal{A}$,
(3) if $H, K \in \mathcal{A}'$ then $\lambda H \prec \lambda K$ implies $H \prec K$.

Write $C = C(\mathcal{A})$, $C' = C(\mathcal{A}')$, and $C'' = C(\mathcal{A}'')$.

Lemma 9.17 $C' \subseteq C$.

Proof. Let $S' \in \mathbf{S}'$ be χ-independent. Note that $\cap S' \neq \emptyset$. Assume that $S' \notin C$. Then S' contains a broken circuit of $\mathcal{A}$, so there exists $H \in \mathcal{A}$ such that $(S', H) \in \mathbf{S}$ contains a circuit and $\max S' \prec H$. Since H_0 is the minimal element of the linear order, $H \in \mathcal{A}'$. It implies that S' contains a broken circuit of $\mathcal{A}'$, which is a contradiction. □

Let $i : C' \to C$ be the inclusion map. Recall the definition of $\theta : E \to E''$ from Lemma 9.15.

Lemma 9.18 $\theta(C) = C''$.

Proof. We claim first that $\theta(C) \subseteq C''$. Otherwise there exists $S \in \mathbf{S}'$ such that $(H_0, S) \in \mathbf{S}$ is χ-independent but $\lambda S \in \mathbf{S}''$ is not. Since $\cap(\lambda S) = H_0 \cap (\cap S) \neq \emptyset$, λS contains a broken circuit. So there exists $K \in \mathcal{A}'$ such that $\max\lambda S \prec \lambda K$ and $(\lambda S, \lambda K)$ is dependent. It follows from our choice of linear orders on $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ that $\max S \prec K$ and (H_0, S, K) is dependent. Thus (H_0, S) contains a broken circuit, which is a contradiction.

Next we show that $\theta(C) \supseteq C''$. Let $S'' \in \mathbf{S}''$ be χ-independent. For each $H'' \in \mathbf{S}''$, let $\mu H'' = \max\{\lambda^{-1}(H'')\} \in \mathcal{A}$. Arrange the $\mu H''$ for $H'' \in S''$ into a standard tuple $S \in \mathbf{S}$. Then obviously $\lambda S = S''$. Suppose that (H_0, S) is not χ-independent. Since $H_0 \cap (\cap S) = \cap(\lambda S) = \cap S'' \neq \emptyset$, (H_0, S) contains a broken circuit. So there exists $K \in \mathcal{A}'$ such that $\max S \prec K$ and (H_0, S, K) is dependent. It follows from the definition of S that $\max\lambda S \prec \lambda K$ and $(\lambda S, \lambda K)$ is dependent. Thus λS contains a broken circuit, which is a contradiction. □

Lemma 9.19 *Let $S_1 \in \mathbf{S}'$ and $S_2 \in \mathbf{S}'$. If (H_0, S_1) and (H_0, S_2) are χ-independent with $\lambda S_1 = \lambda S_2$, then $S_1 = S_2$.*

Proof. Suppose that $S_1 \neq S_2$. Then there exist $H_i \in \mathbf{S}_i$ $(i = 1, 2)$ such that $H_1 \neq H_2$ and $\lambda H_1 = \lambda H_2$. We may assume that $H_1 \prec H_2$. So $(H_0, H_1, H_2) \in \mathbf{S}$ is dependent, and (H_0, H_1) is a broken circuit. This contradicts the χ-independence of (H_0, S_1). □

Proposition 9.20 *Let $j : C \to C''$ be the restriction of θ. The following sequence is exact:*

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0.$$

Proof. By Lemmas 9.17 and 9.18, it is sufficient to show that $\ker(j) \subseteq \operatorname{im}(i)$. Suppose

$$j(\sum c_S e_{H_0} e_S) = \sum c_S e_{\wedge S} = 0,$$

where the sum is over $\{S \in \mathbf{S}' \mid (H_0, S) \text{ is } \chi\text{-independent}\}$ and $c_S \in \mathcal{K}$. By Lemma 9.19, we have $c_S = 0$ for all S. □

Theorem 9.21 *Let $\mathcal{A}$ be an affine arrangement. Let $H_0 \in \mathcal{A}$ and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the corresponding triple. Let $i : A(\mathcal{A}') \to A(\mathcal{A})$ be the natural homomorphism and let $j : A(\mathcal{A}) \to A(\mathcal{A}'')$ be the $\mathcal{K}$-linear map defined by*

$$\begin{aligned} j(a_{H_1} \cdots a_{H_p}) &= 0 \\ j(a_{H_0} a_{H_1} \cdots a_{H_p}) &= a_{H_0 \cap H_1} \cdots a_{H_0 \cap H_p} \end{aligned}$$

for $(H_1, \ldots, H_p) \in \mathbf{S}(\mathcal{A}')$. Then the following sequence is exact:

$$0 \to A(\mathcal{A}') \xrightarrow{i} A(\mathcal{A}) \xrightarrow{j} A(\mathcal{A}'') \to 0.$$

Proof. This follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & C' & \xrightarrow{i} & C & \xrightarrow{j} & C'' & \to & 0 \\ & & \psi' \downarrow & & \psi \downarrow & & \psi'' \downarrow & & \\ 0 & \to & A' & \xrightarrow{i} & A & \xrightarrow{j} & A'' & \to & 0, \end{array}$$

Theorem 9.11, and Proposition 9.20. □

Proposition 9.22 *Let $\mathcal{A}$ be an arrangement and let $\mathcal{B}$ be a subarrangement. The natural homomorphism $i : A(\mathcal{B}) \to A(\mathcal{A})$ is a monomorphism.*

Proof. The assertion is true if $|\mathcal{A}| - |\mathcal{B}| = 1$ by Theorem 9.21. The conclusion follows by induction on $|\mathcal{A}| - |\mathcal{B}|$. □

In Theorem 9.21 the map i has degree zero and j has degree -1. Thus we have:

Corollary 9.23 *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple and let A, A', A'' be the corresponding algebras. Then*

(1) $\operatorname{Poin}(A, t) = \operatorname{Poin}(A', t) + t\operatorname{Poin}(A'', t)$.
(2) $\operatorname{rank} A = \operatorname{rank} A' + \operatorname{rank} A''$. □

Theorem 9.24 $\operatorname{Poin}(A(\mathcal{A}), t) = \pi(\mathcal{A}, t)$.

Proof. We prove this by induction on the cardinality of $|\mathcal{A}|$. When $\mathcal{A}$ is the empty arrangement $A(\mathcal{A}) = \mathcal{K}$. So $\operatorname{Poin}(A(\mathcal{A}), t) = 1 = \pi(\mathcal{A}, t)$. We proved the formula $\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t)$ in Theorem 6.9. This recursion combined with Corollary 9.23.1 completes the proof. □

The Structure of $A(\mathcal{A})$

As in the central case, we define for $X \in L$

$$\mathbf{S}_X = \mathbf{S}_X(\mathcal{A}) = \{S \in \mathbf{S}(\mathcal{A}) \mid \cap S = X\}, \quad E_X = E_X(\mathcal{A}) = \sum_{S \in \mathbf{S}_X} \mathcal{K} e_S,$$

$$A_X(\mathcal{A}) = \phi(E_X), \quad C_X = C_X(\mathcal{A}) = C(\mathcal{A}) \cap E_X(\mathcal{A}).$$

We can generalize Lemmas 8.39 and 8.40 without changing their proofs:

Lemma 9.25 *For $p \geq 0$ we have $C_p = \oplus_{X \in L_p} C_X$ and hence $C = \oplus_{X \in L} C_X$.* □

Lemma 9.26 *If $Y \leq X$ then $C_Y(\mathcal{A}_X) = C_Y(\mathcal{A})$.* □

Let $X \in L$. Since the natural map $i : A(\mathcal{A}_X) \to A(\mathcal{A})$ is injective by Proposition 9.22, we have the affine version of Proposition 8.31:

Proposition 9.27 *If $Y \leq X$ then $A_Y(\mathcal{A}_X) \simeq A_Y(\mathcal{A})$.* □

Theorem 9.28 *Let $\mathcal{A}$ be an affine arrangement and let $A = A(\mathcal{A})$. Then*

$$A = \bigoplus_{X \in L} A_X.$$

Proof. Let $X \in L$. Note that $\mathcal{A}_X$ is a central arrangement. By Lemma 9.26, Theorem 8.43, and Proposition 9.27, we have

$$C_X(\mathcal{A}) = C_X(\mathcal{A}_X) \simeq A_X(\mathcal{A}_X) \simeq A_X(\mathcal{A}).$$

Since $C(\mathcal{A}) = \oplus_{X \in L} C_X(\mathcal{A})$ by Lemma 9.25 and $C(\mathcal{A}) \simeq A(\mathcal{A})$ by Theorem 9.11, we have the desired result. □

Recall Brieskorn's Lemma from the introduction. The next result is its algebraic analog. Its proof is the same as the proof of the central version, Corollary 8.27. It will be used in Theorem 22.15 to give an elementary proof of Brieskorn's Lemma.

Corollary 9.29 *Let $\mathcal{A}$ be an affine arrangement. We have $A_p = \bigoplus_{X \in L_p} A_X$.* □

These results are summarized in the next corollary.

Corollary 9.30 *The algebra $A(\mathcal{A})$ is a free graded $\mathcal{K}$-module. The $\mathcal{K}$-modules $A_X(\mathcal{A})$ for $X \in L$ and $A_p(\mathcal{A})$ for $p \geq 0$ are also free.*

Proof. The $\mathcal{K}$-modules $C_X(\mathcal{A})$ are free by definition. It follows from Lemma 9.26 that $C_X(\mathcal{A}) \simeq A_X(\mathcal{A})$. Thus $A_X(\mathcal{A})$ is a free $\mathcal{K}$-module. We showed in Corollary 9.29 that $A_p = \oplus_{X \in L_p} A_X$. Thus A_p is also free. □

Proposition 9.31 *If $X \in L(\mathcal{A})$ then the rank of the free $\mathcal{K}$-module $A_X(\mathcal{A})$ is equal to $(-1)^{r(X)}\mu(X)$.*

Proof. The leading coefficient of $\pi(\mathcal{A}_X, t)$ is equal to $(-1)^{r(X)}\mu(X)$. Since $\operatorname{Poin}(A(\mathcal{A}_X), t) = \pi(\mathcal{A}_X, t)$, it is also equal to $\operatorname{rank} A_X(\mathcal{A}_X) = \operatorname{rank} A_X(\mathcal{A})$. □

$\mathcal{A}$–equivalence

Definition 9.32 *The arrangements $\mathcal{A}$ and $\mathcal{B}$ are* $\mathcal{K}$**–algebra equivalent**, *or A-equivalent, if there is an isomorphism of graded $\mathcal{K}$-algebras $\phi : A(\mathcal{A}) \to A(\mathcal{B})$.*

Clearly, L–equivalent arrangements are A–equivalent and A–equivalent arrangements are π equivalent. Example 6.14 showed that π-equivalent arrangements are not L-equivalent. In fact these three notions of equivalence are distinct. Falk [63] used his work on minimal models to find an invariant of the algebra A which is different for the two arrangements of Example 6.14. L. Rose and H. Terao constructed the following example.

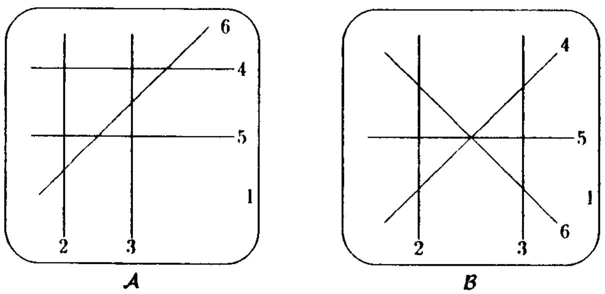


Figure 18: A–equivalent but not L–equivalent

Example 9.33 *The 3-arrangements $\mathcal{A}$ and $\mathcal{B}$ in* Figure 18 *are A-equivalent but not L-equivalent.*

To see that $L(\mathcal{A})$ and $L(\mathcal{B})$ are not isomorphic, note that the two triple points of $\mathcal{A}$ are on the same line, while the two triple points of $\mathcal{B}$ are on different lines. In order to show that $A(\mathcal{A}) \simeq A(\mathcal{B})$ label the hyperplanes as in Figure 18. Let $E(\mathcal{A})$ have generators e_i for $1 \le i \le 6$, and let $E(\mathcal{B})$ have generators f_i for $1 \le i \le 6$. Define $\phi : E(\mathcal{A}) \to E(\mathcal{B})$ by $\phi(e_i) = f_i$ for $i = 1, 2, 3, 6$ and $\phi(e_4) = f_5 - f_4 + f_1$, $\phi(e_5) = f_6 - f_5 + f_1$. Note that $I_2(\mathcal{A})$ is generated by $e_{1,2} - e_{1,3} + e_{2,3}$ and $e_{1,4} - e_{1,5} + e_{4,5}$. We have

$$
\begin{aligned}
\phi(e_{1,2} - e_{1,3} + e_{2,3}) &= f_{1,2} - f_{1,3} + f_{2,3} \\
&\in I_2(\mathcal{B}) \\
\phi(e_{1,4} - e_{1,5} + e_{4,5}) &= \phi((e_1 - e_4)(e_1 - e_5)) \\
&= \phi(e_1 - e_4)\phi(e_1 - e_5) \\
&= (f_4 - f_5)(f_5 - f_6) \\
&= f_{4,5} - f_{4,6} + f_{5,6} \\
&\in I_2(\mathcal{B}).
\end{aligned}
$$

Since these images generate $I_2(\mathcal{B})$, it follows that ϕ induces an isomorphism $A(\mathcal{A}) \simeq A(\mathcal{B})$.

10 Algebra Factorizations

Let $\mathcal{A}$ be an affine arrangement. Let $\mathbf{c}\mathcal{A}$ be the cone over $\mathcal{A}$. Recall that K_0 is the additional hyperplane and we write $e_0 = e_{K_0}$. Let $\phi : E(\mathbf{c}\mathcal{A}) \to A(\mathbf{c}\mathcal{A})$ be the natural surjection and let $a_0 = \phi(e_0)$. It follows from Corollary 9.12 that $A(\mathcal{A})$ and $A(\mathbf{c}\mathcal{A})$ are free $\mathcal{K}$–modules. Thus the short exact sequence

$$0 \to A(\mathcal{A}) \xrightarrow{i} A(\mathbf{c}\mathcal{A}) \xrightarrow{s} A(\mathcal{A}) \to 0$$

of Corollary 9.13 splits. We get:

Theorem 10.1 *Let $\mathbf{c}\mathcal{A}$ be the cone over the affine arrangement $\mathcal{A}$. Let $a_0 \in A(\mathbf{c}\mathcal{A})$ correspond to the additional hyperplane. There is an isomorphism of graded $\mathcal{K}$-modules:*

$$(\mathcal{K} + \mathcal{K}a_0) \otimes A(\mathcal{A}) \simeq A(\mathbf{c}\mathcal{A}). \quad \square$$

It follows from Theorem 9.24 that Proposition 6.4 is a consequence of Theorem 10.1. The topological interpretation of this algebra factorization follows from Proposition 19.1 and Theorem 22.14.

In this section we prove algebra factorization in two more cases. The factorization for supersolvable arrangements also has a topological interpretation. It is described after the Fibration Theorem 23.18. We prove here that the existence of a nice partition is equivalent to algebra factorization. We do not know of a topological interpretation in this case.

Supersolvable Arrangements

Recall that a supersolvable arrangement is central. Thus we may assume that $\mathcal{A}$ is a central arrangement.

Lemma 10.2 *Suppose there exists a modular element $Y \in L(\mathcal{A})$ with $r(Y) = r(\mathcal{A}) - 1$. For every $H \in \mathcal{A} \setminus \mathcal{A}_Y$ there is a $\mathcal{K}$-algebra isomorphism $\rho : A(\mathcal{A}_Y) \to A(\mathcal{A}^H)$ defined by $\rho(a_K) = a_{H \cap K}$ for all $K \in \mathcal{A}_Y$.*

Proof. Since $H \in \mathcal{A} \setminus \mathcal{A}_Y$ we have $L_Y = L(\mathcal{A}_Y) = [H \wedge Y, Y]$ and $L^H = L(\mathcal{A}^H) = [H, H \vee Y]$. It follows from Lemma 4.25 that the map $\tau : L_Y \to L^H$ given by $\tau(Z) = Z \vee H = Z \cap H$ is a lattice isomorphism. If $S = (H_1, \ldots, H_p) \in \mathbf{S}(\mathcal{A}_Y)$ define $\tau S = (\tau H_1, \ldots, \tau H_p) \in \mathbf{S}(\mathcal{A}^H)$. The $\mathcal{K}$-algebra isomorphism from $E(\mathcal{A}_Y)$ to $E(\mathcal{A}^H)$ which sends e_S to $e_{\tau S}$ maps $I(\mathcal{A}_Y)$ to $I(\mathcal{A}^H)$. It induces a $\mathcal{K}$-algebra homomorphism $\rho : A(\mathcal{A}_Y) \to A(\mathcal{A}^H)$ such that $\rho a_S = a_{\tau S}$. The inverse of ρ is constructed using σ, the inverse of τ in Lemma 4.25. $\square$

Lemma 10.3 *Suppose there exists a modular element $Y \in L(\mathcal{A})$ with $r(Y) = r(\mathcal{A}) - 1$. Let $\mathcal{B} = \mathcal{A} \setminus \mathcal{A}_Y$. Then*

$$A(\mathcal{A}) = A(\mathcal{A}_Y) \oplus \Big(\bigoplus_{H \in \mathcal{B}} A(\mathcal{A}_Y)a_H\Big).$$

Proof. It follows from Proposition 9.22 that we may identify $A(\mathcal{A}_Y)$ with the $\mathcal{K}$-subalgebra of $A(\mathcal{A})$ generated by the elements $a_K = e_K + I(\mathcal{A})$ for $K \in \mathcal{A}_Y$. Let $U = \sum_{H \in \mathcal{B}} A(\mathcal{A}_Y)a_H$. Note first that if $H, K \in \mathcal{B}$ then $a_H a_K \in U$. This is clear if $H = K$ since $a_H^2 = 0$. Suppose $H \neq K$. Recall again that the map $\tau : L_Y \to L^H$ given by $\tau Z = Z \cap H$ is an isomorphism. It follows that there exists $M \in L_Y$ such that $M \cap H = K \cap H$. Since $K \neq H$ we have $r(K \cap H) = 2$ so $r(M \cap H) = 2$ and thus $M \in \mathcal{A}_Y$. Since $r(M \cap H \cap K) = 2$, the 3 tuple (M, H, K) is dependent and thus $a_H a_K - a_M a_K + a_M a_H = 0$. This shows that $a_H a_K \in A(\mathcal{A}_Y)a_H + A(\mathcal{A}_Y)a_K \subseteq U$.

Since $A(\mathcal{A}_Y)$ is a $\mathcal{K}$-subalgebra of $A(\mathcal{A})$ containing the identity, it follows that U is closed under multiplication and $A(\mathcal{A}_Y)U \subseteq U$. Thus $A(\mathcal{A}_Y) + U$ is a $\mathcal{K}$-subalgebra of $A(\mathcal{A})$. But $\mathcal{A} = \mathcal{A}_Y \cup \mathcal{B}$ so $a(\mathcal{A}_Y) + U$ contains all the generators a_H, $H \in \mathcal{A}$ of $A(\mathcal{A})$. It follows that

$$A(\mathcal{A}) = A(\mathcal{A}_Y) + U = A(\mathcal{A}_Y) + \Big(\sum_{H \in \mathcal{B}} A(\mathcal{A}_Y)a_H\Big).$$

We show next that this is a direct sum. We need $A(\mathcal{A}_Y) \cap U = 0$. Recall that $A = \oplus_{X \in L} A_X$ where $A_X = \phi(E_X) = \sum_{\cap S = X} \mathcal{K}a_S$. Let $\pi_X : A \to A_X$ be the natural projection. Thus

$$\pi_X a_S = \begin{cases} a_S & \text{if } \cap S = X \\ 0 & \text{otherwise.} \end{cases}$$

We show that $\pi_X(A(\mathcal{A}_Y) \cap U) = 0$ for all $X \in L$ by showing that (i) $\pi_X(A(\mathcal{A}_Y)) = 0$ if $X \not\leq Y$, and (ii) $\pi_X(U) = 0$ if $X \leq Y$. Assertion (i) is immediate from $A(\mathcal{A}_Y) = \sum_{\cap S \leq Y} \mathcal{K}a_S$. To prove (ii) we observe that if $H \in \mathcal{B}$ and $X \leq Y$ then $\pi_X(a_S a_H) = \pi_X(a_{(S,H)}) = 0$ because $(\cap S) \cap H \not\leq Y$. Thus $A(\mathcal{A}_Y) \cap U = 0$.

It remains to show that the sum $\sum_{H \in \mathcal{B}} A(\mathcal{A}_Y)a_H$ is also direct. Fix $H_0 \in \mathcal{B}$ and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the inductive triple with respect to H_0. Let $\lambda : \mathbf{S}' \to \mathbf{S}''$ and $j : A \to A''$ be the maps defined in Lemma 9.15 and Corollary 9.16. Since $H_0 \in \mathcal{B}$ we have $\mathbf{S}(\mathcal{A}_Y) \subseteq \mathbf{S}'$. It follows from Corollary 9.16 that for $S \in \mathbf{S}(\mathcal{A}_Y)$ and $H \in \mathcal{B}$ we have $j(a_H a_S) = a_{\lambda S}$ if $H = H_0$ and $j(a_H a_S) = 0$ otherwise. By Lemma 10.2 with H replaced by H_0, there exists a $\mathcal{K}$-algebra isomorphism $\rho : A(\mathcal{A}_Y) \to A''$ with $\rho(a_S) = a_{\lambda S}$. Thus $j(a_H a_S) = \rho(a_S)$ if $H = H_0$ and $j(a_H a_S) = 0$ otherwise. It follows that for $u \in A(\mathcal{A}_Y)$ we have

$$j(a_H u) = \begin{cases} \rho(u) & \text{if } H = H_0 \\ 0 & \text{if } H \in \mathcal{B} \setminus \{H_0\} \end{cases}$$

Suppose $\sum_{H \in \mathcal{B}} a_H u_H = 0$ where $u_H \in A(\mathcal{A}_Y)$. Then

$$0 = j\Big(\sum_{H \in \mathcal{B}} a_H u_H\Big) = \rho(u_{H_0})$$

and thus $u_{H_0} = 0$. Since H_0 was an arbitrary hyperplane of $\mathcal{B}$, it follows that $u_H = 0$ for all $H \in \mathcal{B}$. □

Theorem 10.4 *Let $\mathcal{A}$ be a supersolvable arrangement with $r = r(\mathcal{A})$ and maximal chain of modular elements*

$$V = Y_0 < Y_1 < \ldots < Y_r = T.$$

Let $A = A(\mathcal{A})$. For $1 \leq i \leq r$ let $\mathcal{B}_i = \mathcal{A}_{Y_i} \setminus \mathcal{A}_{Y_{i-1}}$ and let $B_i = \sum_{H \in \mathcal{B}_i} \mathcal{K}a_H$. The $\mathcal{K}$-linear map

$$(\mathcal{K} + B_1) \otimes \ldots \otimes (\mathcal{K} + B_r) \to A$$

defined by multiplication in A is an isomorphism of graded $\mathcal{K}$-modules. In particular with $b_i = |\mathcal{B}_i|$

$$\mathrm{Poin}(A, t) = (1 + b_1) \ldots (1 + b_r).$$

Proof. Let $Y = Y_{r-1}$. Then Y is a modular element of $L(\mathcal{A})$ with $r(Y) = r - 1$. Let $\mathcal{B} = \mathcal{A} \setminus \mathcal{A}_Y$ and let $B = \sum_{H \in \mathcal{B}} \mathcal{K}a_H$. Lemma 10.3 shows that the $\mathcal{K}$-linear map $A(\mathcal{A}_Y) \otimes (\mathcal{K} + B) \to A$ defined by the multiplication in A is an isomorphism of modules. The result follows by induction on r. □

Nice Partitions of Central Arrangements

Assume first that $\mathcal{A}$ is a central arrangement. We follow [195]. Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_s)$ be a partition of $\mathcal{A}$. Let (π_i) denote the free $\mathcal{K}$-module with basis 1 and the elemenets of π_i. It is graded by $\deg 1 = 0$ and $\deg H = 1$. Define the graded $\mathcal{K}$-module

$$(\boldsymbol{\pi}) = (\pi_1) \otimes (\pi_2) \otimes \cdots \otimes (\pi_s).$$

We agree that $(\boldsymbol{\pi}) = \mathcal{K}$ when $\mathcal{A} = \Phi_\ell$. Since $\mathrm{Poin}((\pi_i), t) = (1 + |\pi_i| t)$, we obtain

$$\mathrm{Poin}((\boldsymbol{\pi}), t) = \prod_{i=1}^{s} (1 + |\pi_i| t).$$

Definition 10.5 *Let $S = (H_1, \ldots, H_k) \in \mathbf{S}_k$. Call S a* **k-section** *of $\boldsymbol{\pi}$ if for $1 \leq i \leq k$*

$$H_i \in \pi_{n(i)}, \quad 1 \leq n(1) < n(2) < \ldots < n(k) \leq s.$$

We agree that the 0-section is $S = (\,)$. Let $\mathcal{S}_k \subset \mathbf{S}_k$ be the set of k-sections of $\boldsymbol{\pi}$, and let $\mathcal{S} = \cup_{k=0}^{s} \mathcal{S}_k$. Given $S \in \mathcal{S}_k$ let $p_S = x_1 \otimes \cdots \otimes x_s \in (\boldsymbol{\pi})$ where

$$x_j = \begin{cases} H_i & \text{if } j = n(i) \\ 1 & \text{if } j \notin \{n(1), \ldots, n(k)\}. \end{cases}$$

Note that $p_{(\,)} = 1$ and p_S is homogeneous of degree k. The graded $\mathcal{K}$-module $(\boldsymbol{\pi})$ is free with basis $\{p_S \mid S \in \mathcal{S}\}$. Recall the notation $e_S \in E(\mathcal{A})$ and $a_S \in A(\mathcal{A})$. Each element of the algebra $A(\mathcal{A})$ may be expressed as a linear combination of elements $\{a_S \mid S \in \mathbf{S}\}$, but this expression is not necessarily unique.

Definition 10.6 *Define $\kappa : (\pi) \to A(\mathcal{A})$ as follows. For $S \in \mathcal{S}$ assign $\kappa(p_S) = a_S$ and let κ be the unique homogeneous $\mathcal{K}$-linear map of degree zero which extends this assignement.*

We will show that the map κ is an isomorphism of graded $\mathcal{K}$-modules if and only if the partition π is nice. Denote the homogeneous part of degree k of (π) by $(\pi)_k$. Then

$$(\pi) = \bigoplus_{k=0}^{\bullet} (\pi)_k.$$

Here $(\pi)_0 = \mathcal{K}$. Given $S = (H_1, \ldots, H_k) \in \mathbf{S}_k$ recall that $\cap S = H_1 \cap \cdots \cap H_k \in L$. For $X \in L$, define a free submodule $(\pi)_X$ of (π) with basis $\{p_S \mid S \in \mathcal{S},\ \cap S = X\}$. It follows that $(\pi)_V = \mathcal{K}$.

Lemma 10.7 *Suppose that π is an independent partition. For each $k \geq 0$, we have*

$$(\pi)_k = \bigoplus_{X \in L_k} (\pi)_X.$$

Proof. Note that $\{p_S \mid S \in \mathcal{S}_k\}$ is a basis for $(\pi)_k$. If $\cap S = X$ then $p_S \in (\pi)_X$. We have $X \in L_k$ because π is independent. □

Lemma 10.8 *For $X, Y \in L$ with $Y \leq X$ the natural map $(\pi_X)_Y \to (\pi)_Y$ is an isomorphism.*

Proof. If $S \in \mathcal{S}$ with $\cap S = Y$ then $S \subseteq \mathcal{A}_Y \subseteq \mathcal{A}_X$. Thus S is also a section of π_X:

$$\{S \mid S \in \mathcal{S},\ \cap S = Y\} = \{S \mid S \text{ is a section of } \pi_X,\ \cap S = Y\}.$$

The isomorphism $p_S \in (\pi_X)_Y \mapsto p_S \in (\pi)_Y$ is obtained by inserting "$1\otimes$" the required number of times. □

Let $S = (H_1, \ldots, H_k) \in \mathbf{S}_k$. Recall that S_j denotes the tuple with H_j deleted. Define a $\mathcal{K}$-linear map $\partial : (\pi)_k \to (\pi)_{k-1}$ by $\partial(p_{()}) = 0$, $\partial(p_H) = 1$, and for $k \geq 2$ and $S \in \mathcal{S}_k$

$$\partial(p_S) = \sum_{j=1}^{k} (-1)^{j-1} p_{S_j}.$$

Then $\partial\partial = 0$ and $((\pi)_\bullet, \partial)$ is a chain complex.

Lemma 10.9 *If the partition π contains a block which is a singleton then the complex $((\pi)_\bullet, \partial)$ is acyclic.*

Proof. We may assume that π_1 is a singleton, $\pi_1 = \{H_1\}$. Suppose that $x \in (\pi)_k$ is a cycle, $\partial x = 0$. Write x as $x = H_1 \otimes x_1 + 1 \otimes x_2$, where $x_1, x_2 \in (\pi_2) \otimes \cdots \otimes (\pi_s)$. Then

$$0 = \partial x = 1 \otimes x_1 - H_1 \otimes (\partial x_1) + 1 \otimes (\partial x_2) = 1 \otimes (x_1 + \partial x_2) - H_1 \otimes (\partial x_1).$$

This implies that $x_1 = -\partial x_2$. Define $y = H_1 \otimes x_2 \in (\pi)_{k+1}$. Then

$$\partial y = 1 \otimes x_2 - H_1 \otimes (\partial x_2) = 1 \otimes x_2 + H_1 \otimes x_1 = x. \quad \square$$

Theorem 10.10 *Let $\mathcal{A}$ be a central arrangement and let π be a partition of $\mathcal{A}$. Define the homogeneous $\mathcal{K}$-linear map κ as Definition 10.6. Then κ is an isomorphism if and only if the partition π is nice.*

Proof. Assume that π is a nice partition. We argue by induction on $r = r(\mathcal{A})$. If $r(\mathcal{A}) = 0$ then $\mathcal{A} = \Phi_\ell$. Thus $(\pi) = \mathcal{K} = A(\mathcal{A})$. Assume that $r = r(\mathcal{A}) > 0$. Note $s \leq r$ because π is independent. Consider the diagram

$$\begin{array}{ccccccccccc}
0 & \to & (\pi)_r & \xrightarrow{\partial} & (\pi)_{r-1} & \xrightarrow{\partial} & \cdots \xrightarrow{\partial} & (\pi)_1 & \xrightarrow{\partial} & (\pi)_0 & \to 0 \\
 & & \downarrow \kappa_r & & \downarrow \kappa_{r-1} & & & \downarrow \kappa_1 & & \downarrow \kappa_0 & \\
0 & \to & A_r(\mathcal{A}) & \xrightarrow{\partial} & A_{r-1}(\mathcal{A}) & \xrightarrow{\partial} & \cdots \xrightarrow{\partial} & A_1(\mathcal{A}) & \xrightarrow{\partial} & A_0(\mathcal{A}) & \to 0.
\end{array}$$

The vertical maps are induced by $\kappa : (\pi) \to A(\mathcal{A})$. The top row is exact by Lemma 10.9. The bottom row is exact by Lemma 8.13. Note that

$$(\pi)_k = \bigoplus_{Y \in L_k} (\pi)_Y \simeq \bigoplus_{Y \in L_k} (\pi_Y)_Y$$

by Lemmas 10.7 and 10.8. Also note that

$$A_k(\mathcal{A}) = \bigoplus_{Y \in L_k} A_Y(\mathcal{A}) \simeq \bigoplus_{Y \in L_k} A_Y(\mathcal{A}_Y)$$

by Corollary 8.27 and Proposition 8.31. By applying the induction assumption to L_Y for $r(Y) < r$, we obtain that κ_i is an isomorphism for $1 \leq i < r$. It follows from the commutative diagram that κ_r is also an isomorphism. Thus $\kappa : (\pi) \to A(\mathcal{A})$ is an isomorphism.

For the converse suppose κ is an isomorphism. First we show that π is independent. Let $S \in \mathcal{S}$. Then $p_S \neq 0$ and $a_S = \kappa(p_S) \neq 0$. This shows that S is indepenedent. Next we show that if $X \neq V$ then π_X contains a block which is a singleton. Since

$$(\pi) = \bigoplus_{Y \in L} (\pi)_Y, \quad A(\mathcal{A}) = \bigoplus_{Y \in L} A_Y(\mathcal{A}),$$

κ induces isomorphisms $(\pi)_Y \to A_Y(\mathcal{A})$. By Lemma 10.8 and Proposition 8.31, we obtain

$$(\pi_X) = \bigoplus_{Y \in L_X} (\pi_X)_Y \simeq \bigoplus_{Y \in L_X} (\pi)_Y \simeq \bigoplus_{Y \in L_X} A_Y(\mathcal{A}) \simeq \bigoplus_{Y \in L_X} A_Y(\mathcal{A}_X) = A(\mathcal{A}_X).$$

Let $X \neq V$. Then

$$0 = \sum_{Y \in L_X} \mu(Y) = \mathrm{Poin}(A(\mathcal{A}_X), -1) = \mathrm{Poin}((\pi_X), -1) = \prod_i (1 - |\pi_i \cap \mathcal{A}_X|).$$

This implies that π_X contains a block which is a singleton. □

Corollary 10.11 *If $\mathcal{A}$ has a nice partition $\pi = (\pi_1, \ldots, \pi_s)$ then $s = r$ and*

$$\operatorname{Poin}(A(\mathcal{A}), t) = \prod_{i=1}^{r}(1 + |\pi_i| t). \quad \square$$

Corollary 10.12 *If $\mathcal{A}$ has a nice partition $\pi = (\pi_1, \ldots, \pi_s)$ then the multiset $\{|\pi_1|, \ldots, |\pi_s|\}$ depends only on $\mathcal{A}$.* $\square$

Corollary 10.13 *If $\mathcal{A}$ has a nice partition $\pi = (\pi_1, \ldots, \pi_s)$ then for all $X \in L$*

$$r(X) = |\{i \mid \pi_i \cap \mathcal{A}_X \neq \emptyset\}|.$$

Proof. We showed in the proof of Theorem 10.10 that the isomorphism κ induces isomorphisms $\kappa_X : (\pi_X) \to A(\mathcal{A}_X)$ for all $X \in L$. Thus π_X is a nice partition of $\mathcal{A}_X$. By Corollary 10.11, we have

$$r(X) = r(\mathcal{A}_X) = |\pi_X| = |\{i \mid \pi_i \cap \mathcal{A}_X \neq \emptyset\}|. \quad \square$$

Nice Partitions of Affine Arrangements

Next assume that $\mathcal{A}$ is an affine arrangement. We generalize Theorem 10.10 to affine arrangements. Recall that $\mathcal{A}_X$ is central for each $X \in L$.

Definition 10.14 *Let $\pi = (\pi_1, \ldots, \pi_s)$ be a partition of the affine arrangement $\mathcal{A}$. It is called* **nice** *if*

(1) for every choice of hyperplanes $H_i \in \pi_i$ for $1 \leq i \leq s$, the intersection $H_1 \cap H_2 \cap \cdots \cap H_s$ is not empty, and

(2) if $X \in L \setminus \{V\}$ then the induced partition π_X of the central arrangement $\mathcal{A}_X$ is a nice partition in the sense of Definition 6.19.

For any partition of $\mathcal{A}$ we can define the graded free $\mathcal{K}$-module $(\pi) = \bigoplus_k (\pi)_k$, the submodule $(\pi)_X$ for $X \in L$, and the map $\kappa : (\pi) \to A(\mathcal{A})$ as before. If the first condition of Definition 10.14 is satisfied then

$$(\pi) = \bigoplus_{X \in L} (\pi)_X.$$

As in Lemma 10.8 for the central case, the natural map $(\pi_X)_Y \to (\pi)_Y$ is an isomorphism for $X, Y \in L$ with $Y \leq X$.

Theorem 10.15 *Let $\mathcal{A}$ be an affine arrangement and let π be a partition of $\mathcal{A}$. Then κ is an isomorphism if and only if the partition π is nice.*

Proof. Suppose π is nice. Theorem 10.10 and Proposition 9.27 give

$$(\pi) \simeq \bigoplus_{X \in L} (\pi)_X \simeq \bigoplus_{X \in L} (\pi_X)_X \simeq \bigoplus_{X \in L} A_X(\mathcal{A}_X) \simeq \bigoplus_{X \in L} A_X(\mathcal{A}) \simeq A(\mathcal{A}).$$

Conversely, assume that the map κ is an isomorphism. If $S = (H_1, \ldots, H_s)$ with $H_i \in \pi_i$ for $1 \leq i \leq s$ then e_S is not zero in $A(\mathcal{A})$ because κ is injective. It follows that $\cap S \neq \emptyset$. This is the first condition. The isomorphism κ induces isomorphisms $(\pi)_Y \simeq A_Y(\mathcal{A})$ for all $Y \in L$. Let $X \in L$. Proposition 9.27 gives

$$(\pi_X) \simeq \bigoplus_{Y \in L_X} (\pi)_Y \simeq \bigoplus_{Y \in L_X} A_Y(\mathcal{A}) \simeq A(\mathcal{A}_X).$$

Theorem 10.10 implies that each partition π_X is nice. $\square$

11 The Algebra $B(\mathcal{A})$

In this section we construct a $\mathcal{K}$-algebra $B(\mathcal{A})$ whose elements are certain $\mathcal{K}$-linear combinations of ordered subsets of $L(\mathcal{A})$ with multiplication defined using a shuffle product. Thus $B(\mathcal{A})$ depends only on $L(\mathcal{A})$. We prove that the algebras $A(\mathcal{A})$ and $B(\mathcal{A})$ are isomorphic. This algebra will reappear in section 17 as the homology of a chain complex based on $L(\mathcal{A})$.

The Shuffle Product

Definition 11.1 *Let $\mathcal{A}$ be an arrangement with lattice $L = L(\mathcal{A})$. For $p \geq 0$ define free $\mathcal{K}$ modules $\mathcal{T}_p$ as follows: $\mathcal{T}_0 = \mathcal{K}$ and for $p > 0$, $\mathcal{T}_p$ has a basis consisting of all p-tuples $(X_1, \ldots, X_p)$ where $X_i \in L \setminus \{V\}$. Let*

$$\mathcal{T} = \bigoplus_{p \geq 0} \mathcal{T}_p.$$

Let $Sym(p)$ be the symmetric group on the letters $1, \ldots, p$. If $\pi \in Sym(p)$ and $u = (X_1, \ldots, X_p)$ let $\pi u = (X_{\pi^{-1}1}, \ldots, X_{\pi^{-1}p})$. This makes $\mathcal{T}_p$ a $Sym(p)$-module.

Definition 11.2 *Define a product $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$, written $*$, as follows. If $u = (X_1, \ldots, X_p)$ and $v = (Y_1, \ldots, Y_q)$ let*

$$w = (Z_1, \ldots, Z_{p+q}) = (X_1, \ldots, X_p, Y_1, \ldots, Y_q).$$

Define

$$u * v = \sum \operatorname{sign}\pi(\pi w)$$

where the sum is over all (p,q)-shuffles π of $1, \ldots, p+q$.

Recall [122, p.243] that a (p,q)-shuffle of $1, \ldots, p+q$ is a permutation $\pi \in Sym(p+q)$ such that $\pi i < \pi j$ whenever $i < j \leq p$ or $p < i < j$. This makes $\mathcal{T}$ into an associative graded anticommutative $\mathcal{K}$-algebra with identity.

Definition 11.3 *Let $\eta : \mathcal{T} \to \mathcal{T}$ be the antisymmetrizer defined for $u = (X_1, \ldots, X_p)$ by*

$$\eta u = \sum \operatorname{sign}(\pi u) = \sum \operatorname{sign}(\pi^{-1} u)$$

summed over all $\pi \in Sym(p)$. Define a $\mathcal{K}$-linear map $\lambda : \mathcal{T} \to \mathcal{T}$ by $\lambda 1 = 1$ and

$$\lambda(X_1, \ldots, X_p) = \begin{cases} (X_1, X_1 \cap X_2, \ldots, X_1 \cap X_2 \cap \ldots \cap X_p) & \text{if } X_1 \cap X_2 \cap \ldots \cap X_p \neq \emptyset \\ 0 & \text{if } X_1 \cap X_2 \cap \ldots \cap X_p = \emptyset. \end{cases}$$

Lemma 11.4 *We have*
(1) $\eta(X_1, \ldots, X_p) = (X_1) * \ldots * (X_p)$,
(2) if $u, v \in \mathcal{T}$ then $\lambda(\lambda u * \lambda v) = \lambda(u * v)$.

Proof. Assertion (1) follows by induction. In (2) if one side is zero so is the other. Otherwise it suffices to check (2) for $u = (X_1, \ldots, X_p)$ and $v = (Y_1, \ldots, Y_q)$. Then $\lambda u = (X'_1, \ldots, X'_p)$ and $\lambda v = (Y'_1, \ldots, Y'_q)$ where $X'_i = X_1 \cap \ldots \cap X_i$ and $Y'_j = Y_1 \cap \ldots \cap Y_j$. Write $(Z_1, \ldots, Z_{p+q}) = (X_1, \ldots, X_p, Y_1, \ldots, Y_q)$ and $(Z'_1, \ldots, Z'_{p+q}) = (X'_1, \ldots, X'_p, Y'_1, \ldots, Y'_q)$. It follows from the idempotence $Z \cap Z = Z$ that $Z'_{\pi 1} \cap \ldots \cap Z'_{\pi i} = Z_{\pi 1} \cap \ldots \cap Z_{\pi i}$ for all $1 \leq i \leq p+q$, and all permutations π of $1, \ldots, p+q$. Thus

$$\begin{aligned} \lambda(\lambda u * \lambda v) &= \sum (\operatorname{sign}\pi) \lambda(Z'_{\pi 1}, \ldots, Z'_{\pi(p+q)}) \\ &= \sum (\operatorname{sign}\pi) \lambda(Z_{\pi 1}, \ldots, Z_{\pi(p+q)}) \\ &= \lambda(u * v). \quad \square \end{aligned}$$

Definition 11.5 *Let $\mathcal{U} = \lambda(\mathcal{T})$. Then $\mathcal{U}$ inherits a grading from $\mathcal{T}$. Since λ is idempotent, $\mathcal{U}$ is spanned by the identity and all $(X_1, \ldots, X_p)$ with $X_1 \leq \ldots \leq X_p$. Define a product in $\mathcal{U}$ by $uv = \lambda(u * v)$ for $u, v \in \mathcal{U}$.*

The multiplication in $\mathcal{U}$ is associative. To see this, let $u, v, w \in \mathcal{U}$. Since $\lambda w = w$ it follows from Lemma 11.4.2 that

$$(uv)w = \lambda(uv * w) = \lambda(\lambda(u * v) * \lambda w) = \lambda((u * v) * w).$$

The conclusion follows since $*$ is associative. Thus $\mathcal{U}$ is an associative, anticommutative algebra with identity.

The Algebra $B(\mathcal{A})$

Recall the notation $S = (H_1, \ldots, H_p) \in \mathbf{S}$. We may view each element $S \in \mathbf{S}$ as an element of $\mathcal{T}$.

Definition 11.6 *For $S \in \mathbf{S}$ define an element $b_S \in \mathcal{U}$ as follows: if $S = (\,)$ let $b_S = 1$ and for $S = (H_1, \ldots, H_p)$ let $b_S = \lambda(\eta S)$. Thus*

$$b_S = \begin{cases} \sum_{\pi \in Sym(p)} \operatorname{sign}\pi (H_{\pi 1}, H_{\pi 1} \cap H_{\pi 2}, \ldots, H_{\pi 1} \cap H_{\pi 2} \cap \ldots \cap H_{\pi p}) & \text{if } \cap S \neq \emptyset \\ 0 & \text{if } \cap S = \emptyset. \end{cases}$$

Lemma 11.7 *Let $S, T \in \mathbf{S}$. Then $b_S b_T = b_{(S,T)}$.*

Proof. Let $S = (H_1, \ldots, H_p)$ and $T = (K_1, \ldots, K_q)$ where $H_i, K_j \in \mathcal{A}$. Using Lemma 11.4 we get:

$$\begin{aligned} b_S b_T &= \lambda(b_S * b_T) \\ &= \lambda(\lambda(\eta S) * \lambda(\eta T)) \\ &= \lambda(\eta S * \eta T) \\ &= \lambda((H_1) * \ldots * (H_p) * (K_1) * \ldots * (K_q)) \\ &= \lambda\eta(H_1, \ldots, H_p, K_1, \ldots, K_q) \\ &= b_{(S,T)}. \quad \square \end{aligned}$$

Definition 11.8 *Let*

$$B = B(\mathcal{A}) = \sum_{S \in \mathbf{S}} \mathcal{K} b_S.$$

Define $B_p(\mathcal{A}) = B_p = B \cap \mathcal{T}_p$. *It follows from Lemma 11.7 that* $B = \oplus_{p \geq 0} B_p$ *is a graded subalgebra of* $\mathcal{U}$.

Lemma 11.9 *If* $S \in \mathbf{S}$ *is dependent then* $b_S = 0$. *In particular,* $b_S = 0$ *if* $|S| > \ell$ *so*

$$B = \bigoplus_{p=0}^{\ell} B_p.$$

Proof. Since S is dependent, $\cap S \neq \emptyset$. Let $S = (H_1, \ldots, H_p)$. If $S_k = (H_1, \ldots, \hat{H}_k, \ldots, H_p)$ is dependent for some k then $b_S = (-1)^{k-1} b_{(H_k, S_k)} = (-1)^{k-1} b_{H_k} b_{S_k}$ and we are done by induction. Thus we may assume that S_k is independent for each k. It follows that $\cap S_k = \cap S$ for all k. If $\pi \in Sym(p)$ let ζ be the permutation defined by $\zeta k = \pi k$ for $1 \leq k \leq p-2$, $\zeta(p-1) = \pi(p)$ and $\zeta(p) = \pi(p-1)$. Then $\text{sign}\zeta = -\text{sign}\pi$ and the terms corresponding to π and ζ in Definition 11.6 cancel. □

Example 11.10 *Consider the central 2-arrangement* $\mathcal{A} = \{H_1, \ldots, H_n\}$. *Write* $b_k = b_{H_k} = (H_k)$. *Then we have*

$$B(\mathcal{A}) = \mathcal{K} \oplus \bigoplus_{p=1}^{n} \mathcal{K} b_p \oplus \bigoplus_{k=1}^{n-1} \mathcal{K} b_k b_n.$$

We know B_0, B_1 and that $B_p = 0$ for $p > 2$. By definition

$$b_i b_j = b_{(H_i, H_j)} = (H_i, H_i \cap H_j) - (H_j, H_j \cap H_i).$$

Thus it is clear that B_2 is spanned by $b_k b_n$ for $1 \leq k < n$. It is equally clear that these generators are linearly independent and hence the sum is direct. The reader should compare this example with Example 8.8.

Example 11.11 *Recall the affine 2-arrangement* $\mathcal{A}$ *defined by* $Q(\mathcal{A}) = xy(x+y-1)$ *in* Example 2.6. *Let* $H_1 = \ker(x)$, $H_2 = \ker(y)$ *and* $H_3 = \ker(x+y-1)$.

Note that $H_1 \cap H_2 \cap H_3 = \emptyset$. Write $e_i = e_{H_i}$ and $b_i = b_{H_i}$. Then $b_1 b_2 b_3 = 0$. We have

$$B(\mathcal{A}) = \mathcal{K} \oplus (\mathcal{K} b_1 \oplus \mathcal{K} b_2 \oplus \mathcal{K} b_3) \oplus (\mathcal{K} b_1 b_2 \oplus \mathcal{K} b_2 b_3 \oplus \mathcal{K} b_3 b_1).$$

The reader should compare this example with Example 9.2.

The Isomorphism of B and A

Recall the algebra $E(\mathcal{A})$. Note that for $S = (H_1, \ldots, H_p)$ and $\pi \in Sym(p)$ we have $\eta \pi S = (\text{sign}\pi) \eta S$ and hence $b_{\pi S} = (\text{sign}\pi) b_S$. This allows us to define the following map.

Definition 11.12 *Define a* $\mathcal{K}$*-linear map* $\psi : E \to B$ *by* $\psi e_S = b_S$. *Since* $e_S e_T = e_{(S,T)}$, *the map* ψ *is a homomorphism of algebras.*

Definition 11.13 *Define a* $\mathcal{K}$*-linear map* $\tau : \mathcal{T} \to \mathcal{T}$ *by* $\tau 1 = 0$, $\tau(X) = 1$ *for* $X \in L \setminus \{V\}$, *and for* $p \geq 2$ *and* $X_i \in L \setminus \{V\}$

$$\tau(X_1, \ldots, X_p) = (-1)^{p-1}(X_1, \ldots, X_{p-1}).$$

Lemma 11.14 *If* $S \in \mathbf{S}$ *and* $\cap S \neq \emptyset$ *then* $\psi \partial e_S = \tau \psi e_S$.

Proof. Suppose $S \in \mathbf{S}_p$ then

$$\begin{aligned} \psi \partial e_S &= \lambda\Big(\sum_{k=1}^{p} (-1)^{k-1} \eta S_k\Big) \\ &= \lambda\Big(\sum_{k=1}^{p} \sum_{\zeta \in W_k} (-1)^{k-1} (\text{sign}\zeta)(H_{\zeta 1}, \ldots, \hat{H}_{\zeta k}, \ldots, H_{\zeta p})\Big) \end{aligned}$$

where W_k is the group of permutations of $1, \ldots, \hat{k}, \ldots, p$. On the other hand, since $\cap S \neq \emptyset$ we have $\tau \lambda = \lambda \tau$ and hence

$$\tau \psi e_S = \lambda\Big(\sum \text{sign}\pi (H_{\pi 1}, \ldots, H_{\pi(p-1)})\Big)$$

where π ranges over $Sym(p)$. If $\pi \in Sym(p)$ and $\pi(p) = k$ define $\zeta \in W_k$ by $\zeta i = \pi i$ for $1 \leq i \leq k-1$, $\zeta i = \pi(i-1)$ for $i > k$. Then $\text{sign}\pi = (-1)^{p-k}\text{sign}\zeta$, and the sums $\psi \partial e_S$ and $\tau \psi e_S$ are equal term for term. □

Lemma 11.15 *The map* $\psi : E \to B$ *induces a surjection of algebras* $\theta : A \to B$ *such that* $\theta a_S = b_S$.

Proof. If $\cap S = \emptyset$ then $\psi(e_S) = b_S = 0$. If S is dependent then $\psi \partial e_S = \tau \psi e_S = \tau b_S = 0$ so $\partial e_S \in \ker \psi$. Thus $I \subseteq \ker \psi$ and ψ induces a surjective map $\theta : A \to B$ such that $\theta a_S = b_S$. Since ψ is an algebra homomorphism, so is θ. □

Lemma 11.16 *If* $X \in L$ *let* $B_X = \sum_{S \in \mathbf{S}_X} \mathcal{K} b_S$. *Then*

(1) $B(\mathcal{A}) = \bigoplus_{X \in L(\mathcal{A})} B_X(\mathcal{A})$.
(2) *If* $Y \leq X$ *then* $B_Y(\mathcal{A}_X) = B_Y(\mathcal{A})$.
(3) $B_p(\mathcal{A}) = \bigoplus_{X \in L_p(\mathcal{A})} B_X(\mathcal{A})$.

Proof. Assertion (1) is immediate from the definition of $\mathcal{B}$. To prove (2) note that there is a natural inclusion $\mathcal{T}(\mathcal{A}_X) \to \mathcal{T}(\mathcal{A})$ and because intersections in L_X are the same as in L there is a natural inclusion $\mathcal{U}(\mathcal{A}_X) \to \mathcal{U}(\mathcal{A})$. Hence $B(\mathcal{A}_X) \to B(\mathcal{A})$ is an inclusion. Assertion (1) and Lemma 11.15 prove (3). □

Lemma 11.17 *Suppose $\mathcal{A}$ is a central arrangement. Then τ induces a map $\tau : B(\mathcal{A}) \to B(\mathcal{A})$ which satisfies*
(1) $\tau^2 = 0$,
(2) If $b \in B_p$ and $u \in B$ then $\tau(bu) = \tau(b)u + (-1)^p b\tau(u)$.
(3) If $\mathcal{A}$ is not empty then the complex (B, τ) is acyclic.

Proof. Properties (1) and (2) follow from the corresponding facts for ∂_E. We argue (3) as in Lemma 8.13.3. □

Theorem 11.18 *Let $\mathcal{A}$ be an arrangement. Then $\theta : A(\mathcal{A}) \to B(\mathcal{A})$ is an isomorphism of graded $\mathcal{K}$-algebras.*

Proof. Assume first that $\mathcal{A}$ is a central arrangement of rank $r = r(\mathcal{A})$ with $T = T(\mathcal{A})$. Clearly $\theta(A_X) \subseteq B_X$ so θ induces a map $\theta_X : A_X \to B_X$. It suffices to show that this map is an isomorphism for all $X \in L(\mathcal{A})$. We use induction on r. The assertion holds for the empty arrangement with $r = 0$ and $A(\mathcal{A}) = \mathcal{K} = B(\mathcal{A})$. Suppose $r > 0$. Let $X \in L(\mathcal{A})$ with $r(X) < r$. Then $r(\mathcal{A}_X) < r$ so by the induction hypothesis $\theta_X : A_X(\mathcal{A}_X) \to B_X(\mathcal{A}_X)$ is an isomorphism. We see from Proposition 8.31 that $A_X(\mathcal{A}_X) \simeq A_X(\mathcal{A})$ and from Lemma 11.16 that $B_X(\mathcal{A}_X) = B_X(\mathcal{A})$. It follows from the commutativity of the diagram

$$\begin{array}{ccc} A_X(\mathcal{A}_X) & \xrightarrow{\theta_X(\mathcal{A}_X)} & B_X(\mathcal{A}_X) \\ \downarrow & & \downarrow \\ A_X(\mathcal{A}) & \xrightarrow{\theta_X(\mathcal{A})} & B_X(\mathcal{A}) \end{array}$$

that $\theta_X(\mathcal{A})$ is an isomorphism for $X \in L$ with $r(X) < r$. It remains to prove the isomorphism for $X = T$. In the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & A_r & \to & A_{r-1} & \to & \dots & \to & A_0 & \to & 0 \\ & & \theta_r \downarrow & & \theta_{r-1} \downarrow & & & & \theta_0 \downarrow & & \\ 0 & \to & B_r & \to & B_{r-1} & \to & \dots & \to & B_0 & \to & 0 \end{array}$$

the horizontal maps are the respective boundary operators in the two acyclic complexes, so the sequences are exact. Since $B_p = \oplus_{X \in L_p} B_X$ and $A_p = \oplus_{X \in L_p} A_X$, the first part of the argument shows that all vertical maps except θ_r are isomorphisms. It follows from the diagram that θ_r is an isomorphism. This completes the argument because $A_r = A_T(\mathcal{A})$, $B_r = B_T(\mathcal{A})$ and $\theta_r = \theta_T(\mathcal{A})$.

Now assume that $\mathcal{A}$ is an affine arrangement. By Proposition 9.27, Corollary 9.29, and Lemma 11.16 we have:

$$A(\mathcal{A}) \simeq \bigoplus_{X \in L} A_X(\mathcal{A}) \simeq \bigoplus_{X \in L} A_X(\mathcal{A}_X) \simeq \bigoplus_{X \in L} B_X(\mathcal{A}_X) = \bigoplus_{X \in L} B_X(\mathcal{A}) = B(\mathcal{A}). \quad \square$$

The following results are consequences of Corollary 9.30, Theorem 9.24 and Proposition 9.31 respectively.

Corollary 11.19 *The algebra $B(\mathcal{A})$ is a free graded $\mathcal{K}$-module. The $\mathcal{K}$-modules $B_X(\mathcal{A})$ for $X \in L$ and $B_p(\mathcal{A})$ for $p \geq 0$ are also free.* □

Corollary 11.20 *The Poincaré polynomial of $B(\mathcal{A})$ is*

$$\mathrm{Poin}(B(\mathcal{A}), t) = \pi(\mathcal{A}, t). \quad \square$$

Corollary 11.21 *If $X \in L(\mathcal{A})$ then* $\mathrm{rank} B_X = (-1)^{r(X)} \mu(X)$. □

12 Differential Forms

In this section we study the algebra of differential forms generated by 1 and the differential forms $\omega_H = d\alpha_H/\alpha_H$ for $H \in \mathcal{A}$. This algebra was first computed by Arnold [6] for the braid arrangement. Brieskorn [33] defined it for all arrangements and showed that it is isomorphic to the cohomology algebra. Its isomorphism with $A(\mathcal{A})$ was established in [142] for central arrangements. In all these topological considerations the field was $\mathbb{C}$. Our presentation is based on [150], where the properties of $R(\mathcal{A})$ over an arbitrary field $\mathbb{K}$ were first studied. The results here extend the results of [150] to affine arrangements. It is important to note that the definition of $R(\mathcal{A})$ involves the polynomials α_H of degree 1, and thus this algebra is not obviously a combinatorial invariant of $\mathcal{A}$. Its combinatorial nature is a consequence of the main theorem of this section which establishes an algebra isomorphism between $A(\mathcal{A})$ and $R(\mathcal{A})$. We also study the properties of $R(\mathcal{A})$ with respect to deletion and restriction. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane $H_0 \in \mathcal{A}$. We construct linear maps $i : R(\mathcal{A}') \to R(\mathcal{A})$ and $j : R(\mathcal{A}) \to R(\mathcal{A}'')$ and prove that there is an exact sequence

$$0 \to R(\mathcal{A}') \xrightarrow{i} R(\mathcal{A}) \xrightarrow{j} R(\mathcal{A}'') \to 0.$$

The corresponding exact sequence for $A(\mathcal{A})$ will allow us to prove the isomorphism $A(\mathcal{A}) \simeq R(\mathcal{A})$ by induction.

The de Rham Complex

Let $(\mathcal{A}, V)$ be an affine arrangement. Let S be the symmetric algebra of V^* and let F be the quotient field of S. Sometimes it will be convenient to indicate the dependence of S and F on V. In this case we write $S = \mathbb{K}[V]$ and $F = \mathbb{K}(V)$. We view $F \otimes_{\mathbb{K}} V^*$ as a vector space over F by defining $f(g \otimes \alpha) = fg \otimes \alpha$ where $f, g \in F$ and $\alpha \in V^*$. There exists a unique $\mathbb{K}$-linear map $d : F \to F \otimes V^*$ such that $d(fg) = f(dg) + g(df)$ for $f, g \in F$ and $d\alpha \in \mathbb{K}$ for $\alpha \in V^*$. Recall that we have chosen a basis $x_1, \ldots, x_\ell$ for V^* so we may identify the symmetric algebra of V^* with the polynomial algebra $S = \mathbb{K}[x_1, \ldots, x_\ell]$ and its quotient field with the field of rational functions $F = \mathbb{K}(x_1, \ldots, x_\ell)$. In terms of this basis the differential df is given by the usual formula

$$df = \sum_{i=1}^{\ell} \frac{\partial f}{\partial x_i} \otimes x_i = \sum_{i=1}^{\ell} \frac{\partial f}{\partial x_i} dx_i.$$

Note that $F \otimes V^* = F dx_1 \oplus \ldots \oplus F dx_\ell$.

Definition 12.1 *Let $\Omega(V)$ be the exterior algebra of the F-vector space $F \otimes V^*$ graded by $\Omega(V) = \oplus_{p=0}^{\ell} \Omega^p(V)$ where*

$$\Omega^p(V) = \bigoplus_{1 \le i_1 < \ldots < i_p \le \ell} F dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

For simplicity of notation we write $\omega\eta = \omega \wedge \eta$ for $\omega, \eta \in \Omega(V)$. In particular we write $dx_1 \ldots dx_p = dx_1 \wedge \ldots \wedge dx_p$. We identify Ω^0 with F. The elements of $\Omega^p(V)$ are called rational differential p-forms on V. We list some well known properties of d for future reference.

Proposition 12.2 *The map $d : F \to F \otimes V^*$ may be extended in a unique way to a $\mathbb{K}$-linear map $d : \Omega(V) \to \Omega(V)$ with the following properties:*
(1) $d^2 = 0$,
(2) if $\omega \in \Omega^p(V)$ and $\eta \in \Omega(V)$ then $d(\omega\eta) = (d\omega)\eta + (-1)^p \omega(d\eta)$,
(3) if $\omega = \sum f_{i_1 \ldots i_p} dx_{i_1} \ldots dx_{i_p}$ where $1 \le i_1 < \ldots < i_p \le \ell$ and $f_{i_1 \ldots i_p} \in F$ then

$$d\omega = \sum_{j=1}^{\ell} \sum (\partial f_{i_1 \ldots i_p} / \partial x_j) dx_j dx_{i_1} \ldots dx_{i_p}. \quad \square$$

The Algebra $R(\mathcal{A})$

Let $\mathcal{K}$ be a commutative subring of the field $\mathbb{K}$.

Definition 12.3 *Let $\mathcal{A}$ be an affine arrangement. For $H \in \mathcal{A}$ let $\alpha_H \in S$ be a polynomial of degree 1 with $H = \ker(\alpha_H)$ and let $\omega_H = d\alpha_H/\alpha_H \in \Omega^1(V)$. Let $R = R(\mathcal{A})$ be the $\mathcal{K}$-subalgebra of $\Omega(V)$ generated by 1 and ω_H for $H \in \mathcal{A}$.*

Let $R_p = R \cap \Omega^p(V)$. Since R is generated by 1 and the 1-forms ω_H, it is naturally graded $R = \oplus_{p=0}^{\ell} R_p$.

To give the reader some intuitive idea why this algebra is again isomorphic to $A(\mathcal{A})$ we work out the analog of Examples 8.8 and 11.10.

Example 12.4 *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central 2-arrangement. Write $\omega_i = \omega_{H_i}$. Then*

$$R(\mathcal{A}) = \mathcal{K} \oplus \bigoplus_{p=1}^{n} \mathcal{K}\omega_p \oplus \bigoplus_{k=1}^{n-1} \mathcal{K}\omega_k\omega_n.$$

We know that $R_0 = \mathcal{K}$ and that $R_p = 0$ for $p > 2$. By definition $\omega_1, \ldots, \omega_n$ span R_1 over $\mathcal{K}$. These 1-forms are linearly independent over $\mathcal{K}$ because the rational functions $1/\alpha_1, \ldots, 1/\alpha_n$ are linearly independent over $\mathcal{K}$. Since $\omega_i^2 = 0$ and $\omega_i\omega_j = -\omega_j\omega_i$, the space R_2 is spanned over $\mathcal{K}$ by the $\omega_i\omega_j$ with $i < j$. In order to discover the remaining relations among these generators let x, y be a basis for V^* and write $\alpha_i = a_i x + b_i y$ with $a_i, b_i \in \mathbb{K}$. Then $\omega_i = (a_i/\alpha_i)dx + (b_i/\alpha_i)dy$ and we have

$$d\alpha_i d\alpha_j = (a_i b_j - b_i a_j) dx dy.$$

Thus for any i, j, k we have

$$\alpha_k d\alpha_i d\alpha_j + \alpha_i d\alpha_j d\alpha_k + \alpha_j d\alpha_k d\alpha_i =$$

$$\det \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ \alpha_i & \alpha_j & \alpha_k \end{bmatrix} dxdy = 0$$

because the third row is a linear combination of the first two. If we multiply this equation by $1/(\alpha_i\alpha_j\alpha_k)$ we get:

$$\omega_i\omega_j + \omega_j\omega_k + \omega_k\omega_i = 0.$$

In particular we have $\omega_i\omega_j = \omega_i\omega_n - \omega_j\omega_n$ if $1 \le i < j \le n$, so R_2 is spanned by the elements $\omega_k\omega_n$ for $1 \le k < n$. It remains to show that these elements are linearly independent over $\mathcal{K}$. Define an F-linear map $\partial : \Omega^2(V) \to \Omega^1(V)$ by $\partial(fdxdy) = fxdy - fydx$. Then $\partial(\omega_i\omega_j) = \omega_j - \omega_i$. If $\sum_{k=1}^{n-1} c_k\omega_k\omega_n = 0$ with $c_k \in \mathcal{K}$ then applying ∂ gives $\sum_{k=1}^{n-1} c_k(\omega_n - \omega_k) = 0$. Since $\omega_1, \ldots, \omega_n$ are linearly independent over $\mathcal{K}$, we get $c_1 = \ldots = c_{n-1} = 0$. This proves the assertion.

Next consider the analog of Examples 9.2 and 11.11.

Example 12.5 *Recall the affine 2-arrangement $\mathcal{A}$ defined by $Q(\mathcal{A}) = xy(x+y-1)$ in* Example 2.6. *Let $H_1 = \ker(x)$, $H_2 = \ker(y)$ and $H_3 = \ker(x+y-1)$.*

Note that $H_1 \cap H_2 \cap H_3 = \emptyset$. Write $\omega_i = \omega_{H_i}$. Then

$$\omega_1 = \frac{dx}{x}, \quad \omega_2 = \frac{dy}{y}, \quad \omega_3 = \frac{d(x+y-1)}{x+y-1}.$$

Note the relation $\omega_1\omega_2\omega_3 = 0$. We have

$$R(\mathcal{A}) = \mathcal{K} \oplus (\mathcal{K}\omega_1 \oplus \mathcal{K}\omega_2 \oplus \mathcal{K}\omega_3) \oplus (\mathcal{K}\omega_1\omega_2 \oplus \mathcal{K}\omega_2\omega_3 \oplus \mathcal{K}\omega_3\omega_1).$$

Lemma 12.6 *There exists a surjective homomorphism $\gamma : A(\mathcal{A}) \to R(\mathcal{A})$ of graded $\mathcal{K}$-algebras such that $\gamma(a_H) = \omega_H$ for all $H \in \mathcal{A}$.*

Proof. Define a $\mathcal{K}$-algebra homomorphism $\nu : E \to R$ by $\nu(e_H) = \omega_H$. To prove that ν induces a homomorphism $\gamma : A \to R$ we must show that $\nu(I) = 0$. Thus we need to show that if $\cap S = \emptyset$ then $\nu(e_S) = 0$ and that if $S = (H_1, \ldots, H_p)$ is dependent then $\nu(\partial e_S) = 0$. In the first case it is easy to see that there exist $c_i \in \mathcal{K}$, not all zero, with $\sum_{i=1}^p c_i\alpha_i = 1$. Thus $\sum_{i=1}^p c_i(d\alpha_i) = 0$ and hence $d\alpha_1, \ldots, d\alpha_p$ are linearly dependent. Thus we have

$$\nu(e_S) = \omega_1 \cdots \omega_p = (d\alpha_1 \cdots d\alpha_p)/(\alpha_1 \cdots \alpha_p) = 0.$$

In the second case since $\alpha_1, \ldots, \alpha_p$ is a linearly dependent set, there exist $c_i \in \mathcal{K}$, not all zero, with $\sum_{i=1}^p c_i\alpha_i = 0$. The following argument, suggested by M. Kervaire, is a simplification of the proof in [142]. We may assume that $c_p = -1$ so we have $\alpha_p = \sum_{k=1}^{p-1} c_k\alpha_k$ and $d\alpha_p = \sum_{k=1}^{p-1} c_k d\alpha_k$. Thus

(1) $$\omega_p = \sum_{k=1}^{p-1} \frac{c_k\alpha_k}{\alpha_p}\omega_k.$$

We get

$$\begin{aligned} \nu(\partial e_S) &= \sum_{k=1}^{p}(-1)^{k-1}\omega_1 \ldots \widehat{\omega_k} \ldots \omega_p \\ &= \sum_{k=1}^{p-1}(-1)^{k-1}\omega_1 \ldots \widehat{\omega_k} \ldots \omega_p + (-1)^{p-1}\omega_1 \ldots \omega_{p-1}. \end{aligned}$$

Substitute (1) to get

$$\begin{aligned} \nu(\partial e_S) &= \Big(\sum_{k=1}^{p-1}(-1)^{k-1}(-1)^{p-(k-1)}\frac{c_k\alpha_k}{\alpha_p}\Big)\omega_1 \ldots \omega_{p-1} + (-1)^{p-1}\omega_1 \ldots \omega_{p-1} \\ &= \Big((-1)^p\frac{\sum_{k=1}^{p-1} c_k\alpha_k}{\alpha_p} + (-1)^{p-1}\Big)\omega_1 \ldots \omega_{p-1} \\ &= ((-1)^p + (-1)^{p-1})\omega_1 \ldots \omega_{p-1} \\ &= 0. \end{aligned}$$

Thus $\nu(I) = 0$ and ν induces a surjective map $\gamma : A \to R$ such that $\gamma(a_H) = \omega_H$. □

Deletion and Restriction

Let $\mathcal{A}$ be a nonempty arrangement, let $H_0 \in \mathcal{A}$, and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the inductive triple with respect to H_0. Note that $R(\mathcal{A}')$ and $R(\mathcal{A})$ are both subalgebras of $\Omega(V)$ and that $R(\mathcal{A}') \subseteq R(\mathcal{A})$. We prove next that there is a short exact sequence of $\mathcal{K}$-modules

$$0 \to R(\mathcal{A}') \xrightarrow{i} R(\mathcal{A}) \xrightarrow{j} R(\mathcal{A}'') \to 0.$$

We define j with the help of the Leray residue map on differential forms. This definition is analogous to Pham's definition [153, Chap.III] in case $\mathbb{K} = \mathbb{C}$ and the forms are holomorphic. Let $\alpha_0 = \alpha_{H_0}$ and let S_0 be the localization of S at α_0. By definition S_0 is the subring of F consisting of all f/g such that $f, g \in S$ and g is prime to α_0. Let $\rho : V^* \to H_0^*$ be the restriction map and let $y_i = \rho(x_i)$. We may extend ρ uniquely to a $\mathbb{K}$-algebra homomorphism $\rho : S_0 \to \mathbb{K}(H_0)$. Both existence and uniqueness follow from the formula

$$\rho(f/g) = f(y_1, \ldots, y_\ell)/g(y_1, \ldots, y_\ell).$$

Note that $g(y_1, \ldots, y_\ell) \neq 0$ because g is prime to α_0. Define a $\mathbb{K}$-subalgebra Ω_0 of $\Omega(V)$ by

$$\Omega_0 = \oplus_{p=0}^{\ell} \bigoplus_{i_1 < \ldots < i_p} S_0 dx_{i_1} \ldots dx_{i_p}.$$

This subalgebra does not depend on the basis for V^*.

Lemma 12.7 *The map $\rho : S_0 \to \mathbb{K}(H_0)$ may be extended in a unique way to a $\mathbb{K}$-linear map $\rho : \Omega_0 \to \Omega(H_0)$ such that for $\omega, \eta \in \Omega_0$, $f \in S_0$ and $\beta \in V^*$ we have:*

(1) $\rho(\omega\eta) = \rho(\omega)\rho(\eta)$,
(2) $\rho(f\omega) = \rho(f)\rho(\omega)$,
(3) $\rho(d\beta) = d\rho(\beta)$,
(4) if $\omega = \sum f_{i_1\ldots i_p} dx_{i_1} \ldots dx_{i_p}$ *then*

$$\rho(\omega) = \sum f_{i_1\ldots i_p}(y_1, \ldots, y_\ell) dy_{i_1} \ldots dy_{i_p}.$$

Proof. If $\omega = \sum f_{i_1\ldots i_p} dx_{i_1} \ldots dx_{i_p}$ and ρ has the properties (1)-(3) then

$$\begin{aligned} \rho(\omega) &= \sum \rho(f_{i_1\ldots i_p})\rho(dx_{i_1})\ldots\rho(dx_{i_p}) \\ &= \sum \rho(f_{i_1\ldots i_p}) dy_{i_1} \ldots dy_{i_p}. \end{aligned}$$

This shows that $\rho(\omega)$ is given by (4) and proves uniqueness. To prove existence define $\rho(\omega)$ by (4) and then (1)-(3) are clear. □

Lemma 12.8 *Suppose $\beta \in V^*$ and $\beta \neq 0$. If $\omega \in \Omega_0$ and $(d\beta)\omega = 0$ then there exists $\psi \in \Omega_0$ with $\omega = (d\beta)\psi$.*

Proof. Choose a basis $x_1, \ldots, x_\ell$ for V^* such that $\beta = x_1$. We may assume that ω is a p-form. Write $\omega = \sum f_{i_1\ldots i_p} dx_{i_1} \ldots dx_{i_p}$ where $f_{i_1\ldots i_p} \in S_0$ and the sum is over all $1 \le i_1 < \ldots < i_p \le \ell$. Then

$$0 = (dx_1)\omega = \sum f_{i_1\ldots i_p} dx_1 dx_{i_1} \ldots dx_{i_p}$$

where the sum is over all $2 \le i_1 < \ldots < i_p \le \ell$. Thus $f_{i_1\ldots i_p} = 0$ if $i_1 \ge 2$. □

Definition 12.9 *Say that $\phi \in \Omega(V)$ has at most a simple pole along H_0 if $\alpha_0\phi \in \Omega_0$.*

Lemma 12.10 *Suppose $\phi \in \Omega(V)$ has at most a simple pole along H_0 and that $d\phi = 0$. Then there exist ψ, $\theta \in \Omega_0$ such that*

$$\phi = (d\alpha_0/\alpha_0)\psi + \theta.$$

The form $\rho(\psi) \in \Omega(H_0)$ is uniquely determined by ϕ.

Proof. For simplicity write $\alpha = \alpha_0$. Let $\beta \in V^*$ be the degree one homogeneous part of α. Then $d\alpha = d\beta$. Since $d\phi = 0$ it follows from Lemma 12.2.2 that $d(\alpha\phi) = (d\alpha)\phi - \alpha(d\phi) = (d\alpha)\phi = (d\beta)\phi$. Since $\alpha\phi \in \Omega_0$ by hypothesis and $d\Omega_0 \subseteq \Omega_0$, it follows from Lemma 12.8 that there exists $\theta \in \Omega_0$ such that $d(\alpha\phi) = (d\beta)\theta$. Thus $(d\beta)\phi = (d\beta)\theta$, which implies $(d\beta)\alpha(\phi - \theta) = 0$. Since $\alpha(\phi - \theta) \in \Omega_0$, it follows from Lemma 12.8 that there exists $\psi \in \Omega_0$ such that $\alpha(\phi - \theta) = (d\beta)\psi = (d\alpha)\psi$. This proves the existence of θ and ψ.

To prove the uniqueness of $\rho(\psi)$ it suffices to show that if $\psi, \theta \in \Omega_0$ and $(d\alpha/\alpha)\psi + \theta = 0$ then $\rho(\psi) = 0$. First note that $(d\beta)\theta = (d\alpha)\theta = 0$. It follows from Lemma 12.8 that there exists $\theta' \in \Omega_0$ such that $\theta = (d\beta)\theta'$. Now $(d\beta)(\psi + \alpha\theta') = (d\beta)\psi + \alpha\theta = (d\alpha)\psi + \alpha\theta = 0$. Since $\psi + \alpha\theta' \in \Omega_0$, we may apply Lemma 12.8 again to conclude that there exists $\theta'' \in \Omega_0$ with $\psi + \alpha\theta' = (d\beta)\theta'' = (d\alpha)\theta''$. Since $\rho(\alpha) = 0$, it follows from Lemma 12.7 that $\rho(\alpha\theta') = 0$ and $\rho((d\alpha)\theta'') = 0$. Thus $\rho(\psi) = 0$. □

Definition 12.11 *The uniquely determined form $\rho(\psi)$ is called the* **residue** *of ϕ along H_0. We denote it* $\mathrm{res}(\phi)$.

If $H \in \mathcal{A}$ then $d\omega_H = 0$ so $d(\omega_{H_1} \cdots \omega_{H_p}) = 0$ for all $H_1, \ldots, H_p \in \mathcal{A}$. Thus $d\phi = 0$ for all $\phi \in R(\mathcal{A})$. It is clear from the definition that each $\phi \in R(\mathcal{A})$ has at most a simple pole along H_0. Thus $\mathrm{res}(\phi)$ is defined for all $\phi \in R(\mathcal{A})$.

Lemma 12.12 *Suppose $H_1, \ldots, H_p \in \mathcal{A}'$. Then*

(1) $\mathrm{res}(\omega_{H_1} \cdots \omega_{H_p}) = 0$,
(2) $\mathrm{res}(\omega_{H_0}\omega_{H_1} \cdots \omega_{H_p}) = \omega_{H_0 \cap H_1} \cdots \omega_{H_0 \cap H_p}$,
(3) $\mathrm{res} R(\mathcal{A}) \subseteq R(\mathcal{A}'')$.

Proof. In case $p = 0$ formulas (1) and (2) are interpreted as $\mathrm{res}(1) = 0$ and $\mathrm{res}(\omega_{H_0}) = 1$. Let $\phi = \omega_{H_1} \cdots \omega_{H_p}$. We may choose $\psi = 0$ and $\theta = \phi$ in Lemma 12.10. This shows that $\mathrm{res}(\phi) = 0$ and proves (1). Now let $\phi = \omega_{H_0}\omega_{H_1} \cdots \omega_{H_p}$. We may choose $\psi = \omega_{H_1} \cdots \omega_{H_p}$ and $\theta = 0$ in Lemma 12.10. This shows that $\mathrm{res}(\phi) = \rho(\omega_{H_1} \cdots \omega_{H_p})$. By Lemma 12.7.1 we have $\rho(\omega_{H_1} \cdots \omega_{H_p}) = \rho(\omega_{H_1}) \cdots \rho(\omega_{H_p})$. It remains to show that $\rho(\omega_{H_i}) = \omega_{H_0 \cap H_i}$. If $H \in \mathcal{A}'$ then it follows from Lemma 12.7.1 and 3 that $\rho(\omega_H) = \rho(d\alpha_H/\alpha_H) = d\rho(\alpha_H)/\rho(\alpha_H)$. Since $\rho(\alpha_H)$ is a polynomial function on H_0 which defines the hyperplane $H_0 \cap H \in \mathcal{A}''$ we have $\rho(\omega_H) = \omega_{H_0 \cap H}$. This proves (2). To prove (3) note that since $\omega_{H_0}^2 = 0$ it follows from the definition of $R(\mathcal{A})$ and $R(\mathcal{A}')$ that $R(\mathcal{A}) = R(\mathcal{A}') + \omega_{H_0} R(\mathcal{A}')$. Thus (3) follows from (1) and (2). □

The Isomorphism of R and A

Theorem 12.13 *Let $\mathcal{A}$ be an arrangement and let $R(\mathcal{A})$ be the algebra of differential forms generated by 1 and $\omega_H = d\alpha_H/\alpha_H$. The map $\gamma : A(\mathcal{A}) \to R(\mathcal{A})$ induces an isomorphism of graded $\mathcal{K}$-algebras such that $\gamma(a_H) = \omega_H$.*

Theorem 12.14 *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with respect to $H_0 \in \mathcal{A}$. Let $i : R(\mathcal{A}') \to R(\mathcal{A})$ be the inclusion map and define $j : R(\mathcal{A}) \to R(\mathcal{A}'')$ by $j(\phi) = \mathrm{res}(\phi)$ for $\phi \in R(\mathcal{A})$, where $\mathrm{res}(\phi)$ is the residue of ϕ along H_0. Then there is an exact sequence:*

$$0 \to R(\mathcal{A}') \xrightarrow{i} R(\mathcal{A}) \xrightarrow{j} R(\mathcal{A}'') \to 0.$$

Proof. We prove Theorems 12.13 and 12.14 simultaneously by induction on $|\mathcal{A}|$. If $\mathcal{A}$ is empty then $A(\mathcal{A}) = \mathcal{K} = R(\mathcal{A})$ and the first result holds. The second assumes that $\mathcal{A}$ is nonempty. If $|\mathcal{A}| = 1$ then $\mathcal{A}'$ and $\mathcal{A}''$ are empty arrangements. Let $\mathcal{A} = \{H\}$. Then $R(\mathcal{A}) = \mathcal{K} + \mathcal{K}\omega_H$ and $R(\mathcal{A}') = \mathcal{K} = R(\mathcal{A}'')$ so both statements are clear. If $|\mathcal{A}| > 1$ then we see from Lemma 12.12.3 that $jR(\mathcal{A}) \subseteq R(\mathcal{A}'')$ and from Lemma 12.12.2 that j is surjective. It follows from Lemma 12.12.1 that $ji = 0$ so $\operatorname{im}(i) \subseteq \ker(j)$. To prove that $\ker(j) \subseteq \operatorname{im}(i)$ consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \rightarrow & A(\mathcal{A}') & \xrightarrow{i_{\mathcal{A}}} & A(\mathcal{A}) & \xrightarrow{j_{\mathcal{A}}} & A(\mathcal{A}'') & \rightarrow & 0 \\ & & \gamma' \downarrow & & \gamma \downarrow & & \gamma'' \downarrow & & \\ 0 & \rightarrow & R(\mathcal{A}') & \xrightarrow{i} & R(\mathcal{A}) & \xrightarrow{j} & R(\mathcal{A}'') & \rightarrow & 0 \end{array}$$

The diagram is commutative. This is clear for the left square by definition of $i_{\mathcal{A}}$ and i. For the right square it follows from Lemma 12.12. The top row is exact by Theorem 9.21. We may assume by the induction hypothesis in Theorem 12.13 that γ' and γ'' are isomorphisms. A diagram chase shows that $\ker(j) \subseteq \operatorname{im}(i)$. This proves that the second row of the diagram is exact. Thus Theorem 12.14 holds for $\mathcal{A}$. It follows from the Five Lemma that γ is an isomorphism, so Theorem 12.13 is also established for $\mathcal{A}$. □

We obtain from Corollary 9.30 and Theorem 9.24:

Corollary 12.15 *The algebra $R(\mathcal{A})$ is a free graded $\mathcal{K}$-module. The $\mathcal{K}$-module $R_p(\mathcal{A})$ is free for $p \geq 0$.* □

Corollary 12.16 *Let $\mathcal{A}$ be an arrangement and let $R(\mathcal{A})$ be the algebra of differential forms generated by 1 and $\omega_H = d\alpha_H/\alpha_H$. The Poincaré polynomial of $R(\mathcal{A})$ is:*

$$\operatorname{Poin}(R(\mathcal{A}), t) = \pi(\mathcal{A}, t).$$

Definition 12.17 *For $X \in L$ let $R_X = R_X(\mathcal{A}) = \sum \mathcal{K}\omega_{H_1} \cdots \omega_{H_p}$ where the sum is over all $(H_1, \ldots, H_p) \in \mathbf{S}_X$.*

Proposition 12.18 *We have*

$$R_p = \bigoplus_{X \in L_p} R_X.$$

Proof. The sum is direct because $R_X = \gamma(A_X)$, γ is an isomorphism and $A_p = \oplus_{X \in L_X} A_X$ by Corollary 9.29. □

Chapter IV

In this chapter we assume that all arrangements are central and use "arrangement" in place of "central hyperplane arrangement." Section 13 contains the basic definitions. In section 14 we define free arrangements and establish their fundamental properties. If $\mathcal{A}$ is free then we can associate with it a collection of non-negative integers, called its exponents, $\exp \mathcal{A} = \{b_1, \ldots, b_\ell\}$. These integers are unique up to order, but they are not necessarily distinct. In section 15 we prove the Addition–Deletion Theorem 15.14 following [186]. It asserts that if $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple then any two of the following statements imply the third:

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp \mathcal{A} &= \{b_1, \ldots, b_{\ell-1}, b_\ell\}, \\ \mathcal{A}' \text{ is free with } \exp \mathcal{A}' &= \{b_1, \ldots, b_{\ell-1}, b_\ell - 1\}, \\ \mathcal{A}'' \text{ is free with } \exp \mathcal{A}'' &= \{b_1, \ldots, b_{\ell-1}\}. \end{aligned}$$

This result leads to the definition of inductively free arrangements. We give several examples and prove that a supersolvable arrangement is inductively free. In section 16 we define the module $\Omega^p(\mathcal{A})$ of logarithmic p-forms with poles on the hypersurface $N(\mathcal{A})$. We show that the complex $\Omega^{\cdot}(\mathcal{A})$ is closed under exterior product and that $\Omega^1(\mathcal{A})$ is the dual of $D(\mathcal{A})$. We also study several lattice homology theories in this chapter. In section 17 we construct a simplicial complex $\mathsf{F}(\mathcal{A})$ associated to $L(\mathcal{A})$ by Folkman [70]. We compute its homology groups and show that $\mathsf{F}(\mathcal{A})$ has the homotopy type of a wedge of spheres. We also construct another chain complex whose homology is naturally isomorphic to the algebra $B(\mathcal{A})$ defined in section 11. We show how these constructions are related. These lattice homology theories are part of a more general theory essentially due to K. Baclawski [14]. In section 18 we generalize these constructions to order complexes with arbitrary functor coefficient. This allows proof of an important technical result due to Yuzvinsky [209]. It is used in the proof of a formula [177] for the characteristic polynomial of any arrangement. When this formula is applied to a free arrangement it yields the Factorization Theorem 18.21 of [188]. It asserts that if $\mathcal{A}$ is a free ℓ-arrangement with $\exp \mathcal{A} = \{b_1, \ldots, b_\ell\}$ then

$$\pi(\mathcal{A}, t) = (1 + b_1 t) \cdots (1 + b_\ell t).$$

The class of free arrangements contains the important class we call **reflection arrangements**, which is the subject of chapter VI.

13 The Module $D_S(\mathcal{A})$

Derivations

Recall that $S = S(V^*)$ is the symmetric algebra of the dual space V^* of V. If $x_1, \ldots, x_\ell$ is a basis for V^* then $S \simeq \mathbf{K}[x_1, \ldots, x_\ell]$. We identify S with the polynomial ring $\mathbf{K}[x_1, \ldots, x_\ell]$ by this isomorphism.

Definition 13.1 *Let* $\mathrm{Der}_{\mathbf{K}}(S)$ *be the set of* $\mathbf{K}$*-linear maps* $\theta : S \to S$ *such that*

$$\theta(fg) = f\theta(g) + g\theta(f) \qquad f, g \in S.$$

An element of $\mathrm{Der}_{\mathbf{K}}(S)$ *is called a* **derivation** *of* S *over* $\mathbf{K}$.

For $f \in S$ and $\theta_1, \theta_2 \in \mathrm{Der}_{\mathbf{K}}(S)$, define $f\theta_1 \in \mathrm{Der}_{\mathbf{K}}(S)$ and $\theta_1 + \theta_2 \in \mathrm{Der}_{\mathbf{K}}(S)$ by $f\theta_1(g) = f(\theta_1(g))$ and $(\theta_1 + \theta_2)(g) = \theta_1(g) + \theta_2(g)$ for any $g \in S$. Any $\mathbf{K}$-linear map from V^* to S can be extended uniquely to a derivation of S over $\mathbf{K}$. In particular, for any $v \in V$ there exists a unique $D_v \in \mathrm{Der}_{\mathbf{K}}(S)$ such that $D_v(\alpha) = \alpha(v)$ for any $\alpha \in V^*$. Let $e_1, \ldots, e_\ell \in V$ be the dual basis of $x_1, \ldots, x_\ell$. Define

$$D_i = D_{e_i} \qquad 1 \le i \le \ell.$$

Then D_i is the usual derivation $\partial/\partial x_i$:

$$D_i(f) = \partial f/\partial x_i, \qquad f \in S.$$

It is easy to see that $D_1, \ldots, D_\ell$ is a basis for $\mathrm{Der}_{\mathbf{K}}(S)$ over S. Thus any derivation θ of S over $\mathbf{K}$ is expressed uniquely as

$$\theta = f_1 D_1 + \cdots + f_\ell D_\ell, \qquad f_1, \ldots, f_\ell \in S.$$

It follows that $\mathrm{Der}_{\mathbf{K}}(S)$ is a free S-module of rank ℓ.

Let S_p denote the $\mathbf{K}$-vector subspace of S consisting of 0 and the homogeneous polynomials of degree p for $p \ge 0$. For $p < 0$ define $S_p = 0$. Then

$$S = \bigoplus_{p \in \mathbf{Z}} S_p$$

is a graded $\mathbf{K}$-algebra. It follows that $\deg x = 1$ for $x \in V^*$ and $x \ne 0$.

Definition 13.2 *A nonzero element* $\theta \in \mathrm{Der}_{\mathbf{K}}(S)$ *is* **homogeneous of polynomial degree** p *if* $\theta = \sum_{k=1}^{\ell} f_k D_k$ *and* $f_k \in S_p$ *for* $1 \le k \le \ell$. *In this case we write* $\mathrm{pdeg}\theta = p$. *Note that* $\mathrm{pdeg} D_i = 0$. *Let* $\mathrm{Der}_{\mathbf{K}}(S)_p$ *denote the vector space consisting of all homogeneous elements of pdegree* p *for* $p \ge 0$. *Let* $\mathrm{Der}_{\mathbf{K}}(S)_p = 0$ *if* $p < 0$.

With this pdegree function $\mathrm{Der}_{\mathbf{K}}(S)$ is a graded S-module:

$$\mathrm{Der}_{\mathbf{K}}(S) = \bigoplus_{p \in \mathbf{Z}} \mathrm{Der}_{\mathbf{K}}(S)_p.$$

If we view derivations as a subset of the set of $\mathbf{K}$-linear endomorphisms of S then there is another natural grading of $\mathrm{Der}_{\mathbf{K}}(S)$.

Definition 13.3 *A nonzero element* $\theta \in \mathrm{Der}_{\mathbf{K}}(S)$ *is* **homogeneous of total degree** r *if* $\theta(S_q) \subseteq S_{r+q}$. *In this case we write* $\mathrm{tdeg}\theta = r$. *Note that* $\mathrm{tdeg}\theta = \mathrm{pdeg}\theta - 1$. *In particular* $\mathrm{tdeg} D_i = -1$.

In this chapter polynomial degree is the natural grading in all formulas and proofs. In chapter VI functoriality requires the use of total degree. In our earlier work we sometimes used degree to denote total degree, and called the polynomial degree of θ its exponent. This led to some confusion. We hope that this new terminology will clarify the issue. See also Definitions 16.1 and 16.2.

Definition 13.4 *For any* $f \in S$, *define*

$$D(f) = \{\theta \in \mathrm{Der}_{\mathbf{K}}(S) \mid \theta(f) \in fS\}.$$

Note that $D(f)$ is an S-submodule of $\mathrm{Der}_{\mathbf{K}}(S)$.

Definition 13.5 *Let* $\mathcal{A}$ *be an arrangement in* V *with defining polynomial*

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

where $H = \ker(\alpha_H)$. *Define the* **module of** $\mathcal{A}$**–derivations** *by*

$$D(\mathcal{A}) = D(Q(\mathcal{A})).$$

Clearly $D(\mathcal{A})$ does not depend on the choice of $Q(\mathcal{A})$. In particular $D(\Phi_\ell) = \mathrm{Der}_{\mathbf{K}}(S)$ because $Q(\Phi_\ell) = 1$. An element of $D(\mathcal{A})$ is called a **derivation tangent to** $\mathcal{A}$. This terminology is justified by the topological significance of the module $D(\mathcal{A})$ in case $\mathbf{K} = \mathbf{C}$, see Proposition 19.16.

Example 13.6 *Let* $\mathcal{A}$ *be the Boolean arragement defined by* $Q(\mathcal{A}) = x_1 \cdots x_\ell$. *Then*

$$\begin{aligned}
\sum_{i=1}^{\ell} f_i D_i \ \in\ D(\mathcal{A}) & \\
\Leftrightarrow\ & \sum_{i=1}^{\ell} f_i(\partial(x_1 \cdots x_\ell)/\partial x_i) \in x_1 \cdots x_\ell S \\
\Leftrightarrow\ & (x_1 \cdots x_\ell) \sum_{i=1}^{\ell} (f_i/x_i) \in x_1 \cdots x_\ell S \\
\Leftrightarrow\ & f_i \in x_i S \qquad (1 \le i \le \ell).
\end{aligned}$$

This implies that $D(\mathcal{A})$ *is a free* S*-module with basis* $\{x_1 D_1, \ldots, x_\ell D_\ell\}$.

Basic Properties

Definition 13.7 *The* **Euler derivation** $\theta_E \in \mathrm{Der}_{\mathbb{K}}(S)$ *is defined by*

$$\theta_E = \sum_{i=1}^{\ell} x_i D_i.$$

For any homogeneous $f \in S$

$$\theta_E(f) = (\deg f) f.$$

Thus θ_E is independent of the choice of $\{x_1, \ldots, x_\ell\}$. Taking $f = Q = Q(\mathcal{A})$ we get $\theta_E(Q) = |\mathcal{A}| Q \in QS$. Thus $\theta_E \in D(\mathcal{A})$ for any arrangement $\mathcal{A}$.

Proposition 13.8

$$D(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D(\alpha_H) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

Proof. It is sufficient to prove

$$D(f_1 f_2) = D(f_1) \cap D(f_2)$$

for any $f_1, f_2 \in S$ such that f_1 and f_2 are coprime. If $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ then

$$\begin{aligned} \theta &\in D(f_1 f_2) \\ &\Leftrightarrow \theta(f_1 f_2) \in f_1 f_2 S \\ &\Leftrightarrow f_1\theta(f_2) + f_2\theta(f_1) \in f_1 f_2 S \\ &\Leftrightarrow \theta(f_i) \in f_i S \quad (i = 1, 2) \\ &\Leftrightarrow \theta \in D(f_1) \cap D(f_2). \quad \square \end{aligned}$$

Corollary 13.9 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two arrangements in V such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Then*

$$D(\mathcal{A}_1) \supseteq D(\mathcal{A}_2). \quad \square$$

Proposition 13.10 *Let $D(\mathcal{A})_p = D(\mathcal{A}) \cap \mathrm{Der}_{\mathbb{K}}(S)_p$. Then*

$$D(\mathcal{A}) = \bigoplus_{p \in \mathbb{Z}} D(\mathcal{A})_p.$$

Thus $D(\mathcal{A})$ is a graded S-submodule of $\mathrm{Der}_{\mathbb{K}}(S)$.

Proof. Decompose $\theta \in D(\mathcal{A})$ into homogeneous components:

$$\theta = \theta_0 + \theta_1 + \cdots,$$

where θ_p is zero or homogeneous of pdegree $p \geq 0$. Since the ideal QS is generated by a homogeneous element Q, each homogeneous component $\theta_p(Q)$ of $\theta(Q)$ also lies in QS. This shows that $\theta_p \in D(\mathcal{A})$ for $p \geq 0$. $\square$

Definition 13.11 *If $\theta \in Der(S)$ then $\theta = \sum \theta(x_i) D_i$. Given derivations $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ define the* **coefficient matrix** $\mathbf{M}(\theta_1, \ldots, \theta_\ell)$ *by $\mathbf{M}_{i,j} = \theta_j(x_i)$.*

Thus

$$\mathbf{M}(\theta_1, \ldots, \theta_\ell) = \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix}$$

and $\theta_j = \sum \mathbf{M}_{i,j} D_i$.

Proposition 13.12 *If $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ then $\det \mathbf{M}(\theta_1, \ldots, \theta_\ell) \in QS$.*

Proof. This is clear for $\mathcal{A} = \Phi_\ell$ since $Q(\mathcal{A}) = 1$. Let $H \in \mathcal{A}$ and let $H = \ker(\alpha_H)$, where $\alpha_H = \sum_{i=1}^{\ell} c_i x_i \in V^*$. We may assume that $c_i = 1$ for some i. Then

$$\det \mathbf{M}(\theta_1, \ldots, \theta_\ell) = \det \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & & \vdots \\ \theta_1(\alpha_H) & \cdots & \theta_\ell(\alpha_H) \\ \vdots & & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix} \in \alpha_H S.$$

Since H is arbitrary, $\det \mathbf{M}(\theta_1, \ldots, \theta_\ell)$ is divisible by all α_H, and hence by Q. $\square$

Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be two arrangements. Let $S_i = S(V_i^*)$ for $i = 1, 2$ and let $V = V_1 \oplus V_2$. Then S_1 and S_2 may be regarded as $\mathbb{K}$-subalgebras of $S = S(V^*)$. An element $\theta \in \mathrm{Der}(S_1)$ is uniquely extended to an element ϑ of $\mathrm{Der}(S)$ such that $\vartheta|_{S_2} = 0$. By this extension $\mathrm{Der}(S_i)$ may be regarded as a subset of $\mathrm{Der}(S)$ for $i = 1, 2$. The following is easy to see.

Proposition 13.13

$$\mathrm{Der}(S) = S\mathrm{Der}(S_1) \oplus S\mathrm{Der}(S_2). \quad \square$$

Proposition 13.14 *For two arrangements $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$, we have*

$$D(\mathcal{A}_1 \times \mathcal{A}_2) = SD(\mathcal{A}_1) \oplus SD(\mathcal{A}_2).$$

Proof. Write $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. For $i = 1, 2$ let $S_i = S(V_i^*)$ and let $Q_i \in S_i$ be defining polynomials for $\mathcal{A}_i$. Then $Q_1 Q_2$ is a defining polynomial for $\mathcal{A}$. Since an element of $\mathrm{Der}(S_1)$ annihilates every element in S_2, we get

$$SD(\mathcal{A}_1) \subseteq D(\mathcal{A}).$$

Thus we obtain

$$SD(\mathcal{A}_1) \oplus SD(\mathcal{A}_2) \subseteq D(\mathcal{A}).$$

By Proposition 13.13 any element $\theta \in D(\mathcal{A})$ can be written as $\theta = \theta_1 + \theta_2$ for some $\theta_i \in S\mathrm{Der}(S_i)$ with $i = 1, 2$. By symmetry we only have to prove $\theta_1 \in SD(\mathcal{A}_1)$. For that purpose we may assume that $\theta = \theta_1$. Since

$$Q_1Q_2S \ni \theta(Q_1Q_2) = Q_2\theta(Q_1) + Q_1\theta(Q_2) = Q_2\theta(Q_1),$$

we have $\theta(Q_1) \in Q_1S$. Let $G = \{g_1, g_2, \ldots\}$ be a $\mathbf{K}$-basis for S_2. For example, take G to be the set of all monomials. Note that G is linearly independent over S_1 also. There is a unique expression

$$\theta = \sum_{i \geq 1} g_i \eta_i$$

with $\eta_i \in \mathrm{Der}(S_1)$. There is also a unique expression

$$\theta(Q_1) = Q_1 \sum_{i \geq 1} g_i h_i$$

with $h_i \in S_1$. Thus we have

$$\sum_{i \geq 1} g_i(h_i Q_1) = \theta(Q_1) = \sum_{i \geq 1} g_i \eta_i(Q_1).$$

By the uniqueness of the expression we have for $i \geq 1$

$$\eta_i(Q_1) = h_i Q_1 \in Q_1 S_1$$

so $\eta_i \in D(\mathcal{A}_1)$. Thus $\theta = \sum_i g_i \eta_i \in SD(\mathcal{A}_1)$. □

14 Free Arrangements

Saito's Criterion

Definition 14.1 *An arrangement $\mathcal{A}$ is called a* **free arrangement** *if $D(\mathcal{A})$ is a free module over S.*

Example 14.2 *Let $\mathcal{A} = \Phi_\ell$. Then $Q(\mathcal{A}) = 1$ and $D_1, \ldots, D_\ell$ is a basis for $D(\mathcal{A}) = \mathrm{Der}_{\mathbf{K}}(S)$. Thus the empty arragement is free.*

Example 14.3 *Let $\mathcal{A}$ be the Boolean arragement defined by $Q(\mathcal{A}) = x_1 \cdots x_\ell$. It follows from the calculation in* Example 13.6 *that $\mathcal{A}$ is a free arrangement.*

Proposition 14.4 *If $\mathcal{A}$ is a free arrangement then $D(\mathcal{A})$ has a basis consisting of ℓ homogeneous elements.*

Proof. Let r be the rank of the free S-module $D(\mathcal{A})$, see Definition 26.4. Note that

$$Q\mathrm{Der}_{\mathbf{K}}(S) \subseteq D(\mathcal{A}) \subseteq \mathrm{Der}_{\mathbf{K}}(S),$$

Since $\mathrm{Der}_{\mathbf{K}}(S)$ contains the ℓ linearly independent elements $D_1, \ldots, D_\ell$, and $Q\mathrm{Der}_{\mathbf{K}}(S)$ contains the ℓ linearly independent elements $QD_1, \ldots, QD_\ell$, it follows from Proposition 26.3.1 that $\ell \leq r \leq \ell$. By Theorem 26.20 we can choose a basis consisting of ℓ homogeneous elements in $D(\mathcal{A})$. □

Let $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$. Recall that the (i,j)-entry of the $\ell \times \ell$ matrix $\mathsf{M}(\theta_1, \ldots, \theta_\ell)$ is $\theta_j(x_i)$. The following criterion is very useful:

Theorem 14.5 (Saito's criterion) *Given $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ the following two conditions are equivalent:*

(1) $\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) = cQ(\mathcal{A})$ *for some $c \in \mathbf{K}^*$,*

(2) $\theta_1, \ldots, \theta_\ell$ form a basis for $D(\mathcal{A})$ over S.

Proof. (1) $\Rightarrow$ (2): First note that the derivations $\theta_1, \ldots, \theta_\ell$ are linearly independent over S because $\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) \neq 0$. We may assume that $\det \mathsf{M}(\theta_1, \ldots, \theta_\ell) = Q$. It suffices to show that $\theta_1, \ldots, \theta_\ell$ generate $D(\mathcal{A})$ over S. Let $\eta \in D(\mathcal{A})$. We shall show that $\eta \in S\theta_1 + \cdots + S\theta_\ell$. Since

$$\theta_i = \sum_{j=1}^{\ell} \theta_i(x_j) D_j,$$

Cramer's rule implies that

$$QD_j \in S\theta_1 + \cdots + S\theta_\ell.$$

Write

$$Q\eta = \sum_{j=1}^{\ell} f_j \theta_j.$$

By Proposition 13.12 $\det \mathbf{M}(\theta_1,\dots,\theta_{i-1},\eta,\theta_{i+1},\dots,\theta_\ell) \in QS$. Thus

$$\begin{aligned} Q\det \mathbf{M}(\theta_1,\dots,\theta_{i-1},\eta,\theta_{i+1},\dots,\theta_\ell) &= \det \mathbf{M}(\theta_1,\dots,\theta_{i-1},Q\eta,\theta_{i+1},\dots,\theta_\ell) \\ &= \det \mathbf{M}(\theta_1,\dots,\theta_{i-1},f_i\theta_i,\theta_{i+1},\dots,\theta_\ell) \\ &= f_i \det \mathbf{M}(\theta_1,\dots,\theta_\ell) \\ &= f_iQ \\ &\in Q^2S. \end{aligned}$$

Thus $f_i \in QS$ for each i. This shows that

$$\eta = \sum_{i=1}^{\ell}(f_i/Q)\theta_i \in S\theta_1 + \cdots + S\theta_\ell.$$

(2) $\Rightarrow$ (1): By Proposition 13.12 we can write $\det \mathbf{M}(\theta_1,\dots,\theta_\ell) = fQ$ for some $f \in S$. Fix $H \in \mathcal{A}$. We can assume that $H = \ker(x_1)$. Then $Q_H = Q/x_1$ is a defining polynomial for $\mathcal{A}\setminus\{H\}$. Define $\eta_1 = QD_1$ and for $2 \le i \le \ell$ let $\eta_i = Q_HD_i$. These derivations are in $D(\mathcal{A})$. Since each η_i is an S-linear combination of $\theta_1,\dots,\theta_\ell$, there exists an $\ell \times \ell$ matrix $\mathbf{N}$ with entries in S such that $\mathbf{M}(\eta_1,\dots,\eta_\ell) = \mathbf{M}(\theta_1,\dots,\theta_\ell)\mathbf{N}$. Thus we have

$$QQ_H^{\ell-1} = \det \mathbf{M}(\eta_1,\dots,\eta_\ell) \in \det \mathbf{M}(\theta_1,\dots,\theta_\ell)S = fQS.$$

Therefore f divides $Q_H^{\ell-1}$. This is true for all $H \in \mathcal{A}$. Since the polynomials $\{Q_H^{\ell-1}\}_{H\in\mathcal{A}}$ have no common factor, $f \in \mathbb{K}^*$. □

Example 14.6 *Let $\ell = 2$. Assume that $\mathcal{A} \neq \Phi_\ell$ and let $Q = Q(\mathcal{A})$. Choose coordinates so that* $\ker(x_1) \in \mathcal{A}$ *and let $Q = x_1Q_0$. Let $\theta_E = x_1D_1 + x_2D_2$ be the Euler derivation* (13.7) *and let $\theta = Q_0D_2$. Then $\theta_E, \theta \in D(\mathcal{A})$. Moreover*

$$\det \mathbf{M}(\theta_E,\theta) = \det \begin{bmatrix} x_1 & 0 \\ x_2 & Q_0 \end{bmatrix} = Q.$$

It follows from Saito's criterion 14.5 *that θ_E and θ form a basis for $D(\mathcal{A})$. This implies that all 2-arrangements are free.*

Example 14.7 *Let $\mathcal{A}$ be the Boolean arrangement defined by $Q(\mathcal{A}) = x_1\cdots x_\ell$. We showed in* Example 13.6 *that $\theta_i = x_iD_i$ for $1 \le i \le \ell$ form a basis for $D(\mathcal{A})$. This can be verifed by applying* Saito's criterion 14.5:

$$\det \mathbf{M}(\theta_1,\dots,\theta_\ell) = \det \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_\ell \end{bmatrix} = x_1\cdots x_\ell = Q.$$

Example 14.8 *Recall the braid arrangement defined in* Example 2.10 *by*

$$Q(\mathcal{A}) = \prod_{1\le i<j\le\ell}(x_i - x_j).$$

For $1 \le k \le \ell$ define

$$\theta_k = \sum_{n=1}^{\ell} x_n^{k-1}D_n.$$

Then

$$\theta_k(x_i - x_j) = x_i^{k-1} - x_j^{k-1} \in (x_i - x_j)S$$

for $1 \le k \le \ell$ and $1 \le i < j \le \ell$. Thus $\theta_1,\dots,\theta_\ell \in D(\mathcal{A})$ by Proposition 13.8. *Moreover*

$$\det \mathbf{M}(\theta_1,\dots,\theta_\ell) = \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{\ell-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_\ell & \cdots & x_\ell^{\ell-1} \end{bmatrix}$$

is the Vandermonde determinant and thus equals $\prod_{1\le i<j\le\ell}(x_i - x_j) = Q$. It follows from Saito's criterion 14.5 *that $\theta_1,\dots,\theta_\ell$ is a basis for $D(\mathcal{A})$. Thus $\mathcal{A}$ is a free arrangement.*

In fact $\mathcal{A}$ is a reflection arrangement and we will show in chapter VI that all reflection arrangements are free.

The Exponents of a Free Arrangement

Theorem 14.9 *Let $\theta_1,\dots,\theta_\ell \in D(\mathcal{A})$ be homogeneous and linearly independent over S. Then $\mathcal{A}$ is free with basis $\theta_1,\dots,\theta_\ell$ if and only if*

$$\sum_{i=1}^{\ell} \mathrm{pdeg}\theta_i = |\mathcal{A}|.$$

Proof. Since $\theta_1,\dots,\theta_\ell$ are linearly independent $\det \mathbf{M}(\theta_1,\dots,\theta_\ell) \neq 0$. By Proposition 13.12 we may write $\det \mathbf{M}(\theta_1,\dots,\theta_\ell) = fQ$ with some nonzero *homogeneous* polynomial $f \in S$. Since $\deg \det \mathbf{M}(\theta_1,\dots,\theta_\ell) = \sum_{i=1}^{\ell} \mathrm{pdeg}\theta_i = |\mathcal{A}| = \deg Q$, we see that $f \in \mathbb{K}^*$. The conclusion follows from Saito's criterion 14.5. □

Example 14.10 *Let $\mathcal{A}$ be the arrangement consisting of all hyperplanes in an ℓ-dimensional vector space over a finite field of q elements, $\mathbb{K} = \mathbb{F}_q$.*

We showed after Definition 2.11 that $|\mathcal{A}| = 1 + q + q^2 + \cdots + q^{\ell-1}$. For $1 \le i \le \ell$ define

$$\theta_i = \sum_{j=1}^{\ell} x_j^{q^{i-1}} D_j.$$

We use the fact that $c = c^q$ for any $c \in \mathbf{K}$ to show that $\theta_i \in D(\mathcal{A})$:

$$\begin{aligned}\theta_i(\textstyle\sum_{j=1}^{\ell} c_j x_j) &= \textstyle\sum_{j=1}^{\ell} c_j x_j^{q^{i-1}} \\ &= \textstyle\sum_{j=1}^{\ell} c_j^{q^{i-1}} x_j^{q^{i-1}} \\ &= (\textstyle\sum_{j=1}^{\ell} c_j x_j)^{q^{i-1}} \quad \in (\textstyle\sum_{j=1}^{\ell} c_j x_j)S\end{aligned}$$

for any $c_1, \ldots, c_\ell \in \mathbf{K}$. In order to prove that $\det \mathbf{M}(\theta_1, \ldots, \theta_\ell) \neq 0$ by induction on ℓ, it is sufficient to consider the coefficient of $x_\ell^{q^{\ell-1}}$ in $\det \mathbf{M}(\theta_1, \ldots, \theta_\ell)$. Note that each θ_i is homogeneous with pdegree q^{i-1}. Thus

$$\sum_{i=1}^{\ell} \text{pdeg}\theta_i = 1 + q + q^2 + \cdots + q^{\ell-1} = |\mathcal{A}|.$$

It follows from Theorem 14.9 that $\mathcal{A}$ is a free arrangement with basis $\theta_1, \ldots, \theta_\ell$.

If $\mathcal{A}$ is free then by Proposition 14.4 there exists a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A})$. It follows from the general result in Proposition 26.24 that the pdegrees

$$\{\text{pdeg}\theta_1, \ldots, \text{pdeg}\theta_\ell\}$$

(with multiplicity but neglecting the order) depend only on $\mathcal{A}$.

Definition 14.11 *Let $\mathcal{A}$ be a free arrangement and let $\{\theta_1, \ldots, \theta_\ell\}$ be a homogeneous basis for $D(\mathcal{A})$. We call* pdeg$\theta_1, \ldots,$ pdegθ_ℓ *the* **exponents** *of $\mathcal{A}$ and write*

$$\exp \mathcal{A} = \{\text{pdeg}\theta_1, \ldots, \text{pdeg}\theta_\ell\}.$$

Note that $\exp \mathcal{A}$ may have repetitions and that the order should be neglected. If the integer m occurs $e \ge 0$ times in the multi-set $\exp \mathcal{A}$ we write $m^e \in \exp \mathcal{A}$. Using this notation there is a unique expression

$$\exp \mathcal{A} = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \ldots\},$$

where $e_i \ge 0$. If $e_i = 0$ then it is understood that $i \notin \exp \mathcal{A}$. For example $\{0,3,1,3,5\} = \{0^1, 1^1, 2^0, 3^2, 4^0, 5^1\}$. The next result follows from Theorem 14.9.

Proposition 14.12 *If $\mathcal{A}$ is a free ℓ-arrangement with $\exp \mathcal{A} = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \ldots\}$ then*

$$\sum_{k \ge 0} e_k = \ell, \qquad \sum_{k \ge 0} k e_k = |\mathcal{A}|. \quad \square$$

Proposition 14.13 *If $\mathcal{A} \neq \Phi$ is free, then there exists a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A})$ such that $\theta_1 = \theta_E$ is the Euler derivation.*

Proof. Let $H \in \mathcal{A}$ and let $\alpha = \alpha_H$. Then $\theta_E(\alpha) = \alpha$. Define

$$\text{Ann}(H) = \{\theta \in D(\mathcal{A}) \mid \theta(\alpha) = 0\}.$$

Then $\text{Ann}(H)$ is a graded submodule of $D(\mathcal{A})$. For any $\theta \in D(\mathcal{A})$

$$\theta - \frac{\theta(\alpha)}{\alpha}\theta_E \in \text{Ann}(H).$$

Also

$$S\theta_E \cap \text{Ann}(H) = 0.$$

Thus

$$D(\mathcal{A}) = S\theta_E \oplus \text{Ann}(H).$$

Let G be a minimal system of homogeneous generators for $\text{Ann}(H)$. Then $\{\theta_E\} \cup G$ is a minimal system of generators for $D(\mathcal{A})$. By Theorem 26.19 it is a homogeneous basis for $D(\mathcal{A})$. $\square$

Proposition 14.14 *Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be two arrangements. The product arrangement $(\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ is free if and only if both $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ are free. In this case*

$$\exp(\mathcal{A}_1 \times \mathcal{A}_2) = \{\exp \mathcal{A}_1, \exp \mathcal{A}_2\}.$$

Proof. Recall from Proposition 13.14 that

$$D(\mathcal{A}_1 \times \mathcal{A}_2) = SD(\mathcal{A}_1) \oplus SD(\mathcal{A}_2).$$

Assume that both $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ are free. Let θ_i for $1 \le i \le \ell_1$ and η_j for $1 \le j \le \ell_2$ be homogeneous bases for $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$ respectively. Then $\{\theta_1, \ldots, \theta_{\ell_1}, \eta_1, \ldots, \eta_{\ell_2}\}$ is linearly independent over S. By Theorem 14.9

$$\sum_{i=1}^{\ell_1} \text{pdeg}\theta_i + \sum_{j=1}^{\ell_2} \text{pdeg}\eta_j = |\mathcal{A}_1| + |\mathcal{A}_2| = |\mathcal{A}_1 \times \mathcal{A}_2|.$$

It follows from Theorem 14.9 that $\{\theta_1, \ldots, \theta_{\ell_1}, \eta_1, \ldots, \eta_{\ell_2}\}$ is a basis for $D(\mathcal{A}_1 \times \mathcal{A}_2)$.

Conversely, assume that $D(\mathcal{A}_1 \times \mathcal{A}_2)$ is free. Let $\{\theta_i \mid 1 \le i \le m_1\}$ and $\{\eta_j \mid 1 \le j \le m_2\}$ be minimal sets of homogeneous generators for $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$ respectively. Then by Proposition 13.14

$$\{\theta_i \mid 1 \le i \le m_1\} \cup \{\eta_j \mid 1 \le j \le m_2\}$$

is also a minimal set of homogeneous generators for $D(\mathcal{A}_1 \times \mathcal{A}_2)$ over S. By Theorem 26.19 they form a basis. In particular they are independent over S. Thus $\{\theta_i \mid 1 \le i \le m_1\}$ and $\{\eta_j \mid 1 \le j \le m_2\}$ are linearly independent over $S(V_1^*)$ and $S(V_2^*)$ respectively. $\square$

Proposition 14.15 *(1) If $\mathcal{A} = \Phi_\ell$ then $\exp\mathcal{A} = \{0^\ell\}$.*

(2) If $\mathcal{A}$ is free of rank $r(\mathcal{A})$ then $\exp\mathcal{A} = \{0^{\ell-r(\mathcal{A})}, 1^{e_1}, 2^{e_2}, \ldots\}$.

(3) If $\mathcal{A} \neq \Phi_\ell$ is free and $\exp\mathcal{A} = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \ldots\}$ then $\mathcal{A}$ is a direct product of e_1 nonempty irreducible arrangements.

(4) If $\ell = 2$ and $\mathcal{A} \neq \Phi_2$ then $\exp\mathcal{A} = \{1, |\mathcal{A}| - 1\}$.

Proof. (1) Let $\mathcal{A} = \Phi_\ell$. We showed in Example 14.2 that the exponents of $\mathcal{A}$ are all zero.

(2) Since $\mathcal{A} = \Phi_{\ell-r(\mathcal{A})} \times \mathcal{A}_0$ with an essential free arrangement $\mathcal{A}_0$, it follows from Proposition 14.14 and from (1) that

$$\exp\mathcal{A} = \{\exp\Phi_{\ell-r(\mathcal{A})}, \exp\mathcal{A}_0\} = \{0^{\ell-r(\mathcal{A})}, \exp\mathcal{A}_0\}.$$

Thus it suffices to show that if $\mathcal{A}$ is essential and free then the integer zero does not appear in $\exp\mathcal{A}$. Assume that $\mathcal{A}$ is essential and free. Let $0 \neq \theta \in D(\mathcal{A})$ be homogeneous with $\operatorname{pdeg}\theta = 0$. Then for every $H \in \mathcal{A}$, $\theta(\alpha_H) \in \alpha_H S$ and $\deg\theta(\alpha_H) = 0$. This implies that $\theta(\alpha_H) = 0$ for every $H \in \mathcal{A}$. Write $\theta = D_v$ for some $v \in V$. Therefore

$$0 = \theta(\alpha_H) = D_v(\alpha_H) = \alpha_H(v).$$

Thus $v \in \bigcap_{H\in\mathcal{A}} H = 0$. Therefore $v = 0$ and $\theta = D_v = 0$, which is a contradiction.

(3) In light of Proposition 14.14 we may assume that $\mathcal{A}$ is irreducible. It follows from Proposition 14.13 that $e_1 \geq 1$. Suppose $\theta \in D(\mathcal{A})$ and $\operatorname{pdeg}\theta = 1$. Then there exists $c_H \in \mathbf{K}$ such that $\theta(\alpha_H) = c_H\alpha_H$. We see from Proposition 13.8 that α_H is an eigenvector of the linear transformation $\theta|_{V^*}$. Since $V^* = \sum_{H\in\mathcal{A}} \mathbf{K}\alpha_H$, we see that $\theta|_{V^*}$ is semisimple. For $\lambda \in \mathbf{K}$ let

$$W_\lambda = \{\alpha \in V^* \mid \theta(\alpha) = \lambda\alpha\}.$$

Since $\theta|_{V^*}$ is semisimple, we have

$$V^* = \bigoplus_{\lambda\in\mathbf{K}} W_\lambda.$$

Choose $\lambda \in \mathbf{K}$ so that $W_\lambda \neq 0$ and let

$$W'_\lambda = \bigoplus_{\mu\neq\lambda} W_\mu.$$

Then $V^* = W_\lambda \oplus W'_\lambda$. If $H \in \mathcal{A}$ then $\alpha_H \in W_\mu$ for some $\mu \in \mathbf{K}$ so $\alpha_H \in W_\lambda \cup W'_\lambda$. Choose a basis $x_1, \ldots, x_k$ for W_λ and a basis $x_{k+1}, \ldots, x_\ell$ for W'_λ. Define

$$Q_1 = \prod_{\alpha_H\in W_\lambda} \alpha_H \in \mathbf{K}[x_1, \ldots, x_k] \quad\text{and}\quad Q_2 = \prod_{\alpha_H\in W'_\lambda} \alpha_H \in \mathbf{K}[x_{k+1}, \ldots, x_\ell].$$

Then $Q = Q_1Q_2$. This implies that $\mathcal{A}$ is a direct product of two nonempty arrangements. This is a contradiction. Therefore $W'_\lambda = 0$ and $V^* = W_\lambda$. Thus $\theta(\alpha) = \lambda\alpha$ for all $\alpha \in V^*$, so $\theta = \lambda\theta_E$, where θ_E is the Euler derivation. This implies that $e_1 = 1$.

(4) A homogeneous basis for $D(\mathcal{A})$ is given in Example 14.6. □

Corollary 14.16 *A free arrangement $\mathcal{A}$ is irreducible if and only if $e_0 = 0$ and $e_1 = 1$.* □

Examples

Example 14.17 *Let $\mathcal{A}$ be a Boolean arrangement defined by $Q(\mathcal{A}) = x_1 \cdots x_\ell$. We showed* in Example 13.6 *that the derivations $\theta_i = x_iD_i$ for $1 \leq i \leq \ell$ form a basis for $D(\mathcal{A})$. Thus $\exp\mathcal{A} = \{1^\ell\}$.*

Example 14.18 *Let $\mathcal{A}$ be the braid arrangement. We showed in* Example 14.8 *that the derivations $\theta_k = \sum_{j=1}^{\ell} x_j^k D_j$ for $0 \leq k \leq (\ell - 1)$ form a homogeneous basis for $D(\mathcal{A})$. Thus*

$$\exp\mathcal{A} = \{0, 1, \ldots, \ell - 1\}.$$

The integer 0 appears once here. On the other hand $\ell - r(\mathcal{A}) = \ell - (\ell - 1) = 1$. This is consistent with the assertion in Proposition 14.15.2. *The sum of the exponents is equal to $\ell(\ell-1)/2$, which is consistent with* Proposition 14.12.

Example 14.19 *Let $\mathcal{A}$ be the arrangement consisting of all hyperplanes in an ℓ-dimensional vector space over a finite field of q elements, $\mathbf{K} = \mathbf{F}_q$. In* Example 14.10 *we showed that for $1 \leq i \leq \ell$*

$$\theta_i = \sum_{j=1}^{\ell} x_j^{q^{i-1}} D_j$$

form a basis for $D(\mathcal{A})$. Thus $\exp\mathcal{A} = \{1, q, q^2, \ldots, q^{\ell-1}\}$. Note that the sum of the exponents $\sum_{i=1}^{\ell} q^{i-1}$ is equal to $|\mathcal{A}|$ by Proposition 14.12. *This is consistent with the calculation following* Definition 2.11.

Example 14.20 *Consider 3-arrangements $\mathcal{A}$ with $|\mathcal{A}| = 4$. If $r(\mathcal{A}) < 3$ then $\mathcal{A}$ is free by* Example 14.6. *Assume that $\mathcal{A}$ is essential. After a linear change of coordinates we may assume that $\mathcal{A}$ is defined by*

$$Q(\mathcal{A}) = x_1x_2x_3(a_1x_1 + a_2x_2 + a_3x_3)$$

where at most one of a_1, a_2, a_3 is zero.

There are two cases to consider.

(i) If one of the parameters is zero, say $a_1 = 0$, then $Q = Q_1Q_2$. Here $Q_1 = x_1$ defines a 1-arrangement $\mathcal{A}_1$ and $Q_2 = x_2x_3(a_2x_2 + a_3x_3)$ defines a 2-arrangement such that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. Clearly, $\mathcal{A}_1$ is free with $\exp\mathcal{A}_1 = \{1\}$. It follows from Proposition 14.15.4 that $\mathcal{A}_2$ is free with $\exp\mathcal{A}_2 = \{1, 2\}$. It follows from Proposition 14.14 that $\mathcal{A}$ is free with $\exp\mathcal{A} = \{1, 1, 2\}$.

(ii) If $a_1a_2a_3 \neq 0$ then $\mathcal{A}$ is not free. These are the simplest examples of non-free arrangements. We argue by assuming that $\mathcal{A}$ is free. Let

$$\exp\mathcal{A} = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \ldots\}.$$

Since $\mathcal{A}$ is irreducible and $r(\mathcal{A}) = 3$ it follows from Corollary 14.16 that $e_0 = 0$, $e_1 = 1$, and from Proposition 14.12 that $\sum_i e_i = 3$ and $\sum_i ie_i = 4$. This is impossible.

This example illustrates the fact that arrangements are in general not free. Consider the parameter space P spanned by a_1, a_2, a_3. Then P is a subset of $\mathbb{K}^3$ where at most one parameter is zero. Let S be the subset of P where one parameter is zero. Then S is the parameter space for free arrangements and $P \setminus S$ is the parameter space for non-free arrangements. Note that S is a *thin* subset of P. In case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ the set $P \setminus S$ is open and dense.

Example 14.21 *Let* $Q(\mathcal{A}) = x_1x_2x_3(x_1 - x_2)(x_2 + x_3)(x_1 + x_3)(x_1 + x_2 + x_3)$. *Then* $\mathcal{A}$ *is free for all* $\mathbb{K}$, *but its exponents depend on the characteristic of* $\mathbb{K}$.

Assume $\operatorname{char}(\mathbb{K}) \neq 2$. Then the derivations

$$\begin{aligned}\theta_1 &= x_1D_1 + x_2D_2 + x_3D_3,\\ \theta_2 &= x_1(x_1+x_3)(x_1+x_2+x_3)D_1 + x_2(x_2+x_3)(x_1+x_2+x_3)D_2,\\ \theta_3 &= x_1(x_1+x_3)(2x_2+x_3)D_1 + x_2(x_2+x_3)(2x_1+x_3)D_2\end{aligned}$$

are in $D(\mathcal{A})$ and $\det \mathbf{M}(\theta_1, \theta_2, \theta_3) = 2Q \in \mathbb{K}^*Q$. By Saito's criterion 14.5 they form a basis for $D(\mathcal{A})$. It follows that $\exp \mathcal{A} = \{1, 3, 3\}$.

For $\operatorname{char}(\mathbb{K}) = 2$ the derivations

$$\begin{aligned}\theta_1 &= x_1D_1 + x_2D_2 + x_3D_3,\\ \theta_2 &= x_1^2D_1 + x_2^2D_2 + x_3^2D_3,\\ \theta_3 &= x_1^4D_1 + x_2^4D_2 + x_3^4D_3\end{aligned}$$

are in $D(\mathcal{A})$. By Saito's criterion 14.5 they form a basis for $D(\mathcal{A})$. It follows that $\exp \mathcal{A} = \{1, 2, 4\}$. Note that

$$|\mathcal{A}| = 7 = 1 + 3 + 3 = 1 + 2 + 4$$

in agreement with Proposition 14.12.

We close the section with a discussion of subarrangements and restrictions. Note first that a free arrangement may have a subarrangement which is not free, or be the subarrangement of an arrangement which is not free.

Example 14.22 *Define 3-arrangements* $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ *by*

$$Q(\mathcal{A}) = xyz(x+y)(x+y-z), \quad Q(\mathcal{B}) = xyz(x+y-z), \quad Q(\mathcal{C}) = xyz.$$

We will show in Example 15.17 *that* $\mathcal{A}$ *is free. It follows from* Example 14.20 *that* $\mathcal{B}$ *is not free. It follows from* Example 13.6 *that* $\mathcal{C}$ *is free.*

Theorem 14.23 *If* $\mathcal{A}$ *is free then* $\mathcal{A}_X$ *is free for all* $X \in L(\mathcal{A})$.

Proof. Let $Q_X = Q(\mathcal{A}_X)$ and let $Q_0 = Q(\mathcal{A})/Q_X$. Choose $w \in M(\mathcal{A}^X)$ and note that by this choice $\alpha_H(w) = 0$ if and only if $X \subseteq H$. Thus $Q_0(w) \neq 0$. Define the affine linear map $\tau : V \to V$ by $\tau(v) = v + w$. If $e_1, \ldots, e_\ell$ is a basis for V dual to $x_1, \ldots, x_\ell$, and $w = \sum w_ie_i$ then τ induces a map, which we again call $\tau : V^* \to V^*$. This map extends to a $\mathbb{K}$-algebra isomorphism $\tau : S \to S$ given by $\tau(x_i) = x_i + w_i$ for $1 \leq i \leq \ell$. It maps the ideal $\mathcal{M}_w = (x_1 - w_1, \ldots, x_\ell - w_\ell)$ to the maximal ideal $\mathcal{M} = (x_1, \ldots, x_\ell)$. Define $\tau : \operatorname{Der}_{\mathbb{K}}(S) \to \operatorname{Der}_{\mathbb{K}}(S)$ by $\tau(\sum_{j=1}^{\ell} h_jD_j) = \sum_{j=1}^{\ell} \tau(h_j)D_j$. Note that $(\tau\theta)(\tau f) = \tau(\theta(f))$ and that $\tau Q_X = Q_X$. Since $Q_0(w) \neq 0$ we get $\tau Q_0(0) \neq 0$. Suppose that $\theta \in D(\mathcal{A})$. Then $(\tau\theta)(Q_X) = (\tau\theta)(\tau Q_X) = \tau(\theta(Q_X)) \in SQ_X$. Thus $\tau(D(\mathcal{A})) \subseteq D(\mathcal{A}_X)$. Let $\theta_1, \ldots, \theta_\ell$ be a basis for $D(\mathcal{A})$ and let $\mathbf{M} = \mathbf{M}(\theta_1, \ldots, \theta_\ell)$. We may assume that $\det \mathbf{M} = Q(\mathcal{A})$. Then

$$\begin{aligned}\det \mathbf{M}(\tau\theta_1, \ldots, \tau\theta_\ell) &= \det[\tau \mathbf{M}_{i,j}]\\ &= \tau Q(\mathcal{A})\\ &= (\tau Q_X)(\tau Q_0)\\ &= Q_X(\tau Q_0).\end{aligned}$$

Write $\tau\theta_i = \sum_{k\geq 0} \phi_i^{(k)}$ where $\operatorname{pdeg}\phi_i^{(k)} = k$. It follows that $\phi_i^{(k)} \in D(\mathcal{A}_X)$. Since $\tau Q_0(0) \neq 0$ there exist $\phi_1^{(k_1)}, \ldots, \phi_\ell^{(k_\ell)}$ such that $\det \mathbf{M}(\phi_1^{(k_1)}, \ldots, \phi_\ell^{(k_\ell)}) = cQ_X$ with $c \in \mathbb{K}^*$. Saito's criterion 14.5 implies that $\phi_1^{(k_1)}, \ldots, \phi_\ell^{(k_\ell)}$ form a basis for $D(\mathcal{A}_X)$. □

Two important questions remain unsolved.

Conjecture 14.24 *If* $\mathcal{A}$ *is free then* $\mathcal{A}^X$ *is free for all* $X \in L(\mathcal{A})$.

Conjecture 14.25 *For fixed* $\mathbb{K}$ *the property that* $\mathcal{A}$ *is free depends only on* $L(\mathcal{A})$.

15 The Addition–Deletion Theorem and Inductively Free Arrangements

In this section we study the properties of free arrangements for triples $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. If Conjecture 14.25 holds then there is no triple where $\mathcal{A}, \mathcal{A}'$ are free but $\mathcal{A}''$ is not. We may use Example 14.22 to construct a triple where $\mathcal{A}$ and $\mathcal{A}''$ are free but $\mathcal{A}'$ is not. The same example provides a triple where $\mathcal{A}'$ and $\mathcal{A}''$ are free but $\mathcal{A}$ is not. The main result of this section is the Addition–Deletion Theorem. It asserts that if $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple then any two of the following statements imply the third:

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp \mathcal{A} &= \{b_1, \dots, b_{\ell-1}, b_\ell\}, \\ \mathcal{A}' \text{ is free with } \exp \mathcal{A}' &= \{b_1, \dots, b_{\ell-1}, b_\ell - 1\}, \\ \mathcal{A}'' \text{ is free with } \exp \mathcal{A}'' &= \{b_1, \dots, b_{\ell-1}\}. \end{aligned}$$

This result leads to the definition of inductively free arrangements introduced in [186]. We give several examples and prove that a supersolvable arrangement is inductively free. This allows us to give an example of a free arrangement which is not inductively free. This example leads to the more general notion of recursively free arrangements defined by Ziegler [218]. It is not known whether every free arrangement is recursively free. We close this section with the result that the Poincaré polynomial of a recursively free arrangement factors. All derivations in this section are homogeneous.

Basis Extension

Let $\mathcal{A}$ be a nonempty arrangement defined by $Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$. Let $H_0 \in \mathcal{A}$ be a distinguished hyperplane. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the corresponding triple. Recall the map $\lambda : \mathcal{A}' \to \mathcal{A}''$ defined in Lemma 9.15 by $\lambda(H) = H_0 \cap H$. Since λ is surjective, we may define a map $\nu : \mathcal{A}'' \to \mathcal{A}'$ as in the paragraph following Corollary 9.16 by the property $\lambda\nu(X) = X$ for all $X \in \mathcal{A}''$.

Definition 15.1 *Define the polynomial*

$$b(\mathcal{A}, \nu) = \frac{Q}{\alpha_0 \prod_{X \in \mathcal{A}''} \alpha_{\nu(X)}}.$$

Lemma 15.2 *(1)* $\deg b(\mathcal{A}, \nu) = |\mathcal{A}'| - |\mathcal{A}''|$.
(2) The ideal $(\alpha_0, b(\mathcal{A}, \nu))$ is independent of the choice of ν. Thus we may write $(\alpha_0, b(\mathcal{A}))$.
(3) $(\alpha_0, b(\mathcal{A})) = (\alpha_0, \prod_{X \in \mathcal{A}''} \alpha_{\nu(X)}^{|\mathcal{A}_X|-2})$.

Proof. (1) $\deg b(\mathcal{A}) = |\mathcal{A}| - (1 + |\mathcal{A}''|) = |\mathcal{A}'| - |\mathcal{A}''|$. (2) Suppose $\rho : \mathcal{A}'' \to \mathcal{A}'$ such that $\lambda\rho(X) = X$ for all $X \in \mathcal{A}''$. Then $H_0 \cap \nu(X) = X = H_0 \cap \rho(X)$ and hence H_0, $\nu(X)$, and $\rho(X)$ are dependent. Thus $(\alpha_0, b(\mathcal{A}, \nu)) = (\alpha_0, b(\mathcal{A}, \rho))$. Part (3) follows from (2). □

Definition 15.3 *Let* $D(\mathcal{A}')\alpha_0 = \{\theta(\alpha_0) \mid \theta \in D(\mathcal{A}')\}$.

Proposition 15.4 *The ideal $D(\mathcal{A}')\alpha_0$ is contained in $(\alpha_0, b(\mathcal{A}))$.*

Proof. Since $D(\mathcal{A}')$ is an S-module, $D(\mathcal{A}')\alpha_0$ is an ideal. Let $X \in \mathcal{A}''$ and $\nu(X) \in \mathcal{A}'$. Let $\mathcal{A}'_X = \mathcal{A}_X \setminus \{H_0\}$. Then $r(\mathcal{A}'_X) \le r(\mathcal{A}_X) = 2$, so $\mathcal{A}'_X$ is a nonempty free arrangement. If we choose coordinates so that $\alpha_0 = x_1$ and $\alpha_{\nu(X)} = x_2$ then $Q(\mathcal{A}_X) \in \mathbb{K}[x_1, x_2]$ is divisible by x_1x_2. Note that $b(\mathcal{A}_X) = Q(\mathcal{A}_X)/x_1x_2$. Saito's criterion 14.5 shows that

$$\{\theta_E, b(\mathcal{A}_X)D_1, D_3, \dots, D_\ell\}$$

is a basis for $D(\mathcal{A}'_X)$. Since $D(\mathcal{A}'_X)\alpha_0 = (x_1, b(\mathcal{A}_X))$ and $D(\mathcal{A}') \subseteq D(\mathcal{A}'_X)$ we conclude that $D(\mathcal{A}')\alpha_0 \subseteq (\alpha_0, \alpha_{\nu(X)}^{|\mathcal{A}_X|-2})$ for all $X \in \mathcal{A}''$. Since the polynomials $\alpha_{\nu(X)}^{|\mathcal{A}_X|-2}$ are coprime, we have

$$\begin{aligned} D(\mathcal{A}')\alpha_0 &\subseteq \bigcap_{X \in \mathcal{A}''} (\alpha_0, \alpha_{\nu(X)}^{|\mathcal{A}_X|-2}) \\ &= (\alpha_0, \prod_{X \in \mathcal{A}''} \alpha_{\nu(X)}^{|\mathcal{A}_X|-2}) \\ &= (\alpha_0, b(X)). \quad \square \end{aligned}$$

Theorem 15.5 *Let $\mathcal{A}$ be a free arrangement with $\exp \mathcal{A} = \{b_1, \dots, b_\ell\}$, where $b_1 \le \cdots \le b_\ell$. If $\theta_1, \dots, \theta_k \in D(\mathcal{A})$ satisfy for $1 \le i \le k$,*
(1) $\operatorname{pdeg}\theta_i = b_i$,
(2) $\theta_i \notin S\theta_1 + \cdots + S\theta_{i-1}$
then $\theta_1, \dots, \theta_k$ may be extended to a basis for $D(\mathcal{A})$.

Proof. We argue by induction on k. The assertion is true for $k = 0$. For $k \ge 1$ we assume that $D(\mathcal{A})$ has a basis $\theta_1, \dots, \theta_{k-1}, \phi_k, \dots, \phi_\ell$ such that $\operatorname{pdeg}\phi_j = b_j$. Write

$$\theta_k = f_1\theta_1 + \cdots + f_{k-1}\theta_{k-1} + f_k\phi_k + \cdots + f_\ell\phi_\ell.$$

Compare homogeneous components of degree b_k. It follows from hypothesis (2) and $b_k \le b_{k+1} \le \cdots \le b_\ell$ that there is a nonzero term $f_j\phi_j$ of pdegree b_k. We may assume that $f_k\phi_k \ne 0$. Since $\operatorname{pdeg}\phi_k = b_k$, this implies that $f_k \in \mathbb{K}^*$ and we may replace ϕ_k by θ_k. □

The Map from $D(\mathcal{A})$ to $D(\mathcal{A}'')$

Definition 15.6 *Let $\bar{S} = S/\alpha_0 S$. If $\theta \in D(\mathcal{A})$ then $\theta(\alpha_0 S) \subseteq \alpha_0 S$. Thus we may define $\bar{\theta} : \bar{S} \to \bar{S}$ by $\bar{\theta}(f + \alpha_0 S) = \theta(f) + \alpha_0 S$.*

Proposition 15.7 *If $\theta \in D(\mathcal{A})$ then $\bar{\theta} \in D(\mathcal{A}'')$. If $\bar{\theta} \ne 0$ then $\operatorname{pdeg}\bar{\theta} = \operatorname{pdeg}\theta$.*

Proof. It follows from the definition that $\bar\theta \in \mathrm{Der}_{\mathbf{K}}(\bar S)$. Suppose $H \in \mathcal{A}'$ and $\alpha = \alpha_H$. Then $H \cap H_0 \in \mathcal{A}''$. If we identify $H_0^* = V^*/\alpha_0$ then $H \cap H_0 = \ker\bar\alpha$. It follows from $\theta(\alpha) \in \alpha S$ that $\bar\theta(\bar\alpha) \in \bar\alpha\bar S$ and therefore $\bar\theta \in D(\{H \cap H_0\})$. Since this holds for all $H \in \mathcal{A}'$ we get

$$\bar\theta \in \bigcap_{H\in\mathcal{A}'} D(\{H \cap H_0\}) = D(\mathcal{A}''). \quad \square$$

Proposition 15.8 *Define* $p : D(\mathcal{A}') \to D(\mathcal{A})$ *by* $p(\theta) = \alpha_0\theta$ *and* $q : D(\mathcal{A}) \to D(\mathcal{A}'')$ *by* $q(\theta) = \bar\theta$. *The sequence*

$$0 \to D(\mathcal{A}') \xrightarrow{p} D(\mathcal{A}) \xrightarrow{q} D(\mathcal{A}'')$$

is exact.

Proof. If $\alpha_0\theta = 0$ then $\theta = 0$ so p is a monomorphism. It is clear that $\mathrm{im}\,p \subseteq \ker q$. To show the reverse inclusion let $\theta \in D(\mathcal{A})$ with $\bar\theta = 0$. Then $\theta(f) \in \alpha_0 S$ for all $f \in S$. Write $\theta = \sum h_j D_j$. Then $\theta(x_j) = h_j \in \alpha_0 S$ shows that $\theta = \alpha_0\eta$ for some $\eta \in \mathrm{Der}_{\mathbf{K}}(S)$. It remains to show that $\eta \in D(\mathcal{A}')$. Let $H \in \mathcal{A}'$. Since $H \in \mathcal{A}$ we have $\theta(\alpha_H) = \alpha_0\eta(\alpha_H) \in \alpha_H S$. Since α_0 and α_H are relatively prime, we have $\eta \in D(\alpha_H)$. Since this holds for all $H \in \mathcal{A}'$, we conclude that $\eta \in D(\mathcal{A}')$. $\square$

It is natural to ask when the map q is surjective. We give an example where q is not surjective following Remark 15.19. The next theorem provides necessary conditions.

Theorem 15.9 *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are free arrangements. Then there is a basis $\{\theta_1, \dots, \theta_\ell\}$ for $D(\mathcal{A}')$ such that*

(1) $\{\theta_1, \dots, \theta_{i-1}, \alpha_0\theta_i, \theta_{i+1}, \dots, \theta_\ell\}$ is a basis for $D(\mathcal{A})$,
(2) $\{\bar\theta_1, \dots, \bar\theta_{i-1}, \bar\theta_{i+1}, \dots, \bar\theta_\ell\}$ is a basis for $D(\mathcal{A}'')$.

Proof. Let $\theta_1, \dots, \theta_\ell$ be a basis for $D(\mathcal{A}')$. We may assume that $D(\mathcal{A})$ has a basis $\{\theta_1, \dots, \theta_{i-1}, \phi_i, \phi_{i+1}, \dots, \phi_\ell\}$, where $1 \le i \le \ell$. Thus if $i = 1$ the two bases have no common element. Let $d_j = \mathrm{pdeg}\theta_j$, $e_j = \mathrm{pdeg}\phi_j$, and assume $d_1 \le \cdots \le d_\ell$ and $d_1 \le \cdots \le d_{i-1} \le e_i \le \cdots \le e_\ell$. Note that $\theta_i \in D(\mathcal{A}')$ implies that $\alpha_0\theta_i \in D(\mathcal{A})$. Thus we can write

$$\alpha_0\theta_i = \sum_{k=1}^{i-1} f_k\theta_k + \sum_{p=i}^{\ell} g_p\phi_p.$$

Since $\{\theta_1, \dots, \theta_\ell\}$ is an S-independent set, some $g_p \ne 0$ for $p \ge i$. Thus $\mathrm{pdeg}\alpha_0\theta_i \ge \mathrm{pdeg}\phi_p \ge \mathrm{pdeg}\phi_i$ and we have $d_i + 1 \ge e_i$. On the other hand $\mathcal{A}' \subset \mathcal{A}$ implies that $D(\mathcal{A}) \subset D(\mathcal{A}')$. Thus we have

$$\phi_i = \sum_{k=1}^{i-1} a_k\theta_k + \sum_{p=i}^{\ell} b_p\theta_p.$$

Since $\{\theta_1, \dots, \theta_{i-1}, \phi_i\}$ is an S-independent set, some $b_p \ne 0$ for $p \ge i$. Thus $\mathrm{pdeg}\phi_i \ge \mathrm{pdeg}\theta_p \ge \mathrm{pdeg}\theta_i$ and we have $e_i \ge d_i$. These two inequalities give

$$d_i \le e_i \le d_i + 1.$$

If $d_i = e_i$ then $\mathrm{pdeg}\theta_i = e_i$ and we may apply Theorem 15.5 to $\theta_1, \dots \theta_i$ and extend it to a basis for $D(\mathcal{A})$: $\theta_1, \dots \theta_i, \phi_{i+1}, \dots, \phi_\ell$. We can then repeat the first part of the proof with $i + 1$ in place of i. Since $\sum_{j=1}^{\ell} d_j = |\mathcal{A}'| < |\mathcal{A}| = \sum_{j=1}^{\ell} e_j$ we cannot have $d_i = e_i$ for $1 \le i \le \ell$. Thus we may assume that i is the index where $d_i + 1 = e_i$.

If $d_i + 1 = e_i$ then we apply Theorem 15.5 to the S-independent set $\theta_1, \dots, \theta_{i-1}, \alpha_0\theta_i$ and extend it to a basis for $D(\mathcal{A})$ by choosing new $\theta_{i+1}, \dots, \theta_\ell$. It follows from Saito's criterion 14.5 that $\theta_1, \dots \theta_{i-1}, \theta_i, \theta_{i+1}, \dots, \theta_\ell$ is a basis for $D(\mathcal{A}')$. This proves assertion (1).

To prove (2) recall that $\deg b(\mathcal{A}) = |\mathcal{A}'| - |\mathcal{A}''|$ and that $\theta_i(\alpha_0) \in (\alpha_0, b(\mathcal{A}))$. If $\mathrm{pdeg}\theta_i < \deg b(\mathcal{A})$ then $\theta_i(\alpha_0) \in \alpha_0 S$ and hence $\theta_i \in D(\mathcal{A})$. This contradicts (1). It follows that $\mathrm{pdeg}\theta_i \ge |\mathcal{A}'| - |\mathcal{A}''|$. Since $\bar\theta_j \ne 0$ for $j \ne i$ we have

$$(\sum_{j\ne i} \mathrm{pdeg}\bar\theta_j) \le |\mathcal{A}'| - (|\mathcal{A}'| - |\mathcal{A}''|) = |\mathcal{A}''|.$$

It remains to show that $\{\bar\theta_1, \dots, \bar\theta_{i-1}, \bar\theta_{i+1}, \dots, \bar\theta_\ell\}$ are independent over $\bar S$. Otherwise there is a dependence. Since $S \to \bar S$ is surjective, we may write this dependence as:

$$\sum_{k=1}^{i-1} \bar a_k\bar\theta_k + \sum_{p=i+1}^{\ell} \bar a_p\bar\theta_p = 0$$

with some $\bar a_m \ne 0$. Thus a_m is not divisible by α_0. It follows that for some $\theta \in \mathrm{Der}_{\mathbf{K}}(S)$

$$\sum_{k=1}^{i-1} a_k\theta_k + \sum_{p=i+1}^{\ell} a_p\theta_p = \alpha_0\theta.$$

Since the left hand side is in $D(\mathcal{A}')$, so is $\alpha_0\theta$. Since α_0 is coprime to α_H for all $H \in \mathcal{A}'$, it follows that $\theta \in D(\mathcal{A}')$. Since $\theta_1, \dots, \theta_\ell$ is an S-independent set, it follows that α_0 divides a_m for all m. This is a contradiction. $\square$

Corollary 15.10 *If $\mathcal{A}$ and $\mathcal{A}'$ are free then $\mathcal{A}''$ is free and there exist nonnegative integers $b_1 \le \cdots \le b_\ell$ such that*

$$\begin{aligned} \exp\mathcal{A} &= \{b_1, \dots, b_{k-1}, b_k + 1, b_{k+1}, \dots, b_\ell\}, \\ \exp\mathcal{A}' &= \{b_1, \dots, b_{k-1}, b_k, b_{k+1}, \dots, b_\ell\}, \\ \exp\mathcal{A}'' &= \{b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_\ell\}. \end{aligned}$$

The Addition–Deletion Theorem

Lemma 15.11 *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple with respect to H_0. Assume that $\mathcal{A}''$ is free with $\exp\mathcal{A}'' = \{b_1, \dots, b_{\ell-1}\}$ where*

$$b_1 \le \cdots \le b_{k-1} < b_k \le \cdots \le b_{\ell-1}.$$

Suppose $\theta_1, \dots, \theta_k \in D(\mathcal{A})$ such that $\mathrm{pdeg}\theta_j = b_j$ for $1 \le j \le k-1$ and $\mathrm{pdeg}\theta_k < b_k$. There exists p with $1 \le p \le k$ such that

(1) $$\theta_p \in S\theta_1 + \cdots + S\theta_{p-1} + \alpha_0 D(\mathcal{A}').$$

Proof. Assume that (1) is false for $1 \le p \le k$. It follows that for $1 \le p < k$

$$\bar{\theta}_p \notin \bar{S}\bar{\theta}_1 + \cdots + \bar{S}\bar{\theta}_{p-1}.$$

By Theorem 15.5 we may extend $\bar{\theta}_1, \ldots, \bar{\theta}_{k-1}$ to a basis for $D(\mathcal{A}'')$. Since $\mathrm{pdeg}\bar{\theta}_k < b_k$ it follows that $\bar{\theta}_k \in \bar{S}\bar{\theta}_1 + \cdots + \bar{S}\bar{\theta}_{k-1}$ and hence $\theta_k \in S\theta_1 + \cdots + S\theta_{k-1} + \alpha_0 D(\mathcal{A}')$. This contradicts the assumption that (1) is false for $1 \le p \le k$. □

Theorem 15.12 (Deletion) *If $\mathcal{A}$ and $\mathcal{A}''$ are free and $\exp\mathcal{A}'' \subset \exp\mathcal{A}$ then $\mathcal{A}'$ is free.*

Proof. Let $\exp\mathcal{A} = \{b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_\ell\}$ where $b_1 \le \cdots \le b_\ell$ and let $\exp\mathcal{A}'' = \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_\ell\}$. We may assume that either $b_i < b_{i+1}$ or $i = \ell$. Let $\theta_1, \ldots, \theta_\ell$ be a basis for $D(\mathcal{A})$ with $\mathrm{pdeg}\theta_k = b_k$ for $1 \le k \le \ell$. Apply Lemma 15.11 to $\theta_1, \ldots, \theta_i$ to find p with $1 \le p \le i$ so that

$$\theta_p \in S\theta_1 + \cdots + S\theta_{p-1} + \alpha_0 D(\mathcal{A}').$$

We may assume that $\theta_p \in \alpha_0 D(\mathcal{A}')$. It follows from Saito's criterion that

$$\theta_1, \ldots, \theta_{p-1}, \frac{\theta_p}{\alpha_0}, \theta_{p+1}, \ldots, \theta_\ell$$

is a basis for $D(\mathcal{A}')$. □

Theorem 15.13 (Addition) *If $\mathcal{A}'$ and $\mathcal{A}''$ are free and $\exp\mathcal{A}'' \subset \exp\mathcal{A}'$ then $\mathcal{A}$ is free.*

Proof. Let $\exp\mathcal{A}' = \{b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_\ell\}$ where $b_1 \le \cdots \le b_\ell$ and let $\exp\mathcal{A}'' = \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_\ell\}$. We may assume that either $b_i < b_{i+1}$ or $i = \ell$. Note that $\deg b(\mathcal{A}) = |\mathcal{A}'| - |\mathcal{A}''| = b_i$. Let $\theta_1, \ldots, \theta_\ell$ be a basis for $D(\mathcal{A}')$ with $\mathrm{pdeg}\theta_k = b_k$ for $1 \le k \le \ell$. It follows from Proposition 15.4 that $\theta_k(\alpha_0) \in (\alpha_0, b(\mathcal{A}))$. This implies that if $\mathrm{pdeg}\theta_k < b_i = \deg b(\mathcal{A})$ then $\theta_k(\alpha_0) \in \alpha_0 S$ and hence $\theta_k \in D(\mathcal{A})$.

We show first the existence of $\theta_r \in \{\theta_1, \ldots, \theta_\ell\}$ with $\mathrm{pdeg}\theta_r = b_i$ so that $\theta_r \notin D(\mathcal{A})$. Suppose not. Then $\theta_1, \ldots, \theta_i \in D(\mathcal{A})$ and we conclude from Lemma 15.11 that there exists p with $1 \le p \le i$ so that

$$\theta_p \in S\theta_1 + \cdots + S\theta_{p-1} + \alpha_0 D(\mathcal{A}').$$

We may assume $\theta_p \in \alpha_0 D(\mathcal{A}')$. Thus α_0 divides $\det \mathbf{M}(\theta_1, \ldots, \theta_\ell) = Q(\mathcal{A}')$. This is a contradiction.

The existence of $\theta_r \in \{\theta_1, \ldots, \theta_\ell\}$ with $\mathrm{pdeg}\theta_r = b_i$ so that $\theta_r \notin D(\mathcal{A})$ implies that $\theta_r(\alpha_0) \notin \alpha_0 S$. Thus $\theta_r(\alpha_0) = c_r b(\mathcal{A}) + f_r\alpha_0$ with $c_r \ne 0$. Since $\mathrm{pdeg}\theta_r = b_i = \deg b(\mathcal{A})$ we may assume $c_r = 1$. Let $\eta_r = \alpha_0\theta_r$. For $k \ne r$ write $\theta_k(\alpha_0) = c_k b(\mathcal{A}) + f_k\alpha_0$ and let $\eta_k = \theta_k - c_k\theta_r$. Then $\eta_k(\alpha_0) \in \alpha_0 S$ and therefore $\eta_k \in D(\mathcal{A})$. It follows from Saito's criterion 14.5 that $\eta_1, \ldots, \eta_\ell$ is a basis for $D(\mathcal{A})$. □

These results may be conveniently summarized in a single statement. Here we return to the convention that the exponents are unordered.

Theorem 15.14 (Addition–Deletion) *Suppose $\mathcal{A}$ is nonempty and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple. Any two of the following statements imply the third:*

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp\mathcal{A} &= \{b_1, \ldots, b_{\ell-1}, b_\ell\}, \\ \mathcal{A}' \text{ is free with } \exp\mathcal{A}' &= \{b_1, \ldots, b_{\ell-1}, b_\ell - 1\}, \\ \mathcal{A}'' \text{ is free with } \exp\mathcal{A}'' &= \{b_1, \ldots, b_{\ell-1}\}. \end{aligned}$$

It follows from Example 14.6 that every 2-arrangement is free. Thus for 3-arrangements the Addition–Deletion Theorem has a special form.

Theorem 15.15 *Let $\mathcal{A}$ be a nonempty 3-arrangement and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple. Any two of the following statements imply the third:*

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp\mathcal{A} &= \{b_1, b_2, b_3\}, \\ \mathcal{A}' \text{ is free with } \exp\mathcal{A}' &= \{b_1, b_2, b_3 - 1\}, \\ |\mathcal{A}''| &= b_1 + b_2. \end{aligned}$$

Inductively Free Arrangements

Definition 15.16 *The class $\mathcal{IF}$ of* **inductively free** *arrangements is the smallest class of arrangements which satisfies:*

(1) $\Phi_\ell \in \mathcal{IF}$ for $\ell \ge 0$.

(2) If there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{IF}$, $\mathcal{A}' \in \mathcal{IF}$ and $\exp\mathcal{A}'' \subset \exp\mathcal{A}'$ then $\mathcal{A} \in \mathcal{IF}$.

Examples

In order to show that a given arrangement $\mathcal{A}$ is inductively free, we must start with some inductively free arrangement and add hyperplanes one at a time satisfying (2). This process may be described conveniently in an **induction table**. Each row is one step in the process. The first column gives $\exp\mathcal{A}'$ of the arrangement which is the $\mathcal{A}'$ of that step. The second column gives α_H, where $H = \ker\alpha_H$ is the hyperplane added to $\mathcal{A}'$. The third column gives $\exp\mathcal{A}''$. The last row displays $\exp\mathcal{A}$. Since $Q(\mathcal{A}')$ is the product of the α_H in the rows above the row in consideration, it is easy to compute $Q(\mathcal{A}'')$. At each step the difficulty lies in showing that $\mathcal{A}''$ is free, and in computing $\exp\mathcal{A}''$.

Example 15.17 *The induction table below shows that the arrangement $\mathcal{A}$ defined by $Q(\mathcal{A}) = xyz(x+y)(x+y-z)$ is inductively free. The most delicate problem is to determine in which order to add the hyperplanes. Even in this simple example the order of the last two hyperplanes could not be reversed.*

$\exp \mathcal{A}'$	α_H	$\exp \mathcal{A}''$
0,0,0	x	0,0
0,0,1	y	0,1
0,1,1	z	1,1
1,1,1	$x+y$	1,1
1,1,2	$x+y-z$	1,2
1,2,2		

This process shows only that $\mathcal{A}$ is free. Construction of a basis for $D(\mathcal{A})$ requires more work. Let $\alpha_0 = x + y - z$. Then $\mathcal{A}'$ defined by $Q(\mathcal{A}') = xyz(x+y)$ is the product of a 1-arrangement and a 2-arrangement and hence $\mathcal{A}'$ is free. Let

$$\begin{aligned} \theta_1 &= xD_x + yD_y, \\ \theta_2 &= y(x+y)D_y, \\ \theta_3 &= zD_z. \end{aligned}$$

It follows from Saito's criterion 14.5 that $\{\theta_1, \theta_2, \theta_3\}$ is a basis for $D(\mathcal{A}')$. Choose coordinates x', y' for H_0^*. Then $Q(\mathcal{A}'') = x'y'(x'+y')$. Let $\nu(x') = x$, $\nu(y') = y$, and $\nu(x'+y') = x+y$. Then $b(\mathcal{A}) = z$. Thus we have

$$\begin{aligned} \theta_1(\alpha_0) &= x + y &&= b(\mathcal{A}) + \alpha_0, \\ \theta_2(\alpha_0) &= y(x+y) &&= yb(\mathcal{A}) + y\alpha_0, \\ \theta_3(\alpha_0) &= -z &&= -b(\mathcal{A}). \end{aligned}$$

It follows from the proof of the Addition Theorem 15.13 that we may choose the following basis for $D(\mathcal{A})$:

$$\begin{aligned} \eta_1 &= \alpha_0\theta_1 &&= (x+y-z)(xD_x + yD_y), \\ \eta_2 &= \theta_2 - y\theta_1 &&= xy(D_y - D_x), \\ \eta_3 &= \theta_3 - (-1)\theta_1 &&= xD_x + yD_y + zD_z. \end{aligned}$$

Example 15.18 *The braid arrangement is inductively free. This is argued by a double induction. For $k \le \ell$ define an ℓ-arrangement $\mathcal{A}_\ell(k)$ by $Q_\ell(k) = \prod_{1 \le i < j \le k}(x_i - x_j)$. We will show that $\mathcal{A}_\ell(k)$ is inductively free with*

$$\exp \mathcal{A}_\ell(k) = \{0^{\ell-k+1}, 1, 2, \ldots, k-1\}.$$

By induction we may assume that $\mathcal{A}_p(q)$ is inductively free with the appropriate exponents for $p < \ell, q \le p$, and that $\mathcal{A}_\ell(q)$ is inductively free with the appropriate exponents for $q < k$. We want to show that $\mathcal{A}_\ell(k)$ is inductively free. We may start with $\mathcal{A}_\ell(k-1)$ and add the hyperplanes $H_{i,k}$ for $1 \le i < k$. The crucial fact is that $\mathcal{A}'' = \mathcal{A}_{\ell-1}(k-1)$ independently of i. Thus we get the following induction table:

$\exp \mathcal{A}'$	α_H	$\exp \mathcal{A}''$
$0^{\ell-k+2}, 1, \ldots, k-2$	$x_1 - x_k$	$0^{\ell-k+1}, 1, \ldots, k-2$
$0^{\ell-k+1}, 1, 1, 2, \ldots, k-2$	$x_2 - x_k$	$0^{\ell-k+1}, 1, \ldots, k-2$
$\vdots$	$x_i - x_k$	$0^{\ell-k+1}, 1, \ldots, k-2$
$1^{\ell-k+1}, 1, \ldots, k-2, k-2$	$x_{k-1} - x_k$	$0^{\ell-k+1}, 1, \ldots, k-2$
$0^{\ell-k+1}, 1, \ldots, k-1$		

In particular we recover the exponents of the braid arrangement

$$\exp \mathcal{A} = \{0, 1, 2, \ldots, (\ell-1)\}.$$

Remark 15.19 *The map $q : D(\mathcal{A}) \to D(\mathcal{A}'')$ is not always surjective.*

Let $Q(\mathcal{A}) = xyz(x+y)(x+y-z)$ define $\mathcal{A}$. In Example 15.17 we showed that $\mathcal{A}$ is free and $\exp \mathcal{A} = \{1, 2, 2\}$. Let $\alpha_0 = x + y$. Choose coordinates x', z' for H_0^*. Then $Q(\mathcal{A}'') = x'z'$. In particular $\mathcal{A}''$ is free with $\exp \mathcal{A}'' = \{1, 1\}$. Thus the map $q : D(\mathcal{A}) \to D(\mathcal{A}'')$ is not surjective.

Proposition 15.20 *Suppose $\mathcal{A}$ and $\mathcal{A}''$ are free. The map $q : D(\mathcal{A}) \to D(\mathcal{A}'')$ is surjective if and only if $\mathcal{A}'$ is free.*

Proof. Since $\mathcal{A}$ and $\mathcal{A}''$ are free, surjectivity of q implies that $\exp \mathcal{A}'' \subset \exp \mathcal{A}$. It follows from the Deletion Theorem 15.12 that $\mathcal{A}'$ is free. If $\mathcal{A}$ and $\mathcal{A}'$ are free then the proof of Theorem 15.5 shows that q is surjective. □

Supersolvable Arrangements

Theorem 15.21 *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with a maximal chain of modular elements*

$$V = X_0 < X_1 < \cdots < X_\ell = T.$$

For $1 \le i \le \ell$ define $b_i = |\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}|$. Then $\mathcal{A}$ is inductively free with

$$\exp \mathcal{A} = \{b_1, \ldots, b_\ell\}.$$

Proof. We argue by induction on $|\mathcal{A}|$. The assertion is clear for $|\mathcal{A}| = 1$. As in Lemma 4.25, we let $H \in \mathcal{A}$ be a complement of $X_{\ell-1}$ in $L(\mathcal{A})$. Then both $\mathcal{A}'$ and $\mathcal{A}''$ are supersolvable and the induction hypothesis applies to them. It follows from Lemma 4.25 that

$$\begin{aligned} \exp \mathcal{A}' &= \{b_1, \ldots, b_\ell - 1\} \\ \exp \mathcal{A}'' &= \{b_1, \ldots, b_{\ell-1}\}. \end{aligned}$$

The conclusion follows from the Addition-Deletion Theorem 15.14. □

Example 15.22 *The central 3-arrangement $\mathcal{A}$ whose projective image $\mathbf{d}\mathcal{A}$ contains the five sides and five diagonals of a regular pentagon, together with the line at infinity, is free but not inductively free.*

The first example of a free arrangement which is not inductively free appeared in [186]. The example presented here is due to K. Brandt. To see that $\mathcal{A}$ is free we consider the arrangement $\mathcal{B}$ obtained by adding the two dotted lines in Figure 19. Then $\mathcal{B}$ is supersolvable. This can be seen directly from $L(\mathcal{B})$ or by using the Fibration Theorem 23.18, see the discussion after Remark 23.19. It follows from Theorem 15.21 that $\mathcal{B}$ is inductively free with $\exp \mathcal{B} = \{1, 5, 7\}$. Apply the Deletion Theorem 15.12 to $\mathcal{B}$ and the horizontal dotted line to get the free arrangement $\mathcal{B}'$ with $\exp \mathcal{B}' = \{1, 5, 6\}$. Apply the Deletion Theorem to $\mathcal{B}'$ and the vertical dotted line to see that $\mathcal{A}$ is free with $\exp \mathcal{A} = \{1, 5, 5\}$.

To see that $\mathcal{A}$ is not inductively free we note that the last addition would require $\exp \mathcal{A}' = \{1, 4, 5\}$. It follows from Theorem 15.15 that $|\mathcal{A}''| = 6$, but each solid line contains five intersection points.

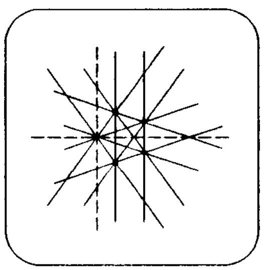


Figure 19: Free but not inductively free

Definition 15.23 *The class $\mathcal{RF}$ of* **recursively free** *arrangements is the smallest class of arrangements which satisfies:*

(1) $\Phi_\ell \in \mathcal{RF}$ for $\ell \geq 0$.

(2) If there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{RF}$, $\mathcal{A}' \in \mathcal{RF}$ and $\exp \mathcal{A}'' \subset \exp \mathcal{A}'$ then $\mathcal{A} \in \mathcal{RF}$.

(3) If there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{RF}$, $\mathcal{A} \in \mathcal{RF}$ and $\exp \mathcal{A}'' \subset \exp \mathcal{A}$ then $\mathcal{A}' \in \mathcal{RF}$.

Example 15.22 shows that $\mathcal{IF} \subseteq \mathcal{RF}$ is a proper inclusion. It is not known whether all free arrangements are recursively free.

Factorization Theorem

Theorem 15.24 *Suppose $\mathcal{A} \in \mathcal{RF}$ with $\exp \mathcal{A} = \{b_1, \ldots, b_\ell\}$. Then*

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (1 + b_i t).$$

Proof. We argue by induction on $|\mathcal{A}|$. If $|\mathcal{A}| = 0$ then $\mathcal{A} = \Phi_\ell$. Since $\pi(\Phi_\ell, t) = 1$ and $\exp \Phi_\ell = \{0^\ell\}$, the assertion holds. The induction step consists of two cases, one for addition and one for deletion. Since they are similar it suffices to show the argument for addition. Let $\mathcal{A}'$ and $\mathcal{A}''$ be recursively free with $\exp \mathcal{A}' = \{b_1, \ldots, b_{\ell-1}, b_\ell - 1\}$ and $\exp \mathcal{A}'' = \{b_1, \ldots, b_{\ell-1}\}$. By the induction hypothesis we have

$$\pi(\mathcal{A}', t) = \prod_{i=1}^{\ell-1} (1 + b_i t)(1 + (b_\ell - 1)t)$$

$$\pi(\mathcal{A}'', t) = \prod_{i=1}^{\ell-1} (1 + b_i t).$$

In Theorem 6.9 we proved the formula:

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t).$$

Thus

$$\begin{aligned} \pi(\mathcal{A}, t) &= \prod_{i=1}^{\ell-1} (1 + b_i t)(1 + (b_\ell - 1)t) + t \prod_{i=1}^{\ell-1} (1 + b_i t) \\ &= \prod_{i=1}^{\ell-1} (1 + b_i t)(1 + b_\ell t) \\ &= \prod_{i=1}^{\ell} (1 + b_i t). \quad \square \end{aligned}$$

16 The Modules $\Omega^p(\mathcal{A})$

In this section we study the S-modules $\Omega^p(\mathcal{A})$ of logarithmic differential p-forms with poles along $N = \bigcup_{H\in\mathcal{A}} H$. Modules of logarithmic differential forms with poles along a divisor with normal crossings were used by P. Deligne in [50] to define mixed Hodge structures. K. Saito [168] generalized the definition to any divisor. His work is in the analytic category. Here we consider the special case when the divisor is a union of hyperplanes, and we work in the algebraic category.

Recall the module $\Omega^p(V)$ of rational differential p-forms on V from Definition 12.1. Let $x_1, x_2, \ldots, x_\ell$ be a basis for V^* and let p be a nonnegative integer throughout this section. Let

$$\Omega^p[V] = \bigoplus_{1\le i_1<\cdots<i_p\le\ell} S(dx_{i_1}\wedge\cdots\wedge dx_{i_p}).$$

We agree that $\Omega^0[V] = S$. The elements of $\Omega^p[V]$ are called **regular** differential p-forms on V.

Definition 16.1 *An element $\omega \in \Omega^p[V]$ is homogeneous of* **polynomial degree** q *if the coefficient of each $dx_{i_1}\wedge\cdots\wedge dx_{i_p}$ $(1\le i_1<i_2<\cdots<i_p\le\ell)$ is homogeneous of degree q. In this case we write $\mathrm{pdeg}\,\omega = q$. Let $\Omega^p_q[V]$ denote the vector space consisting of all homogeneous p-forms of polynomial degree q for $q\in\mathbb{Z}$. Note that $\Omega^p_q[V]=0$ if $q<0$. This gives $\Omega^p[V]$ a graded S-module structure:*

$$\Omega^p[V] = \bigoplus_{q\in\mathbb{Z}} \Omega^p_q[V].$$

This grading of $\Omega^p[V]$ by polynomial degree is similar to the grading of $\mathrm{Der}_{\mathbb{K}}(S)$ in Definition 13.2. For example $\mathrm{pdeg}(dx_1) = 0$, $\mathrm{pdeg}(x_2dx_1) = 1$. We may also define a total degree as in Definition 13.3.

Definition 16.2 *We say that $\omega\in\Omega^p_q[V]$ has* **total degree** $p+q$ *and write $\mathrm{tdeg}\,\omega = p+q$.*

Definition of $\Omega^p(\mathcal{A})$

Definition 16.3 *Let p be a nonnegative integer. Let Q be a defining polynomial for $\mathcal{A}$. The module $\Omega^p(\mathcal{A})$ of* **logarithmic p-forms with poles along** $\mathcal{A}$ *is defined as*

$$\Omega^p(\mathcal{A}) = \{\omega\in\Omega^p(V) \mid Q\omega\in\Omega^p[V] \text{ and } Q(d\omega)\in\Omega^{p+1}[V]\}.$$

It is easy to see that $\Omega^p(\mathcal{A})$ is closed under addition and under multiplication by an element of S. Thus it is an S-module. Let

$$\Omega(\mathcal{A}) = \bigoplus_{p\ge 0}\Omega^p(\mathcal{A}).$$

Example 16.4 *Let $Q = xyz$ define the Boolean 3-arrangement.*

Here $\omega_1 = dx/x \in \Omega^1(\mathcal{A})$ because $Q\omega_1 = yzdx \in \Omega^1[V]$ and $d\omega_1 = 0$. On the other hand, $\omega_2 = dx/y \notin \Omega^1(\mathcal{A})$ because

$$Q(d\omega_2) = Q(dx\wedge dy)/y^2 = (xzdx\wedge dy)/y \notin \Omega^2[V].$$

Proposition 16.5 *For any arrangement $\mathcal{A}$ with defining polynomial Q,*
(1) $dQ/Q\in\Omega^1(\mathcal{A})$.
(2) Let $\alpha\in V^*$. Then $d\alpha/\alpha\in\Omega^1(\mathcal{A}) \iff \ker(\alpha)\in\mathcal{A}$. □

Basic Properties of $\Omega^p(\mathcal{A})$

Lemma 16.6 *Let Q define a nonempty arrangement. Then Q and the partial derivatives $\partial Q/\partial x_1, \partial Q/\partial x_2, \ldots, \partial Q/\partial x_\ell$ have no common factor of positive degree.*

Proof. Suppose f is a nonconstant common factor. Then there exists a linear form $\alpha\in V^*$ which divides Q and f. We may assume that for some i we have $\partial\alpha/\partial x_i = 1$. Write $Q=\alpha Q'$. Since Q is square free, Q' is not divisible by α. But

$$Q' = Q'(\partial\alpha/\partial x_i) = \partial(\alpha Q')/\partial x_i - \alpha(\partial Q'/\partial x_i) = \partial Q/\partial x_i - \alpha(\partial Q'/\partial x_i)$$

shows that Q' is divisible by α. This is a contradiction. □

For the extreme values of p it is easy to compute $\Omega^p(\mathcal{A})$.

Proposition 16.7 *(1) $\Omega^p(\mathcal{A}) = 0$ for $p>\ell$,*
(2) $\Omega^\ell(\mathcal{A}) = (1/Q)\Omega^\ell[V]$,
(3) $\Omega^0(\mathcal{A}) = S$.

Proof. (1) and (2) are obvious from the definition. Let $f/Q\in\Omega^0(\mathcal{A})$ with $f\in S$. If $\mathcal{A}$ is the empty arrangement then $Q=1$ and there is nothing to prove. Suppose $\mathcal{A}$ is nonempty. We have

$$Qd(f/Q) = df - (fdQ/Q)\in\Omega^1[V].$$

Thus $f(\partial Q/\partial x_i)\in QS$ for $1\le i\le\ell$. It follows from Lemma 16.6 that $f\in QS$. □

Proposition 16.8 *For $\omega\in\Omega^p(V)$, the following three conditions are equivalent:*
(1) $\omega\in\Omega^p(\mathcal{A})$,
(2) $Q\omega\in\Omega^p[V]$ and $dQ\wedge\omega\in\Omega^{p+1}[V]$,
(3) $Q\omega\in\Omega^p[V]$ and $Q(d\alpha/\alpha)\wedge\omega\in\Omega^{p+1}[V]$ for all $\alpha\in V^*$ with $\ker(\alpha)\in\mathcal{A}$.

Proof. The equivalence of (1) and (2) follows from the formula

$$d(Q\omega) = Q(d\omega) + (dQ) \wedge \omega.$$

Let $Q = \prod_{i=1}^{n} \alpha_i$ be a defining polynomial for $\mathcal{A}$. Then

$$dQ \wedge \omega = \sum_{i=1}^{n} Q(d\alpha_i/\alpha_i) \wedge \omega.$$

This shows that (3) implies (2). For the converse assume (2) and let $\alpha = \alpha_1$. Write $Q = \alpha Q'$. We have

$$Q(dQ \wedge \omega) = Q'(d\alpha \wedge (Q\omega)) + \alpha(dQ' \wedge (Q\omega)).$$

Thus each coefficient of $Q'(d\alpha \wedge (Q\omega))$ is a polynomial divisible by α. Since Q' is not divisible by α, each coefficient of $d\alpha \wedge (Q\omega)$ is a polynomial divisible by α. This proves (3). □

Corollary 16.9 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two arrangements in V with $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Let $Q_3 \in S$ be a defining polynomial of the arrangement $\mathcal{A}_2 \setminus \mathcal{A}_1$. Then*

$$Q_3\Omega^p(\mathcal{A}_2) \subseteq \Omega^p(\mathcal{A}_1) \subseteq \Omega^p(\mathcal{A}_2).$$

Proof. For $i = 1, 2$ let Q_i be a defining polynomial for $\mathcal{A}_i$. Then $Q_2 = Q_1Q_3$. If $\omega \in \Omega^p(\mathcal{A}_1)$ then

$$dQ_2 \wedge \omega = d(Q_1Q_3) \wedge \omega = dQ_3 \wedge (Q_1\omega) + Q_3(dQ_1 \wedge \omega)$$

is a regular differential form. It follows from Proposition 16.8 that $\omega \in \Omega^p(\mathcal{A}_2)$.

Suppose $\eta \in \Omega^p(\mathcal{A}_2)$. To show that $Q_3\eta \in \Omega^p(\mathcal{A}_1)$ we verify the conditions of Proposition 16.8.3. Clearly $Q_1(Q_3\eta) = Q_2\eta$ is regular. Now suppose that $\alpha \in V^*$ and $\ker(\alpha) \in \mathcal{A}_1$. Then

$$Q_1(d\alpha/\alpha) \wedge (Q_3\eta) = Q_2(d\alpha/\alpha) \wedge \eta.$$

Since $\eta \in \Omega^p(\mathcal{A}_2)$ and $\ker(\alpha) \in \mathcal{A}_2$, the last form is regular by Proposition 16.8. □

Definition 16.10 *An element $\omega \in \Omega^p(\mathcal{A})$ is homogeneous of* **polynomial degree** *q if the regular differential form $Q\omega \in \Omega^p[V]$ is homogeneous of polynomial degree $(q + \deg Q)$. Let $\Omega^p_q(\mathcal{A})$ denote the vector space consisting of all homogeneous p-forms of polynomial degree q for $q \in \mathbb{Z}$. Note that $\Omega^p_q(\mathcal{A}) = 0$ if $q < -\deg Q$.*

Thus for example $\mathrm{pdeg}(dx_1) = 0$, $\mathrm{pdeg}(dQ/Q) = -1$.

Proposition 16.11 *We have*

$$\Omega^p(\mathcal{A}) = \bigoplus_{q \in \mathbb{Z}} \Omega^p_q(\mathcal{A}).$$

Thus $\Omega^p(\mathcal{A})$ is a graded S-module.

Proof. Let $\omega \in \Omega^p(\mathcal{A})$. Decompose $Q\omega \in \Omega^p[V]$ into homogeneous components

$$Q\omega = \sum_{q \in \mathbb{Z}} \omega_q,$$

where $\omega_q \in \Omega^p_q[V]$ for each q. We obtain

$$Q(dQ \wedge \omega) = \sum_{q \in \mathbb{Z}} (dQ \wedge \omega_q).$$

This implies that each coefficient on the left hand side belongs to the ideal QS. Since QS is generated by the homogeneous element Q, each coefficient of $dQ \wedge \omega_q$ belongs to QS. This shows that $(\omega_q/Q) \in \Omega^p(\mathcal{A})$. □

Let $1 \le p \le \ell$. There is an S-bilinear pairing $\langle\, ,\, \rangle$ between $\mathrm{Der}_{\mathbf{K}}(S)$ and $\Omega^1(V)$ called the **interior product**. More generally, the interior product

$$\langle\, ,\, \rangle : \mathrm{Der}_{\mathbf{K}}(S) \times \Omega^p(V) \longrightarrow \Omega^{p-1}(V)$$

is the unique S-bilinear map which satisfies

$$\langle \theta, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \rangle = \sum_{k=1}^{p} (-1)^{k-1} \theta(x_{i_k}) dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_k}} \wedge \cdots \wedge dx_{i_p}$$

for $1 \le i_1 < \cdots < i_p \le \ell$. Here $\widehat{dx_{i_k}}$ denotes the deletion of dx_{i_k}. Restriction of the interior product gives an S-bilinear pairing

$$\langle\, ,\, \rangle : \mathrm{Der}_{\mathbf{K}}(S) \times \Omega^p[V] \longrightarrow \Omega^{p-1}[V].$$

The following properties of the interior product are well known.

Lemma 16.12 *If $\theta \in \mathrm{Der}_{\mathbf{K}}(S)$, $f \in S$, $\omega_1 \in \Omega^p(V)$, and $\omega_2 \in \Omega^q(V)$ then*
(1) $\langle \theta, \omega_1 \wedge \omega_2 \rangle = \langle \theta, \omega_1 \rangle \wedge \omega_2 + (-1)^p \omega_1 \wedge \langle \theta, \omega_2 \rangle$.
(2) $\langle \theta, df \rangle = \theta(f)$.

Proof. Since both sides are S-linear with respect to ω_1 and ω_2, we may assume that $\omega_1 = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $\omega_2 = dx_{j_1} \wedge \cdots \wedge dx_{j_q}$, where $1 \le i_1 < \cdots < i_p \le \ell$ and $1 \le j_1 < \cdots < j_q \le \ell$. Then

$$\begin{aligned} \langle \theta, \omega_1 \wedge \omega_2 \rangle &= \langle \theta, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \rangle \\ &= \langle \theta, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \rangle \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\ &\quad + (-1)^p dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge \langle \theta, dx_{j_1} \wedge \cdots \wedge dx_{j_q} \rangle \\ &= \langle \theta, \omega_1 \rangle \wedge \omega_2 + (-1)^p \omega_1 \wedge \langle \theta, \omega_2 \rangle. \end{aligned}$$

$$\begin{aligned}\langle\theta, df\rangle &= \left\langle\theta, \sum_{i=1}^{\ell}(\partial f/\partial x_i)dx_i\right\rangle \\ &= \sum_{i=1}^{\ell}(\partial f/\partial x_i)\langle\theta, dx_i\rangle = \sum_{i=1}^{\ell}(\partial f/\partial x_i)\theta(x_i) = \theta(f). \quad \square\end{aligned}$$

Proposition 16.13 *For any arrangement $\mathcal{A}$, the interior product induces maps*

$$\langle\ ,\ \rangle : D(\mathcal{A}) \times \Omega^p(\mathcal{A}) \longrightarrow \Omega^{p-1}(\mathcal{A}).$$

Proof. Let $\theta \in D(\mathcal{A})$ and $\omega \in \Omega^p(\mathcal{A})$. It follows from Proposition 16.8 that $dQ \wedge \omega \in \Omega^{p+1}[V]$. Since $\theta(Q) \in QS$, the calculation

$$\begin{aligned}\langle\theta, dQ\wedge\omega\rangle &= \langle\theta, dQ\rangle\,\omega - dQ\wedge\langle\theta,\omega\rangle \\ &= \theta(Q)\omega - dQ\wedge\langle\theta,\omega\rangle\end{aligned}$$

shows that $dQ \wedge \langle\theta,\omega\rangle \in \Omega^p[V]$. $\square$

It follows from Proposition 16.7 that $\Omega^0(\mathcal{A}) = S$. Thus setting $p = 1$ in Proposition 16.13 gives an S-bilinear pairing

$$\langle\ ,\ \rangle : D(\mathcal{A}) \times \Omega^1(\mathcal{A}) \longrightarrow S.$$

Theorem 16.14 *The S-modules $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are duals of each other: $D(\mathcal{A})^* \simeq \Omega^1(\mathcal{A})$ and $\Omega^1(\mathcal{A})^* \simeq D(\mathcal{A})$.*

Proof. The interior product pairing provides natural S-linear maps

$$\alpha : D(\mathcal{A}) \longrightarrow \Omega^1(\mathcal{A})^*,$$

$$\beta : \Omega^1(\mathcal{A}) \longrightarrow D(\mathcal{A})^*.$$

We will prove that they are isomorphisms.

α is injective: Let $\theta \in D(\mathcal{A})$. Suppose $\alpha(\theta) = 0$. This implies $0 = [\alpha(\theta)](\omega) = \langle\theta,\omega\rangle$ for all $\omega \in \Omega^1(\mathcal{A})$. Thus $\theta(f) = \langle\theta, df\rangle = 0$ for all $f \in S$. This shows that $\theta = 0$.

α is surjective: Let $\eta \in \Omega^1(\mathcal{A})^*$. Define a $\mathbb{K}$-linear map $\tilde{\eta} : S \to S$ by $\tilde{\eta}(f) = \eta(df)$ for $f \in S$. It is easy to check that $\tilde{\eta}$ is a derivation. We have

$$\tilde{\eta}(Q) = \eta(dQ) = Q\eta(dQ/Q) \in QS$$

because $dQ/Q \in \Omega^1(\mathcal{A})$. This implies that $\tilde{\eta} \in D(\mathcal{A})$. Moreover, we have

$$[\alpha(\tilde{\eta})](dx_i) = \langle\tilde{\eta}, dx_i\rangle = \tilde{\eta}(x_i) = \eta(dx_i)$$

for $1 \le i \le \ell$. Therefore $\alpha(\tilde{\eta}) = \eta$. Thus α is a bijection.

β is injective: Let $\omega \in \Omega^1(\mathcal{A})$. Suppose $\beta(\omega) = 0$. This implies $0 = [\beta(\omega)](\theta) = \langle\theta,\omega\rangle$ for all $\theta \in D(\mathcal{A})$. Write

$$\omega = (1/Q)\sum_{i=1}^{\ell} a_i dx_i$$

using a basis for V^*. Then $0 = \langle QD_i, \omega\rangle = a_i$ for $1 \le i \le \ell$. This shows that $\omega = 0$.

β is surjective: Let $\omega \in D(\mathcal{A})^*$. Define a rational 1-form $\tilde{\omega}$ by

$$\tilde{\omega} = (1/Q)\sum_{i=1}^{\ell}\omega(QD_i)dx_i.$$

The coefficient of $dx_i \wedge dx_j$ $(i < j)$ in $dQ \wedge \tilde{\omega}$ is equal to:

$$(1/Q)[(\partial Q/\partial x_i)\omega(QD_j) - (\partial Q/\partial x_j)\omega(QD_i)] = \omega((\partial Q/\partial x_i)D_j - (\partial Q/\partial x_j)D_i).$$

Since $(\partial Q/\partial x_i)D_j - (\partial Q/\partial x_j)D_i \in D(\mathcal{A})$, we know that each coefficient of $dQ \wedge \tilde{\omega}$ belongs to S. It follows that $dQ \wedge \tilde{\omega} \in \Omega^2[V]$. This implies that $\tilde{\omega} \in \Omega^1(\mathcal{A})$ by Proposition 16.8. Moreover we have

$$\begin{aligned}[\beta(\tilde{\omega})](\theta) &= \langle\theta, \tilde{\omega}\rangle \\ &= \left\langle\theta, (1/Q)\sum_{i=1}^{\ell}\omega(QD_i)dx_i\right\rangle \\ &= (1/Q)\sum_{i=1}^{\ell}\omega(QD_i)\theta(x_i) = (1/Q)\omega(Q\sum_{i=1}^{\ell}\theta(x_i)D_i) = (1/Q)\omega(Q\theta) = \omega(\theta).\end{aligned}$$

Therefore $\beta(\tilde{\omega}) = \omega$. Thus β is a bijection. $\square$

Since $\Omega^1(\mathcal{A})$ and $D(\mathcal{A})$ are S-duals of each other, we have the following consequences.

Corollary 16.15 *The arrangement $\mathcal{A}$ is free if and only if $\Omega^1(\mathcal{A})$ is a free S-module.* $\square$

Corollary 16.16 *Assume that the arrangement $\mathcal{A}$ is free with $\exp\mathcal{A} = \{b_1, \ldots, b_\ell\}$. Then $\Omega^1(\mathcal{A})$ has a homogeneous basis $\omega_1, \omega_2, \ldots, \omega_\ell$ with $\mathrm{pdeg}(\omega_i) = -b_i$ for $1 \le i \le \ell$.* $\square$

Next we give more characterizations of logarithmic differential forms.

Proposition 16.17 *The following statements are equivalent:*

(1) $\omega \in \Omega^p(\mathcal{A})$,
(2) $Q\omega \in \Omega^p[V]$ and $dQ \wedge \omega \in \Omega^{p+1}[V]$,
(3) $Q\omega \in \Omega^p[V]$ and $Q(d\alpha/\alpha) \wedge \omega \in \Omega^{p+1}[V]$ for all $\alpha \in V^*$ with $\ker(\alpha) \in \mathcal{A}$,
(4) there exist $\xi_j \in \Omega^{p-1}[V]$ and $\eta_j \in \Omega^p[V]$ for $1 \le j \le \ell$ so that

$$(\partial Q/\partial x_j)\omega = (dQ/Q) \wedge \xi_j + \eta_j,$$

(5) there exist polynomials $f_1, \ldots, f_k \in S$ and forms $\xi_j \in \Omega^{p-1}[V]$ and $\eta_j \in \Omega^p[V]$ so that $f_1, \ldots, f_k$ and Q have no common factor of positive degree and $f_j\omega = (dQ/Q) \wedge \xi_j + \eta_j$ for $1 \le j \le k$.

Proof. The equivalence of (1), (2), and (3) was proved in Proposition 16.8.

(2) $\Rightarrow$ (4) Let $\theta = D_j$, $\omega_1 = dQ$, and $\omega_2 = \omega$ in Lemma 16.12.1. Then we have

$$\begin{aligned}\langle D_j, dQ\wedge\omega\rangle &= \langle D_j, dQ\rangle\,\omega - dQ\wedge\langle D_j,\omega\rangle\\ &= (\partial Q/\partial x_j)\omega - (dQ/Q)\wedge\langle D_j, Q\omega\rangle.\end{aligned}$$

Define $\xi_j = \langle D_j, Q\omega\rangle$ and $\eta_j = \langle D_j, dQ\wedge\omega\rangle$ for $1\le j\le \ell$.

(4) $\Rightarrow$ (5) This follows from Lemma 16.6 since the polynomials $\partial Q/\partial x_1,\ldots,\partial Q/\partial x_\ell$ and Q have no common factor of positive degree.

(5) $\Rightarrow$ (2) We have

$$f_j(Q\omega) = dQ\wedge\xi_j + Q\eta_j \in \Omega^p[V],$$

and

$$f_j(dQ\wedge\omega) = dQ\wedge\eta_j \in \Omega^{p+1}[V].$$

Since the polynomials $f_1,\ldots,f_k$, and Q have no common factor of positive degree, this shows that $Q\omega\in\Omega^p[V]$ and $dQ\wedge\omega\in\Omega^{p+1}[V]$. □

Proposition 16.18 *Let $\Omega^{\cdot}(\mathcal{A}) = \bigoplus_{p\ge 0}\Omega^p(\mathcal{A})$. The S-module $\Omega^{\cdot}(\mathcal{A})$ is closed under exterior product:*

$$\Omega^p(\mathcal{A})\times\Omega^q(\mathcal{A}) \longrightarrow \Omega^{p+q}(\mathcal{A}).$$

Proof. Let $\omega_1\in\Omega^p(\mathcal{A})$ and $\omega_2\in\Omega^q(\mathcal{A})$. By Proposition 16.17 there exist $\xi_j^{(i)}\in\Omega^{p-1}[V]$ and $\eta_j^{(i)}\in\Omega^p[V]$ with $i=1,2$ and $1\le j\le\ell$ so that

$$(\partial Q/\partial x_j)\omega_i = (dQ/Q)\wedge\xi_j^{(i)} + \eta_j^{(i)}\quad (i=1,2).$$

Thus we have

$$(\partial Q/\partial x_j)^2\omega_1\wedge\omega_2 = (dQ/Q)\wedge(\xi_j^{(1)}\wedge\eta_j^{(2)} + \xi_j^{(2)}\wedge\eta_j^{(1)}) + (\eta_j^{(1)}\wedge\eta_j^{(2)}).$$

Since $(\partial Q/\partial x_1)^2,\ldots,(\partial Q/\partial x_\ell)^2$ and Q have no common factor of positive degree, we may apply Proposition 16.17 again to conclude that $\omega_1\wedge\omega_2\in\Omega^{p+q}(\mathcal{A})$. □

Recall the $\mathbb{K}$-algebra $R(\mathcal{A})$ from Definition 12.3. We noted in Proposition 16.5 that its generators are in $\Omega^1(\mathcal{A})$. It follows from Proposition 16.18 that $R(\mathcal{A})\subseteq\Omega^{\cdot}(\mathcal{A})$. Since elements of $R(\mathcal{A})$ have total degree 0, we get

$$R(\mathcal{A})\subseteq\bigoplus_{p+q=0}\Omega^p_q(\mathcal{A}).$$

Next we prove the analog of Saito's criterion 14.5 for differential forms.

Proposition 16.19 *Given $\omega_1,\ldots,\omega_\ell$ the following two conditions are equivalent:*
(1) $\omega_1,\ldots,\omega_\ell$ form a basis for $\Omega^1(\mathcal{A})$ over S,
(2) $\omega_1\wedge\cdots\wedge\omega_\ell = (c/Q)(dx_1\wedge\cdots\wedge dx_\ell)$ for some $c\in\mathbb{K}^*$.

Proof. Write

$$\omega_j = \sum_{k=1}^{\ell} a_{j,k}dx_k,$$

and let $\mathsf{N} = [a_{j,k}]_{1\le j,k\le\ell}$. Then we have

$$\omega_1\wedge\cdots\wedge\omega_\ell = (\det\mathsf{N})(dx_1\wedge\cdots\wedge dx_\ell).$$

(1) $\Rightarrow$ (2) Since $\Omega^1(\mathcal{A})^*\simeq D(\mathcal{A})$ by Theorem 16.14, $D(\mathcal{A})$ has the dual basis $\theta_1,\ldots,\theta_\ell$ defined by $\langle\theta_i,\omega_j\rangle = \delta_{i,j}$. Recall the coefficient matrix

$$\mathsf{M} = \mathsf{M}(\theta_1,\ldots,\theta_\ell) = [\theta_j(x_i)]_{1\le i,j\le\ell}$$

from Definition 13.11. Since

$$\delta_{i,j} = \langle\theta_i,\omega_j\rangle = \sum_{k=1}^{\ell}\theta_i(x_k)a_{j,k},$$

we have $\mathsf{NM} = \mathsf{I}$. By Saito's criterion 14.5, $\det\mathsf{M}\in\mathbb{K}^*Q$. Thus $\det\mathsf{N}\in\mathbb{K}^*Q^{-1}$.

(2) $\Rightarrow$ (1) We may assume that

$$\omega_1\wedge\cdots\wedge\omega_\ell = (1/Q)(dx_1\wedge\cdots\wedge dx_\ell).$$

Thus $\det\mathsf{N} = Q^{-1}$. For $1\le i\le\ell$ define $\theta_i\in\Omega^1(\mathcal{A})^*$ by

$$\theta_i(\omega)(1/Q)(dx_1\wedge\cdots\wedge dx_\ell) = \omega_1\wedge\cdots\wedge\omega_{i-1}\wedge\omega\wedge\omega_{i+1}\wedge\cdots\wedge\omega_\ell.$$

By identifying $\Omega^1(\mathcal{A})^*$ with $D(\mathcal{A})$, we may assume that $\theta_i\in D(\mathcal{A})$. Then $\langle\theta_i,\omega_j\rangle = \delta_{i,j}$ and hence

$$\mathsf{NM}(\theta_1,\ldots,\theta_\ell) = \mathsf{I}.$$

Thus $\det\mathsf{M}(\theta_1,\ldots,\theta_\ell) = Q$. Apply Saito's criterion 14.5 again to conclude that $D(\mathcal{A})$ is free. It follows that $\Omega^1(\mathcal{A})\simeq D(\mathcal{A})^*$ is also free. □

Proposition 16.20 *If $\Omega^1(\mathcal{A})$ is a free S-module with basis $\omega_1,\ldots,\omega_\ell$ then for $1\le p\le\ell$ the S-module $\Omega^p(\mathcal{A})$ is free with basis $\{\omega_{i_1}\wedge\ldots\wedge\omega_{i_p}\mid 1\le i_1<\cdots<i_p\le\ell\}$.*

Proof. By Proposition 16.19 we may assume that

$$\omega_1\wedge\cdots\wedge\omega_\ell = (1/Q)(dx_1\wedge\cdots\wedge dx_\ell).$$

Given a multiindex $K = (k_1,\ldots,k_s)$, define

$$\begin{aligned}\omega_K &= \omega_{k_1}\wedge\cdots\wedge\omega_{k_s},\\ dx_K &= dx_{k_1}\wedge\cdots\wedge dx_{k_s}.\end{aligned}$$

Let $I = (i_1, \ldots, i_p)$ be an ordered multiindex of lenth p, $1 \le i_1 < \cdots < i_p \le \ell$. Let $I^c = (j_1, \ldots, j_{\ell-p})$ be the ordered complement of I, $1 \le j_1 < \cdots < j_{\ell-p} \le \ell$. Thus

$$\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{\ell-p}\} = \{1, \ldots, \ell\}.$$

Let $\sigma(I)$ be the sign of the permutation $(i_1, \ldots, i_p, j_1, \ldots, j_{\ell-p})$. Let $\mathcal{I}$ be the set of ordered multiindices of lenght p.

Let $\omega \in \Omega^p(\mathcal{A})$, and let $I \in \mathcal{I}$. Since $\Omega^\cdot(\mathcal{A})$ is closed under exterior product by Proposition 16.18, we have

$$\omega \wedge \omega_{I^c} \in \Omega^\ell(\mathcal{A}) = (1/Q)\Omega^\ell[V].$$

Define $f_I \in S$ by

$$\omega \wedge \omega_{I^c} = (f_I/Q)(dx_1 \wedge \cdots \wedge dx_\ell).$$

Let $\eta = \omega - \sum_{I \in \mathcal{I}} \sigma(I) f_I \omega_I$. Then $\eta \wedge \omega_{I^c} = 0$ for all $I \in \mathcal{I}$. Since $dx_1, \ldots, dx_\ell \in \Omega^1(\mathcal{A})$, each dx_i is a linear combination of $\omega_1, \ldots, \omega_\ell$ over S. Thus for any $K = (k_1, \ldots, k_{\ell-p})$, dx_K is a linear combination of $\{\omega_{I^c} \mid I \in \mathcal{I}\}$. This implies that $\eta \wedge dx_K = 0$. Therefore $\eta = 0$ and

$$\omega = \sum_{I \in \mathcal{I}} \sigma(I) f_I \omega_I.$$

This shows that $\{\omega_{i_1} \wedge \ldots \wedge \omega_{i_p} \mid 1 \le i_1 < \cdots < i_p \le \ell\}$ spans $\Omega^p(\mathcal{A})$. To show that this set is S-independent, assume $\sum_{I \in \mathcal{I}} f_I \omega_I = 0$. By taking exterior product with ω_{I^c} we get $f_I = 0$. □

Next we prove some facts about product arrangements. Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be arrangements. Recall their product $\mathcal{A}_1 \times \mathcal{A}_2$ from Definition 4.13. Let $V = V_1 \oplus V_2$, $S = S(V^*)$, and for $i = 1, 2$ let $S_i = S(V_i^*)$. We may identify $S = S_1 \otimes_{\mathbf{K}} S_2$. For $i = 1, 2$ let $Q_i \in S_i$ be a defining polynomial for $\mathcal{A}_i$.

Lemma 16.21 *For $1 \le i \le m$ let $\eta_i \in (1/Q_1)\Omega^p[V_1]$ and let $\xi_i \in (1/Q_2)\Omega^q[V_2]$. If ξ_i are linearly independent over $\mathbf{K}$ and $\sum_{i=1}^m \eta_i \wedge \xi_i \in \Omega^n(\mathcal{A}_1 \times \mathcal{A}_2)$ then $\eta_i \in \Omega^p(\mathcal{A}_1)$ for $1 \le i \le m$.*

Proof. Note that

$$Q(dQ) \wedge (\sum_{i=1}^m \eta_i \wedge \xi_i) = \sum_{i=1}^m (dQ_1 \wedge Q_1\eta_i) \wedge Q_2(Q_2\xi_i) + (-1)^p \sum_{i=1}^m Q_1(Q_1\eta_i) \wedge (dQ_2 \wedge Q_2\xi_i).$$

Since $\sum_{i=1}^m \eta_i \wedge \xi_i \in \Omega^n(\mathcal{A}_1 \times \mathcal{A}_2)$, we have $Q(dQ) \wedge (\sum_{i=1}^m \eta_i \wedge \xi_i) \in Q\Omega^{n+1}[V]$. Since $Q_2^2\xi_i$ are linearly independent over $\mathbf{K}$, they are linearly independent over S_1. Thus for $1 \le i \le m$ we have $dQ_1 \wedge Q_1\eta_i \in Q_1\Omega^{p+1}[V_1]$ and hence $\eta_i \in \Omega^p(\mathcal{A}_1)$. □

Proposition 16.22 $\Omega^n(\mathcal{A}_1 \times \mathcal{A}_2) \simeq \bigoplus_{p+q=n} \Omega^p(\mathcal{A}_1) \otimes_{\mathbf{K}} \Omega^q(\mathcal{A}_2)$.

Proof. Let $\omega \in \Omega^n(\mathcal{A}_1 \times \mathcal{A}_2)$. To show that $\omega \in \bigoplus_{p+q=n} \Omega^p(\mathcal{A}_1) \otimes_{\mathbf{K}} \Omega^q(\mathcal{A}_2)$ we may assume that $\omega \in (1/Q_1)\Omega^p[V_1] \otimes (1/Q_2)\Omega^q[V_2]$. Write $\omega = \sum_{i=1}^m \eta_i \wedge \xi_i$ where for $1 \le i \le m$ we have $\eta_i \in (1/Q_1)\Omega^p[V_1]$, $\xi_i \in (1/Q_2)\Omega^q[V_2]$, and the ξ_i are linearly independent over $\mathbf{K}$. It follows from Lemma 16.21 that $\eta_i \in \Omega^p(\mathcal{A}_1)$ for $1 \le i \le m$. Thus $\omega \in \Omega^p(\mathcal{A}_1) \otimes_{\mathbf{K}} (1/Q_2)\Omega^q[V_2]$. Write $\omega = \sum_{i=1}^k \sigma_i \otimes \tau_i$ where for $1 \le i \le k$ we have $\sigma_i \in \Omega^p(\mathcal{A}_1)$, $\tau_i \in (1/Q_2)\Omega^q[V_2]$, and the σ_i are linearly independent over $\mathbf{K}$. It follows from Lemma 16.21 that $\tau_i \in \Omega^q(\mathcal{A}_2)$ for $1 \le i \le k$. Thus $\omega \in \Omega^p(\mathcal{A}_1) \otimes_{\mathbf{K}} \Omega^q(\mathcal{A}_2)$. □

The Acyclic Complex $(\Omega^\cdot(\mathcal{A}), \partial)$

Assume that $\mathcal{A}$ is a nonempty arrangement. Fix a hyperplane $H \in \mathcal{A}$. Let $\alpha = \alpha_H \in V^*$ with $H = \ker(\alpha)$. Recall that $d\alpha/\alpha \in \Omega^1(\mathcal{A})$. We showed in Proposition 16.18 that $\Omega^\cdot(\mathcal{A})$ is closed under exterior product.

Definition 16.23 *Define $\partial : \Omega^p \to \Omega^{p+1}(\mathcal{A})$ by $\partial(\omega) = (d\alpha/\alpha) \wedge \omega$. This map is homogeneous of pdegree -1. Since $\partial \cdot \partial = 0$, we have a complex*

$$0 \to \Omega^0(\mathcal{A}) \xrightarrow{\partial} \Omega^1(\mathcal{A}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^\ell(\mathcal{A}) \to 0.$$

Proposition 16.24 *The complex $(\Omega^\cdot(\mathcal{A}), \partial)$ is acyclic.*

Proof. Recall the Euler derivation $\theta_E = \sum_{i=1}^\ell x_i D_i$ from Definition 13.7. Let $\omega \in \Omega^p(\mathcal{A})$ be a cocycle, $\partial(\omega) = (d\alpha/\alpha) \wedge \omega = 0$. It follows from Proposition 16.13 that $\langle \theta_E, \omega \rangle \in \Omega^{p-1}(\mathcal{A})$. By Lemma 16.12.1 we have

$$\begin{aligned} \partial(\langle \theta_E, \omega \rangle) &= (d\alpha/\alpha) \wedge \langle \theta_E, \omega \rangle \\ &= -\langle \theta_E, (d\alpha/\alpha) \wedge \omega \rangle + \langle \theta_E, (d\alpha/\alpha) \rangle \wedge \omega \\ &= (\theta_E(d\alpha)/\alpha) \wedge \omega \\ &= \omega. \ \square \end{aligned}$$

The η-Complex $(\Omega^\cdot(\mathcal{A}), \partial_h)$

We will define other boundary maps in $\Omega^\cdot(\mathcal{A})$. Let $\mathcal{A}$ be a possibly empty arrangement. Let d be a positive integer. Let $\eta \in \Omega_d^1[V]$ be a homogeneous regular differential 1-form of pdegree d on V. Note that $\eta \in \Omega^1(\mathcal{A})$ because $\Omega^1[V] \subseteq \Omega^1(\mathcal{A})$.

Definition 16.25 *Define $\partial_\eta : \Omega^p(\mathcal{A}) \to \Omega^{p+1}(\mathcal{A})$ by $\partial_\eta(\omega) = \eta \wedge \omega$. This map is homogeneous of pdegree d. Since $\partial_\eta \cdot \partial_\eta = 0$, we have a complex*

$$0 \to \Omega^0(\mathcal{A}) \xrightarrow{\partial_\eta} \Omega^1(\mathcal{A}) \xrightarrow{\partial_\eta} \cdots \xrightarrow{\partial_\eta} \Omega^\ell(\mathcal{A}) \to 0.$$

We call it the **η-complex** $(\Omega^\cdot(\mathcal{A}), \partial_\eta)$.

In the rest of this section we assume that the field $\mathbf{K}$ is infinite. Our aim is to prove that the cohomology groups of the η-complex are finite dimensional over $\mathbf{K}$ for a **generic** $\eta \in \Omega_d^1[V]$. In order to define the term generic, note that we can introduce the Zariski topology in S_d and $\Omega_d^1[V]$ because they can be identified with affine spaces:

$$\begin{aligned} S_d &\simeq \mathbf{K}^{\dim S_d}, \\ \Omega_d^1[V] &\simeq S_d^\ell \simeq \mathbf{K}^{\ell \dim S_d}. \end{aligned}$$

Let Y be a topological space. We say that property P is true for a **generic** $\omega \in Y$ if Y has an open dense subset $Z \subseteq Y$ such that property P is true for all $\omega \in Z$.

Lemma 16.26 *Let W be a k-dimensional vector space over $\mathbb{K}$. A generic $\omega \in \Omega^1_d[W]$ vanishes only at the origin.*

Proof. Let $x_1, \dots, x_k$ be a basis for the dual space W^*. Write

$$\omega = f_1 dx_1 + \cdots + f_k dx_k$$

for $f_i \in \mathbb{K}[x_1, \dots, x_k]$ with $\deg f_i = d$ for $1 \le i \le k$. It is well-known that there is an open dense set of polynomials $(f_1, \dots, f_k) \in S_d^k$ such that the system of equations

$$f_1 = 0, f_2 = 0, \dots, f_k = 0$$

has only the trivial solution. □

Suppose $\mathcal{A}$ is a nonempty arrangement. Consider $X \in L(\mathcal{A})$ with $\dim X > 0$. Define

$$\Omega^1_d[X]^\circ = \{\omega \in \Omega^1_d[X] \mid \omega \text{ vanishes only at the origin } \}.$$

It follows from Lemma 16.26 that $\Omega^1_d[X]^\circ$ is an open dense set in $\Omega^1_d[X]$. Let S^X be the symmetric algebra of the dual space X^* of X. The natural restriction map $S_d \to S_d^X$ is continuous with respect to the Zariski topology. Denote the image of $f \in S_d$ under this restriction map by $\bar{f} \in S_d^X$. We also have the restriction map $r_{V,X} : \Omega^1_d[V] \to \Omega^1_d[X]$. If we choose a basis $x_1, \dots, x_\ell$ for V^* so that X is defined by $x_{k+1} = \cdots = x_\ell = 0$ then $r_{V,X}$ is given by

$$r_{V,X}(f_1 dx_1 + \cdots + f_\ell dx_\ell) = \bar{f}_1 d\bar{x}_1 + \cdots \bar{f}_k d\bar{x}_k.$$

The next result is immediate from the definitions.

Lemma 16.27 *For $X \in L(\mathcal{A})$ the restriction map $r_{V,X} : \Omega^1_d[V] \to \Omega^1_d[X]$ is continuous with respect to the Zariski topology.* □

Let $N_d^X = r_{V,X}^{-1}(\Omega^1_d[X]^\circ)$. It follows from Lemmas 16.26 and 16.27 that N_d^X is an open dense set in $\Omega^1_d[V]$. Let

$$N_d = \bigcap_{\substack{X \in L(\mathcal{A}) \\ \dim X > 0}} N_d^X.$$

Since $L(\mathcal{A})$ is a finite set, N_d is an open dense set in $\Omega^1_d[V]$. In particular it is nonempty.

Lemma 16.28 *Let $\eta \in N_d$. Then the radical of the ideal*

$$I(\eta) = \{\langle \theta, \eta \rangle \mid \theta \in D(\mathcal{A})\}$$

contains the maximal ideal $S_+ = \bigoplus_{p>0} S_p$.

Proof. By Hilbert's Nullstellensatz it suffices to show that the zero locus $V(I(\eta))$ of $I(\eta)$ is contained in $\{0\}$. Let $v \in V \setminus \{0\}$ and let $X = \bigcap_{v \in H \in \mathcal{A}} H$. Thus $v \in X$ and $v \notin Y$ for any $Y \in L(\mathcal{A})$ with $Y \subset X$. Choose a basis $x_1, \dots, x_\ell$ for V^* so that X is defined by $x_{k+1} = \cdots = x_\ell = 0$. Let $\mathcal{A}_1 = \{H \in \mathcal{A} \mid H \not\supseteq X\}$. Let Q_1 be a defining polynomial for $\mathcal{A}_1$. Note that $Q_1(\partial/\partial x_i) \in D(\mathcal{A})$ for $1 \le i \le k$. Write $\eta = f_1 dx_1 + \cdots + f_\ell dx_\ell$ with $f_i \in S$ and $1 \le i \le \ell$. Then $\langle Q_1(\partial/\partial x_i), \eta \rangle = Q_1 f_i$. Since $\eta \in N_d^X$,

$$r_{V,X}(\eta) = \bar{f}_1 d\bar{x}_1 + \cdots \bar{f}_k d\bar{x}_k$$

vanishes only at the origin. Thus there exists i with $1 \le i \le k$ so that $f_i(v) = \bar{f}_i(v) \ne 0$. Since $Q_1(v) \ne 0$, we have $v \notin V(I(\eta))$. □

Proposition 16.29 *If $\eta \in \Omega^1_d[V]$ is generic then the cohomology groups of the η-complex are finite dimensional over $\mathbb{K}$.*

Proof. Let $\eta \in N_d$. Let H^p denote the p-th cohomology group of the complex in Definition 16.25. Note that H^p is an S-module. Since $\Omega^p(\mathcal{A})$ is a finitely generated S-module, so is H^p. By Lemma 16.28 and Proposition 26.25, it suffices to prove that $I(\eta)$ annihilates H^p. Let $\omega \in \Omega^p(\mathcal{A})$ be a cocycle and let $\theta \in D(\mathcal{A})$. We have

$$0 = \langle \theta, \eta \wedge \omega \rangle = \langle \theta, \eta \rangle \omega - \eta \wedge \langle \theta, \omega \rangle = \langle \theta, \eta \rangle \omega - \partial_\eta(\langle \theta, \omega \rangle).$$

Thus $\langle \theta, \eta \rangle \omega = \partial_\eta(\langle \theta, \omega \rangle)$ is a coboundary since $\langle \theta, \omega \rangle \in \Omega^{p-1}(\mathcal{A})$ by Proposition 16.13. □

17 Lattice Homology

In this section we associate to the lattice $L(\mathcal{A})$ a simplicial complex $\mathsf{F}(\mathcal{A})$ first studied by Folkman [70] and Rota [163], compute its homology groups, and determine its homotopy type. There is an active area of research concerned with the topological properties of complexes obtained from partially ordered sets, such as the poset of all subgroups of a group. Since $L(\mathcal{A})$ is a geometric lattice we need not be concerned with the deeper aspects of that theory. We shall present only as much as we need for arrangements. We use books by Dold [54] and Spanier [179] as general references in topology and papers by Rota [163], Folkman [70], Björner [24] and Quillen [154] as general references for combinatorial topology.

The Order Complex

Definition 17.1 *Let P be a partially ordered set. Let $K = K(P)$ be the simplicial complex associated to P as follows:*

(1) the vertices of K are the elements of P,

(2) a set of vertices $\{X_0, \ldots, X_q\}$ spans a q-simplex if and only if it is a linearly ordered subset of P; so after relabeling

$$X_0 < \cdots < X_q.$$

Definition 17.2 *Given a poset P and the associated simplicial complex $K(P)$, let $\mathsf{K}(P)$ be the corresponding geometric complex called the* **order complex**.

If P_1 and P_2 are posets then there is a natural partial order on the set $P_1 \times P_2$ given in Remark 4.14:

$$(X_1, X_2) \leq (Y_1, Y_2) \iff X_1 \leq Y_1 \text{ and } X_2 \leq Y_2.$$

Proposition 17.3 *Let P be any poset and let $Q = \{0 < 1\}$ be a poset with two elements. Then $\mathsf{K}(P \times Q)$ is a subdivision of $\mathsf{K}(P) \times I$.*

Proof. The space $\mathsf{K}(Q) = I$ is the unit interval. It is sufficient to prove the special case when $P = \{X_0 < \cdots < X_p\}$ is a linearly ordered set, so $\mathsf{K}(P) = \Delta^p$ is a simplex. Consider Δ^p as a simplex of the standard simplicial subdivision of the p-cube I^p, see [54, p.118]. The complex $\mathsf{K}(P \times Q)$ is a simplicial subdivision of the subspace $\Delta^p \times I$ in $I^p \times I = I^{p+1}$. Figure 20 illustrates the case $p = 2$. □

Corollary 17.4 *Let P be a poset and let $f : P \to P$ be an order preserving map with the property that $f(X) \leq X$ for all $X \in P$. Then the induced map of topological spaces $f : \mathsf{K}(P) \to \mathsf{K}(P)$ is homotopic to the identity. If $P_0 \subseteq P$ is a subset such that $f|_{P_0} = id_{P_0}$ then the homotopy is relative $\mathsf{K}(P_0)$.*

Proof. Let $Q = \{0 < 1\}$. Define $F : P \times Q \to P$ by $F(X,0) = f(X)$ and $F(X,1) = X$. Since $f(X) \leq X$, F is order preserving. It induces a map

$$F : \mathsf{K}(P \times Q) \to \mathsf{K}(P).$$

By Proposition 17.3 we may view F as a homotopy between f and the identity:

$$F : \mathsf{K}(P) \times I \to \mathsf{K}(P).$$

If $f|_{P_0} = id_{P_0}$ then F is the identity on $\mathsf{K}(P_0)$. □

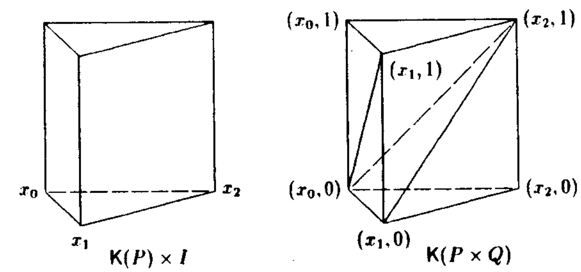


Figure 20: Subdivision of $\Delta^2 \times I$

Lemma 17.5 *(1) Suppose P has a unique minimal element V. Then $\mathsf{K}(P)$ is a cone with base $\mathsf{K}(P \setminus \{V\})$. Thus $\mathsf{K}(P)$ is a contractible space.*

(2) Suppose P has a unique maximal element T. Then $\mathsf{K}(P)$ is a cone with base $\mathsf{K}(P \setminus \{T\})$. Thus $\mathsf{K}(P)$ is a contractible space.

Proof. We prove (1), the argument is similar for (2). If $\sigma^p = [X_0, \ldots, X_p] \in K(P \setminus \{V\})$ then $\tau^{p+1} = [V, X_0, \ldots, X_p] \in K(P)$. Moreover, every simplex of $K(P) \setminus K(P \setminus \{V\})$ has the form $V\sigma$ for some $\sigma \in K(P \setminus \{V\})$. □

The Folkman Complex

Definition 17.6 *Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. Suppose $r(\mathcal{A}) \geq 2$. Let $K(L \setminus \{V, T\})$ be the simplicial complex associated to the poset obtained from L by deleting its minimal and its maximal elements. Let the* **Folkman complex** $\mathsf{F}(\mathcal{A}) = \mathsf{K}(L \setminus \{V, T\})$ *be the corresponding geometric complex.*

Note that $\dim \mathsf{F}(\mathcal{A}) = r(\mathcal{A}) - 2$. If $r(\mathcal{A}) = 2$ then $\mathsf{F}(\mathcal{A})$ consists of $|\mathcal{A}|$ points.

Example 17.7 *Let $\mathcal{B}(\ell + 1)$ denote the Boolean arrangement defined by $Q = x_0 x_1 \cdots x_\ell$. Let $H_i = \ker(x_i)$. Then F is the $(\ell - 1)$-complex consisting of the barycentric subdivision of the boundary of an ℓ-simplex with vertices $H_0, \ldots, H_\ell$. Thus F is homeomorphic to $S^{\ell-1}$.*

Definition 17.8 *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple with respect to* $H_0 \in \mathcal{A}$. *Let* $L' = L(\mathcal{A}')$, $L'' = L(\mathcal{A}'')$ *and* $T = T(\mathcal{A})$. *Define*

$$\mathsf{F}'' = \mathsf{F}''(\mathcal{A}) = \mathsf{F}(\mathcal{A}'')$$

and

$$\mathsf{F}' = \mathsf{F}'(\mathcal{A}) = \begin{cases} |K(L' \setminus \{V, T\})| & \text{if } T \in L', \\ |K(L' \setminus \{V\})| & \text{if } T \notin L'. \end{cases}$$

This case distinction is essential in several proofs. Recall from Definition 6.11 that H_0 is a separator if $T \notin L(\mathcal{A}')$. The poset $L' \setminus \{V\}$ has a unique maximal element $T' = T(\mathcal{A}')$. Thus by Lemma 17.5 the space $\mathsf{F}'(\mathcal{A})$ is contractible if H_0 is a separator. If H_0 is not a separator then $\mathsf{F}'(\mathcal{A}) = \mathsf{F}(\mathcal{A}')$.

Example 17.9 *Let $\mathcal{A}$ be the 3-arrangement in* Example 8.37 *defined by*

$$Q(\mathcal{A}) = xyz(x+y)(x+y-z).$$

Recall the notation of Example 8.37: $H_0 = \ker(x+y+z)$, $H_1 = \ker(x)$, $H_2 = \ker(y)$, $H_3 = \ker(z)$, $H_4 = \ker(x+y)$. The 1-complex $\mathsf{F}(\mathcal{A})$ is illustrated in Figure 21. The 0-complex $\mathsf{F}''(\mathcal{A})$ consists of the three points $H_0 \cap H_1$, $H_0 \cap H_2$, $H_0 \cap H_3 \cap H_4$. Here $T \in L'$ so H_0 is not a separator. The 1-complex F' is also illustrated in Figure 21.

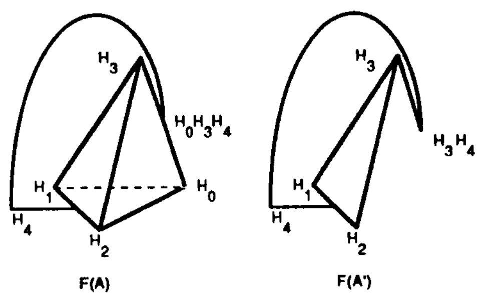


Figure 21: Folkman complexes for $Q = xyz(x+y)(x+y-z)$

Example 17.10 *In* $\mathcal{B}(\ell+1)$ *the complex* F'' *is homeomorphic to* $S^{\ell-2}$. *Note here that* $T \notin L'$ *so* H_0 *is a separator. The complex* F' *is the* $(\ell-1)$*-simplex opposite the vertex* H_0. *These complexes are illustrated for* $\ell = 3$ *in* Figure 22.

Lemma 17.11 *If $\mathcal{A}$ is an arrangement of rank 2 then* $\mathsf{F}(\mathcal{A})$ *consists of* $\mu(\mathcal{A}) + 1$ *points.*

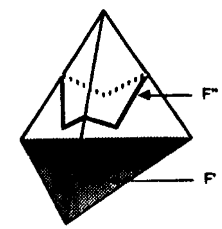


Figure 22: Complexes for the Boolean arrangement

Proof. We observed that F consists of $|\mathcal{A}|$ points and the Möbius function gives $1 - |\mathcal{A}| + \mu(\mathcal{A}) = 0$. □

Lemma 17.12 *If $\mathcal{A}$ is an arrangement with* $r(\mathcal{A}) \geq 3$ *then* $\mathsf{F}(\mathcal{A})$ *is path connected.*

Proof. Suppose X is a vertex of $\mathsf{F}(\mathcal{A})$. There exists $H \in \mathcal{A}$ such that $X \geq H$ and $H \in \mathsf{F}(\mathcal{A})$. If $X \neq H$ then the 1-simplex $[H, X] \subseteq \mathsf{F}(\mathcal{A})$. Thus every vertex and hence every point of $\mathsf{F}(\mathcal{A})$ is connected by a path to some vertex of $\mathsf{F}(\mathcal{A})$ which corresponds to a hyperplane. It remains to show that vertices corresponding to distinct hyperplanes $H_1, H_2 \in \mathcal{A}$ are connected. Since $r(H_1 \cap H_2) = 2 < r(\mathcal{A})$ we have $H_1 \cap H_2 \in \mathsf{F}(\mathcal{A})$. Thus the 1-simplexes $[H_1, H_1 \cap H_2]$ and $[H_2, H_1 \cap H_2]$ are in $\mathsf{F}(\mathcal{A})$. □

Let K be a simplicial complex and let $|K|$ be its geometric complex. Let v be a vertex of K. Recall that the **star** of v is a subset of $|K|$ consisting of all open simplexes whose closure contains v:

$$st(v) = \{|\mathring{\sigma}| \mid v \in \sigma\}.$$

Note that its closure, $\overline{st(v)}$ is a cone with cone point v.

Proposition 17.13 *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple with respect to* H_0 *and let* $\mathsf{F} = \mathsf{F}(\mathcal{A}), \mathsf{F}' = \mathsf{F}'(\mathcal{A})$. *There is a strong deformation retraction*

$$\rho : \mathsf{F} \setminus st(H_0) \to \mathsf{F}'.$$

Proof. Note first that $\mathsf{F} \setminus st(H_0) = |K(L \setminus \{V, H_0, T\})|$. Define a poset map

$$\rho : L \setminus \{V, H_0, T\} \to L \setminus \{V, H_0, T\}$$

by

$$\rho(X) = \bigcap_{H \in \mathcal{A}_X \setminus \{H_0\}} H.$$

If H_0 is a separator then $\mathrm{im}(\rho) \subseteq L' \setminus \{V\}$. If H_0 is not a separator then $\mathrm{im}(\rho) \subseteq L' \setminus \{V, T\}$. Extend ρ linearly to $\mathsf{F} \setminus st(H_0)$ and call the resulting map again ρ. It follows that $\mathrm{im}(\rho) \subseteq \mathsf{F}'$. Clearly $r(\rho(X)) \leq r(X)$ and $\rho|_{\mathsf{F}'} = id_{\mathsf{F}'}$. Thus by Corollary 17.4 ρ is a strong deformation retraction. □

Theorem 17.14 *If $\mathcal{A}$ is an arrangement with $r(\mathcal{A}) \geq 4$ then $\mathsf{F}(\mathcal{A})$ is simply connected.*

Proof. We use induction on $|\mathcal{A}|$. Since $|\mathcal{A}| \geq r(\mathcal{A})$, the induction starts with $|\mathcal{A}| = r(\mathcal{A})$. In this case $\mathcal{A}$ is isomorphic to the Boolean arrangement $\mathcal{B}(q)$ for $q = r(\mathcal{A})$. Example 17.7 showed that $\mathsf{F}(\mathcal{B}(q))$ is homeomorphic to S^{q-2}. Since $q = r(\mathcal{A}) \geq 4$, the assertion holds for $|\mathcal{A}| = r(\mathcal{A})$. For the induction step choose $H_0 \in \mathcal{A}$ and consider the associated spaces $\mathsf{F}, \mathsf{F}', \mathsf{F}''$. We have

(1) $$\mathsf{F} = \overline{st(H_0)} \cup (\mathsf{F} \setminus st(H_0))$$

and

(2) $$\mathsf{F}'' = \overline{st(H_0)} \cap (\mathsf{F} \setminus st(H_0)).$$

Now $\overline{st(H_0)}$ is a cone over F'' with cone point H_0. In particular it is simply connected. We showed in Proposition 17.13 that $\mathsf{F} \setminus st(H_0)$ has the homotopy type of F'. If H_0 is a separator then F' is contractible. If H_0 is not a separator then $\mathsf{F}' = \mathsf{F}(\mathcal{A}')$ and $r(\mathcal{A}') = r(\mathcal{A})$. Since $|\mathcal{A}'| < |\mathcal{A}|$ the induction hypothesis implies that F' is simply connected. Finally, $r(\mathcal{A}'') = r(\mathcal{A}) - 1 \geq 3$ so it follows from Lemma 17.12 that F'' is path connected. The van Kampen theorem implies that F is simply connected. □

The Homology Groups

Next we want to compute the homology groups of $\mathsf{F}(\mathcal{A})$. Integer coefficients are understood unless otherwise indicated, and $\tilde{H}$ denotes reduced homology. Consider the Mayer-Vietoris sequence of reduced homology for the excisive couple $\{\mathsf{F} \setminus st(H_0), \overline{st(H_0)}\}$. Using (1) and (2) we get the long exact sequence:

$$\cdots \to \tilde{H}_p(\mathsf{F} \setminus st(H_0)) \oplus \tilde{H}_p(\overline{st(H_0)}) \xrightarrow{(i_1,i_2)_*} \tilde{H}_p(\mathsf{F}) \xrightarrow{\partial_*}$$

$$\tilde{H}_{p-1}(\mathsf{F}'') \xrightarrow{(j_1,-j_2)_*} \tilde{H}_{p-1}(\mathsf{F} \setminus st(H_0)) \oplus \tilde{H}_{p-1}(\overline{st(H_0)}) \to \cdots$$

The fact that $\overline{st(H_0)}$ is contractible and Proposition 17.13 give

(3) $$\cdots \to \tilde{H}_p(\mathsf{F}') \xrightarrow{i_*} \tilde{H}_p(\mathsf{F}) \xrightarrow{\partial_*} \tilde{H}_{p-1}(\mathsf{F}'') \xrightarrow{j_*} \tilde{H}_{p-1}(\mathsf{F}') \to \cdots$$

The next result is due to Folkman [70].

Theorem 17.15 *Let $\mathcal{A}$ be an arrangement with Folkman complex $\mathsf{F} = \mathsf{F}(\mathcal{A})$. Then*

$$\tilde{H}_i(\mathsf{F}) = \begin{cases} 0 & \text{if } i \neq r(\mathcal{A}) - 2 \\ \text{free of rank } |\mu(\mathcal{A})| & \text{if } i = r(\mathcal{A}) - 2. \end{cases}$$

Proof. We use induction on $r(\mathcal{A})$, and for fixed $r(\mathcal{A})$ on $|\mathcal{A}|$. The assertion is correct for $r(\mathcal{A}) = 2$ and arbitrary $|\mathcal{A}|$ by Lemma 17.11. The assertion is also correct for arbitrary $r(\mathcal{A})$ when $|\mathcal{A}| = r(\mathcal{A})$, since in that case $\mathcal{A}$ is the Boolean arrangement and we noted in Example 17.7 that $\mathsf{F}(\mathcal{A})$ is an $(r(\mathcal{A})-2)$-sphere, while it follows from Proposition 5.11 that $|\mu(\mathcal{A})| = 1$. For the induction step we assume that the result holds for all arrangements $\mathcal{B}$ with $r(\mathcal{B}) < r(\mathcal{A})$ and for all arrangements $\mathcal{B}$ with $r(\mathcal{A}) = r(\mathcal{B})$ and $|\mathcal{B}| < |\mathcal{A}|$. Fix $H_0 \in \mathcal{A}$. Consider the exact sequence (3). For $p \neq r(\mathcal{A}) - 2$ the induction hypothesis implies that $\tilde{H}_p(\mathsf{F}') = \tilde{H}_{p-1}(\mathsf{F}'') = 0$ and hence $\tilde{H}_p(\mathsf{F}) = 0$. For $p = r(\mathcal{A}) - 2$ the induction hypothesis implies that $\tilde{H}_{p-1}(\mathsf{F}'')$ is free of rank $|\mu(\mathcal{A}'')|$. If H_0 is not a separator then $\mathsf{F}' = \mathsf{F}(\mathcal{A}')$ so the induction hypothesis implies that $\tilde{H}_p(\mathsf{F}')$ is free of rank $|\mu(\mathcal{A}')|$. Thus $\tilde{H}_p(\mathsf{F})$ is free of rank $|\mu(\mathcal{A}')| + |\mu(\mathcal{A}'')|$. If H_0 is a separator then F' is contractible so $\tilde{H}_p(\mathsf{F})$ is free of rank $|\mu(\mathcal{A}'')|$. The conclusion follows from Corollary 6.12. □

Corollary 17.16 *Let $\mathcal{A}$ be an arrangement with Folkman complex $\mathsf{F} = \mathsf{F}(\mathcal{A})$. Let $\mathcal{K}$ be a commutative ring. Then*

$$\tilde{H}_i(\mathsf{F}; \mathcal{K}) = \begin{cases} 0 & \text{if } i \neq r(\mathcal{A}) - 2 \\ \text{free } \mathcal{K}\text{-module of rank } |\mu(\mathcal{A})| & \text{if } i = r(\mathcal{A}) - 2. \end{cases} \quad \square$$

The Homotopy Type

Definition 17.17 *Let $(S_1^k, P_1), \ldots, (S_m^k, P_m)$ be m disjoint k-spheres with base points P_j. Their* **wedge** *is the based space $(\vee_m S^k, P)$ obtained by identifying the base points $P_1 = \ldots = P_m = P$.*

It is clear that $(\vee_m S^k, P)$ is a cell complex. We may write $\vee_m S^k$ for brevity. We have

$$\pi_i(\vee_m S^k) = \begin{cases} 0 & \text{for } i < k \\ \text{free of rank m} & \text{for } i = k, \end{cases}$$

and

$$\tilde{H}_i(\vee_m S^k; \mathbb{Z}) = \begin{cases} 0 & \text{for } i \neq k \\ \text{free of rank m} & \text{for } i = k. \end{cases}$$

The next result was stated by Quillen [154] without proof. It is reasonable to assume that he had in mind the argument below. Björner and Walker [29] proved the result without appeal to facts in homotopy theory.

Theorem 17.18 *Let $\mathcal{A}$ be an arrangement with $r(\mathcal{A}) \geq 2$. Then its Folkman complex $\mathsf{F} = \mathsf{F}(\mathcal{A})$ has the homotopy type of $\vee_m S^k$ with $k = r(\mathcal{A}) - 2$ and $m = |\mu(\mathcal{A})|$.*

Proof. For $r(\mathcal{A}) = 2$ this follows from Lemma 17.11. For $r(\mathcal{A}) = 3$ the complex F is 1-dimensional and hence it has the homotopy type of a wedge of circles. Their number equals the rank of $H_1(\mathsf{F})$. We showed in Theorem 17.15 that this rank is m. For $r(\mathcal{A}) \geq 4$ the complex F is simply connected by Theorem 17.14. It follows from Theorem 17.15

and the Hurewicz isomorphism theorem [179, p.398] that $\pi_i(\mathsf{F}) = 0$ for $1 \le i < k$ and $\pi_k(\mathsf{F}) \simeq H_k(\mathsf{F};\mathbb{Z})$. The last group is free of rank m by Theorem 17.15. For $1 \le i \le m$ let $\rho_i : S^k_i \to \mathsf{F}$ be generators of $\pi_k(\mathsf{F})$. Let $\rho : \vee_m S^k \to \mathsf{F}$ be the sum of ρ_i. Then ρ induces an isomorphism in homology by construction. Since the spaces are simply connected cell complexes, it follows from standard results in homotopy theory [179, pp.405-406] that ρ is a homotopy equivalence. □

Whitney Homology

Next we associate to $L(\mathcal{A})$ another chain complex, studied by Deheuvels [48] and Baclawski [14], compute its homology and relate it to the homology of $\mathsf{F}(\mathcal{A})$ and to the algebra $B(\mathcal{A})$. Let $\mathcal{K}$ be a commutative ring. Recall the spaces $\mathcal{T}_p$ from Definition 11.1 and the set of chains $ch\,L$ from Definition 5.3.

Definition 17.19 *Let $\mathcal{A}$ be an arrangement with lattice $L = L(\mathcal{A})$. Define a chain complex (C,δ) as follows. Let $C_0 = \mathcal{K}$ and for $p > 0$ let $C_p \subseteq \mathcal{T}_p$ have a $\mathcal{K}$-basis consisting of all p-chains $(X_1,\ldots,X_p) \in ch(L \setminus \{V\})$. Let $C = \oplus_{p=0}^{\ell} C_p$. Define a $\mathcal{K}$-linear map $\delta : C \to C$ by $\delta(1) = 0$, $\delta(X) = 0$ for $X \in L - \{V\}$ and for $p \ge 2$*

$$\delta(X_1,\ldots,X_p) = \sum_{k=1}^{p-1}(-1)^{k-1}(X_1,\ldots,\hat{X}_k,\ldots,X_p).$$

The map δ differs from the usual boundary operator in that X_p is never deleted. We still have $\delta^2 = 0$ so (C,δ) is a chain complex. We call it the **Whitney complex** of $\mathcal{A}$. The Poincaré polynomial of its homology was first computed by Baclawski [14]. We give a generalization of his result.

For $1 \le k \le p$ define $\mathcal{K}$-linear maps $\delta_k : C_p \to C_{p-1}$ by

$$\delta_k(X_1,\ldots,X_p) = (-1)^{k-1}(X_1,\ldots,\hat{X}_k,\ldots,X_p).$$

We agree that $\delta_1 : C_1 \to C_0$ is the zero map. Recall the map $\tau : \mathcal{T} \to \mathcal{T}$ of Definition 11.13. If $x \in C_p$ then $\tau(x) = \delta_p(x)$ and $\delta(x) = \delta_1(x) + \cdots \delta_{p-1}(x)$. Recall the $\mathcal{K}$-algebra $B = B(\mathcal{A})$ of section 11 and note that $B \subseteq C$.

Lemma 17.20 *(1) The elements of B are cycles of C so $\delta B = 0$.*
(2) Let $r = r(\mathcal{A})$. If $x \in C_r$ is a cycle then $\tau\tau x = \delta\tau x = 0$.

Proof. (1) It suffices to show that $\delta b_S = 0$ for all $S \in \mathbf{S}$. Let $S = (H_1,\ldots,H_p)$. In $\delta b_{(H_1,\ldots,H_p)}$ each term

$$(\mathrm{sign}\,\pi)(-1)^{k-1}(H_{\pi 1},\ldots,H_{\pi 1}\cap\widehat{\cdots}\cap H_{\pi k},\ldots,H_{\pi 1}\cap\cdots\cap H_{\pi p})$$

is cancelled against the term in which πk and $\pi(k+1)$ are transposed. For (2) note that x is a linear combination of chains consisting of elements of ranks $1,2,\ldots,r$. So $\delta_i x$ is a

linear combination of chains consisting of elements of ranks $1,2,\ldots,i-1,i+1,\ldots,r$. Thus $0 = \delta x = \delta_1 x + \cdots + \delta_{r-1}x$ implies $\delta_i x = 0$ for $1 \le i \le r-1$. We have

$$\tau\tau x = \delta_{r-1}\delta_r x = \delta_{r-1}\delta_{r-1}x = 0$$

and

$$\delta\tau x = (\delta_1 + \cdots \delta_{r-2})\delta_r x = \delta_{r-1}(\delta_1 + \cdots \delta_{r-2})x = 0. \ \square$$

For $X \in L \setminus \{V\}$ let C_X be the subspace of C spanned by all $(X_1,\ldots,X_p)$ with $X_p = X$ and let $C_V = \mathcal{K}$. Then $C = \oplus_{X \in L} C_X$ and $\delta : C_X \to C_X$. Thus (C_X,δ) is a subcomplex. Let $\mathcal{H}_X$ be its homology. Then

(1) $$\mathcal{H} = \bigoplus_{X \in L}\mathcal{H}_X.$$

The natural identification $C_X(\mathcal{A}_X) \simeq C_X(\mathcal{A})$ implies

(2) $$\mathcal{H}_X(\mathcal{A}_X) \simeq \mathcal{H}_X(\mathcal{A}).$$

Theorem 17.21 *The map $B(\mathcal{A}) \to \mathcal{H}(\mathcal{A};\mathcal{K})$ which sends b_S to its homology class $[b_S]$ is an isomorphism of free $\mathcal{K}$-modules.*

Proof. By Lemma 11.16, (1) and (2), it suffices to show that the natural map $B_T(\mathcal{A}) \to \mathcal{H}_T(\mathcal{A};\mathcal{K})$ is an isomorphism. Let $\ell = r(T)$. Since the map is induced by inclusion, it is injective because $C_{\ell+1} = 0$. We will show surjectivity by induction on $r(\mathcal{A}) = \ell$. The assertion is clear for $\ell = 1$. Denote the natural projection from $C = \bigoplus_{X \in L} C_X$ onto C_X by π_X. Let $x \in C_\ell$ with $\delta x = 0$ and write

$$\tau x = \sum_{X \in L_{\ell-1}} \pi_X(\tau x).$$

Lemma 17.20.2 implies that

$$0 = \delta\tau x = \sum_{X \in L_{\ell-1}} \delta\pi_X(\tau x) = \sum_{X \in L_{\ell-1}} \pi_X(\delta\tau x).$$

Thus $0 = \pi_X(\delta\tau x) = \delta(\pi_X \tau x)$, and therefore $\pi_X(\tau x) \in \mathcal{H}_X(\mathcal{A}_X)$ for all $X \in L_{\ell-1}$. By the induction assumption $\mathcal{H}_X(\mathcal{A}_X) = B_X(\mathcal{A}_X)$ for $X < T$. Thus $\tau x \in B_{\ell-1}(\mathcal{A})$. Recall that $\tau\tau x = 0$ by Lemma 17.20.2. Since the complex (B,τ) is acyclic by Lemma 11.17.3, we have $\tau x \in \tau B_\ell = \tau B_T$. Since $\tau : C_T \to C$ is injective, we conclude that $x \in B_T$. □

The following results are consequences of Corollaries 11.19, 11.20 and 11.21 respectively.

Corollary 17.22 *The module $\mathcal{H}(\mathcal{A};\mathcal{K}) = \bigoplus_p \mathcal{H}_p(\mathcal{A};\mathcal{K})$ is a free graded $\mathcal{K}$-module.* □

Corollary 17.23 *The Poincaré polynomial of $\mathcal{H}(\mathcal{A};\mathcal{K})$ is*

$$\mathrm{Poin}(\mathcal{H}(\mathcal{A};\mathcal{K}),t) = \pi(\mathcal{A},t). \ \square$$

Corollary 17.24 *If $X \in L(\mathcal{A})$ then* $\mathrm{rank}\,\mathcal{H}_X(\mathcal{A};\mathcal{K}) = (-1)^{r(X)}\mu(X)$. □

Connection with the Folkman Complex

These constructions are closely related. In fact the homology groups of the Whitney complex equal direct sums of the homology groups of all Folkman subcomplexes of $\mathcal{A}$ in the following sense.

Theorem 17.25 *Let $\mathcal{A}$ be an arrangement. Then*

$$\mathcal{H}_0(\mathcal{A};\mathcal{K}) = \mathcal{K}, \quad \mathcal{H}_1(\mathcal{A};\mathcal{K}) = \bigoplus_{H\in\mathcal{A}} \mathcal{K}(H),$$

and for $p \geq 2$ the map $\tau : \mathcal{T} \to \mathcal{T}$ of Definition 11.13 *induces isomorphisms*

$$\mathcal{H}_p(\mathcal{A};\mathcal{K}) \simeq \bigoplus_{X\in L_p(\mathcal{A})} \tilde{H}_{p-2}(\mathsf{F}(\mathcal{A}_X);\mathcal{K}).$$

Proof. Recall that we identified B_p with the group of p-cycles of $\mathcal{C}$. Since $B_p = \oplus_{X\in L_p} B_X$ and $B_X(\mathcal{A}) = B_X(\mathcal{A}_X)$ it suffices to prove the assertion for $X = T(\mathcal{A})$ and $r(\mathcal{A}) \geq 2$. Write $T = T(\mathcal{A})$ and $r(\mathcal{A}) = \ell$. If $\ell = 2$ then

$$B_T \simeq (\bigoplus_{H\in\mathcal{A}} \mathcal{K}(H))/\mathcal{K}(\sum_{H\in\mathcal{A}} H) \simeq \tilde{H}_0(\mathsf{F}(\mathcal{A});\mathcal{K}).$$

Assume $\ell \geq 3$. Since there are no $(\ell-2)$-boundaries we have $\tilde{H}_{\ell-2}(\mathsf{F};\mathcal{K}) = H_{\ell-2}(\mathsf{F};\mathcal{K}) = Z_{\ell-2}(\mathsf{F};\mathcal{K})$. We identify the cycle group $Z_{\ell-2}$ with a subspace of $\mathcal{T}_{\ell-1}$. Note that $B_T \subseteq \mathcal{T}_\ell$. We show that $\tau B_T \subseteq Z_{\ell-2}(\mathsf{F};\mathcal{K})$. Let $S = (H_1,\ldots,H_\ell) \in \mathbf{S}_\ell$. If S is dependent then $\tau b_S = \tau 0 = 0$. Suppose S is independent. Let $\mathcal{B}$ denote the subarrangement of $\mathcal{A}$ whose elements are the hyperplanes of S. This is a fine but useful distinction: $\mathcal{B}$ is a set, S is an ordered set with the same elements. Then $L(\mathcal{B})$ is a Boolean lattice. We showed that $H_{\ell-2}(\mathsf{F}(\mathcal{B});\mathcal{K})$ is one dimensional, generated by the cycle

$$z_S = \sum_{\pi\in Sym(\ell)} (-1)^{\ell-1}(\operatorname{sign}\pi)(H_{\pi 1}, H_{\pi 1}\cap H_{\pi 2},\ldots,H_{\pi 1}\cap\cdots\cap H_{\pi(\ell-1)}).$$

Since $\tau b_S = z_S$, this shows that $\tau B_T \subseteq Z_{\ell-2}(\mathsf{F};\mathcal{K})$. But $\tau : \mathcal{C}_T \to \mathcal{C}$ is a monomorphism and thus $\tau : B_T \to Z_{\ell-2}(\mathsf{F};\mathcal{K})$ is a monomorphism. Let $x \in Z_{\ell-2}(\mathsf{F};\mathcal{K})$. Then $x \in \mathcal{C}_{\ell-1}$. Define $y \in \mathcal{C}_\ell$ by adding T at the end of each chain appearing in x. Then $\tau y = x$ and $\delta y = 0$. Thus $y \in \mathcal{H}_\ell$. Since $B_\ell = \mathcal{H}_\ell$ by Theorem 17.21, we have $y \in B_T$. This proves that $\tau : B_T \to Z_{\ell-2}(\mathsf{F};\mathcal{K})$ is surjective. $\square$

18 The Characteristic Polynomial and Factorization

Recall the characteristic polynomial of an arrangement from Definition 6.5 and the graded S-module $\Omega^p(\mathcal{A})$ from Proposition 16.11. The central object of this section is the Poincaré series of $\Omega^p(\mathcal{A})$.

Definition 18.1 *Suppose M is a finitely generated graded S-module and each M_p is finite dimensional over* $\mathbf{K}$. *The* **Poincaré series** Poin $(M,x) \in \mathbb{Z}[x^{-1}][[x]]$ *of the graded S-module M is*

$$\text{Poin}\,(M,x) = \sum_{p\in\mathbb{Z}} (\dim_{\mathbf{K}} M_p)x^p.$$

First we study a generalization of the Folkman complex and Whitney complex, called the order complex with functors. It could be stated in terms of sheaf theory on ordered sets, but we choose not to use any sheaf theory here. It is used to prove Theorem 18.20, which amounts to the following formula for the characteristic polynomial:

$$\chi(\mathcal{A},t) = \lim_{x\to 1} \sum_{p=0}^{\ell} \text{Poin}(\Omega^p(\mathcal{A}),x)(t(1-x)-1)^p.$$

When this formula is applied to a free arrangement we obtain the Factorization Theorem 18.21. It asserts that if $\mathcal{A}$ is a free arrangement with $\exp\mathcal{A} = \{b_1,\ldots,b_\ell\}$ then

$$\pi(\mathcal{A},t) = \prod_{i=1}^{\ell}(1+b_i t).$$

In Theorem 15.24 we proved this factorization for recursively free arrangements.

The Order Complex with Functors

A poset L may be regarded as a category. Its objects are the elements of L. Its morphisms are induced by the partial order:

$$\text{Hom}(X,Y) = \begin{cases} \{X \leq Y\} & \text{if } X \leq Y \\ \emptyset & \text{otherwise.} \end{cases}$$

The composition of two morphisms $X \leq Y$ and $Y \leq Z$ is $X \leq Z$.

Let R be a commutative ring and let (R-Mod) be the category of R-modules. Suppose F is a covariant functor from L to (R-Mod). When $X,Y \in L$ with $X \leq Y$, the induced morphism from $F(X)$ to $F(Y)$ is denoted $\nu_{X,Y} : F(X) \to F(Y)$. When we are working with a covariant functor F we assume that L has a unique maximal element T and set $P = L \setminus \{T\}$. Suppose F is a contravariant functor from L to (R-Mod). Then we have the induced morphism $\nu_{X,Y} : F(Y) \to F(X)$. When we are working with a contravariant functor F we assume that L has a unique minimal element V and set $P = L \setminus \{V\}$.

Let p be a positive integer. Recall from Definition 5.3 that a p-chain in P is a p-tuple $c = (X_1, \dots, X_p)$ of elements $X_i \in P$ satisfying $X_1 < \cdots < X_p$. Let $C_p(P)$ be the free R-module with basis the set of all p-chains in P.

Definition 18.2 *Suppose $F : L \to$ (R-Mod) is a covariant functor. We extend* Definition 17.19 *to the* **order complex with** F **coefficients**, $(C.(L,F), \partial)$ *as follows. Let $C_0(L,F) = F(T)$ and for $p \geq 1$ let $C_p(L,F)$ be the R-module spanned by*

$$\{y \otimes (X_1, \dots, X_p) \in (\bigoplus_{X \in P} F(X)) \otimes_R C_p(P) \mid y \in F(X_1),\ X_1 < \dots < X_p < T\}.$$

The boundary operator

$$\partial : C_p(L,F) \to C_{p-1}(L,F)$$

is given by $\partial(y \otimes (X_1)) = \nu_{X_1,T}(y)$ and for $p > 1$

$$\partial(y \otimes (X_1, \dots, X_p)) = \nu_{X_1,X_2}(y) \otimes (X_2, \dots, X_p) + \sum_{k=2}^{p} (-1)^{k-1} y \otimes (X_1, \dots, \widehat{X_k}, \dots, X_p).$$

In case $F : L \to$ (R-Mod) is a contravariant functor, the following modifications are necessary. Let $C_0(L,F) = F(V)$ and for $p \geq 1$ let $C_p(L,F)$ be the R-module spanned by

$$\{y \otimes (X_1, \dots, X_p) \in (\bigoplus_{X \in P} F(X)) \otimes_R C_p(P) \mid y \in F(X_p),\ V < X_1 < \dots < X_p\}.$$

The boundary operator

$$\partial : C_p(L,F) \to C_{p-1}(L,F)$$

is given by $\partial(y \otimes (X_1)) = \nu_{V,X_1}(y)$ and for $p > 1$ we have $\partial(y \otimes (X_1, \dots, X_p)) =$

$$\sum_{k=1}^{p-1} (-1)^{k-1} y \otimes (X_1, \dots, \widehat{X_k}, \dots, X_p) + (-1)^{p-1} \nu_{X_{p-1},X_p}(y) \otimes (X_1, \dots, X_{p-1}).$$

The following two examples were studied in the last section.

Example 18.3 *Let $\mathcal{A}$ be an arrangement in V. Let $P = L(\mathcal{A}) \setminus \{T(\mathcal{A})\}$. Define the contravariant functor F by $F(X) = \mathbb{Z}$ for $X \in P$ and let $\nu_{X,Y}$ be the identity map for all X, Y with $X \leq Y$. In this case the chain complex $(C.(P,F), \partial)$ is equal to the simplicial chain complex associated with the Folkman complex.*

Example 18.4 *Let $\mathcal{A}$ be an arrangement in V. Let $L = L(\mathcal{A})$. Let $R = \mathcal{K}$ be a commutative ring. Define the contravariant functor F by $F(X) = \mathcal{K}$. Here $\nu_{X,X}$ is the identity map, and $\nu_{X,Y} = 0$ for $X < Y$. In this case the chain complex $(C.(L,F), \partial)$ is equal to the Whitney complex.*

Local Functors

Let $\mathcal{A}$ be an arrangement in V. Write $L = L(\mathcal{A})$ and recall that $S = S(V^*)$ is the symmetric algebra of the dual space V^* of V. Given a prime ideal $\wp \in \mathrm{Spec}(S)$ and an element $X \in L$, define $X(\wp) \in L$ by

$$X(\wp) = \bigcap_{\substack{H \in \mathcal{A}_X \\ \alpha_H \in \wp}} H.$$

It follows from the definition that $X(\wp) \supseteq X$, so $X(\wp) \leq X$.

Definition 18.5 *A covariant functor $F : L(\mathcal{A}) \to$ (S-Mod) is called* **local** *if the localization at $\wp$ of the map*

$$\nu_{X(\wp),X} : F(X(\wp)) \to F(X)$$

is an isomorphism for all $\wp \in$ Spec(S) and for all $X \in L$.

If F is a contravariant functor then F is local if the localization at $\wp$ of the map $\nu_{X(\wp),X} : F(X) \to F(X(\wp))$ is an isomorphism for all $\wp \in \mathrm{Spec}(S)$ and for all $X \in L$.

Example 18.6 *Recall the S-modules $\Omega^q(\mathcal{A})$ from* Definition 16.3. *Let q be a nonnegative integer. Define the covariant functor $F : L \to$ (S-Mod) by $F(X) = \Omega^q(\mathcal{A}_X)$. If $X \leq Y$ then $\mathcal{A}_X \subseteq \mathcal{A}_Y$. It follows from* Proposition 16.9 *that there are inclusions*

$$\nu_{X,Y} : \Omega^q(\mathcal{A}_X) \hookrightarrow \Omega^q(\mathcal{A}_Y).$$

To show that F is a local functor let $\wp \in \mathrm{Spec}(S)$ and let $X \in L$. The localization at $\wp$ of the inclusion

$$\nu_{X(\wp),X} : \Omega^q(\mathcal{A}_{X(\wp)}) \hookrightarrow \Omega^q(\mathcal{A}_X)$$

is injective because localization is an exact functor, see Theorem 26.9. Let $\omega \in \Omega^q(\mathcal{A}_X)$. It follows from Proposition 16.9 that if f is a defining polynomial for the arrangement $\mathcal{A}_X \setminus \mathcal{A}_{X(\wp)}$ then $f\omega \in \Omega^q(\mathcal{A}_{X(\wp)})$. Since $f \notin \wp$, we have $\omega = (f\omega)/f \in \Omega^q(\mathcal{A}_{X(\wp)})_\wp$. This implies that the localization of $\nu_{X(\wp),X}$ is bijective.

Example 18.7 *Recall the S-modules $D(\mathcal{A}_X)$ from* Definition 13.12. *Let q be a nonnegative integer. Define the contravariant functor $F : L \to$ (S-Mod) by $F(X) = D(\mathcal{A}_X)$. If $X \leq Y$ then $\mathcal{A}_X \subseteq \mathcal{A}_Y$. It follows from* Proposition 13.9 *that there are inclusions*

$$\nu_{X,Y} : D(\mathcal{A}_Y) \hookrightarrow D(\mathcal{A}_X).$$

The functor F is local. The proof is similar to the argument above.

The Homology $H_p(\mathcal{A}, F)$

Let $\mathcal{A}$ be an ℓ-arrangement in V of rank $r(\mathcal{A})$. Let $T = T(\mathcal{A})$ and let $P = L(\mathcal{A}) \setminus \{T\}$. Let F be a local covariant functor from $L(\mathcal{A})$ to (S-Mod). Let $(C_.(\mathcal{A}, F), \partial) = (C_.(L(\mathcal{A}), F), \partial)$ denote the order complex with F coefficients. Let $q \geq 0$ and let $H_q(\mathcal{A}, F)$ denote the q-th homology of $(C_.(\mathcal{A}, F), \partial)$. It is an S-module. Suppose that $\wp \in \mathrm{Spec}(S)$ satisfies $\mathrm{ht}\wp < r(\mathcal{A})$. We want to show that $H_q(\mathcal{A}, F)_\wp = 0$. In order to prove this result, we modify the chain complex $(C_.(\mathcal{A}, F), \partial)$ to obtain a new chain complex $(\tilde{C}_.(\mathcal{A}, F), \tilde{\partial})$ by allowing chains with repetitions. Let $\tilde{C}_0(\mathcal{A}, F) = C_0(\mathcal{A}, F) = F(T)$. For $q > 0$ the free S-module $\tilde{C}_q(\mathcal{A}, F)$ is spanned by

$$\{y \otimes (X_1, \ldots, X_q) \mid y \in F(X_1),\ X_1 \leq \cdots \leq X_q < T\}.$$

Clearly $C_q(\mathcal{A}, F) \subseteq \tilde{C}_q(\mathcal{A}, F)$. The boundary maps

$$\tilde{\partial} : \tilde{C}_q(\mathcal{A}, F) \to \tilde{C}_{q-1}(\mathcal{A}, F)$$

are defined by the same formulas as the boundary maps in $(C_.(\mathcal{A}, F), \partial)$. Define the S-linear map $\pi : \tilde{C}_q(\mathcal{A}, F) \to C_q(\mathcal{A}, F)$ by

$$\pi(y \otimes (X_1, \ldots, X_q)) = \begin{cases} y \otimes (X_1, \ldots, X_q) & \text{if } (X_1, \ldots, X_q) \text{ has no repetition,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 18.8 *The map π is a chain map: $\pi\tilde{\partial} = \partial\pi$.*

Proof. If $(X_1, \ldots, X_q)$ has no repetition then the terms of

$$\sum_{k=1}^{q} (-1)^{k-1} (X_1, \ldots, \widehat{X_k}, \ldots, X_q)$$

have no repetition. Thus

$$\pi\tilde{\partial}(y \otimes (X_1, \ldots, X_q)) = \partial\pi(y \otimes (X_1, \ldots, X_q))$$

for $y \otimes (X_1, \ldots, X_q) \in \tilde{C}_q(\mathcal{A}, F)$. Suppose that $(X_1, \ldots, X_q)$ has a repetition, $X_i = X_{i+1}$ for $1 \leq i \leq q-1$. Then

$$\sum_{k=1}^{q} (-1)^{k-1} (X_1, \ldots, \widehat{X_k}, \ldots, X_q) = \sum_{k=1}^{i-1} (-1)^{k-1} (X_1, \ldots, \widehat{X_k}, \ldots, X_q) + \sum_{k=i+2}^{q} (-1)^{k-1} (X_1, \ldots, \widehat{X_k}, \ldots, X_q).$$

Since $(X_1, \ldots, \widehat{X_k}, \ldots, X_q)$ with $i \neq k \neq i+1$ has the repetition $X_i = X_{i+1}$, we have $\pi\tilde{\partial}(y \otimes (X_1, \ldots, X_q)) = 0 = \partial\pi(y \otimes (X_1, \ldots, X_q))$. □

Proposition 18.9 *Let $\mathcal{A}$ be an arrangement of rank r and let $F : L(\mathcal{A}) \to$ (S-Mod) be a covariant local functor. Suppose that $\wp \in \mathrm{Spec}(S)$ satisfies $\mathrm{ht}\wp < r$. Then $H_q(\mathcal{A}, F)_\wp = 0$.*

Proof. Let $\wp \in \mathrm{Spec} S$. Recall that $T = T(\mathcal{A})$ and $r = r(T)$. By assumption $\mathrm{ht}\wp < r(T)$. Note that $r(T(\wp)) \leq \mathrm{ht}\wp < r(T)$. Thus $T(\wp) \in P = L \setminus \{T\}$. Identify $F(X)_\wp$ with $F(X(\wp))_\wp$ for all $X \in L$. Define the S-linear map $h : C_q(\mathcal{A}, F)_\wp \to \tilde{C}_{q+1}(\mathcal{A}, F)_\wp$ as follows. For $q = 0$ let $h(y) = y \otimes (T(\wp))$, where $y \in F(T)_\wp = F(T(\wp))_\wp$. For $q > 0$ let

$$\begin{aligned} & h(y \otimes (X_1, \ldots, X_q)) \\ = {} & (-1)^q y \otimes (X_1(\wp), \ldots, X_q(\wp), T(\wp)) + \sum_{j=1}^{q} (-1)^{j-1} y \otimes (X_1(\wp), \ldots, X_j(\wp), X_j, \ldots, X_q), \end{aligned}$$

where $y \in F(X_1)_\wp = F(X_1(\wp))_\wp$. A lengthy computation, which we omit, yields $\tilde{\partial}h + h\partial = id$ and hence $\pi\tilde{\partial}h + \pi h\partial = id$. Using Lemma 18.8, we have $\partial(\pi h) + (\pi h)\partial = id$. This shows that πh is a chain homotopy between the identity map and the zero map. Thus $(C_.(\mathcal{A}, F)_\wp, \partial)$ is an acyclic complex. It follows from Theorem 26.9 that localization is an exact functor. Thus

$$H_q(\mathcal{A}, F)_\wp = H_q(C_.(\mathcal{A}, F), \partial)_\wp = H_q(C_.(\mathcal{A}, F)_\wp, \partial) = 0. \quad \square$$

Theorem 18.10 *Let $q \geq 0$. If $F : L(\mathcal{A}) \to$ (S-Mod) is a local covariant functor then*

$$\dim_S H_q(\mathcal{A}, F) \leq \dim T.$$

Proof. Let $M = H_q(\mathcal{A}, F)$. By Theorem 26.18, we have

$$\dim_S M = \ell - \min_{\wp \in \mathrm{Supp}(M)} \mathrm{ht}\wp.$$

Proposition 18.9 asserts that $\wp \notin \mathrm{Supp}(M)$ if $\mathrm{ht}\wp < r(T)$. Consequently, if $\wp \in \mathrm{Supp}(M)$ then $\mathrm{ht}\wp \geq r(T)$. Thus $\dim_S M \leq \ell - r(T) = \dim T$. □

The next result follows from Theorems 18.10 and 26.22.

Corollary 18.11 *Let $q \geq 0$. If $F : L(\mathcal{A}) \to$ (S-Mod) is a local covariant functor then* $\mathrm{Poin}(H_q(\mathcal{A}, F), x)$ *has a pole at $x = 1$ of order at most* $\dim T$. □

Let $ch_q(P)$ be the set of all q-chains in P:

$$ch_q(P) = \{(X_1, \ldots, X_q) \mid X_i \in P,\ X_1 < \cdots < X_q\}.$$

Then $ch(P) = \bigcup_q ch_q(P)$. Recall from Definition 5.3 that for $c \in ch(P)$, the first element of c is $\underline{c}$, the last element of c is $\bar{c}$, and the cardinality of c is $|c|$.

Theorem 18.12 *Let $X \in L$. If $F : L \to$ (S-Mod) is a local covariant functor then*

$$\sum_{Y \in L_X} \mu(Y, X) \mathrm{Poin}(F(Y), x)$$

has a pole at $x = 1$ of order at most $\dim X$.

Proof. Since the sum involves elements $Y \leq X$, we may assume that $X = T$. Let $P = L \setminus \{T\}$. Note that $C_0(\mathcal{A}, F) = F(T)$ and $C_q(\mathcal{A}, F) \simeq \bigoplus_{c \in ch_q(P)} F(\underline{c})$. Therefore we have

$$\begin{aligned}\sum_{q=0}^{\ell}(-1)^q \mathrm{Poin}(H_q(\mathcal{A}, F), x) &= \sum_{q=0}^{\ell}(-1)^q \mathrm{Poin}(C_q(\mathcal{A}, F), x) \\ &= \mathrm{Poin}(F(T), x) + \sum_{q=0}^{\ell}(-1)^q \sum_{c \in ch_q(P)} \mathrm{Poin}(F(\underline{c}), x) \\ &= \mathrm{Poin}(F(T), x) + \sum_{c \in ch(P)}^{\ell} (-1)^{|c|} \mathrm{Poin}(F(\underline{c}), x) \\ &= \mathrm{Poin}(F(T), x) + \sum_{Y \in P} \mathrm{Poin}(F(Y), x) \sum_{\substack{c \in ch(P) \\ \underline{c} = Y}} (-1)^{|c|}.\end{aligned}$$

Let $ch(L)$ be the set of all chains in L. It follows from Proposition 5.4 that for all $Y \in P$

$$\sum_{\substack{c \in ch(P) \\ \underline{c} = Y}} (-1)^{|c|} = \sum_{c \in ch[Y, T]} (-1)^{|c|-1} = \mu(Y, T).$$

Thus we have

$$\sum_{q=0}^{\ell}(-1)^q \mathrm{Poin}(H_q(\mathcal{A}, F), x) = \sum_{Y \in L} \mu(Y, T) \mathrm{Poin}(F(Y), x),$$

which has a pole at $x = 1$ of order at most $\dim T$ by Corollary 18.11. □

The following theorem is an immediate consequence of Theorem 18.12 and Example 18.6.

Theorem 18.13 *Let $X \in L$ and let p be a nonnegative integer. Then*

$$\sum_{Y \in L_X} \mu(Y, X) \mathrm{Poin}(\Omega^p(\mathcal{A}_Y), x)$$

has a pole at $x = 1$ of order at most $\dim X$. □

The Polynomial $\Psi(\mathcal{A}, x, t)$

Next we prove the main result of this section. It is the following formula for the characteristic polynomial:

$$\chi(\mathcal{A}, t) = \lim_{x \to 1} \sum_{p=0}^{\ell} \mathrm{Poin}(\Omega^p(\mathcal{A}), x)(t(1 - x) - 1)^p. \tag{1}$$

Note first that in the proof of (1) we may assume that the field $\mathbb{K}$ is infinite. Let $\mathbb{L}$ be a field extension of $\mathbb{K}$. Let $V_{\mathbb{L}} = \mathbb{L} \oplus_{\mathbb{K}} V$ and let $\mathcal{A}_{\mathbb{L}}$ be the corresponding arrangement in $V_{\mathbb{L}}$. Clearly $\chi(\mathcal{A}_{\mathbb{L}}, t) = \chi(\mathcal{A}, t)$. Moreover, $\mathbb{L} \oplus_{\mathbb{K}} \Omega^p(\mathcal{A})$ is isomorphic to $\Omega^p(\mathcal{A}_{\mathbb{L}})$. Thus the right-hand side of (1) is also independent of field extension.

Definition 18.14 *Let*

$$\Psi(\mathcal{A}; x, t) = \sum_{p=0}^{\ell} \mathrm{Poin}(\Omega^p(\mathcal{A}), x)(t(1 - x) - 1)^p.$$

Proposition 18.15 *If $\mathcal{A}$ is a free arrangement with* $\exp \mathcal{A} = b_1, \ldots, b_\ell$ *then*

$$\Psi(\mathcal{A}; x, t) = \prod_{i=1}^{\ell}(tx^{-b_i} - (x^{-1} + x^{-2} + \cdots + x^{-b_i})).$$

In particular for the empty ℓ-arrangement $\Psi(\Phi_\ell; x, t) = t^\ell$.

Proof. It follows from Corollary 16.16 that

$$\mathrm{Poin}(\Omega^p(\mathcal{A}), x) = \sum x^{-b_{i_1} - \cdots - b_{i_p}}/(1 - x)^\ell,$$

where the sum is over the set $\{(i_1, \ldots, i_p) \mid 1 \leq i_1 < \cdots < i_p \leq \ell\}$. Let y be an indeterminate. Then

$$\sum_{p=0}^{\ell} \mathrm{Poin}(\Omega^p(\mathcal{A}), x) y^p = \prod_{i=1}^{\ell}((1 + x^{-b_i} y)/(1 - x)).$$

Now set $y = t(1 - x) - 1$ and divide by $1 - x$ in each factor. □

A priori $\Psi(\mathcal{A}; x, t)$ may have a pole at $x = 1$ because each $\Omega^p(\mathcal{A})$ is a finite S-module. The order of the pole is at most ℓ by Proposition 26.22. We will prove in Proposition 18.17 that $\Psi(\mathcal{A}; x, t)$ has no pole at $x = 1$ and that $\Psi(\mathcal{A}; x, t) \in \mathbb{Z}[x, x^{-1}, t]$. Thus we will be able to rewrite (1) as

$$\chi(\mathcal{A}, t) = \Psi(\mathcal{A}; 1, t). \tag{2}$$

Proposition 18.16 *If $\mathcal{A}$ is nonempty then $\Psi(\mathcal{A}; x, 1) = 0$.*

Proof. In Definition 16.23 we constructed for a nonempty arrangement $\mathcal{A}$ the complex $(\Omega^{\cdot}(\mathcal{A}), \partial)$. Its boundary operator $\partial\omega = (d\alpha/\alpha) \wedge \omega$ with $\ker(\alpha) \in \mathcal{A}$ has pdegree -1. In Proposition 16.24 we showed that this complex is acyclic. Thus

$$\Psi(\mathcal{A}; x, 1) = \sum_{p=0}^{\ell} \mathrm{Poin}(\Omega^p(\mathcal{A}), x)(-x)^p = 0. \quad □$$

Proposition 18.17 *For any arrangement $\Psi(\mathcal{A};x,t) \in \mathbb{Z}[x,x^{-1},t]$. In particular $\Psi(\mathcal{A};x,t)$ has no pole at $x=1$.*

Proof. Let m and n be minimal nonnegative integers so that

$$P(x,t) = x^n(1-x)^m\Psi(x,t)$$

is a polynomial in x and t. Note that $m \le \ell$ because the order of the pole at $x=1$ of $\mathrm{Poin}(\Omega^p(\mathcal{A}),x)$ is at most ℓ by Proposition 26.22. Thus for $d>0$

$$\begin{aligned} P(x,(1-x^{-d})/(1-x)) &= x^n(1-x)^m\Psi(x,(1-x^{-d})/(1-x)) \\ &= x^n(1-x)^m\sum_{p=0}^{\ell}\mathrm{Poin}(\Omega^p(\mathcal{A}),x)(-x^{-d})^p. \end{aligned}$$

Choose a generic $\eta \in \Omega^1_d[V]$ as in Proposition 16.29. Recall the η-complex from Definition 16.25

$$0 \to \Omega^0(\mathcal{A}) \overset{\partial_\eta}{\to} \Omega^1(\mathcal{A}) \overset{\partial_\eta}{\to} \cdots \overset{\partial_\eta}{\to} \Omega^\ell(\mathcal{A}) \to 0$$

Here $\partial_\eta\omega = \eta\wedge\omega$. Proposition 16.29 asserts that its cohomology groups are finite dimensional over $\mathbb{K}$. Since each boundary map ∂_η is of pdegree d,

$$\sum_{p=0}^{\ell}\mathrm{Poin}(\Omega^p(\mathcal{A}),x)(-x^{-d})^p = \sum_{p=0}^{\ell}\mathrm{Poin}(H^p(\Omega^\cdot(\mathcal{A})),x)(-x^{-d})^p$$

has finite value at $x=1$. If $m>0$ then

$$P(x,(1-x^{-d})/(1-x)) = x^n(1-x)^m\sum_{p=0}^{\ell}\mathrm{Poin}(\Omega^p(\mathcal{A}),x)(-x^{-d})^p$$

vanishes at $x=1$. Thus

$$P(1,-d) = P(x,(1-x^{-d})/(1-x))|_{x=1} = 0.$$

This is true for infinitely many positive integers d. It follows that $P(1,t)$ is identically zero. This contradicts the minimality of m. Therefore $m=0$. □

Proposition 18.18 *If $X \in L(\mathcal{A})$ then $t^{\dim X}$ divides $\Psi(\mathcal{A}_X;x,t)$.*

Proof. Let $d = \dim X$. Note that $\mathcal{A}_X = \mathcal{A}_1 \times \Phi_d$ for some $(\ell-d)$-arrangement $\mathcal{A}_1$ and the empty arrangement Φ_d in X. By 16.22, we have

$$\begin{aligned} \Psi(\mathcal{A}_X;x,t) &= \sum_{n=0}^{\ell}\mathrm{Poin}(\Omega^n(\mathcal{A}_X),x)(t(1-x)-1)^n \\ &= \Big(\sum_{p=0}^{\ell-d}\mathrm{Poin}(\Omega^p(\mathcal{A}_1),x)(t(1-x)-1)^p\Big)\cdot\Big(\sum_{q=0}^{d}\mathrm{Poin}(\Omega^q(\Phi_d),x)(t(1-x)-1)^q\Big) \\ &= \Psi(\mathcal{A}_1;x,t)\Psi(\Phi_d;x,t). \end{aligned}$$

By Proposition 18.15, we have $\Psi(\Phi_d;x,t) = t^d$. □

We need the following characterization of $\chi(\mathcal{A},t)$.

Proposition 18.19 *Let $L = L(\mathcal{A})$. Suppose that a map $G: L \to \mathbb{Z}[t]$ satisfies the following four conditions:*
(1) $G(V) = t^\ell$,
(2) $G(X)|_{t=1} = 0$ for $X \ne V$,
(3) $t^{\dim X}$ divides $G(X)$ for all $X \in L$,
(4) the degree of t in

$$\sum_{Y\in L_X}\mu(Y,X)G(Y)$$

does not exceed $\dim X$ *for any $X \in L$. Then $G(X) = \chi(\mathcal{A}_X,t)$ for all $X \in L$.*

Proof. Let

$$G'(X) = \sum_{Y\in L_X}\mu(Y,X)G(Y).$$

If $Y \le X$ then $\dim Y \ge \dim X$. By (3) $t^{\dim Y}$ divides $G(Y)$, so $t^{\dim X}$ divides $G'(X)$. On the other hand, it follows from (4) that $\deg G'(X) \le \dim X$. Therefore we can write $G'(X) = g(X)t^{\dim X}$ for some map $g: L \to \mathbb{Z}$. We get

$$g(X) = G'(X)|_{t=1} = \sum_{Y\in L_X}\mu(Y,X)G(Y)|_{t=1} = \mu(V,X)$$

by using (1) and (2). Thus $G'(X) = \mu(V,X)t^{\dim X}$. It follows from the Möbius inversion formulas of Proposition 5.6 and from Definition 6.5 that

$$G(X) = \sum_{Y\in L_X}G'(Y) = \sum_{Y\in L_X}\mu(V,Y)t^{\dim Y} = \chi(\mathcal{A}_X,t). \quad \square$$

Theorem 18.20 *The characteristic polynomial of an ℓ-arrangement $\mathcal{A}$ is given by*

$$\chi(\mathcal{A},t) = \Psi(\mathcal{A};1,t).$$

Proof. We verify conditions (1)-(4) of Proposition 18.19 for $G(X) = \Psi(\mathcal{A}_X;1,t)$.
(1) It follows from Proposition 18.15 that $G(V) = \Psi(\Phi_\ell;1,t) = t^\ell$.
(2) It follows from Proposition 18.16 that for $X \ne V$

$$G(X)|_{t=1} = \Psi(\mathcal{A}_X;1,1) = \Psi(\mathcal{A}_X;x,1)|_{x=1} = 0.$$

(3) This follows from Proposition 18.18 with $x=1$.

(4) Fix $X \in L$. We compute

$$\begin{aligned}\sum_{Y \in L_X} \mu(Y,X)G(Y) &= \sum_{Y \in L_X} \mu(Y,X)\Psi(\mathcal{A}_Y;1,t) \\ &= \sum_{p=0}^{\ell} \sum_{Y \in L_X} \mu(Y,X)\mathrm{Poin}(\Omega^p(\mathcal{A}_Y),x)(t(1-x)-1)^p|_{x=1} \\ &= \sum_{p=0}^{\ell} M_p(x)(t(1-x)-1)^p|_{x=1},\end{aligned}$$

where

$$M_p(x) = \sum_{Y \in L_X} \mu(Y,X)\mathrm{Poin}(\Omega^p(\mathcal{A}_Y),x).$$

By Theorem 18.13, $(1-x)^{\dim X} M_p(x)$ has no pole at $x = 1$. Thus the coefficient of t^n in $M_p(x)(t(1-x)-1)^p$, which is equal to $(-1)^{p-n}\binom{p}{n} M_p(x)(1-x)^n$, lies in $(1-x)\mathbb{Z}[x,x^{-1}]$ if $n > \dim X$. Thus for each p, the degree of t in $M_p(x)(t(1-x)-1)^p|_{x=1}$ does not exceed $\dim X$. Therefore $\sum_{Y \in L_X} \mu(Y,X)G(Y)$ has the same property. □

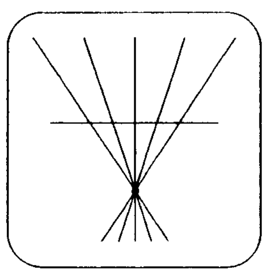


Figure 23: $\pi(\mathcal{A},t)$ factors but $\mathcal{A}$ is not free

Factorization Theorem for Free Arrangements

Theorem 18.21 (Factorization) *If $\mathcal{A}$ is a free arrangement with* $\exp \mathcal{A} = \{b_1,\dots,b_\ell\}$ *then*

$$\pi(\mathcal{A},t) = \prod_{i=1}^{\ell}(1+b_i t).$$

Proof. We computed $\Psi(\mathcal{A};x,t)$ in Proposition 18.15. Set $x = 1$ and use Theorem 18.20 to obtain the equivalent statement $\chi(\mathcal{A},t) = \prod_{i=1}^{\ell}(t-b_i)$. □

We say that $\pi(\mathcal{A},t)$ factors if $\pi(\mathcal{A},t) = \prod_{i=1}^{\ell}(1+b_i t)$ where the $b_i \in \mathbb{Z}$. The next example is due to Falk and Randell [66]. It shows that the implication in Theorem 18.21 cannot be reversed.

Example 18.22 *Let $\mathcal{A}$ be the 3-arrangement in* Figure 23. *Then $\pi(\mathcal{A},t)$ factors but $\mathcal{A}$ is not free.*

Direct computation gives $\pi(\mathcal{A},t) = (1+t)(1+3t)(1+3t)$. Remove the horizontal line to obtain $\mathcal{A}'$. Both $\mathcal{A}'$ and $\mathcal{A}''$ are free with $\exp \mathcal{A}' = \{1,1,4\}$ and $\exp \mathcal{A}'' = \{1,5\}$. If we assume that $\mathcal{A}$ is free then we contradict Theorem 15.9.

Chapter V

In this chapter we return to the convention that an arrangement is not necessarily central. The subject of this chapter is the topology of the complement of a complex arrangement, $M(\mathcal{A})$. Call the complex arrangements $\mathcal{A} = (\mathcal{A}, V)$ and $\mathcal{B} = (\mathcal{B}, V)$ **diffeomorphic**, **homeomorphic**, or **homotopy equivalent** if $M(\mathcal{A})$ and $M(\mathcal{B})$ are diffeomorphic, homeomorphic, or homotopy equivalent. It is natural to ask how these topological equivalence classes relate to the combinatorial equivalence classes defined earlier. For example we will show in section 22 that $M(\mathcal{A})$ and $M(\mathcal{B})$ have the same Betti numbers if and only if $\mathcal{A}$ and $\mathcal{B}$ are π-equivalent, and that $M(\mathcal{A})$ and $M(\mathcal{B})$ have isomorphic cohomology rings if and only if $\mathcal{A}$ and $\mathcal{B}$ are A-equivalent.

In section 19 we prove some elementary facts about $M = M(\mathcal{A})$ and discuss a few examples. We also give a review of fundamental work of Arnold, Brieskorn, Deligne and Hattori. The rest of the chapter does not follow the chronology of discovery. In section 20 we construct a finite simplicial complex M of the homotopy type of M. The construction uses an embedding in V of the order complex of the face poset of a real arrangement. In principle, M contains all information about the homotopy type of M. In the special case of a complexified real arrangement, Salvetti [170] constructed a smaller complex W of the homotopy type of M. Arvola [13] constructed a simplicial map $\mathsf{M} \to \mathsf{W}$ which is a homotopy equivalence. In practice, M and W are very large and unsuited for explicit calculations. It is therefore desirable to find simple algorithms to compute topological invariants of M.

Arvola's presentation of the fundamental group of M is in section 21. It generalizes Randell's presentation of the fundamental group of the complexification of a real arrangement. In section 22 we consider the cohomology groups of $M(\mathcal{A})$. We use our results on $R(\mathcal{A})$ from section 12 to prove that given a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, there are split short exact sequences for all $k \geq 0$

$$0 \to H^{k+1}(M(\mathcal{A}')) \to H^{k+1}(M(\mathcal{A})) \to H^{k}(M(\mathcal{A}'')) \to 0.$$

It follows that the map $R(\mathcal{A}) \to H^{*}(M(\mathcal{A}))$ induced by $\omega_H \mapsto [(1/2\pi i)\omega_H]$ is an algebra isomorphism. Together with the algebra isomorphism $R(\mathcal{A}) \simeq A(\mathcal{A})$ established in section 12, this provides a presentation of the cohomology algebra in terms of generators and relations. This is the topological interpretation of $A(\mathcal{A})$. Thus the cohomology algebra of $M(\mathcal{A})$ depends only on $L(\mathcal{A})$.

We give an elementary proof of Brieskorn's Lemma 22.15. It also follows that the Poincaré polynomial of the complement is

$$\mathrm{Poin}(M(\mathcal{A}), t) = \pi(\mathcal{A}, t).$$

Thus the coefficients of $\pi(\mathcal{A}, t)$ are also the Betti numbers of the complement. It also shows that if $\mathcal{A}$ is a real arrangement then $M(\mathcal{A})$ has the M-property. In section 23 we prove that the complement of a supersolvable arrangement admits a strictly linear fibration. In section 24 we describe some related recent results: work of Falk and Kohno on minimal models, Manin and Schechtman's work on discriminantal arrangements, Falk's geometric linking,

the cohomology of the Milnor fiber of a generic arrangement, and the results of Goresky and MacPherson on arrangements of subspaces of arbitrary codimension.

We have been using the basis $x_1, \ldots, x_\ell$ for V^*. We continue to use it except when we want to introduce real variables in complex space. Then we use $z_1, \ldots, z_\ell$ as a basis for V^* and write $z_k = x_k + iy_k$. Note also that depending on the application we may write these real variables ordered either $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$, or $x_1, y_1, \ldots, x_\ell, y_\ell$.

19 The Complement $M(\mathcal{A})$

In this section we prove some elementary facts about the topology of the complement of an arrangement over the complex numbers. In addition we outline some of the work which started the recent activity in the area by Arnold [6], Brieskorn [33], Deligne [49] and Hattori [91]. The coning construction of Definition 2.15 has a topological interpretation. Recall the Hopf bundle $p : \mathbf{C}^{\ell+1} \setminus \{0\} \to P^{\ell}_{\mathbf{C}}$ with fiber $\mathbf{C}^*$, which identifies $z \in \mathbf{C}^{\ell+1}$ with λz for $\lambda \in \mathbf{C}^*$.

Proposition 19.1 *Let $\mathcal{A}$ be an affine arrangement and let $\mathbf{c}\mathcal{A}$ be the cone over $\mathcal{A}$. The restriction of the Hopf bundle $p : M(\mathbf{c}\mathcal{A}) \to M(\mathcal{A})$ is a trivial bundle so*

$$M(\mathbf{c}\mathcal{A}) \approx M(\mathcal{A}) \times \mathbf{C}^*.$$

Similarly, if $\mathcal{A}$ is a central arrangement then $M(\mathcal{A}) \approx M(\mathbf{d}\mathcal{A}) \times \mathbf{C}^*$.

Proof. The identification $p(M(\mathbf{c}\mathcal{A})) = M(\mathcal{A})$ is immediate from Definition 2.15. Let $K_0 \in \mathbf{c}\mathcal{A}$. The restriction of p to $M_{K_0} = \mathbf{c}V \setminus \{K_0\}$ has base space $P^{\ell}_{\mathbf{C}} \setminus P^{\ell-1}_{\mathbf{C}} \approx \mathbf{C}^{\ell}$. Thus $p : M_{K_0} \to \mathbf{C}^{\ell}$ is a trivial bundle and $p : M(\mathbf{c}\mathcal{A}) \to M(\mathcal{A})$ is a restriction. □

Proposition 19.2 *Let $\mathcal{A}$ be a complex central arrangement defined by $Q = Q(\mathcal{A})$. The map $Q : M \to \mathbf{C}^*$ is the projection of a smooth fiber bundle, called the* **Milnor fibration**. *The typical fiber $F = Q^{-1}(1)$ is called the* **Milnor fiber**.

Proof. It follows from work of Milnor [128] that the restriction of Q to a suitable neighborhood of the origin is a fibration. Since Q is homogeneous, we may take this neighborhood to be all of M. □

Proposition 19.3 *The complement $M = M(\mathcal{A})$ of a complex ℓ-arrangement $\mathcal{A}$ is an open, smooth, parallelizable manifold of real dimension 2ℓ which has the homotopy type of a finite cell complex.*

Proof. The vector space V is an open, smooth, parallelizable manifold of real dimension 2ℓ and M is open in V. By Proposition 19.1 it suffices to prove the last assertion for central arrangements. It is clear if $\mathcal{A}$ is empty. Otherwise we use the Milnor fibration 19.2. Milnor [128] proved that the fiber $F = Q^{-1}(1)$ has the homotopy type of a finite cell complex. It follows that the same holds for M. □

Suppose $\mathcal{A}$ is a central ℓ-arrangement. Its Milnor fiber admits a free action by the cyclic group of order $n = |\mathcal{A}|$. The quotient space is naturally identified with $M(\mathbf{d}\mathcal{A})$. The map $p : M(\mathcal{A}) \to M(\mathbf{d}\mathcal{A})$ is the orbit map of the standard $\mathbb{C}^*$-action

$$t(x_1, \dots, x_\ell) = (tx_1, \dots, tx_\ell).$$

Let $G(n)$ denote the cyclic subgroup of $\mathbb{C}^*$ of order n. Since Q is homogeneous of degree n, $G(n)F \subseteq F$. It follows that $M(\mathbf{d}\mathcal{A}) = M(\mathcal{A})/\mathbb{C}^* = F/G(n)$. Let $\zeta = e^{2\pi i/n}$ be a generator of $G(n)$. The map $F \to F$ induced by multiplication by ζ is called the **monodromy** of the Milnor fiber. Let $\pi : F \to F/G(n)$ and let $\pi' : \mathbb{C}^* \to \mathbb{C}^*/G(n)$. The commutative diagram below connects the two fibrations. Here id denotes the identity map.

$$\begin{array}{ccccc} & & F & \xrightarrow{\pi} & F/G(n) \\ & & \downarrow i & & \downarrow id \\ \mathbb{C}^* & \to & M(\mathcal{A}) & \xrightarrow{p} & M(\mathbf{d}\mathcal{A}) \\ \downarrow \pi' & & \downarrow Q & & \\ \mathbb{C}^*/G(n) & \xrightarrow{id} & \mathbb{C}^* & & \end{array}$$

We return to the calculation of the cohomology of the Milnor fiber for certain arrangements in section 24.

$K(\pi,1)$–arrangements

Example 19.4 *Let $\mathcal{A}$ be the arrangement of* Example 2.5 *defined by $Q(\mathcal{A}) = xy(x+y)$. The complement $M(\mathcal{A})$ has the homotopy type of $(S^1 \vee S^1) \times S^1$.*

We use Proposition 19.1. If we let $K_0 = \ker(y)$ go to the line at infinity then the corresponding 1-arrangement $\mathbf{d}\mathcal{A}$ is defined by $Q(\mathbf{d}\mathcal{A}) = x(x+1)$. Thus $M(\mathbf{d}\mathcal{A})$ is the complement of the two points $x = 0$ and $x = -1$ in $\mathbb{C}$. It follows that $M(\mathbf{d}\mathcal{A})$ has the homotopy type of $S^1 \vee S^1$. Since $\mathbb{C}^*$ has the homotopy type of S^1, the conclusion follows from Proposition 19.1. Note that $\pi_i(M(\mathcal{A})) = 0$ for $i \geq 2$ so $M(\mathcal{A})$ is a $K(\pi,1)$-space.

Definition 19.5 *Call $\mathcal{A}$ a* **$K(\pi,1)$–arrangement** *if $M(\mathcal{A})$ is a $K(\pi,1)$-space.*

Proposition 19.6 *Every central 2-arrangement is $K(\pi,1)$.*

Proof. The argument in Example 19.4 shows that if $\mathcal{A}$ is a central 2-arrangement with $|\mathcal{A}| = n$ then $M(\mathcal{A})$ has the homotopy type of $(\vee_{n-1}S^1) \times S^1$. □

Proposition 19.7 *The Boolean arrangement is $K(\pi,1)$.*

Proof. Since M is the complement of the coordinate hyperplanes $M = (\mathbb{C}^*)^\ell$, it has the homotopy type of an ℓ-torus. □

Remark 19.8 *More on the braid arrangement.*

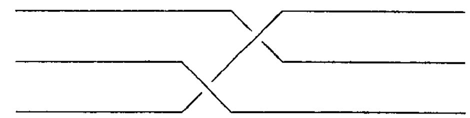


Figure 24: A braid on three strands

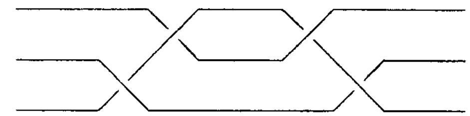


Figure 25: A pure braid on three strands

Before we show that the braid arrangement is also $K(\pi,1)$, it is appropriate to justify its name and describe some of its history. Braids and the braid group were defined by Artin [10]. Figure 24 shows a braid on 3 strands. Braids with ℓ strands may be composed by juxtaposition. There is a suitable notion of isotopy of braids which makes this an associative multiplication. The inverse of a braid is the braid which untangles it. Isotopy classes of braids on ℓ strands form a group called the **braid group**, $B(\ell)$. For details see Birman's book [23]. There is a natural surjection $B(\ell) \to Sym(\ell)$ which sends each braid to the permutation of its ends. The image of the braid in Figure 24 is the 3-cycle (1,2,3). The kernel of this map is called the **pure braid group** $PB(\ell)$. The corresponding pure braids have the property that each strand returns to its point of origin. Figure 25 shows a pure braid on 3 strands. The braid group is generated by the braids a_i for $1 \leq i < \ell - 1$ indicated in Figure 26. It is known [23, p.11] that the following relations are sufficient to give a presentation:

$$\begin{array}{ll} a_i a_j = a_j a_i & \text{if } |i-j| \geq 2 \\ a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} & 1 \leq i \leq \ell - 2. \end{array}$$

The fact that the pure braid group is the fundamental group of the pure braid space as we described it in section 2 first appeared in a paper by Fox and Neuwirth [71]. To make this statement precise let $\mathcal{A}_\ell$ denote the complexified braid arrangement of Definition 2.9. Let

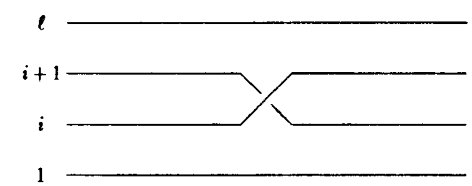


Figure 26: The generator a_i

$M_\ell = M(\mathcal{A}_\ell)$. We described in section 2 how a pure braid gives rise to a map of the circle into M_ℓ and hence to an element of its fundamental group. For the converse choose a base point $x \in M_\ell$. An element of $\pi_1(M_\ell, x)$ is represented by a map $f : (I, \{0,1\}) \to (M_\ell, x)$ which we may assume to be a smooth embedding. The coordinate functions of f are the strands of the pure braid. Thus $\pi_1(M_\ell) = PB(\ell)$. Note that M_ℓ admits a free action of $Sym(\ell)$ by permuting the coordinates. Let $B_\ell = M_\ell/Sym(\ell)$ be the orbit space and let $p : M_\ell \to B_\ell$ be the projection of this covering. A similar argument shows that $\pi_1(B_\ell) = B(\ell)$.

The covering $p : M_\ell \to B_\ell$ is a fibration with discrete fiber F of cardinality $|Sym(\ell)|$. The homotopy long exact sequence of this fibration gives the short exact sequence

$$1 \to \pi_1(M_\ell) \to \pi_1(B_\ell) \to \pi_0(F) \to 1$$

which we may identify for $k = 1$ with

$$1 \to PB(\ell) \to B(\ell) \to Sym(\ell) \to 1.$$

For $k \geq 2$ it gives isomorphisms $\pi_k(M_\ell) \simeq \pi_k(B_\ell)$. Fadell and Neuwirth [58] showed that M_ℓ is a $K(\pi,1)$ space.

Theorem 19.9 *The braid arrangement $\mathcal{A}_\ell$ is $K(\pi,1)$.*

Proof. The projection map $\mathbf{C}^\ell \to \mathbf{C}^{\ell-1}$ defined by $(x_1,\ldots,x_\ell) \to (x_1,\ldots,x_{\ell-1})$ induces a locally trivial fibration $M_\ell \to M_{\ell-1}$. The fiber over $(\xi_1,\ldots,\xi_{\ell-1})$ is $\mathbf{C} \setminus \{\xi_1,\ldots,\xi_{\ell-1}\}$. The fiber retracts onto a wedge of $(\ell-1)$ circles, so it is a $K(\pi,1)$-space. Since $M_2 = \{(x_1,x_2) \mid x_1 \neq x_2\} = \mathbf{C} \times \mathbf{C}^*$, we are done by induction. □

The representation of M as the total space of a sequence of fibrations is an imporant tool. The next two definitions and the following results are due to Falk and Randell [65]. In section 23 the existence of a fibration is proved from the existence of modular elements in $L(\mathcal{A})$.

Definition 19.10 *Let $\mathcal{A}$ be an ℓ-arrangement. Call $\mathcal{A}$* **strictly linearly fibered** *if after a suitable linear change of coordinates the restriction of the projection of $M(\mathcal{A})$ to the first $(\ell-1)$ coordinates is a fiber bundle projection whose base space B is the complement of an arrangement in $\mathbf{C}^{\ell-1}$, and whose fiber is the complex line $\mathbf{C}$ with finitely many points removed.*

Definition 19.11 *(1) The 1-arrangement $\{0\}$ is* **fiber type**.
(2) For $\ell \geq 2$ the ℓ-arrangement $\mathcal{A}$ is **fiber type** *if $\mathcal{A}$ is strictly linearly fibered with base $B = M(\mathcal{B})$ and $\mathcal{B}$ is an $(\ell-1)$-arrangement of fiber type.*

Proposition 19.12 *If $\mathcal{A}$ is fiber type then $\mathcal{A}$ is $K(\pi,1)$.*

Proof. It follows from the definition that there exist k-arrangements $\mathcal{A}_k$ for $1 \leq k \leq \ell$ with $\mathcal{A} = \mathcal{A}_\ell$ and a tower of fibrations

$$M(\mathcal{A}_\ell) \xrightarrow{\pi_{\ell-1}} M(\mathcal{A}_{\ell-1}) \xrightarrow{\pi_{\ell-2}} \cdots \xrightarrow{\pi_2} M(\mathcal{A}_2) \xrightarrow{\pi_1} M(\mathcal{A}_1) = \mathbf{C}^*$$

with the fiber F_k of π_k homeomorphic to $\mathbf{C}$ with d_k points removed. The conclusion follows by repeated application of the homotopy exact sequence of a fibration. □

The arrangement $\mathcal{A}$ of Example 6.14 is strictly linearly fibered. In fact it is easy to see from Figure 11 that $M(\mathcal{A})$ is homeomorphic to $C_3 \times C_3 \times C_1$, where C_k denotes the complex line with k points removed. Falk and Randell [65] proved that in fiber type arrangements each projection map admits a section and in each fibration the fundamental group of the base acts trivially in the cohomology of the fiber. This is sufficient to show:

Theorem 19.13 *If $\mathcal{A}$ is a fiber type ℓ-arrangement then*

$$H^*(M(\mathcal{A})) \simeq H^*(F_1) \otimes \cdots \otimes H^*(F_\ell). \quad \square$$

In a 1971 Bourbaki Seminar talk Brieskorn [33] generalized Arnold's results. He replaced the symmetric group and the braid arrangement by a Coxeter group W acting in an ℓ-dimensional real vector space $V_{\mathbf{R}}$. Let V be the complexification of $V_{\mathbf{R}}$. Then W acts as a reflection group in V. Let $\mathcal{A} = \mathcal{A}(W)$ be its reflection arrangement. Brieskorn conjectured that $\mathcal{A}(W)$ is a $K(\pi,1)$-arrangement for all Coxeter groups W. He proved this for some of the groups by representing M as the total space of a sequence of fibrations. Some of these fibrations are not strictly linear. Deligne [49] settled the question by proving the much stronger result stated below.

Definition 19.14 *Let $(\mathcal{A}_{\mathbf{R}}, V_{\mathbf{R}})$ be a real arrangement. Call $\mathcal{A}_{\mathbf{R}}$ a* **simplicial** *arrangement if every component of $M(\mathcal{A}_{\mathbf{R}})$ is an open simplicial cone.*

Theorem 19.15 *Let $(\mathcal{A}_{\mathbf{R}}, V_{\mathbf{R}})$ be a simplicial arrangement. Then its complexification $(\mathcal{A}, V)$ is $K(\pi,1)$.* □

This result proves Brieskorn's conjecture because the arrangement of a Coxeter group is simplicial [31]. There exist complex reflection groups [176] which are not Coxeter groups. It is natural to ask if their reflection arrangements are $K(\pi,1)$. For a subclass of complex reflection groups called Shephard groups this was proved in [149]. We give an outline of the argument in section 25. The conjecture is still open for some remaining complex reflection groups. The arrangement $\mathcal{A}$ of Example 6.14 shows that Theorem 19.12 is not a consequence of Theorem 19.15.

Free Arrangements

The module $D(\mathcal{A})$ has topological interpretation in case $\mathbb{K} = \mathbb{C}$. Since $N = \cup_{H\in\mathcal{A}} H$ is a singular variety, it has no tangent space. But V has a stratification induced by N and each stratum has a tangent space. We show next that at every point of V the evaluation of $D(\mathcal{A})$ at the point spans the tangent space there.

For $X \in L(\mathcal{A})$ let $M^X = M(\mathcal{A}^X)$. Then M^X is an open submanifold of X and we have a disjoint union:

$$V = \bigcup_{X\in L} M^X.$$

For $v \in V$ let TV_v denote the tangent space of V at v. Then TV_v is a $\mathbb{C}$ vector space with basis D_i. Thus we can define an evaluation map $\rho_v : D(\mathcal{A}) \to TV_v$ as follows. Given $\theta \in D(\mathcal{A})$ write $\theta = \sum h_i D_i$ and let $\rho_v(\theta) = \sum h_i(v) D_i$. Write $D(\mathcal{A})_v = \rho_v(D(\mathcal{A}))$.

Proposition 19.16 *If $v \in M^X$ then $D(\mathcal{A})_v = TM_v^X$.*

Proof. Suppose $r(X) = p$. Choose coordinates so that $X = H_1 \cap \cdots \cap H_p$ where $H_j = \ker x_j$ for $1 \le j \le p$. Note that this makes $Q(\mathcal{A})$ divisible by $x_1 \cdots x_p$. We may choose D_i for $p+1 \le i \le \ell$ as a basis for $TX_v = TM_v^X$. In the rest of this paragraph let $1 \le j \le p$. If $v \in M^X$ then $x_j(v) = 0$. If $\theta \in D(\mathcal{A})$ then by Lemma 13.8 $\theta \in D(x_j)$ and hence $\theta(x_j) \in x_j S$. Write $\theta = \sum_{i=1}^{\ell} h_i D_i$. It follows that h_j is divisible by x_j and hence $h_j(v) = 0$. Thus $\rho_v(\theta) = \sum_{i=p+1}^{\ell} h_i(v) D_i \in TM_v^X$.

For the converse let $Q = Q_1 Q_2$ where $Q_1 = Q(\mathcal{A}_X)$ and $Q_2 = Q(\mathcal{A} \setminus \mathcal{A}_X)$. By our choice of coordinates above, $Q_1 \in \mathbb{C}[x_1, \ldots, x_p]$ and if $v \in M^X \subset X$ then $Q_2(v) = c \neq 0$. It is easy to check that for $p+1 \le i \le \ell$ we have $\theta_i = Q_2 D_i \in D(\mathcal{A})$. Since $\rho_v \theta_i = c D_i$ it follows that $TM_v^X \subset D(\mathcal{A})_v$. □

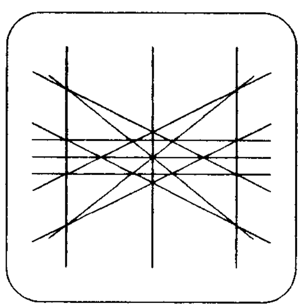


Figure 27: $K(\pi, 1)$ but not free

It is not known what additional topological properties may be implied if $\mathcal{A}$ is free. Grünbaum [83] gives a list of simplicial 3-arrangements. It follows from Theorem 19.15 that these are $K(\pi,1)$. This list was studied in [186] and several of these arrangements are not free. The smallest example is labeled $A_4(13)$ by Grünbaum [83]. It is shown in Figure 27. To see that it is not free we compute $\pi(\mathcal{A}, t) = (1+t)(1+12t+39t^2)$ and use the Factorization Theorem 18.21. The conjecture below concerns the reverse implication.

Conjecture 19.17 (Saito) *If $\mathcal{A}$ is free then $\mathcal{A}$ is $K(\pi,1)$.*

Generic Arrangements

So far we have listed a number of ways for an arrangement to be $K(\pi,1)$. However, this is not generic behavior. In fact it follows from a theorem of Hattori [91] which we state below, that most arrangements are not $K(\pi,1)$. First we need two definitions.

Definition 19.18 *An arrangement $\mathcal{A}$ is called a* **general position** *arrangement if for every subset $\{H_1, \ldots, H_p\} \subseteq \mathcal{A}$ with $p \le \ell$*

$$r(H_1 \cap \cdots \cap H_p) = p$$

and when $p > \ell$

$$H_1 \cap \cdots \cap H_p = \emptyset.$$

Note that if $\mathcal{A}$ is a central general position ℓ-arrangement then $|\mathcal{A}| \le \ell$. Thus the only interesting general position arrangements are centerless.

Definition 19.19 *Let $\mathbf{n} = \{1, \ldots, n\}$. If $I \subseteq \mathbf{n}$ let $|I|$ be its cardinality. Define the subtorus T_I of T^n by*

$$T_I = \{(z_1, \ldots, z_n) \in T^n \mid z_j = 1 \text{ for } j \notin I\}.$$

Theorem 19.20 *Let $\mathcal{A}$ be an ℓ-arrangement in general position and assume that $n = |\mathcal{A}| \ge \ell + 1$. Then $M = M(\mathcal{A})$ has the homotopy type of*

$$M_0 = \bigcup_{|I|=\ell} T_I. \quad \square$$

Hattori [91] also proved that $\pi_1(M)$ is free abelian of rank n, and that the universal covering space $\tilde{M}$ of M has trivial homology in dimensions $\neq 0, \ell$. He also gave a free $\mathbb{Z}[\pi_1(M)]$ resolution of $H_\ell(\tilde{M}; \mathbb{Z})$. In particular if $n = \ell + 1$ then $H_\ell(\tilde{M}; \mathbb{Z})$ is a free $\mathbb{Z}[\pi_1(M)]$-module of rank 1.

Definition 19.21 *Let $\mathcal{A}$ be a central ℓ-arrangement with $\ell \ge 2$. Call $\mathcal{A}$ a* **generic arrangement** *if the hyperplanes of every subarrangement $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \ell$ are linearly independent.*

Thus a central arrangement is generic if and only if it is the cone over a general position arrangement.

Corollary 19.22 *Generic arrangements are not $K(\pi,1)$.*

Proof. We may assume that the given arrangement is $\mathbf{c}\mathcal{A}$, the cone over a general position affine arrangement $\mathcal{A}$. Recall the Hopf bundle $p : M(\mathbf{c}\mathcal{A}) \to M(\mathcal{A})$ from Proposition 19.1. Since the universal cover $\tilde{M}(\mathcal{A})$ is simply connected, it follows from Hattori's results and the Hurewicz isomorphism theorem that $\pi_i(\tilde{M}(\mathcal{A})) = H_i(\tilde{M}(\mathcal{A}); \mathbb{Z}) = 0$ for $1 \le i < \ell - 1$ and $\pi_{\ell-1}(\tilde{M}(\mathcal{A})) = H_{\ell-1}(\tilde{M}(\mathcal{A}); \mathbb{Z})$. By Hattori's theorem $\pi_{\ell-1}(M(\mathcal{A})) \neq 0$ and $M(\mathcal{A})$ is not a $K(\pi,1)$-space. The conclusion follows from the fact that $\pi_i(M(\mathbf{c}\mathcal{A})) = \pi_i(M(\mathcal{A}))$ for $i \ge 2$. □

Example 19.23 *Define the generic arrangement $\mathbf{c}\mathcal{A}$ by $Q = xyz(x+y-z)$. Then $\mathbf{c}\mathcal{A}$ is not a $K(\pi,1)$-arrangement.*

Setting $K_0 = \ker(z)$, the 2-arrangement $\mathcal{A}$ is defined by $Q(\mathcal{A}) = xy(x+y-1)$. This is the arrangement in Example 2.6. By Proposition 19.1 we have $M(\mathbf{c}\mathcal{A}) = M(\mathcal{A}) \times \mathbb{C}^*$. Clearly $\mathcal{A}$ is a general position arrangement, so we may use Theorem 19.20. Here $n = 3$ and $\ell = 2$. Thus

$$M(\mathcal{A})_0 = S^1 \times S^1 \times 1 \cup S^1 \times 1 \times S^1 \cup 1 \times S^1 \times S^1 \subset S^1 \times S^1 \times S^1.$$

We can visualize $M(\mathcal{A})_0$ as the identification space obtained from the boundary of a cube by identifying opposite faces and some of their edges. Figure 28 shows the edge identifications on the front three faces of the cube. Each face becomes a torus. Since $n = 3 = \ell + 1$ it follows that $\pi_2(M(\mathcal{A})) = \mathbb{Z}$. A nontrivial element of $\pi_2(M(\mathcal{A})_0) = \pi_2(M(\mathbf{c}\mathcal{A}))$ is obtained by any map which sends S^2 onto the boundary of the cube, followed by the identifications.

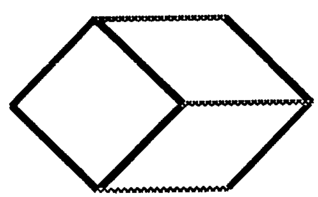


Figure 28: Three lines in general position

Deformation

Part of Hattori's argument consists of showing that every general position arrangement may be deformed through general position arrangements into the complexification of a real general position arrangement. The idea of such a deformation was generalized by Randell [158].

Definition 19.24 *Let $a_j : \mathbb{R} \to \mathbb{C}$ be smooth functions, and let $\alpha_t(x) = \sum_{j=1}^{\ell} a_j(t)x_j$. A smooth 1-parameter family of hyperplanes is defined by $H_t = \ker \alpha_t$. A smooth 1-parameter family of arrangements $\mathcal{A}_t$ in V is a finite collection of smooth 1-parameter families of hyperplanes.*

Definition 19.25 *Let $\mathcal{A} = \{H_1, \dots, H_n\}$, $\mathcal{A}' = \{H'_1, \dots, H'_n\}$ be arrangements in V of the same cardinality. We say that they have the* **same lattice** *if for all $I \subseteq \{1, \dots, n\}$*

$$\dim \cap_{i \in I} H_i = \dim \cap_{i \in I} H'_i.$$

An L-equivalence asserts that there is a linear order of the hyperplanes which gives rise to an isomorphism of the lattices. The present notion includes a fixed linear order of the hyperplanes. Thus it may be viewed as an explicit L-equivalence.

Definition 19.26 *A 1-parameter family $\mathcal{A}_t$ is a* **lattice isotopy** *in V if for any p, q the arrangements $\mathcal{A}_p$ and $\mathcal{A}_q$ have the same lattice.*

Theorem 19.27 *Let $\mathcal{A}_t$ be a lattice isotopy in V. Then $M(\mathcal{A}_0)$ is diffeomorphic to $M(\mathcal{A}_1)$, and the pair $(V, N(\mathcal{A}_0))$ is homeomorphic to the pair $(V, N(\mathcal{A}_1))$.* □

Example 19.28 *The two arrangements in* Figure 9 *have different face posets, but they fit in the 1-parameter family $\mathcal{A}_t$ defined by*

$$Q(\mathcal{A}_t) = (x-z)(x+z)(y-z)(y+z)(y-x+(2+e^{i\pi t})z).$$

It is not hard to check that this is a lattice isotopy. Thus the complements of their complexifications are diffeomorphic.

Arnold's Conjectures

The main result of Arnold's paper [6] was the calculation of the Poincaré polynomial of the pure braid space M_ℓ and the cohomology ring structure of $H^*(M_\ell)$. Arnold showed that

$$\mathrm{Poin}(M_\ell, t) = (1+t)(1+2t)\cdots(1+(\ell-1)t).$$

The reader should compare this formula with the Poincaré polynomial of the braid arrangement computed in Proposition 6.7. Arnold also showed that $H^*(M_\ell)$ is generated by the 1-dimensional elements

$$\omega_{p,q} = \frac{1}{2\pi i}\frac{dz_p - dz_q}{z_p - z_q},$$

and that all relations among these generators are consequences of the relations:

$$\omega_{p,q}\omega_{q,r} + \omega_{q,r}\omega_{r,p} + \omega_{r,p}\omega_{p,q} = 0.$$

Arnold stated two conjectures for an arbitrary arrangement $\mathcal{A}$. The first said that $H^*(M(\mathcal{A});\mathbb{Z})$ is torsion free. The second may be stated as follows. Define holomorphic differential forms $\omega_H = d\alpha_H/\alpha_H$ for $H \in \mathcal{A}$ and let $[\omega_H]$ denote the corresponding cohomology class. Let $R(\mathcal{A}) = \oplus_{p=0}^{\ell} R_p$ be the graded $\mathbb{C}$-algebra of holomorphic differential forms on $M(\mathcal{A})$ generated by the ω_H and 1. Arnold conjectured that the natural map $\omega \to [(1/2\pi i)\omega_H]$ of $R(\mathcal{A}) \to H^*(M(\mathcal{A});\mathbb{Z})$ is an isomorphism of graded algebras. This was proved by Brieskorn [33], who showed in fact that the $\mathbb{Z}$-subalgebra of $R(\mathcal{A})$ generated by the forms $(1/2\pi i)\omega_H$ and 1 is isomorphic to the singular cohomology $H^*(M(\mathcal{A});\mathbb{Z})$. In [142] it was shown that for an arbitrary arrangement $\mathcal{A}$ the Poincaré polynomial of $M(\mathcal{A})$ equals the Poincaré polynomial of $\mathcal{A}$. We prove this in section 22. The structure of the algebra $R(\mathcal{A})$ was also obtained in [142]. We presented this work in section 12.

20 The Homotopy Type of the Complement

In this section we construct a finite simplicial complex $\mathsf{M}(\mathcal{A})$ in the complement $M(\mathcal{A})$ and prove that $\mathsf{M}(\mathcal{A})$ is a strong deformation retract of $M(\mathcal{A})$. The construction applies equally well to all subspace arrangements, see [140]. It combines two ideas. The first is the use of the face poset for real hyperplane arrangements. On the combinatorial side this approach was pioneered by Zaslavsky [211] and continued in the wider setting of oriented matroids by Folkman, Lawrence and others, see [26]. On the topological side it appears in the work of Salvetti [170]. The second is due to Goresky and MacPherson [76]. It shows that from the point of view of singularity theory the natural objects of study are not just hyperplane arrangements but arrangements of arbitrary affine subspaces. In this setting complex arrangements are just special cases of real arrangements.

For central complex arrangements it is interesting to note that although we prove here that the homotopy type of the total space of the Milnor fibration is determined by combinatorial data, the Milnor fiber is a more subtle analytic invariant. When $\mathcal{A}$ is the complexification of a real arrangement, Salvetti [170] constructed a simplicial complex $\mathsf{W}(\mathcal{A})$ which has the homotopy type of $M(\mathcal{A})$. The complexes $\mathsf{M}(\mathcal{A})$ and $\mathsf{W}(\mathcal{A})$ are different. Arvola [13] has found a simplicial map between these complexes which is a homotopy equivalence.

Real Arrangements

Assume that $\mathcal{H}$ is a nonempty real arrangement.

Lemma 20.1 *Maximal elements of $L(\mathcal{H})$ are parallel subspaces.*

Proof. We showed in Lemma 4.4 that maximal elements have the same dimension. If $T \in L(\mathcal{H})$ is a maximal element and $H \in \mathcal{H}$ does not contain T then H is parallel to T. The same holds for the maximal element T'. Thus every hyperplane which contains T' either contains T or is parallel to T. It follows that T and T' are parallel. □

Thus we may assume that $\mathcal{H}$ is essential. Recall the face poset $\mathcal{L} = \mathcal{L}(\mathcal{H})$ associated to $\mathcal{H}$ in Definition 4.18 and the order complex $\mathsf{K}(P)$ associated to any poset P in Definition 17.2. We follow Salvetti [170] to show that there is a natural embedding of the order complex $\mathsf{K}(\mathcal{L}(\mathcal{H}))$ in V. Given two points $u_0, u_1 \in V$, their **join** $u_0 * u_1$ is the line segment between u_0 and u_1:

$$u_0 * u_1 = \{(1-\lambda)u_0 + \lambda u_1,\ \lambda \in [0,1]\}.$$

This may be iterated for the affine independent points $u_0, \ldots, u_k$, $k \leq \ell$ to obtain their convex hull, a k-simplex denoted $u_0 * \cdots * u_k$.

Definition 20.2 *Let $\mathcal{H}$ be an essential real hyperplane arrangement. Define a map $\phi : \mathcal{L} \to V$ which sends each face Q to a point $\mathsf{v}(Q) \in Q$. Let $\mathsf{V} = \phi(\mathcal{L})$. Then $\phi : \mathcal{L} \to \mathsf{V}$ is a bijection. Extend it to a map $\phi : \mathsf{K}(\mathcal{L}) \to V$ as follows. Given a k-simplex $[Q_0, \ldots, Q_k]$ of $\mathsf{K}(\mathcal{L})$, let*

$$\phi([Q_0, \ldots, Q_k]) = \mathsf{v}(Q_0) * \cdots * \mathsf{v}(Q_k).$$

In order to prove that ϕ is an embedding with very good properties, we need to consider central arrangements first.

Lemma 20.3 *Let $\mathcal{H}$ be a central arrangement. Let $Q \neq \{0\}$ be a face and let $q \in Q$. For every $x \in Q$ there exists a number $t_0 > 0$ such that the ray from q through tx, $t \geq 0$, meets ∂Q if $t < t_0$, and does not meet any hyperplane of $\mathcal{H} \setminus \mathcal{H}_{|Q|}$ if $t \geq t_0$.*

Proof. Let $\mathcal{B} = \mathcal{H} \setminus \mathcal{H}_{|Q|}$. By assumption $\{0\} \in \mathcal{L}$ and $Q \neq \{0\}$. Thus $\mathcal{B} \neq \emptyset$. Let $H \in \mathcal{B}$. There exists $t(H) \in \mathbf{R}$, $0 < t(H) < \infty$ such that the ray from q to tx intersects H if $t < t(H)$, it is parallel to H if $t = t(H)$, and it is disjoint from H if $t > t(H)$. In the special case when x is in the ray $\{tq \mid t > 0\}$ the ray from q to tx degenerates to a point when $t = t(H)$. Let $t_0 = \max\{t(H) \mid H \in \mathcal{B}\}$, and let $t_0 = t(H_0)$. Then the segment $[0, t_0x)$ projects from q onto a ray contained in $H_0 \cap \partial Q$. □

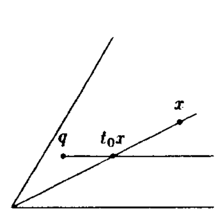


Figure 29: The critical half-line

Lemma 20.4 *Let $\mathcal{H}$ be a central arrangement. Suppose $x \in V \setminus \{0\}$. Then*

(1) there exists a unique linearly ordered subset $[Q_0, \ldots, Q_k]$ of $\mathcal{L}$ such that the ray $s(x) = \{tx \mid t > 0\}$ intersects the simplex $\mathsf{v}(Q_0) * \cdots * \mathsf{v}(Q_k)$, and

(2) the intersection is a single point x', which may be expressed in barycentric coordinates as $x' = \sum \lambda_i \mathsf{v}(Q_i)$ where $\sum \lambda_i = 1$, and $\lambda_i \in (0, 1]$ for $0 < i < k$.

Proof. The interior of the ray $s(x)$ is in some face, say Q_0. If $\mathsf{v}(Q_0) \in s(x)$ then the linearly ordered subset is $[Q_0]$, $x' = \mathsf{v}(Q_0)$ and both assertions hold. Otherwise use Lemma 20.3 with $q = \mathsf{v}(Q_0)$ to project $s(x)$ onto a ray $s_1(x) \subseteq \partial Q_0$. The interior of $s_1(x)$ is in some face, say Q_1. Since $Q_1 \subseteq \partial Q_0$, we have $Q_0 < Q_1$. The argument is repeated until $\mathsf{v}(Q_k) \in s_k(x)$. Since $\dim Q_{i+1} < \dim Q_i$, the process is finite. □

Theorem 20.5 *Let $\mathcal{H}$ be an essential real hyperplane arrangement.*

(1) The map ϕ is an embedding. Let $\mathsf{K}(\mathcal{H}) = \mathrm{im}\phi$. Then $\mathsf{K}(\mathcal{H})$ is a simplicial triangulation of a disk $D(\mathcal{H})$.

(2) If $X \in L(\mathcal{H})$ then $\mathsf{K}(\mathcal{H}) \cap X = \mathsf{K}(\mathcal{H}^X)$ is a full subcomplex which is a simplicial triangulation of the disk $D(\mathcal{H}) \cap X$.

(3) There is a strong deformation retraction of V onto $D(\mathcal{H})$ which respects the stratification: each $X \in L(\mathcal{H})$ is retracted in X to $D(\mathcal{H}) \cap X$.

Proof. Assume first that $\mathcal{H}$ is a central arrangement. The map $\phi : \mathsf{K}(\mathcal{L}) \to \mathsf{K}(\mathcal{H})$ is PL and surjective by definition. It follows from Lemma 20.4 that $\mathsf{K}(\mathcal{H})$ is a simplicial complex and ϕ is injective. The bijection $\phi : \mathcal{L} \to \mathsf{V}$ induces a partial order on V which makes ϕ order preserving. It is natural to view $\mathsf{K}(\mathcal{H}) = \mathsf{K}(\mathsf{V})$. Since $\mathcal{H}$ is central, the origin $\{0\}$ is the unique maximal element of V. It follows from Lemma 17.5 that $\mathsf{K}(\mathcal{H})$ is the cone over $\mathsf{K}(\mathsf{V} \setminus \{0\})$. The map $x' \mapsto x'/|x'|$ applied to the points x' constructed in Lemma 20.4 shows that radial projection of $\mathsf{K}(\mathsf{V} \setminus \{0\})$ is a simplicial triangulation of a sphere $S^{\ell-1}$. Thus radial projection of $\mathsf{K}(\mathcal{H})$ is a simplicial triangulation of a disk $D(\mathcal{H})$. Part (2) is immediate from the construction. The strong deformation retraction of (3) follows the rays considered in Lemma 20.4.

Now suppose $\mathcal{H}$ is centerless. Let $\mathcal{B} = \mathbf{c}\mathcal{H}$ be the cone over $\mathcal{H}$, see Definition 2.15. Let $K_0 = \ker(x_0)$. Since $\mathcal{B}$ is central, we may assume that $\mathsf{K}(\mathcal{B})$ is a simplicial trangulation of a disk. Its boundary S is a simplicial triangulation of S^{ℓ}. Note that $\mathsf{E} = \mathsf{S} \cap \{x_0 \leq 0\}$ is a subcomplex which is again a disk. Identify the total space of $\mathcal{H}$ with $\ker(x_0 - 1)$ and by radial projection with $U = \mathsf{S} \setminus \mathsf{E}$. Then $\mathsf{K}(\mathcal{H}) \subset U$, and the closure of $\mathsf{S} \setminus \mathsf{K}(\mathcal{H})$ is a regular neighborhood of E. It follows that $\mathsf{K}(\mathcal{H})$ is also a simplicial disk. An alternate proof may be given using the fact that the union of two balls which intersect in a codimension 1 face is a ball. Properties (2) and (3) are immediate as before. □

Lemma 20.6 *Let $P_k \in \mathcal{L}$ be a face of codimension k. Let S be the set of all saturated chains $Q_0 < \cdots < Q_{k-1} < P_k$. Then*

$$D_P^k = \bigcup_S \mathsf{v}(Q_0) * \cdots * \mathsf{v}(Q_{k-1}) * \mathsf{v}(P_k)$$

is a triangulated k-cell in V whose boundary is

$$S_P^{k-1} = \bigcup_S \mathsf{v}(Q_0) * \cdots * \mathsf{v}(Q_{k-1}).$$

Proof. Given $P \in \mathcal{L}$, let $\mathrm{pr}_P : V \to V/|P|$ be the affine projection. It sends the affine hyperplanes which contain P into hyperplanes of a central arrangement $(\mathcal{H}/P, V/|P|)$. Let $\mathcal{L}/P$ be the set of faces of $\mathcal{H}/P$. Let $\pi_P : \mathcal{L} \to \mathcal{L}/P$ be the map which sends $Q \in \mathcal{L}$ to the smallest face of $\mathcal{L}/P$ which contains $\mathrm{pr}_P(Q)$. If $Q \in \mathcal{L}$ then $\mathrm{codim}_V(Q) \geq \mathrm{codim}_{V/|P|}(\pi_P(Q))$. Let $\mathcal{L}_P \subseteq \mathcal{L}$ be the set of faces which contain P. Then the restriction $\pi_P : \mathcal{L}_P \to \mathcal{L}/P$ is a bijection which preserves codimension. The assertion follows from the corresponding fact proved for a central arrangement in Theorem 20.5. □

Note that we assigned to the face P_k of codimension k a cell D_P^k of dimension k. This is quite natural. We may view P_k as an $(\ell - k)$-cell of the cell complex $\mathcal{L}$ whose union is V. The cell dual to P_k is D_P^k and $\cup_{P_k \in \mathcal{L}} D_P^k$ is a cell decomposition of $\mathsf{K}(\mathcal{H})$. Figure 30 indicates some of these dual cells.

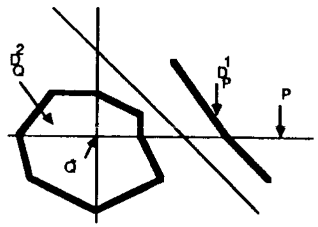


Figure 30: Dual cells

The Homotopy Type of the Complement

Definition 20.7 *Let $\mathcal{A}$ be an arrangement in V and let $\mathcal{H}$ be a real hyperplane arrangement in V. We say that $\mathcal{A}$ is* **embedded** *in $\mathcal{H}$ if $\mathcal{A} \subseteq L(\mathcal{H})$, and write $\mathcal{A} \sqsubset \mathcal{H}$.*

Proposition 20.8 *Every arrangement $\mathcal{A}$ in V is embedded in some real hyperplane arrangement $\mathcal{H}$ in V.*

Proof. Define the real hyperplane arrangement $\mathcal{H}$ by the kernels of the real and imaginary parts of the α_H for $H \in \mathcal{A}$. □

Definition 20.9 *If $\mathcal{H}$ is a real hyperplane arrangement, define for $X \in L(\mathcal{H})$*

$$\mathcal{N}_{\mathcal{H}}(X) = \bigcup \{Q \in \mathcal{L}(\mathcal{H}) \mid X \leq |Q|\}.$$

If $\mathcal{A}$ is a complex arrangement in V then embed $\mathcal{A}$ in a real hyperplane arrangement $\mathcal{H}$ in V. Since $\mathcal{A} \subseteq L(\mathcal{H})$, we may define $\mathcal{N}_{\mathcal{H}}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$ and $\mathcal{M}_{\mathcal{H}}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{H})$ by

$$\mathcal{N}_{\mathcal{H}}(\mathcal{A}) = \bigcup_{X \in \mathcal{A}} \mathcal{N}_{\mathcal{H}}(X), \qquad \mathcal{M}_{\mathcal{H}}(\mathcal{A}) = \mathcal{L}(\mathcal{H}) \setminus \mathcal{N}_{\mathcal{H}}(\mathcal{A}).$$

Note that $\mathcal{N}_{\mathcal{H}}(X)$ is the set of faces whose union is X. Thus $\mathcal{N}_{\mathcal{H}}(\mathcal{A})$ is the set of faces whose union is $N(\mathcal{A})$ and $\mathcal{M}_{\mathcal{H}}(\mathcal{A})$ is the set of faces whose union is $M(\mathcal{A})$.

Proposition 20.10 *Let $D = D(\mathcal{H})$ be the disk in V with triangulation $\mathsf{K}(\mathcal{H})$ constructed in* Theorem 20.5. *Let $\mathsf{N}_{\mathcal{H}}(\mathcal{A}) = D \cap N(\mathcal{A})$. It is a full subcomplex of $\mathsf{K}(\mathcal{H})$ which is a geometric realization of $\mathsf{K}(\mathcal{N}_{\mathcal{H}}(\mathcal{A}))$.*

Proof. In general the union of full subcomplexes may not be a full subcomplex. In our case minimal elements are incomparable and we are taking saturated intervals above them. □

Theorem 20.11 *Let $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ be the largest subcomplex of $\mathsf{K}(\mathcal{H})$ disjoint from $\mathsf{N}_{\mathcal{H}}(\mathcal{A})$. Then*

(1) the finite simplicial complex $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ is a geometric realization of $\mathsf{K}(\mathcal{M}_{\mathcal{H}}(\mathcal{A}))$,
(2) $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ is a strong deformation retract of the complement $M(\mathcal{A})$,
(3) the homotopy type of $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ is independent of $\mathcal{H}$.

Proof. Part (1) follows from Proposition 20.10. The deformation retraction of $M(\mathcal{A})$ onto $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ is done in two stages. First use Theorem 20.5.3 to get a strong deformation retraction of the pair $(V, N(\mathcal{A}))$ onto $(D, D \cap N(\mathcal{A}))$. This shows that $M(\mathcal{A})$ has the same homotopy type as $D \setminus D \cap N(\mathcal{A})$. It follows from Proposition 20.10 that $(D, D \cap N(\mathcal{A}))$ is a polyhedral pair with triangulation $(\mathsf{K}(\mathcal{H}), \mathsf{N}_{\mathcal{H}}(\mathcal{A}))$. Since $\mathsf{N}_{\mathcal{H}}(\mathcal{A})$ is a full subcomplex, it follows from a standard result [133, Lemma 70.1] that $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ is a strong deformation retract of $D \setminus D \cap N(\mathcal{A})$. Part (3) follows from (2). □

Definition 20.12 *For $P \in \mathcal{L}$ let $\mathcal{L}_P = \{Q \in \mathcal{L} \mid Q \leq P\}$ be the segment below P. It consists of all faces whose closure contains P. Let $\mathsf{E}(P) = \mathsf{K}(\mathcal{L}_P)$. It follows from* Lemma *20.6 that $\mathsf{E}(P)$ is a triangulated disk with center $\mathsf{v}(P)$, whose dimension equals* codim$P$.

Proposition 20.13 *The space $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ is the total space of a regular cell complex*

$$\mathsf{M}_{\mathcal{H}}(\mathcal{A}) = \bigcup_{P \in \mathcal{M}_{\mathcal{H}}(\mathcal{A})} \mathsf{E}(P)$$

with $\partial \mathsf{E}(P) = \cup_{Q<P} \mathsf{E}(Q)$. □

Although the description of the complex $\mathsf{M}_{\mathcal{H}}(\mathcal{A})$ is easy, even the simplest examples contain far too many cells to be of much use for explicit calculations, see Examples 20.17 and 20.18 below.

Complexified Real Arrangements

We turn to the special case of the complexification of a real arrangement. We show first that it suffices to consider central arrangements. Suppose $\mathcal{A}$ is a complex hyperplane arrangement. If $\mathcal{A}$ is not central then consider $\mathbf{c}\mathcal{A}$ and recall that $M(\mathbf{c}\mathcal{A}) = M(\mathcal{A}) \times \mathbb{C}^*$. Thus for the topology of the complement of a complex arrangement it suffices to study central arrangements. Now suppose $\mathcal{A}$ is a real arrangement and $\mathcal{A}_{\mathbb{C}}$ is the complexification of $\mathcal{A}$. Then $\mathcal{A}_{\mathbb{C}}$ is central if and only if $\mathcal{A}$ is central. Thus we may assume that $\mathcal{A}$ is central.

Proposition 20.14 *Let $\mathcal{A}$ be a central real arrangement in V. There is a natural embedding of its complexification $\mathcal{A}_{\mathbb{C}}$ in $\mathcal{H} = \mathcal{A} \times \mathcal{A}$.*

Proof. View $V \otimes \mathbb{C} = V \oplus iV$. Let $H \in \mathcal{A}$ be the kernel of the real linear form α_H. Since $\alpha_H(x + iy) = \alpha_H(x) + i\alpha_H(y)$, it follows that $x + iy \in \ker \alpha_H$ if and only if $x \in \ker \alpha_H$ and $y \in \ker \alpha_H$. □

Proposition 20.15 *Let $\mathcal{A}$ be a central real arrangement in V. Consider the natural embedding of its complexification $\mathcal{A}_{\mathbf{C}}$ in $\mathcal{H} = \mathcal{A} \times \mathcal{A}$. Identify $\mathcal{L}(\mathcal{H})$ with $\mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A})$. Then $\mathcal{M}_{\mathcal{H}}(\mathcal{A}_{\mathbf{C}})$ consists of elements $(P, P') \in \mathcal{L}(\mathcal{H})$ with $\zeta(P) \cap \zeta(P') = \emptyset$.*

Proof. Recall from Definition 4.20 that $\zeta(P)$ determines the hyperplanes which contain the face P. Thus $k \in \zeta(P) \cap \zeta(P')$ if and only if $|P| \times |P'| \subset H_k \times H_k$. Since $H_k \times H_k \in L(\mathcal{A}_{\mathbf{C}})$, it follows that $(P, P') \in \mathcal{N}_{\mathcal{H}}(\mathcal{A}_{\mathbf{C}})$. The converse is clear. □

Given a central real arrangement $\mathcal{A}$ with canonical embedding of $\mathcal{A}_{\mathbf{C}} \sqsubset \mathcal{A} \times \mathcal{A}$ we agree to write $\mathcal{M}(\mathcal{A}) = \mathcal{M}_{\mathcal{A}\times\mathcal{A}}(\mathcal{A}_{\mathbf{C}})$, $\mathsf{M}(\mathcal{A}) = \mathsf{M}_{\mathcal{A}\times\mathcal{A}}(\mathcal{A}_{\mathbf{C}})$, etc. When $\mathcal{A}$ is fixed we write $\mathcal{M}$, M, etc. For a complexified real arrangement if $(P, Q) \in \mathcal{M}$ then $\mathcal{L}_P \times \mathcal{L}_Q \subset \mathcal{M}$. Let $\mathsf{E}(P,Q) = \mathsf{K}(\mathcal{L}_P \times \mathcal{L}_Q)$. Thus $\mathsf{E}(P,Q)$ is a triangulated disk with center $\mathbf{v}(P,Q)$, whose dimension equals $\operatorname{codim} P + \operatorname{codim} Q$. It follows from Lemma 20.6 that $\partial\mathsf{E}(P,Q) = \cup_{(R,S)<(P,Q)}\mathsf{E}(R,S)$.

Proposition 20.16 *The complex M has the structure of a regular cell complex:*

$$\mathsf{M} = \bigcup_{(P,Q)\in\mathcal{M}} \mathsf{E}(P,Q). \quad \Box$$

Example 20.17 *Let $V = \mathbf{R}^2$ and let $\mathcal{A}$ consist of the two coordinate lines. Then $\mathcal{A}_{\mathbf{C}}$ consists of the two complex coordinate lines in $V \otimes \mathbf{C} = \mathbf{C}^2$.*

Note that $\mathsf{K}(\mathcal{A})$ is the barycentric triangulation of a square. It is a geometric realization of $\mathsf{K}(J^2)$. Similarly $\mathsf{K}(\mathcal{H})$ is the barycentric triangulation of a four dimensional cube. It is a geometric realization of $\mathsf{K}(J^4)$. The complex M has 16 2-cells. There are eight cells $\mathsf{E}(0, C)$, $\mathsf{E}(C, 0)$ where C is a chamber and eight cells $\mathsf{E}(P,Q)$ where P, Q are faces of dimension 1 with $|P| \neq |Q|$. To check how these cells fit together we must compute their boundaries. It is easy to see that they form a torus which is known to have the homotopy type of the complement $M(\mathcal{A}_{\mathbf{C}})$.

Example 20.18 *Let $V = \mathbf{R}^2$ and let $\mathcal{A}$ consist of three concurrent lines. Then $\mathcal{A}_{\mathbf{C}}$ consists of three concurrent complex lines in $V \otimes \mathbf{C} = \mathbf{C}^2$.*

Label the faces of $\mathcal{L}(\mathcal{A})$ as in Figure 31. The complex $\mathsf{K}(\mathcal{A})$ is the barycentric triangulation of a hexagon. Thus $\mathsf{K}(\mathcal{H})$ is the product triangulation of a 4-ball. The number of 2-cells in M is 36. There are 24 barycentrically triangulated squares $\mathsf{E}(P_i, P_j)$, $|P_i| \neq |P_j|$ and 12 barycentrically triangulated hexagons $\mathsf{E}(0, C_i)$, $\mathsf{E}(C_i, 0)$. These cells fit together as indicated in Figure 31. The sides of the 6×6 square are identified to form a torus T. Of the 36 squares in T, 24 are filled with $\mathsf{E}(P_i, P_j)$. The remaining 12 squares are missing. These are marked with circles. The 12 hexagons are attached to 12 circles in T, which appear as the six horizontal and six vertical lines in Figure 31. We showed in Example 19.4 that the complement has the homotopy type of $(S^1 \vee S^1) \times S^1$. This is not obvious from our description of M.

Arvola [13] gave another description of $\mathcal{M}$. Call two faces $P, Q \in \mathcal{L}$ **noncoplanar** if no hyperplane in $\mathcal{A}$ contains both. It is immediate from Proposition 20.15 that $\mathcal{M}$ consists of noncoplanar pairs. Recall the vector product operation in $\mathcal{L}$ from Definition 4.21.

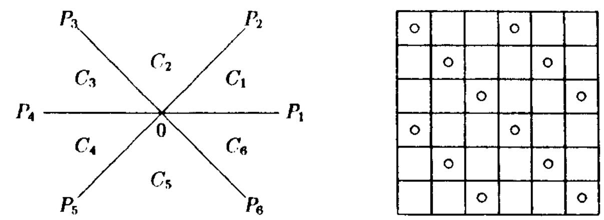


Figure 31: Three concurrent lines in $\mathbf{R}^2$ and in $\mathbf{C}^2$

Proposition 20.19 *Two faces $P, Q \in \mathcal{L}$ are noncoplanar if and only if their vector product PQ is a chamber. Thus*

$$\mathcal{M} = \{(P,Q) \in \mathcal{L} \times \mathcal{L} \mid PQ \in \mathcal{C}\}. \quad \Box$$

The Salvetti Complex

Arvola [13] described the Salvetti complex $\mathbf{W}$ as follows. For $P \in \mathcal{L}$ choose a point $\mathbf{w}(P) \in P$. For each pair $P \times C \in \mathcal{L} \times \mathcal{C}$ define a point $\mathbf{w}(P, C) \in V \oplus V$ by

$$\mathbf{w}(P,C) = \mathbf{w}(P) \oplus (\mathbf{w}(C) - \mathbf{w}(P))$$

where minus indicates vector subtraction. The vertex set of $\mathbf{W}$ is the collection of points

$$\{\mathbf{w}(P,C) \mid P \times C \in \mathcal{L} \times \mathcal{C},\ P \geq C\}.$$

We assign a simplex $\Delta = \Delta(P_1 \geq \cdots \geq P_k; C)$ to a chain of faces $P_1 \geq \cdots \geq P_k$ and a chamber C such that $P_1 \geq C$. The chamber C need not be comparable to any of the faces P_i other than P_1. Recall from Proposition 4.22 that $P_iC \in \mathcal{C}$ and $P_i \geq P_iC$. Then Δ is the convex hull in $V \oplus V$ of the vertices

$$\{\mathbf{w}(P_1, P_1C), \mathbf{w}(P_2, P_2C), \ldots, \mathbf{w}(P_k, P_kC)\}.$$

and

$$\mathbf{W} = \{\Delta(P_1 \geq \cdots \geq P_k; C) \mid P_i \in \mathcal{L},\ C \in \mathcal{C},\ P_1 \geq C\}.$$

Salvetti's theorem [170] may be stated with this notation.

Theorem 20.20 *Let $\mathcal{A}$ be a central real arrangement. The collection $\mathbf{W}(\mathcal{A})$ of convex sets in $V \oplus V$ is the geometric realization of a simplicial complex. There is a strong deformation retraction of the complement $M(\mathcal{A}_{\mathbf{C}})$ onto $\mathbf{W}(\mathcal{A})$.* □

The complex W is also a regular cell complex. Given $(P,C) \in \mathcal{L} \times \mathcal{C}$ with $P \geq C$, let

$$\mathcal{D}(P,C) = \{\mathbf{w}(Q,QC) \mid Q \leq P\}$$

viewed as a set of vertices of W. Introduce a partial order in $\mathcal{D}(P,C)$ by $\mathbf{w}(R,RC) \leq \mathbf{w}(S,SC)$ when $R \leq S$. Let $\mathsf{K}(\mathcal{D}(P,C))$ be its order complex. It follows from the definition of W that the inclusion of vertices $\mathsf{K}(\mathcal{D}(P,C)) \to \mathsf{W}$ extends to a simplicial map which is an embedding. Let $\mathsf{D}(P,C)$ denote its image. It follows from Lemma 20.6 that $\mathsf{D}(P,C)$ is a triangulated disk with center $\mathbf{w}(P,C)$ whose dimension is $\operatorname{codim} P$, and $\partial\mathsf{D}(P,C) = \cup_{Q<P}\mathsf{D}(Q,QC)$.

Proposition 20.21 *The complex* W *has the structure of a regular cell complex:*

$$\mathsf{W} = \bigcup_{P \geq C} \mathsf{D}(P,C). \quad \square$$

We return to Example 20.17 of two lines in the plane. Here W has four 2-cells of the form $\mathsf{D}(0,C)$, where 0 is the origin and C is one of the four chambers of $\mathcal{L}$. To check how these cells fit together we must compute their boundaries. It is easy to see that they form a torus, which is known to have the homotopy type of the complement.

Example 20.18 of three concurrent lines in the plane is more interesting. Here W has six 2-cells $\mathsf{D}(0,C_i)$, 12 1-cells $\mathsf{D}(P_i,C_j)$ with $P_i > C_j$, and six 0-cells $\mathsf{D}(C_i,C_i)$. The attaching maps are computed using the boundary operator: $\partial\mathsf{D}(0,C_k)$ is the union of six 1-cells and six 0-cells. The 1-cells are in cyclic order $\mathsf{D}(P_{k+1},C_k)$, $\mathsf{D}(P_{k+2},C_{k+1})$, $\mathsf{D}(P_{k+3},C_{k+2})$, $\mathsf{D}(P_{k+4},C_{k+4})$, $\mathsf{D}(P_{k+5},C_{k+5})$, $\mathsf{D}(P_k,C_k)$. Recall from Example 19.4 that the complement has the homotopy type of $(S^1 \vee S^1) \times S^1$. This is not even obvious from this complex W.

The Homotopy Equivalence

Arvola [13] constructed an explicit simplicial map $\mathsf{M} \to \mathsf{W}$ which is a homotopy equivalence.

Lemma 20.22 *For each noncoplanar pair* $(P,Q) \in \mathcal{L}\times\mathcal{L}$ *let* $\phi(\mathbf{v}(P,Q)) = \mathbf{w}(P,PQ)$. *Linear extension of* ϕ *defines a surjective simplicial map* $\phi : \mathsf{M} \to \mathsf{W}$.

Proof. Note that ϕ is defined for every vertex of M. It follows from Proposition 20.19 that its image is in the vertex set of W. To show that ϕ is simplicial let

$$\delta = \{\mathbf{v}(P_1,Q_1),\ldots,\mathbf{v}(P_k,Q_k)\}$$

be a simplex in M. Here $P_1 \geq \cdots \geq P_k$, $Q_1 \geq \cdots \geq Q_k$, and (P_i,Q_i) are noncoplanar for $1 \leq i \leq k$. Then $\phi(\delta)$ has vertices

$$\{\mathbf{w}(P_1,P_1Q_1),\ldots,\mathbf{w}(P_k,P_kQ_k)\}.$$

Let $C = P_1Q_1$. We claim that $\phi(\delta) = \Delta(P_1 \geq \cdots \geq P_k; C)$. It suffices to prove that $P_iC = P_iQ_i$ for $1 \leq i \leq k$. Since $P_1 \geq P_i$ it follows that $\zeta(P_i) \subseteq \zeta(P_1)$ and hence $P_iP_1 = P_i$. Thus $P_iC = P_iP_1Q_1 = P_iQ_1$. It remains to show that $P_iQ_1 = P_iQ_i$. This is clear for $k \notin \zeta(P_i)$. If $k \in \zeta(P_i)$ then $k \notin \zeta(Q_i)$ because $\zeta(P_i) \cap \zeta(Q_i) = \emptyset$. Suppose $k \in \zeta(Q_1)$. Then $k \notin \zeta(P_1)$ because $\zeta(P_1) \cap \zeta(Q_1) = \emptyset$, contradicting $\zeta(P_i) \subseteq \zeta(P_1)$.

Given the simplex $\Delta = \Delta(P_1 \geq \cdots \geq P_k; C)$ in W with $P_1 \geq C$, let δ be the simplex of M with vertices $\{\mathbf{v}(P_1,C),\ldots,\mathbf{v}(P_k,C)\}$. Then $\phi(\delta) = \Delta$. $\square$

In the next result we need the following theorem of M. Cohen [40].

Theorem 20.23 *A simplicial map of a finite complex onto another which has contractible fibers is a homotopy equivalence.* $\square$

Theorem 20.24 *The simplicial map* $\phi : \mathsf{M} \to \mathsf{W}$ *is a homotopy equivalece.*

Proof. By Theorem 20.23 it suffices to show that ϕ has contractible fibers. Fix a simplex $\Delta = \Delta(P_1 \geq \cdots \geq P_k; C)$ with $P_1 \geq C$ in W. Let $\mathsf{M}_\Delta = \phi^{-1}(\Delta)$. Let $\mathcal{L}_\Delta = \{P_1,\ldots,P_k\}$ be the corresponding linearly ordered subset of $\mathcal{L}$. The subarrangement $\mathcal{A}_{|P_1|}$ consists of those hyperplanes which contain P_1. There is a unique chamber $D \in \mathcal{C}(\mathcal{A}_{|P_1|})$ with $C \subseteq D$. Let $\mathcal{T}_\Delta = \{Q \in \mathcal{L} \mid Q \subseteq D\}$. It is important to note here that in general $D \notin \mathcal{L}$ so Q and D are related only as subsets of V. Recall that each face Q may be viewed as a map $Q : \mathcal{A} \to \{+,-,0\}$. With this notation $\mathcal{T}_\Delta = \{Q \in \mathcal{L} \mid Q(H) = C(H) \text{ for all } H \in \mathcal{A}_{|P_1|}\}$. Let $\mathcal{M}_\Delta = \mathcal{L}_\Delta \times \mathcal{T}_\Delta$. Note that $\mathcal{M}_\Delta \subset \mathcal{M}$. It suffices to show that $(P_i,Q) \in \mathcal{M}_\Delta$ are noncoplanar. Since Q is contained in a chamber of $\mathcal{A}_{|P_1|}$, $\zeta(P_1) \cap \zeta(Q) = \emptyset$. The conclusion follows from $\zeta(P_i) \subseteq \zeta(P_1)$. Thus $\mathsf{K}(\mathcal{M}_\Delta) \subseteq \mathsf{M}$.

We show next that $\mathsf{K}(\mathcal{M}_\Delta) = \mathsf{M}_\Delta$. Suppose $Q_1 \geq \cdots \geq Q_k$, and (P_i,Q_i) are noncoplanar for $1 \leq i \leq k$, and $P_1Q_1 = C$. Let $\delta = \{\mathbf{v}(P_1,Q_1),\ldots,\mathbf{v}(P_k,Q_k)\}$. To show that $\phi(\delta) = \Delta$ it suffices to prove that for all $H \in \mathcal{A}_{|P_1|}$ we have $P_iQ_i(H) = P_iC(H)$ for $1 \leq i \leq k$ if and only if $Q_i(H) = C(H)$ for $1 \leq i \leq k$. Fix $H \in \mathcal{A}_{|P_1|}$. For those values of i with $P_i(H) = 0$ we have $Q_i(H) = P_iQ_i(H)$ and $P_iC(H) = C(H)$ so the equations are equivalent. For those values of i with $P_i(H) \neq 0$ both equations hold. We have $P_iQ_i(H) = P_i(H) = P_iC(H)$ from the definition. On the other hand $P_1(H) = 0$. Since $C(H) = P_1Q_1(H) = Q_1(H) \neq 0$ and $Q_1 \geq Q_i$, we have $Q_i(H) = Q_1(H) = C(H)$.

Set $\mathsf{T}_\Delta = \mathsf{K}(\mathcal{T}_\Delta)$. Our argument shows that there is a homeomorphism of underlying topological spaces: $|\mathsf{M}_\Delta| \simeq |\Delta| \times |\mathsf{T}_\Delta|$. Since $\mathcal{T}_\Delta$ is the face poset of an arrangement in the affine space D, the complex T_Δ is the triangulation of a disk by Lemma 20.5. In particular it is contractible. $\square$

Corollary 20.25 *If* $\mathsf{E}(P,Q)$ *is a cell in* M *then* $\phi\mathsf{E}(P,Q) \subseteq \mathsf{D}(P,PQ)$. *Thus the map* $\phi : \mathsf{M} \to \mathsf{W}$ *is a cellular homotopy equivalence. Given* $C \in \mathcal{C}$ *and* $P \geq C$ *we have*

$$\phi^{-1}(\mathsf{D}(P,C)) = \{\mathsf{E}(P,Q) \mid Q \in \mathcal{T}_{\mathbf{w}(P,Q)}\}. \quad \square$$

Recall that in the complexification of two lines in $\mathbb{R}^2$ the complex **M** has 16 2-cells and the complex **W** has four 2-cells. The map ϕ sends the four 2-cells $\mathsf{E}(0, C)$ onto the four 2-cells $\mathsf{D}(0, C)$. It collapses the 2-cells $\mathsf{E}(C, 0)$ onto the 0-cells $\mathsf{D}(C, C)$. The remaining eight 2-cells are of the form $\mathsf{E}(P, Q)$ where P, Q are noncoplanar faces of dimension 1. These are sent to the 1-cells $\mathsf{D}(P, PQ)$.

In the complexification of three concurrent lines in $\mathbb{R}^2$ the six 2-cells $\mathsf{E}(0, C_i)$ map onto the six 2-cells $\mathsf{D}(0, C_i)$. The six 2-cells $\mathsf{E}(C_i, 0)$ are collapsed to the 0-cells $\mathsf{D}(C_i, C_i)$. The 24 2-cells $\mathsf{E}(P_i, P_j)$ are collapsed to the 12 1-cells $\mathsf{D}(P_i, P_iP_j)$ in pairs. The cells $\mathsf{E}(P_i, P_j)$ and $\mathsf{E}(P_i, P_k)$ have the same image if and only if $P_iP_j = P_iP_k$. Equivalently, $P_j(|P_i|) = P_k(|P_i|)$, so P_j and P_k are on the same side of the support of P_i.

21 The Fundamental Group

In this section our aim is to obtain a presentation for the fundamental group of the complement. It follows from a theorem of Zariski [89], [76] that we may assume that $\mathcal{A}$ is an arrangement of lines in $\mathbb{C}^2$. Nevertheless, it is convenient to continue calling the complex lines $H \in \mathcal{A}$ "hyperplanes" in order to distinguish them from various other "lines" in our constructions. Write $M = M(\mathcal{A})$.

In the special case when $\mathcal{A}$ is a complexified real arrangement, a presentation of $\pi_1(M)$ was obtained by Randell [157] and Salvetti [170, 171]. Their main result asserts that a presentation may be obtained from the underlying real arrangement of $\mathcal{A}$, which we refer to as the **real graph** $\Gamma(\mathcal{A})$. In his PhD thesis W. Arvola [11] gave a presentation for the fundamental group of the complement of an arbitrary arrangement. He constructed a generalization of the real graph called the **admissible** graph, and he proved that a presentation for $\pi_1(M)$ may be obtained from an admissible graph. This section is a modified version of his paper [12].

Choose coordinates z_1, z_2 for $\mathbb{C}^2$ and coordinates x_1, y_1, x_2, y_2 for $\mathbb{R}^4$. Identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2.$$

Identify $\mathbb{R}^2$ as the span of the coordinates x_1, x_2 in $\mathbb{R}^4$ and let $\mathbb{R}^3$ be the span of x_1, x_2, y_2. Let ϕ^2, ϕ^3, and ϕ denote the natural projections, where $\phi^2 = \phi\phi^3$.

$$\begin{array}{rrcl} \phi^2 : \mathbb{R}^4 \to \mathbb{R}^2 & \phi^2(x_1, y_1, x_2, y_2) & = & (x_1, x_2), \\ \phi^3 : \mathbb{R}^4 \to \mathbb{R}^3 & \phi^3(x_1, y_1, x_2, y_2) & = & (x_1, x_2, y_2), \\ \phi : \mathbb{R}^3 \to \mathbb{R}^2 & \phi(x_1, x_2, y_2) & = & (x_1, x_2). \end{array}$$

Definition 21.1 *A* **multiple point** *is the non-empty intersection of two or more distinct hyperplanes of $\mathcal{A}$. Let P denote the set of multiple points.*

Definition 21.2 *If u and v are words in a free group we set $u^v = v^{-1}uv$. If a group Π is given by a set of generators G and a set of relators R then Π has presentation $\Pi = \langle G \mid R \rangle$. If $w_1, \ldots, w_k$ are words in a free group we set*

$$[w_1, \ldots, w_k] = \{w_1 \cdots w_k = w_{\sigma(1)} \cdots w_{\sigma(k)} \mid \sigma \in C\}$$

where C is the set of cyclic permutations of the tuple $(1, \ldots, k)$. Thus the symbol on the left represents a **set** *of relators.*

With this notation we can describe Randell's result [157].

Theorem 21.3 *Let $\mathcal{A}$ be a complexified real 2-arrangement. Let M be the complement of $\mathcal{A}$. Let $\Gamma(\mathcal{A})$ be the real graph of $\mathcal{A}$. Let $G = \{g_H \mid H \in \mathcal{A}\}$. Then*

$$\pi_1(M) = \langle G \mid \bigcup_{p \in P} R_p \rangle.$$

Here R_p is a set of relators in Definition 21.2 *on suitable conjugates of the generators associated to those hyperplanes which pass through p in the order indicated by $\Gamma(\mathcal{A})$.* □

Admissible Graphs

Let $\mathcal{A}$ be an arbitrary complex 2-arrangement.

Lemma 21.4 *After a suitable linear transformation we may assume that $\mathcal{A}$ satisfies:*
(1) There is no hyperplane in $\mathcal{A}$ of the form $\ker(z_1 - c)$ for any constant $c \in \mathbb{C}$.
(2) If $p, p' \in P$ are distinct multiple points then $x_1(p) \neq x_1(p')$. □

Thus we may introduce a linear order in P by

$$p < p' \iff x_1(p) < x_1(p'), \qquad p, p' \in P.$$

The main tool in the proof of Theorem 21.3 is to sweep $\mathbb{R}^4$ with a family of real 2-planes parametrized by $t \in \mathbb{R}$:

$$K_t = \{q \in \mathbb{C}^2 \mid z_1(q) = t\}.$$

The fact that $\mathcal{A}$ is a complexified real arrangement in Theorem 21.3 implies that the planes K_t pass through all the multiple points. This turns out to be crucial in the argument. For an arbitrary complex 2-arrangement we may not assume that all the multiple points have real first coordinates. Thus the planes K_t may miss some multiple points. If we consider the same family parametrized by $t \in \mathbb{C}$ then we lose the advantages of a one-dimensional parameter space. The right generalization is to let the parameter space be the graph of a piecewise linear (PL) map $f : \mathbb{R} \to \mathbb{R}$ in the complex line parametrized by z_1. We assign to f the one-parameter family of 2-planes defined for $t \in \mathbb{R}$ by

$$\begin{aligned} K_t(f) &= \{q \in \mathbb{C}^2 \mid z_1(q) = t + if(t)\} \\ &= \{q \in \mathbb{R}^4 \mid x_1(q) = t,\ y_1(q) = f(t)\}. \end{aligned}$$

We call f the **graphing map**. Choosing $f = 0$ we get $K_t = K_t(0)$. Let $H \in \mathcal{A}$ be a hyperplane. By Lemma 21.4.1 the set $H \cap K_t(f)$ is a single point in $\mathbb{R}^4$ whose coordinates are continuous in the parameter t. Recall that $N = \bigcup_{H \in \mathcal{A}} H$ and $|\mathcal{A}| = n$. Consider the set $N \cap K_t(f)$ for a fixed value of t. This set consists of n distinct points unless there is a multiple point $p \in P$ with $p \in K_t(f)$. This occurs when $t = x_1(p)$ and $f(t) = y_1(p)$. In this case the set consists of $n - v(p) + 1$ points where $v(p)$ is the number of hyperplanes through p. It follows from Lemma 21.4.2 that for a fixed value of t at most one multiple point may lie in $N \cap K_t(f)$.

Definition 21.5 *Let $\Gamma_f^4 = \bigcup_{t \in \mathbb{R}} N \cap K_t(f)$ and define its projected images*

$$\Gamma_f^3 = \phi^3(\Gamma_f^4), \qquad \Gamma_f^2 = \phi^2(\Gamma_f^4).$$

Lemma 21.6 *For $H \in \mathcal{A}$ the set $H \cap \Gamma_f^4$ is PL homeomorphic to $\mathbb{R}$.*

Proof. Suppose $H = \ker(a_1 z_1 + a_2 z_2 + c)$, where $a_1, a_2, c \in \mathbb{C}$. Write $a_k = \alpha_k + i\beta_k$ for $k = 1, 2$, and $c = \gamma + i\delta$. Separate real and imaginary parts and substitute $x_1 = t$, $y_1 = f(t)$. By Lemma 21.4.1 we can solve for x_2, y_2. □

As in the case of complexified real arrangements, it is necessary for the family of planes $K_t(f)$ to contain all multiple points.

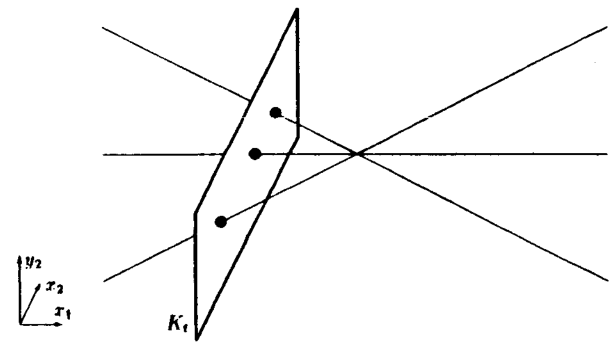


Figure 32: A 3-graph

Definition 21.7 *The graphing map f is* **proper** *if every multiple point is in Γ_f^4. Thus*

$$p \in P \Longrightarrow y_1(p) = f(x_1(p)).$$

We shall assume that all graphing maps are proper.

Suppose that $\mathcal{A}$ is a complexified real arrangement. Then we may choose the graphing map $f = 0$. Note that f is proper. In this case $(\mathbb{R}^2, \Gamma_f^2)$ is precisely the real graph of $\mathcal{A}$ and $(\mathbb{R}^3, \Gamma_f^3)$ is the natural embedding of the real graph into $\mathbb{R}^3$.

Since f is proper there is a natural graph structure on Γ_f^4 with vertex set $\mathsf{V}^4 = P$. The edges E^4 are of two types: bounded and unbounded. Each bounded edge joins two distinct vertices with a piecewise linear path. The unbounded edges are paths starting at one vertex and running off to infinity. Addition of a vertex at infinity would make this a graph in the sense of Definition 7.1. Since the roles of the bounded and unbounded edges are different, it is more convenient to think of Γ_f^4 this way. Two distinct edges are either disjoint or meet in one vertex. It follows from Lemma 21.6 that the real line $H \cap \Gamma_f^4$ is a union of edges. Conversely, each edge is contained in a unique $H \in \mathcal{A}$. We call the pair $(\mathbb{R}^4, \Gamma_f^4)$ with this graph structure the **4-graph** of $\mathcal{A}$ relative to f.

Since $y_1 = f(x_1)$ on Γ_f^4, the projection ϕ^3 induces an isomorphism $\phi^3 : \Gamma_f^4 \simeq \Gamma_f^3$. Let $\mathsf{V}^3 = \phi^3(\mathsf{V}^4)$ denote the vertices of Γ_f^3 and let $\mathsf{E}^3 = \phi^3(\mathsf{E}^4)$ denote the edges of Γ_f^3. Thus the pair $\Gamma_f^3 \subseteq \mathbb{R}^3$ is also an embedded combinatorial graph. We call the pair $(\mathbb{R}^3, \Gamma_f^3)$ with this graph structure the **3-graph** of $\mathcal{A}$ relative to f.

It follows from Lemma 21.4.2 that the restriction of the projection $\phi : \Gamma_f^3 \to \Gamma_f^2$ to V^3 is a monomorphism. Call $\mathsf{P} = \phi(\mathsf{V}^3)$ the set of **actual vertices** of Γ_f^2. Note that $\phi^2 : P \to \mathsf{P}$ is a bijection. The projection ϕ is not necessarily a monomorphism on the interiors of the edges of Γ_f^3. If $E \in \mathsf{E}^3$ let $\text{int}E$ denote its interior. In order to impose a graph structure on Γ_f^2 best suited for the presentation of $\pi_1(M)$, we want ϕ and f to be well behaved.

Lemma 21.8 *By a suitable choice of coordinates and graphing map we may assume that ϕ, f satisfy the following conditions:*

(1) $|\phi^{-1}(\mathsf{m})| \leq 2$ for all $\mathsf{m} \in \Gamma_f^2$.

(2) If $\mathsf{p} \in \mathsf{P}$ then $|\phi^{-1}(\mathsf{p})| = 1$.

(3) If $E, E' \in \mathsf{E}^3$ and $\mathsf{q} \in \phi(\text{int}E) \cap \phi(\text{int}E')$ then $\phi(E)$ and $\phi(E')$ meet transversely at q.

(4) The set of **virtual** *vertices Q of Γ_f^2 defined by*

$$\mathsf{Q} = \{\mathsf{q} \in \Gamma_f^2 \mid |\phi^{-1}(\mathsf{q})| = 2\}$$

is finite.

Proof. The first three assertions follow by transversality. It is clear that there are only a finite number of virtual vertices in the image of the bounded edges. Recall that the multiple points are linearly ordered. If we choose the graphing map to be constant before the smallest multiple point and after the largest multiple point then only a finite number of additional virtual vertices arise. □

Definition 21.9 *Call Γ_f^2* **regular** *provided ϕ, f satisfy the conditions of* Lemma 21.8.

If Γ_f^2 is regular then we may give $\Gamma_f^2 \subseteq \mathbb{R}^2$ the structure of a planar graph as follows. The vertex set is $\mathsf{V} = \mathsf{P} \cup \mathsf{Q}$. The set of edges E consists of the images of those edges of E^3 where ϕ is a monomorphism, together with the images of the edges where ϕ is not a monomorphism, subdivided by the virtual vertices. Call $(\mathbb{R}^2, \Gamma_f^2)$ with this graph structure the **regular 2–graph** of $\mathcal{A}$ relative to f. Note that if Γ_f^2 has virtual vertices then ϕ does not respect the graph structures. It could be made into a map of graphs by refining the structure of Γ_f^3 but we will not need this.

Definition 21.10 *For each $H \in \mathcal{A}$ let $\mathsf{H} = \phi^2(H \cap \Gamma_f^4)$ be the* **trace** *of H in Γ_f^2. Let $\mathsf{A} = \{\mathsf{H} \mid H \in \mathcal{A}\}$ be the set of traces.*

It follows from Lemma 21.6 that each trace H is a PL line. Two traces intersect only in a vertex of Γ_f^2. The map $\tau : \mathcal{A} \to \mathsf{A}$ given by $\tau(H) = \phi^2(H \cap \Gamma_f^4)$ is a bijection and $\Gamma_f^2 = \cup_{\mathsf{H} \in \mathsf{A}} \mathsf{H}$.

Corollary 21.11 *If Γ_f^2 is regular then*

(1) The actual vertex $\mathsf{p} \in \mathsf{P}$ is contained in the traces $\mathsf{H}_1, \dots, \mathsf{H}_k$ if and only if the unique multiple point $p \in P$ with $\mathsf{p} = \phi^2(p)$ is contained in the hyperplanes $H_1, \dots, H_k$, where $\mathsf{H}_i = \tau(H_i)$.

(2) If $\mathsf{q} \in \mathsf{Q}$ is a virtual vertex then it is contained in exactly two traces H, H' and these traces meet transversely at q. □

In order to make the construction explicit, we choose a special collection of graphing maps. Since the multiple points play a principal role, we want f to be constant in a neighborhood of each point $x_1(p)$ for $p \in P$. Suppose $|P| = r$. Choose real numbers u_i, v_i for $1 \leq i \leq r-1$ so that

$$x_1(p_i) < v_i < u_i < x_1(p_{i+1}) \qquad 1 \leq i \leq r-1.$$

Define a **flat** graphing map $f = f_{(u_i, v_i)}$ as follows.

$$\begin{array}{lllllll} -\infty & < & t & \leq & v_1 & \Rightarrow & f(t) = y_1(p_1) \\ u_{i-1} & \leq & t & \leq & v_i & \Rightarrow & f(t) = y_1(p_i), \qquad 2 \leq i \leq r-1 \\ u_{r-1} & \leq & t & < & \infty & \Rightarrow & f(t) = y_1(p_r). \end{array}$$

On the complementary intervals we interpolate linearly.

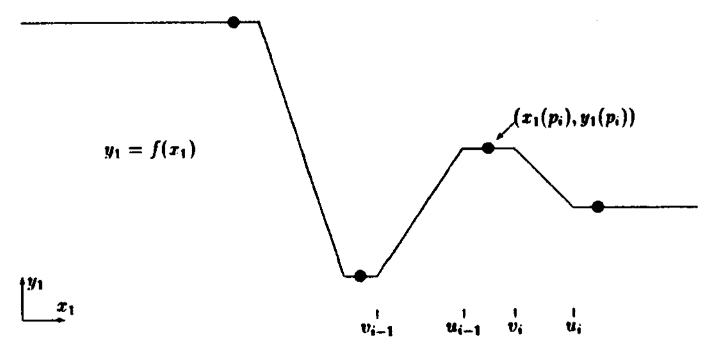


Figure 33: A flat graphing map

The slope of H changes when the slope of the graph of f changes. In a flat graphing map this occurs at $t = u_i$ and $t = v_i$. Call the corresponding points on the trace H **nodes**. Let N denote the set of nodes in Γ_f^2. The next modification is designed to adjust the nodes and vertices of a regular 2-graph.

Lemma 21.12 *By a suitable choice of coordinates and real numbers u_i, v_i for the flat graphing map f we may assume that Γ_f^2 is a regular 2-graph whose vertex set V and set of nodes N satisfy the following conditions:*

(1) If $\mathsf{v}, \mathsf{v}' \in \mathsf{V}$ are distinct vertices then $x_1(\mathsf{v}) \neq x_1(\mathsf{v}')$.

(2) If $\mathsf{v} \in \mathsf{V}$ and $\mathsf{n} \in \mathsf{N}$ then $x_1(\mathsf{v}) \neq x_1(\mathsf{n})$.

Proof. We have already assumed that the actual vertices have distinct x_1 coordinates. Since the set of virtual vertices is also finite, transversality applies. Condition (2) is satisfied by suitable choice of u_i, v_i. □

By Lemma 21.12 we may introduce a linear order in V as follows:

$$\mathsf{v} < \mathsf{v}' \Longleftrightarrow x_1(\mathsf{v}) < x_1(\mathsf{v}'), \qquad \mathsf{v}, \mathsf{v}' \in \mathsf{V}.$$

Our next aim is to describe the behavior of the edges near a vertex. Let $\mathsf{v} \in \mathsf{V}$ be a vertex. Since the vertices are linearly ordered, we may choose real numbers $t_1, t_2 \in \mathbb{R}$, $t_1 < x_1(\mathsf{v}) < t_2$ so that v is the only vertex in the strip $\mathsf{S} = \{\mathsf{m} \in \mathbb{R}^2 \mid t_1 \le x_1(\mathsf{m}) \le t_2\}$. Let $\mathsf{C}_i = \{\mathsf{m} \in \mathbb{R}^2 \mid x_1(\mathsf{m}) = t_i\}$ be the boundaries of S. By choice neither boundary contains a vertex of Γ^2_f. Thus these lines intersect each of the n traces of A exactly once in the interior of an edge. For $i = 1, 2$ let $\mathsf{E}(i) = (\mathsf{e}_1(i), \ldots, \mathsf{e}_n(i))$ be the ordering of those edges which meet C_i, ordered by increasing x_2-coordinates of the intersection points $\{\mathsf{C}_i \cap \mathsf{H} \mid \mathsf{H} \in \mathsf{A}\}$. If $\mathsf{e}_k(1) \in \mathsf{E}(1)$ does not contain v then $\mathsf{e}_k(1)$ intersects both C_1 and C_2 so $\mathsf{e}_k(1) \in \mathsf{E}(2)$. If $\mathsf{e}_k(1)$ contains v then there is a unique edge $\mathsf{e}'_k \in \mathsf{E}(2)$ which also contains v and **lies on the same trace** as $\mathsf{e}_k(1)$.

Proposition 21.13 *Let j be the first index for which $\mathsf{e}_j(1)$ contains v and let k be the last such index. Then*

$$\mathsf{E}(2) = (\mathsf{e}_1(1), \ldots, \mathsf{e}_{j-1}(1), \mathsf{e}'_k, \mathsf{e}'_{k-1}, \ldots, \mathsf{e}'_{j+1}, \mathsf{e}'_j, \mathsf{e}_{k+1}(1), \ldots, \mathsf{e}_n(1)).$$

Proof. The only vertex between C_1 and C_2 is v. Since the graphing map is flat, there is a neighborhood of v in S where the traces which contain v form a linear pencil. Thus $\{\mathsf{e}_i(1) \mid j \le i \le k\}$ are precisely those edges of E(1) which contain v and the subscripts of the corresponding edges $\{\mathsf{e}'_i \mid j \le i \le k\}$ which meet C_2 and lie on the same trace occur in reverse order: $\mathsf{e}_j(2) = \mathsf{e}'_k$, $\mathsf{e}_{j+1}(2) = \mathsf{e}'_{k-1}$, ..., $\mathsf{e}_k(2) = \mathsf{e}'_j$. The edges $\mathsf{e}_i(1)$ for $i < j$ and $i > k$ do not contain v. Thus they also occur in E(2) and retain the position they had in E(1): $\mathsf{e}_i(2) = \mathsf{e}_i(1)$ for $i < j$ and $i > k$. In particular if v is a virtual vertex then $k = j + 1$ since only two traces contain v. □

A regular 2-graph contains almost all the data necessary to obtain a presentation for $\pi_1(M)$. The only place where information is lost in the projection $\phi : \Gamma^3_f \to \Gamma^2_f$ occurs at the virtual vertices. Here the images of disjoint edges intersect and we cannot reconstruct their relative positions in the 3-graph from the 2-graph. Suppose $\mathsf{q} \in \mathsf{Q}$ is a virtual vertex in an admissible 2-graph. Let $E, E' \in \mathsf{E}^3$ such that $\mathsf{q} \in \phi(\mathrm{int}E) \cap \phi(\mathrm{int}E')$. We want to mark q to indicate whether it represents an undercrossing or an overcrossing of E and E' in Γ^3_f. Since $\phi^2 = \phi\phi^3$ and ϕ^3 is an isomorphism, we may determine the relative positions of the corresponding edges in Γ^4_f. The advantage is that these edges lie in unique hyperplanes of $\mathcal{A}$. Suppose $\mathsf{q} \in \mathsf{H} \cap \mathsf{H}'$. Let $H = \tau^{-1}(\mathsf{H})$ and $H' = \tau^{-1}(\mathsf{H}')$. Then $H, H' \in \mathcal{A}$ are two hyperplanes such that there exist $q, q' \in \Gamma^4_f$, $q \neq q'$, $q \in H$, $q' \in H'$ and $\phi^2(q) = \mathsf{q} = \phi^2(q')$. It follows that $x_k(q) = x_k(\mathsf{q}) = x_k(q')$ for $k = 1, 2$. Since $y_1(q) = f(x_1(\mathsf{q})) = y_1(q')$, the points q, q' differ only in their y_2 coordinates. We adopt the following convention.

Definition 21.14 *Suppose $\mathsf{q} \in \mathsf{Q}$ is a virtual vertex in a regular 2-graph Γ^2_f. Choose a real number $c < x_1(\mathsf{q})$ sufficiently close to $x_1(\mathsf{q})$. Recall that $H \cap K_c(f)$ and $H' \cap K_c(f)$ are points in Γ^4_f. Assume that H and H' are labeled so that $x_2(H \cap K_c(f)) < x_2(H' \cap K_c(f))$. Call the virtual vertex* q **positive** *if $y_2(q) < y_2(q')$, and* **negative** *otherwise.*

Call a regular 2-graph Γ^2_f **admissible** *if its graphing map f is flat, its vertices and nodes satisfy the conditions of* Lemma 21.12, *and its virtual vertices are marked by ± according to this convention.* **We shall assume that all 2–graphs are admissible** *and simplify notation by writing $\Gamma = \Gamma^2_f$.*

Note that in case $\mathcal{A}$ is a complexified real arrangement, the choice of $f = 0$ provides a regular 2-graph with no virtual vertices or nodes. Thus $\Gamma(\mathcal{A}) = \Gamma^2_0$ is an admissible graph.

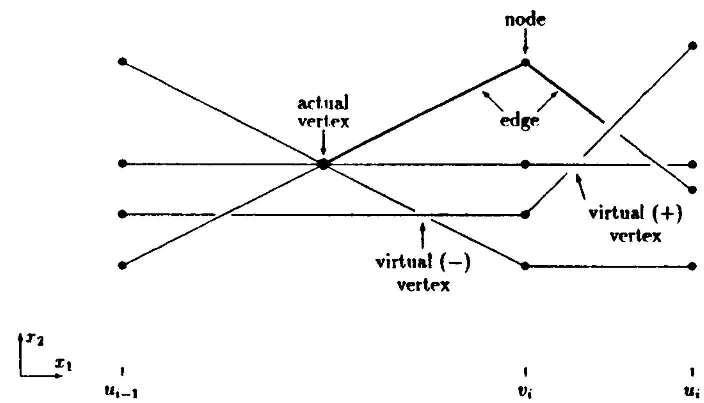


Figure 34: An admissible graph

Arvola's Presentation

First we represent the manifold M as a union of submanifolds suitable to apply van Kampen's theorem. This decomposition is done using Γ. Choose a finite set of real numbers $T \subset \mathbb{R}$ such that

(1) $\{u_j\} \cup \{v_j\} \subset T$, and

(2) if $\mathsf{v}, \mathsf{v}' \in \mathsf{V}$ with $x_1(\mathsf{v}) < x_1(\mathsf{v}')$ then there exists $t \in T$ such that $x_1(\mathsf{v}) < t < x_1(\mathsf{v}')$.

Suppose $T = \{t_0, \ldots, t_s\}$, where $t_0 < \cdots < t_s$.

Definition 21.15 *Let $L_i = \{m \in M \mid x_1(m) = t_i\}$ and define submanifolds of M by*

$$\begin{aligned} M_{-\infty} &= \{m \in M \mid x_1(m) \leq t_0\} \\ M_i &= \{m \in M \mid t_{i-1} \leq x_1(m) \leq t_i\} \quad 1 \leq i \leq s, \\ M_\infty &= \{m \in M \mid t_s \leq x_1(m)\}. \end{aligned}$$

Note that $L_i = M_i \cap M_{i+1}$. Since $M_{-\infty}$ and M_∞ contain no multiple points, we have:

Proposition 21.16 *L_0 is a strong deformation retract of $M_{-\infty}$ and L_s is a strong deformation retract of M_∞. Thus there is a homotopy equivalence*

$$M \sim M_1 \bigcup_{L_1} \cdots \bigcup_{L_{s-1}} M_s. \quad \square$$

We apply van Kampen's theorem to obtain:

Corollary 21.17 *The fundamental group is an amalgamated product*

$$\pi_1(M) \simeq \pi_1(M_1) *_{\pi_1(L_1)} \pi_1(M_2) *_{\pi_1(L_2)} \cdots *_{\pi_1(L_{s-1})} \pi_1(M_s). \quad \square$$

It remains to compute $\pi_1(M_i)$, $\pi_1(L_i)$, and the amalgamation maps. Specify a basepoint for all loops as follows. Choose a real number J sufficiently large so that $-J < y_2(m)$ for all points m in the compact set $\{m \in \Gamma^4 \mid t_0 \leq x_1(m) \leq t_s\}$. Now define

$$B = \{m \in M \mid t_0 \leq x_1(m) \leq t_s,\ y_1(m) = f(x_1(m)),\ y_2(m) = -J\}.$$

The set B is a contractible subspace of the complement M and thus any point in B may serve as basepoint. Let $K_i = K_{t_i}(f)$, $U_i = K_i \cap M$, and $B_i = B \cap U_i$. Then U_i is the complement of the n points $\{K_i \cap H \mid H \in \mathcal{A}\}$ and B_i is a line in U_i.

Lemma 21.18 *The set L_i is the complement of n skew lines in $\mathbb{R}^3$. Its subset U_i is the complement of n points in the real plane K_i. There is a strong deformation retraction $L_i \to U_i$. Thus $\pi_1(L_i)$ is a free group on n generators.*

Proof. The space $L_i' = \{m \in \mathbb{R}^4 \mid x_1(m) = t_i\}$ may be identified with $\mathbb{R}^3$. It contains no multiple point. Thus the lines $\{H \cap L_i' \mid H \in \mathcal{A}\}$ are disjoint. There is a homeomorphism of L_i' onto itself which fixes K_i and makes these lines parallel to each other, but still transverse to K_i. Projection onto K_i along the direction of the lines, when restricted to L_i, provides the strong deformation retraction. $\square$

Choose generators $g_1(i), \ldots, g_n(i)$ for $\pi_1(U_i)$ as indicated in Figure 35. Order these generators by increasing x_2-coordinates of the points they loop in K_i. Let $G(i) = (g_1(i), \ldots, g_n(i))$ denote the ordered n-tuple. Identify these loops with their images in $\pi_1(L_i)$ under the homotopy equivalence $U_i \hookrightarrow L_i$. As $K_t(f)$ moves from t_0 to t_s these loops move around in a continuous fashion. The admissible graph Γ is constructed so that knowledge of their order

$G(i)$ at the finite number of values $t = t_i$ is sufficient to compute $\pi_1(M)$. In order to find an explicit algorithm we must determine $G(i)$ from $G(i-1)$ and we must compute $\pi_1(M_i)$ from Γ.

Extend the notation of Proposition 21.13 to T. Observe that $\phi^2(K_i) = \mathsf{C}_i$. There is a bijection between the ordered sets $E(i)$ and $G(i)$. The generator $g_k(i)$ loops the point $H \cap K_i$ if and only if $\phi^2(H \cap K_i) \subset \mathsf{e}_k(i)$. Equivalently, the hyperplane $H \in \mathcal{A}$ which is linked by the loop $g_k(i)$ has trace $\mathsf{H} = \tau(H)$ which contains the edge $\mathsf{e}_k(i)$. The key to the algorithm is this correspondence between loops and edges, familiar from knot theory, together with Proposition 21.13. Let $\mathsf{S}_i = \phi^2(M_i)$ be the strip between C_{i-1} and C_i in $\mathbb{R}^2$. By our choice of the set T, this strip may contain at most one vertex of Γ. There are four possibilities: S_i may contain no vertex, a positive virtual vertex, a negative virtual vertex, or an actual vertex. If S_i contains no vertex then an argument similar to Lemma 21.18 shows that $M_i = L_{i-1} \times [0,1]$ and $G(i) = G(i-1)$. Identify the loops $G(i-1)$ with their images in $\pi_1(M_i)$ under the inclusion $L_{i-1} \hookrightarrow M_i$. When S_i contains the vertex v we say that $G(i-1)$ is **adapted** to v and write $G(i-1) = (g_1(\mathsf{v}), \ldots, g_n(\mathsf{v}))$. First we need a general result about moving a loop past another loop in the plane.

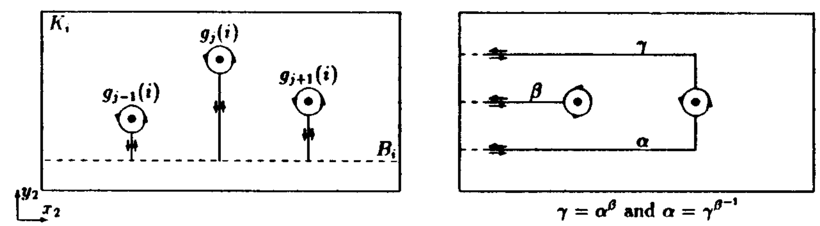


Figure 35: Generators and loop passing

Lemma 21.19 *Consider the three loops in $\mathbb{R}^2 - \{2 \text{ points}\}$ which are depicted in Figure 35. Among these we have the two relations $\gamma = \alpha^\beta$ and $\alpha = \gamma^{\beta^{-1}}$.*

Proof. Recall that the fundamental group consists of based homotopy classes of loops and that the composition of loops is from left to right. Draw the loop $\beta^{-1}\alpha\beta$ and observe that it may be deformed to γ in the complement of the points. $\square$

Lemma 21.20 *Suppose S_i contains the virtual vertex $\mathsf{q} \in \mathsf{Q}$. Then $\pi_1(M_i)$ is a free group generated by the elements of $G(i-1)$. If q is a positive virtual vertex contained in $\mathsf{e}_j(i-1)$ and $\mathsf{e}_{j+1}(i-1)$ then we get*

$$G(i) = (g_1(\mathsf{q}), \ldots, g_{j-1}(\mathsf{q}), g_{j+1}', g_j', g_{j+2}(\mathsf{q}), \ldots, g_n(\mathsf{q}))$$

where

$$\begin{aligned} g_j' &= g_j(\mathbf{q}) \\ g_{j+1}' &= g_{j+1}(\mathbf{q})^{g_j(\mathbf{q})}. \end{aligned}$$

If $\mathbf{q}$ *is a negative virtual vertex contained in* $\mathbf{e}_j(i-1), \mathbf{e}_{j+1}(i-1)$ *then we get*

$$\begin{aligned} g_j' &= g_j(\mathbf{q})^{g_{j+1}(\mathbf{q})^{-1}} \\ g_{j+1}' &= g_{j+1}(\mathbf{q}). \end{aligned}$$

Proof. Recall that $\mathbb{R}^3$ is the span of the coordinates x_1, x_2, y_2 and define $M' = M_i \cap \mathbb{R}^3$. Then M' is a strong deformation retract of M_i. The subspace M' is the complement of a braid which has a single crossing between the strands numbered $j, j+1$. If $\mathbf{q}$ is a positive vertex then as x_1 goes from t_{i-1} to t_i the loop labeled $g_j(i-1)$ passes under the loop labeled $g_{j+1}(i-1)$ in the x_2, y_2 plane. If $\mathbf{q}$ is a negative vertex then the loop labeled $g_j(i-1)$ passes over the loop labeled $g_{j+1}(i-1)$ in the x_2, y_2 plane. The required relationship between $G(i-1)$ and $G(i)$ is a consequence of Lemma 21.19. □

Lemma 21.21 *Suppose* S_i *contains the actual vertex* $\mathbf{p} \in \mathsf{P}$. *Let* j *be the first index for which* $\mathbf{e}_j(i-1)$ *contains* $\mathbf{p}$ *and let* k *be the last such index. Then*

$$G(i) = (g_1(\mathbf{p}), \ldots, g_{j-1}(\mathbf{p}), g_k', g_{k-1}', \ldots, g_{j+1}', g_j', g_{k+1}(\mathbf{p}), \ldots, g_n(\mathbf{p}))$$

where

$$\begin{aligned} g_j' &= g_j(\mathbf{p}) \\ g_{j+1}' &= g_{j+1}(\mathbf{p})^{g_j(\mathbf{p})} \\ g_{j+2}' &= g_{j+2}(\mathbf{p})^{g_{j+1}(\mathbf{p})g_j(\mathbf{p})} \\ &\vdots \\ g_k' &= g_k(\mathbf{p})^{g_{k-1}(\mathbf{p})\cdots g_j(\mathbf{p})}. \end{aligned}$$

Let $R_{\mathbf{p}} = [g_k(\mathbf{p}), \ldots, g_j(\mathbf{p})]$ *be the set of relations of* Definition 21.2. *Observe that the generators adapted to* $\mathbf{p}$ *occur in the relation symbol* $R_{\mathbf{p}}$ *in the order reverse to their order in the generator symbol* $G(i-1)$. *Then*

$$\pi_1(M_i) = (g_1(\mathbf{p}), \ldots, g_n(\mathbf{p}) \mid R_{\mathbf{p}}).$$

Proof. After a suitable isotopy of the strip we may assume that we have the corresponding strip of a real arrangement. For simplicity of notation we analyze the situation illustrated in Figure 36 where the traces are $\mathsf{H}_a, \ldots, \mathsf{H}_f$ and the generators adapted to p are $G(i-1) = (a, b, c, d, e, f)$. It follows from Proposition 21.13 that the general case differs from it only in the number of traces. Write $G(i) = (w_1, \ldots, w_6)$ and $M = M_i$. We want to show first that

$$w_1 = a, \quad w_2 = e^{dcb}, \quad w_3 = d^{cb}, \quad w_4 = c^b, \quad w_5 = b, \quad w_6 = f.$$

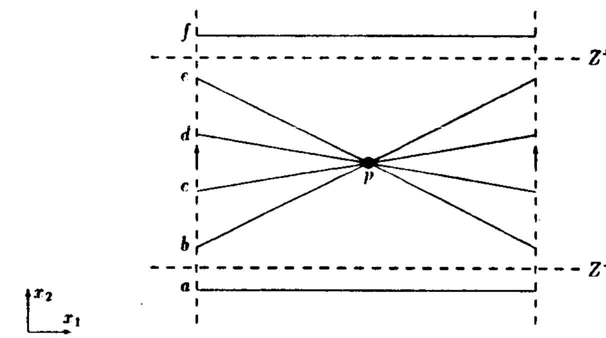


Figure 36: A pencil of lines

Recall that $\mathbb{R}^3$ is spanned by x_1, x_2, y_2. It follows that $M \cap \mathbb{R}^3$ is the complement of the lines in Figure 36. From this it is evident that the loops a and f slide to the right without hindrance. Thus $w_1 = a$ and $w_6 = f$.

Choose real numbers s^- and s^+ so that the lines $Z^- = \ker(x_2 - s^-)$ and $Z^+ = \ker(x_2 - s^+)$ together with C_{i-1} and C_i form a box around the vertex $\mathbf{p}$ separating the pencil of traces $\{\mathsf{H}_b, \mathsf{H}_c, \mathsf{H}_d, \mathsf{H}_e\}$ from the traces H_a and H_f as indicated by the dotted lines in Figure 36. The subspaces

$$\{m \in M \mid x_2(m) = s^-\}, \qquad \{m \in M \mid x_2(m) = s^+\}$$

are contractible. Thus the three sets of loops $\{a\}$, $\{b, c, d, e\}$ and $\{f\}$ are independent in $\pi_1(M)$. Let $p \in P$ be the unique multiple point with $\phi^2(p) = \mathbf{p}$. Set $M_p = \{m \in M \mid s^- \leq x_2(m) \leq s^+\}$. Let S_λ be the 3-sphere centered at p of sufficiently small radius λ so that S_λ is contained in the closure of M_p. Assume without loss of generality that p is the origin and $\lambda = 1$. Consider the map $\psi : \mathbb{R}^4 - \{0\} \to S_\lambda$ given by $\psi(m) = m/|m|$. It induces a strong deformation retraction of M_p onto $M_p \cap S_\lambda$. For $\sigma = b, c, d, e$ let H_σ denote the hyperplane linked by the loop σ, and set $\bar{H}_\sigma = H_\sigma \cap S_\lambda$. Then $\bar{H}_\sigma$ are mutually disjoint embedded circles in S_λ and thus $M_p \cap S_\lambda$ is the link complement

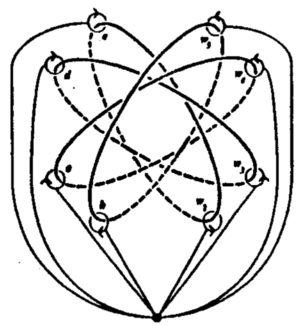
$$S_\lambda - (\bar{H}_b \cup \bar{H}_c \cup \bar{H}_d \cup \bar{H}_e).$$

Consider the stereographic projection $S_\lambda - (0,0,0,1) \to \mathbb{R}^3$ given by

$$(x_1, y_1, x_2, y_2) \mapsto \left(\frac{x_1}{1-y_2}, \frac{y_1}{1-y_2}, \frac{x_2}{1-y_2}\right).$$

The point $(0,0,0,1) \in S_\lambda$ may be deleted without changing the fundamental group and thus we may consider the situation in $\mathbb{R}^3$. This is illustrated in Figure 37. The solid half-circles pass above the plane of the figure. The dashed half-circles pass below the plane of the figure.

In order to determine the w_i we move the loops b, c, d, e along the solid (upper) half-circles. The relations arise by identifying the loops obtained by passing along the dashed (lower) half-circles with the w_i. If we move b along its upper half-circle it is evident that $w_5 = b$. If we move c along its upper half-circle and compare it to w_4 we are obstructed by H_b. Thus by Lemma 21.19 we have $w_4 = c^b$. If we move d along its upper half-circle and compare it to w_3 we are obstructed by first H_c then H_b. Thus by Lemma 21.19 we have $w_3 = d^{cb}$. Similarly $w_2 = e^{dcb}$. This illustrates the general calculation of $G(i)$.



x_2, x_1 The y_1-axis points up out of the plane of the figure.

Figure 37: Linking

Next we want to show that

$$\pi_1(M) = \langle a, b, c, d, e, f \mid [e, d, c, b] \rangle.$$

Move b, c, d, e along their lower half-circles and compare the result with the w_i we found. Again apply (21.19) as needed to get:

$$\begin{array}{rcccl} \underline{\text{lower}} & & & & \underline{\text{upper}} \\ e & = & w_2 & = & e^{dcb} \\ d^{e^{-1}} & = & w_3 & = & d^{cb} \\ c^{d^{-1}e^{-1}} & = & w_4 & = & c^b \\ b^{c^{-1}d^{-1}e^{-1}} & = & w_5 & = & b. \end{array}$$

Move all conjugations to the right hand side of each of the equations above. This set of relators is equivalent to the set $[e, d, c, b]$ as required. This illustrates the general calculation of R_{p}. □

We have collected the necessary information to make Corollary 21.17 explicit and give Arvola's presentation.

Theorem 21.22 *Let $\mathcal{A}$ be an arrangement of complex hyperplanes in $\mathbb{C}^2$. Let M be the complement of $\mathcal{A}$. Let Γ be an admissible 2-graph for $\mathcal{A}$. Let $G = \{g_H \mid H \in \mathcal{A}\}$. Then*

$$\pi_1(M) = \langle G \mid \bigcup_{\mathrm{p} \in \mathrm{P}} R_{\mathrm{p}} \rangle$$

where R_{p} is the set of relators in Lemma 21.21.

Proof. We sweep Γ from t_0 to t_s. There is a bijection between G and the elements of $G(0)$. By successive application of Lemmas 21.20 and 21.21 we can express every $G(i)$ for $1 \leq i \leq s$ as ordered n-tuples of words in G. This shows that G generates $\pi_1(M)$. We apply Lemmas 21.20 and 21.21 again to see that the only relations arise at actual vertices. Finally, at every actual vertex $\mathrm{p} \in \mathrm{P}$ we have generators adapted to p so we can use Lemma 21.21 to calculate the relators. □

Remark 21.23 *It follows from the algorithm that only virtual vertices which are between actual vertices influence $\pi_1(M)$.*

22 The Cohomology Groups of $M(\mathcal{A})$

In this section we determine the cohomology groups $H^k(M;\mathbb{Z})$ and the ring structure of $H^\bullet(M;\mathbb{Z})$. These results prove Arnold's conjectures. Brieskorn [33] showed that the groups $H^k(M;\mathbb{Z})$ are free and obtained the direct sum decomposition of Lemma 22.15 using techniques from algebraic geometry. He also proved that the $\mathbb{Z}$-algebra generated by the forms $(1/2\pi i)\omega_H$ and 1 is isomorphic to the singular cohomology $H^\bullet(M;\mathbb{Z})$. In [142] Brieskorn's Lemma 22.15 was used to show that the Poincaré polynomial of $M(\mathcal{A})$ equals the Poincaré polynomial of $\mathcal{A}$. The Poincaré polynomial was then used to prove the isomorphism $R(\mathcal{A}) \simeq A(\mathcal{A})$. This provides a description of $H^\bullet(M)$ in terms of generators and relators. In [139] we used Brieskorn's Lemma 22.15 to prove that for an inductive triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ the long exact sequence in homology of the pair $(M(\mathcal{A}'), M(\mathcal{A}))$ splits into short exact sequences

$$0 \to H^{k+1}(M(\mathcal{A}')) \to H^{k+1}(M(\mathcal{A})) \to H^k(M(\mathcal{A}'')) \to 0,$$

and that this splitting is equivalent to Brieskorn's Lemma. We remarked there that a direct proof of this splitting would provide a conceptually simpler argument for Brieskorn's Lemma, which avoids references to algebraic geometry. Such an argument appeared in a recent paper by Jozsa and Rice [105]. They use an exact sequence in de Rham cohomology. We give another elementary proof here. It is based on our work in the algebra $R(\mathcal{A})$. In this section all homology and cohomology groups have integer coefficients.

The Thom Isomorphism

We begin with a topological interpretation of restriction and deletion.

Definition 22.1 *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane $H_0 \in \mathcal{A}$. Let $M = M(\mathcal{A})$, $M' = M(\mathcal{A}')$, $M'' = M(\mathcal{A}'')$.*

Since M and M' are open subsets of V, they are complex manifolds of complex dimension ℓ.

Lemma 22.2 *(1) $M = M' \setminus M''$,*
(2) $M'' = M' \cap H_0$,
(3) M'' is a submanifold of M' of complex codimension one.

Proof. Assertions (1) and (2) are clear. For (3) note that H_0 has codimension one in V. The conclusion follows since M'' is open in H_0 and M' is open in V. □

Lemma 22.3 *The submanifold $M'' \subset M'$ has a tubular neighborhood $E \subseteq M'$ which has the structure of a trivial $\mathbb{C}$-bundle over M''.*

Proof. View M' as an open smooth manifold of complex dimension ℓ and M'' a smooth submanifold of complex codimension 1. The existence of the tubular neighborhood is a general fact, see [95, p.110]. The bundle is trivial because it is the restriction of a bundle over a contractible space. □

Call the complex line bundle $\xi = (E, M'', p)$ and view E as a subset of M' with inclusion map $q : E \to M'$. Let E_0 be the complement of the zero section in E. We may identify the zero section with M'' and $E_0 = E \setminus M''$. By Lemma 22.2.1 $M = M' \setminus M''$ so we have

$$E \cap M = E \cap M' \setminus E \cap M'' = E \setminus M'' = E_0.$$

Lemma 22.4 *Let $\xi = (E, M'', p)$ be a tubular neighborhood of M'' in M'. Then there are isomorphisms for $k \geq 1$*

$$\tau : H^{k+1}(M', M) \to H^{k-1}(M'').$$

Proof. Since $E_0 = E \cap M$ the embedding $q : E \to M'$ is an inclusion of pairs $q : (E, E_0) \to (M', M)$. This inclusion is excision of the closed subset $M' \setminus E \subseteq M$ and therefore q^* is an isomorphism. Let $z : M'' \to E$ be the zero section. Since $(E, E_0) \simeq M'' \times (\mathbb{C}, \mathbb{C}^*)$ we have isomorphisms $H^{k-1}(M'') \overset{z^*}{\leftarrow} H^{k-1}(E) \overset{r}{\to} H^{k+1}(E, E_0)$. Here r is the Thom isomorphism for the trivial bundle ξ. Then $\tau = z^* r^{-1} q^*$. □

Corollary 22.5 *For $k \geq 0$ there is a cohomology long exact sequence*

$$\cdots \to H^k(M') \overset{i^*}{\to} H^k(M) \overset{\phi}{\to} H^{k-1}(M'') \overset{\psi}{\to} H^{k+1}(M') \to \cdots$$

where $\phi = \tau\delta$ and $\psi = j^*\tau^{-1}$.

Proof. Consider the long exact sequence of the pair (M', M) in cohomology:

$$\cdots \to H^k(M') \overset{i^*}{\to} H^k(M) \overset{\delta}{\to} H^{k+1}(M', M) \overset{j^*}{\to} H^{k+1}(M') \to \cdots$$

We may use the isomorphism of Lemma 22.4 to replace $H^{k+1}(M', M)$ by $H^{k-1}(M'')$. This provides the required long exact sequence. □

Definition 22.6 *Let $H = \ker \alpha_H$ and let $M_H = V \setminus H$. The map $\alpha_H : V \to \mathbb{C}$ restricts to $\alpha_H : M_H \to \mathbb{C}^*$. Choose the canonical generator of $H^1(\mathbb{C}^*)$ as $(1/2\pi i)(dz/z)$. Define a rational 1-form*

$$\eta_H = \frac{1}{2\pi i}\frac{d\alpha_H}{\alpha_H}$$

on V. Let $\langle \eta_H \rangle$ be the cohomology class of η_H in $H^1(M_H)$. Then

$$\langle \eta_H \rangle = \alpha_H^*\left(\frac{1}{2\pi i}\frac{dz}{z}\right) \in H^1(M_H).$$

Denote the cohomology class of η_H in $H^1(M)$ by $[\eta_H]$. Let $i_H : M \to M_H$ be the inclusion map. Then $[\eta_H] = i_H^*\langle \eta_H \rangle$.

Lemma 22.7 *The natural orientation of the* $\mathbb{C}$*-bundle* $\xi = (E, M'', p)$ *has Thom class* $u \in H^2(E, E_0)$ *given by* $u = q^*\delta[\eta_{H_0}]$.

Proof. Write $M_0 = M_{H_0}$, $\alpha_0 = \alpha_{H_0}$, $i_0 = i_{H_0}$ and $\eta_0 = \eta_{H_0}$. In the cohomology exact sequence of the pair (V, M_0) we have $H^1(V) = H^2(V) = 0$ so $\delta : H^1(M_0) \to H^2(V, M_0)$ is an isomorphism. Since (η_0) generates $H^1(M_0)$, $\delta(\eta_0)$ generates $H^2(V, M_0)$. Let $x \in M''$ be any point and let $F = p^{-1}(x)$. Let $F_0 = F \cap E_0$. Let $k : (F, F_0) \to (V, M_0)$ be the inclusion of the fiber. Since $\alpha_0 : (F, F_0) \to (\mathbb{C}, \mathbb{C}^*)$ is a homotopy equivalence of pairs, k^* is an isomorphism. If ξ is given the natural orientation then k^* sends a positive generator to a positive generator. We have the following inclusion of pairs with $k = jqi$:

$$(F, F_0) \xrightarrow{i} (E, E_0) \xrightarrow{q} (M', M) \xrightarrow{j} (V, M_0).$$

In cohomology we get the commutative diagram:

$$\begin{array}{ccccccc} H^2(V, M_0) & \xrightarrow{j^*} & H^2(M', M) & \xrightarrow{q^*} & H^2(E, E_0) & \xrightarrow{i^*} & H^2(F, F_0) \\ \uparrow \delta & & \uparrow \delta & & & & \\ H^1(M_0) & \xrightarrow{i_0^*} & H^1(M) & & & & \end{array}$$

Since $i^*q^*j^* = k^*$ is an isomorphism and $\delta(\eta_0)$ generates $H^2(V, M_0)$, it follows that $i^*q^*j^*\delta(\eta_0) = i^*q^*\delta[\eta_0]$ generates $H^2(F, F_0)$. Thus the restriction of $q^*\delta[\eta_0]$ is a generator of $H^2(F, F_0)$ for every fiber. This characterizes the Thom class u of the bundle ξ. □

Let $H \in \mathcal{A}'$. Then η_H gives a cohomology class $[\eta_H]'$ in $H^1(M')$. Naturality implies that for $i : M \to M'$ we have $i^*[\eta_H]' = [\eta_H]$. Let $\alpha_{H_0 \cap H}$ be the restriction of $\alpha_H : V \to \mathbb{C}^*$ to H_0: $\alpha_{H_0 \cap H} = \alpha_H|_{H_0}$. Let $[\eta_{H_0 \cap H}] \in H^1(M'')$ be the cohomology class of the rational 1-form

$$\eta_{H_0 \cap H} = \frac{1}{2\pi i}\frac{d\alpha_{H_0 \cap H}}{\alpha_{H_0 \cap H}}.$$

Lemma 22.8 *Let* $H \in \mathcal{A}'$. *Then* $z^*q^*[\eta_H]' = [\eta_{H_0 \cap H}]$. *Here* $q : E \to M'$ *is the tubular neighborhood and* $z : M'' \to E$ *is its zero section.*

Proof. Note that $qz : M'' \to M'$ is the inclusion map. So the pull-back of η_H by qz is equal to $\eta_{H_0 \cap H}$. □

Lemma 22.9 *Let* $H \in \mathcal{A}'$. *Then for any* $[a] \in H^k(M)$ *we have*

$$\delta([a] \cup [\eta_H]) = \delta[a] \cup [\eta_H]'.$$

Proof. Stability [54, p.220] of the diagram

$$\begin{array}{ccccc} H^k(M) \otimes H^1(M') & \xrightarrow{id \otimes i^*} & H^k(M) \otimes H^1(M) & \xrightarrow{\cup} & H^{k+1}(M) \\ \downarrow \delta \otimes id & & & & \downarrow \delta \\ H^{k+1}(M', M) \otimes H^1(M') & \to & \xrightarrow{\cup} & \to & H^{k+2}(M', M) \end{array}$$

gives $\delta([a] \cup i^*[\eta_H]') = \delta[a] \cup [\eta_H]'$ and we noted that $i^*[\eta_H]' = [\eta_H]$ by naturality. □

Our next aim is to determine the structure of $H^*(M)$. Recall the modules $R_k(\mathcal{A})$ from Definition 12.3, the residue map from Lemma 12.12, and the exact sequence in Theorem 12.14. Let $\mathcal{K} = \mathbb{Z}$.

Lemma 22.10 *There is a commutative diagram of exact sequences whose vertical maps* $\eta : R_k(\mathcal{A}) \to H^k(M)$ *are given by* $\eta(\omega_H) = [\eta_H]$

$$\begin{array}{ccccccccc} 0 & \to & R_{k+1}(\mathcal{A}') & \xrightarrow{i} & R_{k+1}(\mathcal{A}) & \xrightarrow{j} & R_k(\mathcal{A}'') & \to & 0 \\ & & \downarrow \eta' & & \downarrow \eta & & \downarrow \eta'' & & \\ \cdots & \to & H^{k+1}(M') & \xrightarrow{i^*} & H^{k+1}(M) & \xrightarrow{\phi} & H^k(M'') & \to & \cdots \end{array}$$

Proof. The commutativity is clear in the left square. For commutativity in the right square let $\gamma = \omega_{H_0}\omega_{H_1}\ldots\omega_{H_k}$. We want $\phi\eta(\gamma) = \eta''j(\gamma)$. Recall that $\phi = z^*r^{-1}q^*\delta$. Thus we have

$$\begin{aligned} \phi\eta(\gamma) &= \phi([\eta_{H_0}] \cup [\eta_{H_1} \ldots \eta_{H_k}]) \\ &= z^*r^{-1}q^*(\delta[\eta_0] \cup [\eta_{H_1} \ldots \eta_{H_k}]') \\ &= z^*r^{-1}(q^*\delta[\eta_0] \cup q^*[\eta_{H_1} \ldots \eta_{H_k}]') \\ &= z^*r^{-1}(u \cup q^*[\eta_{H_1} \ldots \eta_{H_k}]') \\ &= z^*q^*[\eta_{H_1} \ldots \eta_{H_k}]' \\ &= [\eta_{H_0 \cap H_1} \ldots \eta_{H_0 \cap H_k}]. \end{aligned}$$

Here we used naturality and the fact that cupping with u is r. □

Theorem 22.11 *Let* $\mathcal{A}$ *be a nonempty complex arrangement.*

(1) The map $\eta : R_k(\mathcal{A}) \to H^k(M)$ *is an isomorphism for* $k \geq 0$.
(2) $H^k(M)$ *are free abelian groups.*
(3) For $k \geq 0$ *there exist split short exact sequences:*

$$0 \to H^{k+1}(M') \xrightarrow{i^*} H^{k+1}(M) \xrightarrow{\phi} H^k(M'') \to 0.$$

Proof. We argue (1) by a double induction on ℓ and $n = |\mathcal{A}|$. The assertion holds for all 1-arrangements and for all ℓ-arrangements with $|\mathcal{A}| = 1$. The induction step assumes both assertions for all m-arrangements with $m < \ell$ and for all ℓ-arrangements with $|\mathcal{A}| < n$. Thus η' and η'' are isomorphisms. It follows from Lemma 22.10 that η is an isomorphism. This completes the induction step. Corollary 12.15 implies (2). Lemma 22.10 and (1) provide (3). □

Corollary 22.12 *The integral cohomology ring* $H^*(M)$ *is generated by 1 and the classes* $[\eta_H]$ *for* $H \in \mathcal{A}$. □

Theorem 22.13 *The surjective map* $\omega_H \to [(1/2\pi i)\omega_H]$ *induces an isomorphism of graded algebras* $R(\mathcal{A}) \simeq H^*(M(\mathcal{A}))$. □

This shows that there are no relations in cohomology other than what is imposed by the algebraic relations. We showed in Theorem 12.13 that there is an isomorphism of algebras $A(\mathcal{A}) \simeq R(\mathcal{A})$ which sends a_H to ω_H. We may apply this result when $K = \mathbb{Z}$ to obtain a structure theorem for $H^*(M;\mathbb{Z})$ in terms of generators and the relation ideal.

Theorem 22.14 *Let $\mathcal{A}$ be a complex arrangement and let $A = A(\mathcal{A})$. The map $a_H \to [(1/2\pi i)\omega_H]$ induces an isomorphism $A \to H^*(M)$ of graded $\mathbb{Z}$-algebras.* □

This isomorphism is the common thread between the algebra factorization $A(c\mathcal{A}) \simeq (K + Ka_0) \otimes A(\mathcal{A})$ of Theorem 10.1 and the topological factorization $M(c\mathcal{A}) \approx M(\mathcal{A}) \times \mathbb{C}^*$ of Proposition 19.1.

The next result is due to Brieskorn [33]. His proof involves some Lefschetz type results from algebraic geometry. Alternate arguments were given by Falk [59], by Goresky and MacPherson [76], and by Jozsa and Rice [105].

Lemma 22.15 (Brieskorn) *Let $\mathcal{A}$ be a nonempty complex arrangement. For $X \in L(\mathcal{A})$ let $M_X = M(\mathcal{A}_X)$. For $k \geq 0$ there are isomorphisms*

$$\theta_k : \bigoplus_{X \in L_k} H^k(M_X) \to H^k(M)$$

induced by the inclusion maps $i_X : M \to M_X$.

Proof. By Theorem 22.14 there exists an isomorphism $A(\mathcal{A}) \to H^*(M(\mathcal{A}))$ of graded $\mathbb{Z}$-algebras. By Corollary 8.27 we have natural isomorphisms for $k \geq 0$

$$\bigoplus_{X \in L_k} A_k(\mathcal{A}_X) \to A_k(\mathcal{A}).$$

Therefore we have natural isomorphisms for $k \geq 0$

$$\bigoplus_{X \in L_k} H^k(M(\mathcal{A}_X)) \to H^k(M(\mathcal{A})). \quad \square$$

Definition 22.16 *Let $b_p(M) = \operatorname{rank} H^p(M)$ be the Betti numbers of M. The Poincaré polynomial of the complement is*

$$\operatorname{Poin}(M(\mathcal{A}), t) = \sum_{p \geq 0} b_p(M) t^p.$$

Theorem 22.17 *Let $\mathcal{A}$ be a complex arrangement. Then*

$$\operatorname{Poin}(M(\mathcal{A}), t) = \pi(\mathcal{A}, t).$$

Proof. If $\mathcal{A}$ is the empty arrangement then $M(\mathcal{A}) = V$ and $\operatorname{Poin}(M(\mathcal{A}), t) = 1$. It follows from Theorem 22.11 that $\operatorname{Poin}(M(\mathcal{A}), t)$ satisfies the same recursion under deletion and restriction as $\pi(\mathcal{A}, t)$. Thus they are equal. □

Proposition 22.18 *Let $\mathcal{A}$ be a real ℓ-arrangement defined by $Q(\mathcal{A})$. The complement $M(\mathcal{A})$ is an algebraic variety.*

Proof. Consider $\mathbb{R}^{\ell+1}$ with coordinates $x_0, x_1, \ldots, x_\ell$. Then

$$M(\mathcal{A}) \approx \{x \in \mathbb{R}^{\ell+1} \mid x_0 Q(\mathcal{A}) = 1\}. \quad \square$$

Corollary 22.19 (M-property) *Let $(\mathcal{A}_{\mathbf{R}}, V_{\mathbf{R}})$ be a real arrangement and let $(\mathcal{A}_{\mathbf{C}}, V_{\mathbf{C}})$ be its complexification. Let $M_{\mathbf{R}} = M(\mathcal{A}_{\mathbf{R}})$ and $M_{\mathbf{C}} = M(\mathcal{A}_{\mathbf{C}})$ be the real and complex complements. Let $b_i(M_{\mathbf{R}})$ and $b_i(M_{\mathbf{C}})$ be their respective Betti numbers with coefficients in $\mathbb{Z}/2$. Then $M_{\mathbf{R}}$ has the M-property:*

$$\sum_{i \geq 0} b_i(M_{\mathbf{R}}) = \sum_{i \geq 0} b_i(M_{\mathbf{C}}).$$

Proof. Since $M_{\mathbf{R}}$ is a disjoint union of chambers and each chamber is contractible, $b_i(M_{\mathbf{R}}) = 0$ for $i > 0$ and $b_0(M_{\mathbf{R}})$ is the number of chambers. By Theorem 6.21 we have $b_0(M_{\mathbf{R}}) = \pi(\mathcal{A}_{\mathbf{R}}, 1)$. By Theorem 22.17 we have

$$\sum_{i \geq 0} b_i(M_{\mathbf{C}}) = \operatorname{Poin}(M(\mathcal{A}_{\mathbf{C}}), 1) = \pi(\mathcal{A}_{\mathbf{C}}, 1).$$

The result follows from the fact that $L(\mathcal{A}_{\mathbf{R}}) = L(\mathcal{A}_{\mathbf{C}})$. □

23 The Fibration Theorem

Recall strictly linearly fibered arrangements from Definition 19.10 and fiber type arrangements from Definition 19.11. In this section we prove that $\mathcal{A}$ is fiber type if and only if $\mathcal{A}$ is supersolvable. This provides a topological interpretation for the notion of a supersolvable arrangement. Since supersolvable arrangements are central, we assume in this section that $\mathcal{A}$ is a central arrangement. The presentation follows [194].

Let $(\mathcal{A}, V)$ be an ℓ arrangement and let $Y \subseteq V$ be a subspace which is **not necessarily in** $L = L(\mathcal{A})$. Througout this section we will use spaces, maps, and properties whose dependence on Y will be suppressed.

Definition 23.1 *Let $p = p_Y : V \to V/Y$ be the natural projection. Extend the definition of $\mathcal{A}_Y$ to subspaces which are not in L by $\mathcal{A}_Y = \{H \in \mathcal{A} \mid Y \subseteq H\}$. Then $\mathcal{A}/Y = \{p(H) \mid H \in \mathcal{A}_Y\}$ is an arrangement in V/Y. Let $\bar{Y} = \cap_{H \in \mathcal{A}_Y} H$ be the smallest element of L which contains Y.*

Note that $\mathcal{A}_Y = \mathcal{A}_{\bar{Y}}$ so $L(\mathcal{A}_Y) = L(\mathcal{A}_{\bar{Y}})$. There is a natural bijection $\mathcal{A}_Y \to \mathcal{A}/Y$ by $H \to H/Y$. This induces an isomorphism $L(\mathcal{A}_Y) \simeq L(\mathcal{A}/Y)$. The projection p_Y induces a smooth surjection $\pi = \pi_Y : M(\mathcal{A}) \to M(\mathcal{A}/Y)$. We want to determine when π is a bundle map. If $\mathcal{A}$ is not essential then we may choose $Y = T(\mathcal{A})$ to get a bundle $\pi_{T(\mathcal{A})} : M(\mathcal{A}) \to M(\mathcal{A}/T(\mathcal{A}))$ with fiber $T(\mathcal{A})$. Thus we may assume that $\mathcal{A}$ is an essential central ℓ-arrangement.

Horizontal Subspaces

Definition 23.2 *Call $X \in L$* **horizontal** *if $p(X) = V/Y$. Let $\mathrm{Hor} = \mathrm{Hor}_Y$ be the set of horizontal elements of L. Define the* **bad set** *B by*

$$B = B_Y = \bigcup_{X \in L \setminus \mathrm{Hor}} p(X) \cap M(\mathcal{A}/Y).$$

Example 23.3 *Consider $\mathcal{A}$ defined by $Q = xyz(x + y - z)$. Let $H_1 = \ker(x)$, $H_2 = \ker(y)$, $H_3 = \ker(z)$, and $H_4 = \ker(x + y - z)$. Let $Y = H_1 \cap H_2$ so Y is the z-axis. Here $\mathcal{A}_Y = \{H_1, H_2\}$, $\mathrm{Hor} = \{V, H_3, H_4\}$ and $B = p(H_3 \cap H_4)$. We may identify V/Y with the x,y-plane. Then $\mathcal{A}/Y$ consists of the x-axis and the y-axis. The bad set B is the line $x + y = 0$ minus the origin. Note that $\pi^{-1}(u)$ is a twice punctured line for $u \in M(\mathcal{A}/Y) \setminus B$, and a once punctured line for $u \in B$.*

Definition 23.4 *Fix $v \in V$. Define $L_v \subseteq L$ by*

$$L_v = \{X \in L \mid (v + Y) \cap X \neq \emptyset\}.$$

Define $A_v \subseteq v + Y$ by

$$A_v = \{(v + Y) \cap X \mid X \in L_v\}.$$

Let $\psi = \psi_v : L_v \to A_v$ be the natural surjection defined for $X \in L_v$ by $\psi(X) = (v + Y) \cap X$.

In Example 23.3 let $v = (1, -1, 1)$. Then $L_v = \{V, H_3, H_4, H_3 \cap H_4\}$. Here $\psi_v(V) = \{(1, -1, c) \mid c \in \mathbb{C}\}$ and $\psi_v(H_3) = \psi_v(H_4) = \psi_v(H_3 \cap H_4) = (1, -1, 0)$.

Definition 23.5 *Define $C = C_Y \subseteq L$ by*

$$C_Y = \{X \in L \mid X \wedge \bar{Y} = V\}.$$

Lemma 23.6 *For all $v \in M(\mathcal{A})$ we have $\mathrm{Hor} \subseteq L_v \subseteq C_Y$.*

Proof. If $X \in \mathrm{Hor}$ then $p(X) = V/Y$ and hence $X + Y = V$. Thus $(v + Y) \cap X \neq \emptyset$ and $X \in L_v$. If $X \in L_v$ then

$$v \in X + Y \subseteq X + \bar{Y} \subseteq X \wedge \bar{Y}.$$

Since $X \wedge \bar{Y} \in L$ and the only element of L which contains a point of $M(\mathcal{A})$ is V, we have $X \wedge \bar{Y} = V$. Therefore $X \in C_Y$. □

In Example 23.3 we have $C_Y = \{H_3, H_4, H_3 \cap H_4\} = L_v$. Note that the inclusion $\mathrm{Hor} \subset L_v$ is proper.

Lemma 23.7 $C_Y = \{X \in L \mid p(X) \cap M(\mathcal{A}/Y) \neq \emptyset\}$.

Proof.

$$\begin{aligned} p(X) \cap M(\mathcal{A}/Y) = \emptyset &\iff p(X) \subseteq p(H) && \text{for some } H \in \mathcal{A}_Y \\ &\iff X + Y \subseteq H && \text{for some } H \in \mathcal{A}_Y \\ &\iff X + \bar{Y} \subseteq H && \text{for some } H \in \mathcal{A}_Y \\ &\iff X \wedge \bar{Y} \neq V. && \square \end{aligned}$$

Proposition 23.8 *The following four conditions on Y are equivalent.*
(1) $L_v = \mathrm{Hor}$ for all $v \in M(\mathcal{A})$.
(2) L_v is independent of $v \in M(\mathcal{A})$.
(3) The bad set is empty, $B = \emptyset$.
(4) $\mathrm{Hor} = C_Y$.

Proof. (1) $\Rightarrow$ (2) is clear. We argue (2) $\Rightarrow$ (3) by contradiction. If $B \neq \emptyset$ then there exists $v \in M(\mathcal{A})$ such that $\pi(v) \in p(X)$ for some non-horizontal $X \in L$. Since X is not horizontal, there exists $w \in M(\mathcal{A})$ such that $\pi(w) \notin p(X)$. Then $X \in L_v$ but $X \in L_w$. This contradicts (2). For (3) $\Rightarrow$ (4) note that $B = \emptyset$ implies that

$$\{X \in L \mid p(X) \cap M(\mathcal{A}/Y) \neq \emptyset\} \subseteq \mathrm{Hor}.$$

The reverse inclusion follows from Lemmas 23.6 and 23.7. Finally, (4) $\Rightarrow$ (1) follows from Lemma 23.6. □

Good Subspaces

Definition 23.9 *Call Y a* **good** *subspace of V if it satisfies the conditions of* Proposition 23.8.

Proposition 23.10 *Let $v \in M(\mathcal{A})$. If $\pi(v) \in M(\mathcal{A}/Y) \setminus B$ then*
(1) $\mathrm{Hor} = L_v$, and
(2) the map $\psi_v : L_v \to A_v$ is bijective.

Proof. (1) It follows from Lemma 23.6 that $\mathrm{Hor} \subseteq L_v$. If $X \in L_v$ then $v \in X + Y$ and $\pi(v) \in p(X)$. It follows from $\pi(v) \notin B$ that $X \in \mathrm{Hor}$. Thus $L_v \subseteq \mathrm{Hor}$.
(2) The map is surjective by Definition 23.4. Let $X_1, X_2 \in L_v$ with $\psi(X_1) = \psi(X_2)$. Then

$$(v+Y) \cap X_1 = (v+Y) \cap X_2 = (v+Y) \cap (X_1 \cap X_2) \neq \emptyset.$$

Thus $X_1 \cap X_2 \in L_v = \mathrm{Hor}$ and $Y + (X_1 \cap X_2) = V$. We also have

$$\dim(Y \cap X_1) = \dim(Y \cap X_2) = \dim(Y \cap X_1 \cap X_2).$$

Hence

$$\begin{aligned}\dim(X_1 \cap X_2) &= \dim Y + \dim(Y \cap X_1 \cap X_2) - \dim(Y + (X_1 \cap X_2)) \\ &= \dim Y + \dim(Y \cap X_1) - \dim V \\ &= \dim Y + \dim(Y \cap X_1) - \dim(Y + X_1) \\ &= \dim X_1.\end{aligned}$$

This implies that $X_1 = X_1 \cap X_2$. A similar argument shows that $X_2 = X_1 \cap X_2$ and hence $X_1 = X_2$. Thus ψ is injective. $\square$

Corollary 23.11 *If Y is good then $\psi_v : L_v \to A_v$ is a bijection for all $v \in M(\mathcal{A})$.* $\square$

Proposition 23.12 *Suppose $Y \in L$. Then Y is good if and only if Y is a modular element.*

Proof. Since $Y \in L$ we have $Y = \bar{Y}$. Suppose $X \in C_Y$. If Y is good then by Lemma 23.6 we have $X \in \mathrm{Hor}$ and $V = X + Y$. Thus $X \wedge Y = V = X + Y$. It follows from Lemma 4.26 that (X, Y) is a modular pair. Since this holds for all $X \in C_Y$ it follows from Lemma 4.30 that Y is a modular element.
If Y is modular then $X + Y = X \wedge Y$ for all $X \in L$. In particular $\mathrm{Hor} = C_Y$ and Y is good by Lemma 23.6. $\square$

Good Lines

In the rest of this section assume that $\dim Y = 1$.

Lemma 23.13 *For all $v \in M(\mathcal{A})$*

$$\mathrm{Hor} = \{V\} \cup \{\mathcal{A} \setminus \mathcal{A}_Y\} = \{X \in L_v \mid r(X) \leq 1\}.$$

Proof.

$$\begin{aligned}X \in L_v \text{ and } r(X) \leq 1 &\iff v \in X + Y \text{ and } \dim X \geq \ell - 1 \\ &\iff X = V \text{ or } (Y \not\subseteq X \text{ and } \dim X = \ell - 1) \\ &\iff X = V \text{ or } X \in \mathcal{A} \setminus \mathcal{A}_Y \\ &\iff X + Y = V \\ &\iff X \in \mathrm{Hor}. \quad \square\end{aligned}$$

Lemma 23.14 *For all $v \in M(\mathcal{A})$ we have $\psi(\mathrm{Hor}) = A_v$.*

Proof. Suppose $X \in L_v$. Then $\psi(X) = (v + Y) \cap X \in A_v$. There exists $H \in \mathcal{A}$ such that $X \subseteq H$. If $H \in \mathcal{A}_Y$ then $v \in X + Y \subseteq H$. This is a contradiction because $v \in M(\mathcal{A})$. Thus $H \in \mathcal{A} \setminus \mathcal{A}_Y$. It follows from Lemma 23.13 that $H \in \mathrm{Hor}$ and $\psi(H) \supseteq \psi(X) \neq \emptyset$. Since $\dim Y = 1$ and $Y \not\subseteq H$ we see that $\psi(H)$ is a point. It follows that $\psi(H) = \psi(X)$. $\square$

Remark 23.15 *Let $C_k = \mathbb{C} \setminus \{0, 1, \ldots, (k-1)\}$ be the complex line with k integer points removed. It is known that C_k is homeomorphic to the complement of any other k points in $\mathbb{C}$, thus we may abuse notation and use C_k to denote any one of these spaces. It is also known that C_k is homeomorphic to C_m if and only if $k = m$.*

Theorem 23.16 *Let $(\mathcal{A}, V)$ be an essential central ℓ-arrangement and let Y be a subspace of V with $\dim Y = 1$. The following four conditions are equivalent:*
(1) The map $\pi : M(\mathcal{A}) \to M(\mathcal{A}/Y)$ is a fiber bundle projection.
(2) Each fiber of π is homeomorphic to C_k where $k = |\mathcal{A} \setminus \mathcal{A}_Y|$.
(3) Y is good.
(4) $\bar{Y}$ is a modular element with $r(\bar{Y}) = \ell - 1$. In particular $Y = \bar{Y} \in L$.

Proof. (1) $\Leftrightarrow$ (2) Since $\dim Y = 1$, if $X \in L_v \setminus \{V\}$ then $(v + Y) \cap X$ is a point. Thus the fiber $F_v = \pi^{-1}\pi(v)$ is C_m where $m = |A_v| - 1$. It follows from Proposition 23.10 and Lemma 23.13 that if $v \in M(\mathcal{A})$ and $\pi(V) \in M(\mathcal{A}/Y) \setminus B$ then

$$(*) \qquad |A_v| = |L_v| = |\mathrm{Hor}| = |\mathcal{A} \setminus \mathcal{A}_Y| + 1.$$

Since B is a proper Zariski closed subset of $M(\mathcal{A}/Y)$, the generic fiber is C_k where $k = |\mathcal{A} \setminus \mathcal{A}_Y|$. Remark 23.15 completes the argument.
(3) $\Rightarrow$ (2) This follows from $(*)$ and Corollary 23.11.

(2) $\Rightarrow$ (3) By (2) we have $|A_v| = |\mathrm{Hor}|$ for all $v \in M(\mathcal{A})$. It follows from Lemma 23.14 that the restriction map $\psi|_{\mathrm{Hor}} : \mathrm{Hor} \to A_v$ is bijective. Suppose $X \in L_v$. If $r(X) = 1$ then Lemma 23.13 implies that $X \in \mathrm{Hor}$. If $r(X) > 1$ then there exist distinct hyperplanes $H_1, H_2 \in \mathcal{A}$ containing X. First note that $H_i \in \mathcal{A} \setminus \mathcal{A}_Y$ for $i = 1, 2$ by the argument used in the proof of Lemma 23.14 and thus $H_i \in \mathrm{Hor}$. But then we have

$$(v + Y) \cap H_1 = (v + Y) \cap X = (v + Y) \cap H_2$$

contradicting the bijectivity of $\psi|_{\mathrm{Hor}}$. This shows that $r(X) = 1$ and $L_v = \mathrm{Hor}$. It follows from Proposition 23.8 that Y is good.

(3) $\Rightarrow$ (4) Suppose $X \in C_Y$. Then Proposition 23.8 implies that $X \in \mathrm{Hor}$ and hence $X \wedge Y = V = X + Y \subseteq X + \bar{Y}$. Thus $X \wedge Y = X + \bar{Y}$ and (X, Y) is a modular pair. It follows from Lemma 4.30 that $\bar{Y}$ is a modular element. Now suppose that $X \in L$ is a complement of Y. Then $V = X \wedge \bar{Y} = X + \bar{Y} = X + Y$ and $\{0\} = X \vee Y = X \cap Y \supseteq X \cap \bar{Y}$. It follows that $\dim X + \dim \bar{Y} = \ell = \dim X + \dim Y$ and $\dim \bar{Y} = 1$. This shows that $Y = \bar{Y} \in L$.

(4) $\Rightarrow$ (3) Suppose $Y = \bar{Y}$ is a modular element with $r(Y) = \ell - 1$. Then $r(X \wedge Y) + r(X \vee Y) = r(X) + r(Y)$. If $X \in C_Y$ then either $X = V$ and $X \in \mathrm{Hor}$, or $X \vee Y = \{0\}$ and hence $r(X) = 1$. It follows that $Y \not\subseteq X$ and hence $\dim(X + Y) > \dim X = \ell - 1$. Therefore $X + Y = V$ and $X \in \mathrm{Hor}$. It follows from Proposition 23.8 that Y is good. □

Corollary 23.17 *Let $\mathcal{A}$ be an essential central ℓ-arrangement. Then $\mathcal{A}$ is strictly linearly fibered if and only if there is a modular element $Y \in L(\mathcal{A})$ with $r(Y) = \ell - 1$.* □

Theorem 23.18 (Fibration) *Let $\mathcal{A}$ be an essential central ℓ-arrangement. Then $\mathcal{A}$ is fiber type if and only if $\mathcal{A}$ is supersolvable.*

Proof. Suppose $\mathcal{A}$ is fiber type. Then there is a tower of fibrations

$$M(\mathcal{A}) = M_\ell \overset{\pi_{\ell-1}}{\to} M_{\ell-2} \overset{\pi_{\ell-1}}{\to} \cdots \to M_2 \overset{\pi_1}{\to} M_1 = \mathbb{C}^*$$

such that the fiber of π_k is C_{m_k}. This tower is the restriction of a tower of vector space projections

$$V = V_\ell \overset{p_{\ell-1}}{\to} V_{\ell-2} \overset{p_{\ell-1}}{\to} \cdots \to V_2 \overset{p_1}{\to} V_1 = \mathbb{C}.$$

For $1 \le k \le \ell - 1$ let $Y_k = \ker(p_k)$ and let $X_k = \ker(p_{\ell-1}p_{\ell-2}\ldots p_k)$. Let $X_0 = V$ and $X_\ell = \{0\}$. Then we have a chain in $L(\mathcal{A})$

$$V = X_0 < X_1 < \cdots < X_{\ell-1} < X_\ell = \{0\}. \qquad (1)$$

For $1 \le k \le \ell - 1$ we have $V_k = V_{k+1}/Y_k = V/X_k$. Define k-arrangements $(\mathcal{A}_k, V_k)$ inductively by $\mathcal{A}_\ell = \mathcal{A}$ and $\mathcal{A}_k = \mathcal{A}_{k+1}/Y_k$ for $1 \le k \le \ell - 1$. Then $\mathcal{A}_k$ is an essential k-arrangement and $L(\mathcal{A}_k) \simeq L(\mathcal{A}_{k+1})_{Y_k} \simeq L(\mathcal{A}_{X_k})$. It follows from Corollary 23.17 that Y_k is modular in $L(\mathcal{A}_{k+1})$. Thus X_k is modular in $L(\mathcal{A}_{k+1})$. It follows from Lemma 4.31 that $\mathcal{A}$ is supersolvable.

If $\mathcal{A}$ is supersolvable then we have a maximal chain (1) of modular elements. Choose coordinates so that $X_k = \ker(p_{\ell-1}p_{\ell-2}\ldots p_k)$ for $1 \le k \le \ell - 1$. Repeated application of Corollary 23.17 shows that each restriction map π_k is a strictly linear fibration. Thus $\mathcal{A}$ is fiber type. □

This provides a topological interpretation of the algebra factorization of a supersolvable arrangement $\mathcal{A}$ of rank r and maximal chain of modular elements $V = Y_0 < Y_1 < \ldots < Y_r = T$ proved in Theorem 10.4:

$$(\mathcal{K} + B_1) \otimes \ldots \otimes (\mathcal{K} + B_r) \simeq A(\mathcal{A}).$$

Here $\mathcal{B}_i = \mathcal{A}_{Y_i} \setminus \mathcal{A}_{Y_{i-1}}$ and $B_i = \sum_{H \in \mathcal{B}_i} \mathcal{K} a_H$. It follows from Theorem 10.4 that $\mathcal{A}$ is fiber type. Let $\mathcal{K} = \mathbb{Z}$ and let $b_i = |\mathcal{B}_i|$. The fiber F_k is the complex line with b_k points removed. It follows from Theorem 19.13 that the cohomology of the complement is the tensor product of the $H^*(F_k)$. The latter is isomorphic to $\mathbb{Z} + B_k$. The naturality of the construction is a consequence of Theorem 22.14.

Remark 23.19 *Let $\mathcal{A}$ be a real central 3-arrangement. Its complexification $\mathcal{A}_{\mathbb{C}}$ is fiber type if either*

(1) there is a direction so that all multiple points in the affine part of $\mathbf{d}\mathcal{A}$ are contained in lines of $\mathbf{d}\mathcal{A}$ parallel to this direction, or

(2) there is a multiple point P of $\mathbf{d}\mathcal{A}$ so that the pencil of lines of $\mathbf{d}\mathcal{A}$ which contains P contains all the remaining multiple points of $\mathbf{d}\mathcal{A}$, including those on the line at infinity.

These conditions are equivalent and correspond to different deconings of $\mathcal{A}$.

Condition (1) is convenient to see that the complexification of $\mathcal{A}$ defined by

$$Q(\mathcal{A}) = xyz(x - z)(x + z)(y - z)(y + z)$$

is fiber type. Figure 11 shows $\mathbf{d}\mathcal{A}$. Both vertical and horizontal directions satisfy (1). In fact $M(\mathcal{A}_{\mathbb{C}}) = C_3 \times C_3 \times C_1$. The B_3-arrangement provides a more interesting example. Figure 3 shows its decone. Both vertical and horizontal directions satisfy (1). The fiber is C_5 but the total space is not a product.

Condition (2) is convenient to see that the arrangement $\mathcal{B}$ defined in Example 15.22 is supersolvable. Figure 19 shows $\mathbf{d}\mathcal{B}$. It contains the two dotted lines. Let P be their intersection. These dotted lines were added precisely to satisfy condition (2). Thus $\mathcal{B}_{\mathbb{C}}$ is fiber type. It follows from Theorem 23.19 that $\mathcal{B}$ is supersolvable.

24 Related Research

Minimal Models

We start with work of Kohno [108], [110] and Falk [61] who use the theory of minimal models to establish a connection between $\pi_1(M(\mathcal{A}))$ and $H^*(M(\mathcal{A}))$. First we define the terminology and describe some general results of Sullivan [185] and Morgan [131]. Griffiths and Morgan [81] gave a detailed exposition of this work.

All vector spaces and algebras are over the rational numbers $\mathbf{Q}$. A differential graded algebra (DG algebra) A is a graded vector space $A = \oplus_{i\geq 0} A^i$ with a degree 1 coboundary operator $d : A^i \to A^{i+1}$ and a product $\wedge : A^i \otimes A^j \to A^{i+j}$. These satisfy:

(1) $d^2 = 0$,

(2) $d(x \wedge y) = dx \wedge y + (-1)^i x \wedge dy$ for $x \in A^i$,

(3) $x \wedge y = (-1)^{ij} y \wedge x$ for $x \in A^i$ and $y \in A^j$,

(4) $\wedge$ makes A an associative algebra with unit $1 \in A^0$.

If $A^0 = \mathbf{Q}$ then A is called connected. The augmentation ideal of a connected algebra is $\mathcal{I}(A) = \oplus_{i>0} A^i$. The quotient $I(A) = \mathcal{I}(A)/\mathcal{I}(A) \wedge \mathcal{I}(A)$ is the set of **indecomposable elements.**

Let V be a finite dimensional vector space. Let $\Lambda_r(V)$ be the graded-commutative algebra over V where $\deg v = r$ for $v \in V$. Thus $\Lambda_r(V)$ is a polynomial algebra if r is even and an exterior algebra if r is odd. Note that in this case $I(A) = V$.

Let B be a DG algebra. Let A be a DG subalgebra of B. The inclusion $A \subseteq B$ is called a **Hirsch extension** of degree r if for some V there is an isomorphism of graded-commutative algebras $B \simeq A \otimes \Lambda_r(V)$ and the differential d of B satisfies $dV \subseteq A^{r+1}$.

Let (M, d) be a DG algebra. It is called **minimal** if

(1) $M^0 = \mathbf{Q}$,

(2) there is an increasing filtration of DG subalgebras

$$\mathbf{Q} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

such that $M = \cup M_i$ and each inclusion $M_i \subseteq M_{i+1}$ is a Hirsch extension,

(3) d is decomposable: $dM \subseteq \mathcal{I}(M) \wedge \mathcal{I}(M)$.

Let A be a DG algebra. An i**–minimal** model for A is a map $\rho : M \to A$ of DG algebras such that:

(1) M is minimal,

(2) M is generated by elements of degree $\leq i$,

(3) $\rho^* : H^p(M) \to H^p(A)$ is an isomorphism for $p \leq i$ and injective for $p = i + 1$.

In case $i = \infty$, $\rho : M \to A$ is called a **minimal model** for A. It follows from the work of Sullivan [185] and Morgan [131] that if A is a connected DG algebra then an i–minimal model for A exists for each i, and it is unique up to isomorphism.

Suppose X is a connected polyhedron. Sullivan defined the DG algebra $A = A(X)$ of $\mathbf{Q}$-polynomial forms on X. The 1-minimal model $\mathcal{M} \to A$ is an increasing union of Hirsch extensions of degree 1

$$\mathbf{Q} = \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \ldots$$

$$\mathcal{M} = \cup_{n\geq 0} \mathcal{M}_n.$$

Let V_n be the degree 1 part of $\mathcal{M}_n$. The differential in $\mathcal{M}_n$ is determined by its restriction to V_n. Since $\mathcal{M}$ is minimal we have:

$$d|_{V_n} : V_n \to V_n \wedge V_n.$$

The Lie algebra $\mathcal{L}_n$ dual to $\mathcal{M}_n$ has underlying vector space V_n^*. The bracket is dual to $d|_{V_n}$

$$[\,,\,] : V_n^* \wedge V_n^* \to V_n^*.$$

The inclusions $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$ give rise to maps $\mathcal{L}_n \leftarrow \mathcal{L}_{n+1}$. Induction shows that the $\mathcal{L}_n$ are nilpotent. This constructs from the filtration of the 1-minimal model $\mathcal{M}$ a tower of nilpotent Lie algebras:

$$0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \cdots$$

Let G be a finitely presented group. The **lower central series** G_n of G is defined by setting $G_0 = G$ and $G_n = [G_{n-1}, G]$ for $n \geq 1$. Here $[G_k, G]$ denotes the subgroup generated by elements of the form $xyx^{-1}y^{-1}$ with $x \in G_k$, $y \in G$. The quotients G_{n-1}/G_n are finitely generated abelian groups. Let $\phi_n(G) = \mathrm{rank}(G_{n-1}/G_n)$. The quotients G/G_n are nilpotent groups. By a construction of Malcev, see [81, pp.142-145], it is possible to "tensor" these nilpotent groups by $\mathbf{Q}$ and use the central extensions

$$0 \to G_{n-1}/G_n \to G/G_n \to G/G_{n-1} \to 1$$

to define a Lie algebra structure on $(G/G_n) \otimes \mathbf{Q}$, called $\mathcal{L}_n(G)$. If $G = \pi_1(X)$ this leads to Sullivan's theorem [185], [131, (5.11)], [81, p.145].

Theorem 24.1 *Let X be a connected polyhedron, let $\rho : \mathcal{M} \to A(X)$ be a 1-minimal model, and let*

$$0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \cdots$$

be the tower of dual nilpotent Lie algebras. Let $G = \pi_1(X)$. Then $\mathcal{L}_n \simeq \mathcal{L}_n(G)$ for $n \geq 0$.

Since $\mathcal{M}_n$ is a degree 1 Hirsch extension of $\mathcal{M}_{n-1}$, we have $\mathcal{M}_n \simeq \mathcal{M}_{n-1} \otimes \Lambda_1(W_n)$ for some vector space W_n. The following is a direct consequence of Theorem 24.1.

Corollary 24.2 *Let X be a connected polyhedron with finitely generated rational cohomology. Let $\rho : \mathcal{M} \to A(X)$ be a 1-minimal model and let $G = \pi_1(X)$. Then $\phi_n(G) = \dim W_n$.*

Now suppose $\mathcal{A}$ is an arrangement and $M = M(\mathcal{A})$. Since M is a formal space in the sense of Sullivan [185, p.315], we may replace the algebra $A(M)$ of $\mathbf{Q}$-polynomial forms on M with the algebra $A = A(\mathcal{A})$ defined in section 9. We may view A as a DG algebra with zero differential so $H^*(A) = A$. Let $\rho : \mathcal{M} \to A$ be a 1-minimal model.

Definition 24.3 *Call $\mathcal{A}$ a* **rational** *$K(\pi, 1)$-arrangement if $\rho^* : H^*(\mathcal{M}) \to A$ is an isomorphism.*

Falk [61] and Kohno [114] used different methods to prove the following:

Theorem 24.4 *Let $\mathcal{A}$ be a rational $K(\pi,1)$-arrangement and write $\phi_n = \phi_n(\pi_1(M))$. Then*

$$\prod_{n\geq 1}(1-t^n)^{\phi_n} = \mathrm{Poin}(M,-t). \quad \square$$

This formula is called the LCS (lower central series) formula. It connects the ranks of the successive quotients in the lower central series of the fundamental group of M with the Poincaré polynomial of M. It is natural to ask for the largest class of arrangements for which the LCS formula is valid. Falk and Randell [65] showed that the LCS formula holds for fiber type arrangements. It is also known to hold for certain reflection arrangements and to be false for some reflection arrangements.

Discriminantal Arrangements

Next we describe work of Manin and Schechtman [124]. They constructed a family of arrangements which generalize the braid arrangements. Let $W = \mathbb{K}^k$ and let $\mathcal{A}^0 = \{H_1^0, \ldots, H_n^0\}$ with $n > k$ be a general position arrangement in W. Let $U(n,k)$ be the set of k-arrangements $\mathcal{A} = \{H_1, \ldots, H_n\}$ in W which satisfy:

(1) H_i is parallel to H_i^0 for $1 \leq i \leq n$,

(2) $\mathcal{A}$ is a general position arrangement.

Manin and Schechtman showed that $U(n,k)$ is itself the complement of an arrangement. Let $V = \mathbb{K}^n$ be the space of parallel translations of the hyperplanes of $\mathcal{A}^0$. Denote by $C(n,a)$ the set of subsets of $\{1,\ldots,n\}$ of cardinality a. For $K \in C(n,a)$ let D_K be the set of parallel translations $(H_1,\ldots,H_n) \in V$ such that $\cap_{i\in K} H_i \neq \emptyset$. If $|K| \leq k$ then $D_K = V$ and if $|K| \geq k+1$ then $\mathrm{codim} D_K = |K| - k$. In particular for $J \in C(n,k+1)$ the set D_J is a hyperplane in V and these hyperplanes are pairwise distinct. Let $\mathcal{B}(n,k) = \{D_J \mid J \in C(n,k+1)\}$. The arrangement $\mathcal{B}(n,1)$ is the braid arrangement. Manin and Schechtman called $\mathcal{B}(n,k)$ a **discriminantal arrangement**. In our terminology they proved that

$$U(n,k) = M(\mathcal{B}(n,k)).$$

Although $\mathcal{B}(n,k)$ also depends on $\mathcal{A}^0$, its combinatorial properties do not. Let $L(n,k) = L(\mathcal{B}(n,k))$ and let $\ell = n-k$. It is easy to see that $L(n,k)$ has a unique maximal element $D_{(1,\ldots,n)}$ of codimension ℓ and hence $\mathrm{r}(\mathcal{B}(k+\ell,k)) = \ell$. In fact we may identify $D_{(1,\ldots,n)}$ with W. Assume that $H_1^0,\ldots,H_k^0$ contain the origin of W. For every $w \in W$ there is a parallel translation such that every hyperplane contains the endpoint of w. Thus for $\ell = 1$ and $\ell = 2$ the lattices $L(k+1,k)$ and $L(k+2,k)$ are easy to describe.

Manin and Schechtman gave the following description of $L(k+3,k)$. For $J = (1,\ldots,k+3)-(i,j)$ write $D_J = (i,j)$. For $K = (1,\ldots,k+3)-(i)$ write $D_K = (i)$. If i,j,l,m are distinct indices write $(ij,lm) = (i,j)\cap(l,m)$. The hyperplanes of $\mathcal{B}(k+3,k)$ are the $(k+2)(k+3)/2$ sets (i,j). The elements of $L(k+3,k)$ of rank 2 are the $(k+3)$ sets (i) and the $k(k+1)(k+2)(k+3)/8$ sets (ij,lm). Each (i) is contained in $(k+2)$ hyperplanes and

each (ij,lm) is contained in two hyperplanes. Let T denote the unique maximal element of rank three. This gives:

$$\begin{aligned} \mu(V) &= 1, \\ \mu((i,j)) &= -1, \\ \mu((i)) &= k+1, \\ \mu((ij,lm)) &= 1, \\ \mu(T) &= -(1/8)k(k+3)(k^2+3k+6)-1. \end{aligned}$$

Proposition 24.5 *The Poincaré polynomial of $\mathcal{B}(k+3,k)$ is:*

$$\pi(\mathcal{B}(k+3,k),t) = 1 + (1/2)(k+2)(k+3)t + (1/8)(k+1)(k+3)(k^2+2k+8)t^2$$
$$+((1/8)k(k+3)(k^2+3k+6)+1)t^3.$$

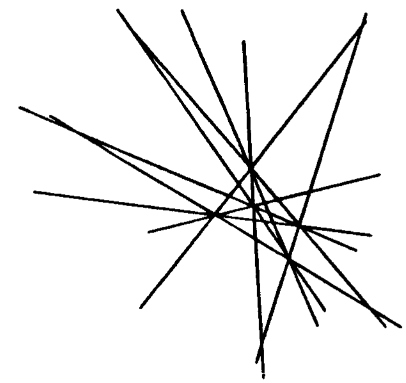


Figure 38: The arrangement $C^*(5)$

The first nontrivial cases are the arrangements $\mathcal{B}(k+3,k)$ for $k \geq 2$. It is immediate from the description above that they may be visualized as follows. Let $C^*(n)$ be an affine real 2-arrangement consisting of n points in the plane labeled (i) for $1 \leq i \leq n$ with the following properties:

(1) no three points are collinear,

(2) if $(i),(j),(l),(m)$ are distinct points then the line through $(i),(j)$ is not parallel to the line through $(l),(m)$.

Let $C(n)$ be the central 3-arrangement obtained by embedding $C^*(n)$ as the affine set $z=1$ and coning over the origin. (This is not the cone over $C^*(n)$ in the sense of Definition 2.15. The defining polynomial of $C(n)$ is the homogenization of the defining polynomial of $C^*(n)$. The difference is that here we do not add the hyperplane $\ker(x_0)$.)

It follows from the description that

$$\mathcal{B}(k+3,k) \simeq \mathcal{C}(k+3) \times \Phi_k.$$

The arrangement $\mathcal{C}^*(5)$ is illustrated in Figure 38. It is not known whether the arrangements $\mathcal{B}(n,k)$ are $K(\pi,1)$.

Proposition 24.6 *The discriminantal arrangements $\mathcal{B}(k+3,k)$ are not free for $k \geq 2$.*

Proof. For $k = 1$ we have a braid arrangement which is free. Write $\mathcal{B}_k = \mathcal{B}(k+3,k)$. It follows from the Factorization Theorem 18.21 that it suffices to show that if there exist integers a, b such that

$$\pi(\mathcal{B}_k, t) = (1+t)(1+at)(1+bt)$$

then $k = 1$. If we factor the Poincaré polynomial given in Proposition 24.5 we get $\pi(\mathcal{B}_k) = (1+t)p_k(t)$ where

$$p_k(t) = 1 + (1/2)(k^2+5k+4)t + (1/8)(k^4+6k^3+15k^2+18k+8)t^2.$$

The discriminant of $p_k(t)$ is

$$D(k) = -(1/4)k(k+1)(k^2+k-4).$$

Thus $D(1) = 1$ and $D(k) < 0$ for $k \geq 2$. □

Alexander Duality

The algebra $A(\mathcal{A})$ describes the cohomology of $M(\mathcal{A})$. Its classes are "torical" in the sense that they are products of one-dimensional classes. The elements of the algebra $B(\mathcal{A})$ are "spherical" in the sense of Theorem 17.25. It is natural to ask whether these classes also have a topological interpretation.

Consider the unit sphere $S^{2\ell-1} \subset V$ and let $\hat{M} = S^{2\ell-1} \cap M$, $\hat{N} = S^{2\ell-1} \cap N$. Clearly $\hat{M}$ is a strong deformation retract of M. Alexander duality [179, p.296] in the compact $(2\ell-1)$-manifold $S^{2\ell-1}$ for the compact polyhedron $\hat{N}$ gives:

$$H_{q+1}(S^{2\ell-1}, \hat{M}) \simeq H^{2\ell-q-2}(\hat{N}).$$

Thus for $1 \leq q \leq 2\ell - 3$ we have

$$H_q(\hat{M}) \simeq H^{2\ell-q-2}(\hat{N}).$$

This gives rise to a linking pairing in $S^{2\ell-1}$. Falk [64] interpreted this as a geometric linking. He constructed an embedding of the complex $\mathsf{F}(\mathcal{A})$ in $\hat{N}$ which induces isomorphism in homology and in homotopy through dimension $(\ell - 2)$. It follows from Theorems 17.20 and 17.25 that this is a geometric representation of $\mathcal{H}_\ell(\mathcal{A}) = B_\ell(\mathcal{A})$. He also constructed embeddings of ℓ tori in $\hat{M}$ which represent the ℓ-dimensional homology of $\hat{M}$. Then he showed that these classes link appropriately.

Theorem 24.7 *Suppose $S = (H_1, \ldots H_\ell)$ is independent so the cycle*

$$z_S = \sum_{\pi \in Sym(\ell)} (-1)^{\ell-1}(\operatorname{sign}\pi)(H_{\pi 1}, H_{\pi 1} \cap H_{\pi 2}, \ldots, H_{\pi 1} \cap \cdots \cap H_{\pi(\ell-1)})$$

is a generator of $\tilde{H}_{\ell-2}(\hat{N})$. Let $\tau_S \in H_\ell(\hat{M})$ be the algebraic dual of $\omega_S \in H^\ell(\hat{M})$. The linking number of z_S and τ_S is ± 1. □

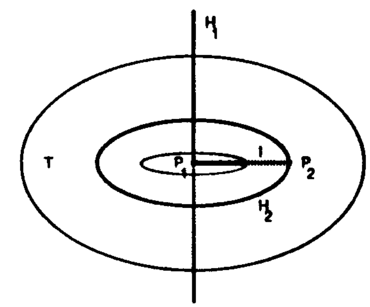


Figure 39: Falk's Linking

The case $\ell = 2$ is illustrated in Figure 39. Let $H_i = \ker(x_i)$ for $i = 1, 2$. Stereographic projection from $(0,1)$ sends S^3 onto R^3. The image of $\hat{N}$ consists of the vertical axis and the unit circle in the horizontal plane. The embedded torus

$$T = \{(x_1, x_2) \in S^3 \mid |x_1| = 1, |x_2| = 1\}$$

is a generator of $H_2(\hat{M}) \simeq \mathbb{Z}$. The Folkman complex consists of one point for each hyperplane, $\mathsf{F}(\mathcal{A}) = \{P_1, P_2\}$. Embed $\mathsf{F}(\mathcal{A}) \to \hat{N}$ by letting $P_i \in H_i \cap S^3$. The group $\tilde{H}_0(\hat{N}) \simeq \mathbb{Z}$ is generated by the cycle $z = (P_2 - P_1)$. The fact that T and z have linking number ± 1 follows because we can choose an embedded 1–chain I with $\partial I = z$ such that the geometric intersection $I \cap T$ is a single point.

Alexander duality in the open 2ℓ–manifold $V = \mathbb{C}^\ell = \mathbb{R}^{2\ell}$ for the closed polyhedron N and its complement M gives for $1 \leq q \leq 2\ell - 1$:

$$H_q(M) \simeq H_c^{2\ell-q-1}(N)$$

where H_c denotes cohomology with compact supports. It would be interesting to give a geometric interpretation of Alexander duality for all q.

The Milnor Fiber of a Generic Arrangement

Let $\mathcal{A}$ be a central complex arrangement defined by $Q = Q(\mathcal{A})$. Recall the Milnor fibration $Q : M \to \mathbb{C}^*$ and the Milnor fiber $F = Q^{-1}(1)$ from Proposition 19.2. It admits a free action by the cyclic group $G(n)$ of order $n = |\mathcal{A}|$. Let $\zeta = e^{2\pi i/n}$ be a generator of $G(n)$. The map $F \to F$ induced by multiplication by ζ is called the **monodromy** of the Milnor fiber. It follows that $B = M/\mathbb{C}^* = F/G(n)$. If we use cohomology with complex coefficients we get $[H^k(F)]^{G(n)} = H^k(B)$. This describes the 1-eigenspace of the monodromy. The eigenspaces of the other n-th roots of unity are harder to detect in general but we get lower bounds:

$$b_k(F) \geq b_k(B).$$

Recall general position arrangements from Definition 19.18, generic arrangements from Definition 19.21, and Hattori's Theorem 19.20. We compute the cohomology groups and the monodromy of the Milnor fiber of a generic arrangement following [141].

Lemma 24.8 *Let $\mathcal{B}$ be a general position $(\ell - 1)$-arrangement with $|\mathcal{B}| = n - 1$, where $n > \ell \geq 3$, and let $B = M(\mathcal{B})$. Then*
(1) $\pi_1(B)$ is free abelian of rank $n - 1$,
(2) $\pi_k(B) = 0$ for $2 \leq k \leq \ell - 2$,
(3) for $0 \leq k \leq \ell - 1$

$$b_k(B) = \binom{n-1}{k},$$

(4) the Euler characteristic of B is

$$\chi(B) = (-1)^{\ell-1}\binom{n-2}{\ell-2}.$$

Proof. We think of T^{n-1} as the $(n-1)$-dimensional hypercube with opposite faces identified. Then Hattori's subspace B_0 is obtained from T^{n-1} by removing cells in dimensions $n-1, n-2, \ldots, \ell$ corresponding to the interior of the cube and to pairs of faces of the cube. Thus B_0 has the same $(\ell-1)$-skeleton as T^{n-1}. The boundaries of the removed ℓ-cells give rise to nonvanishing homotopy classes but they are nullhomologous. This proves parts (1), (2), and (3). Part (4) follows from Lemma 24.9 below. □

Lemma 24.9 *For $m > k$ we have*

$$\binom{m-1}{k} = \binom{m}{k} - \binom{m}{k-1} + \cdots + (-1)^k\binom{m}{0}.$$

Proof. We use induction on k. The formula holds for $k = 1$. If we assume it for $k - 1$ then it follows for k because we have

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}. \quad \square$$

Theorem 24.10 *Assume that $\ell \geq 3$. Let $\mathcal{A}$ be a generic arrangement with total space $M = M(\mathcal{A})$. Let $p_M : M \to B$ be the restriction of the Hopf bundle. Let $Q : M \to \mathbb{C}^*$ be the Milnor fibration and let F be the associated Milnor fiber. Let $\xi = \exp(2\pi i/n)$. Let $h^* : H^*(F) \to H^*(F)$ be the monodromy induced by $h(z_1, \ldots, z_\ell) = (\xi z_1, \ldots, \xi z_\ell)$. Let $u = \binom{n-2}{\ell-2}$ and let $v = \binom{n-2}{\ell-1}$. Then*
(1) $\pi_1(F)$ is a free abelian group of rank $(n-1)$,
(2) $b_k(F) = b_k(B)$ for $0 \leq k \leq \ell - 2$ and hence the monodromy is trivial in this range,
(3) $b_{\ell-1}(F) = u + nv$,
(4) the characteristic polynomial of the monodromy on $H^{\ell-1}(F)$ is

$$\Delta_{\ell-1}(t) = (1-t)^u(1-t^n)^v.$$

Proof. If we think of the universal cover of T^{n-1} as $\mathbb{R}^{n-1}$ subdivided into hypercubes by the integer lattice then the universal cover of B is a giant "Swiss cheese" since in each hypercube the same cells are removed as the cells removed to get B_0. Since the restriction of the Hopf map $p_F : F \to B$ is an n-fold covering, F has the homotopy type of the union of n such hypercubes with the appropriate identifications. This proves (1) and (2). Part (3) follows from (2) together with the formula for the Euler characteristic of a covering $\chi(F) = n\chi(B)$ and the calculation of $\chi(B)$ in Lemma 24.8.4.

To prove (4) we use Milnor's work [128, pp.76–77]. The Weil ζ function of the mapping h can be expressed as a product

$$\zeta(t) = \prod_{d|n}(1-t^d)^{-r_d}$$

where the exponents $-r_d$ can be computed from the formula

$$\chi_j = \sum_{d|j} dr_d.$$

Here χ_j is the Lefschetz number of the mapping h^j, the j-fold iterate of h. Milnor showed that χ_j is the Euler characteristic of the fixed point manifold of h^j. Since h^j has no fixed points for $1 \leq j < n$ and $\chi(F) = n\chi(B)$ we conclude that

$$\text{(i)} \qquad \zeta(t) = (1-t^n)^{-\chi(B)}.$$

The zeta function can be expressed as an alternating product of polynomials

$$\text{(ii)} \qquad \zeta(t) = \Delta_0(t)^{-1}\Delta_1(t)\Delta_2(t)^{-1}\cdots\Delta_{\ell-1}(t)^{\pm 1}$$

where $\Delta_k(t)$ is the characteristic polynomial of the monodromy on $H^k(F)$. Part (4) now follows from equations (i), (ii), and the fact that $\Delta_k(t) = (1-t)^{b_k(F)}$ for $0 \leq k \leq \ell - 2$, which is a consequence of (2). □

Remark 24.11 *If $\ell = 2$ then $\pi_1(F)$ is a free group of rank $(n-1)^2$. Conclusions (2)–(4) of the theorem are valid.*

A central 2-arrangement is always generic. In this case Q has an isolated singularity at the origin. Thus $b_0(F) = 1$ and $b_1(F) = (n-1)^2$. In fact F has the homotopy type of a wedge of $(n-1)^2$ circles. In this case B is the complex line with $(n-1)$ points removed. Thus $b_0(B) = 1$ and $b_1(B) = n-1$. This agrees with assertions (2) and (3). The characteristic polynomial of the monodromy on $H^1(F)$ may be computed using the divisor formula in [129]:

$$\delta(h) = (nE_n - 1)^2 = n(n-2)E_n + 1 = (n-2)\Lambda_n + 1.$$

Thus $\Delta_1(t) = (1-t)(1-t^n)^{n-2}$, which agrees with (4).

Remark 24.12 *It is shown in [10] that the complexification of the D_3 arrangement defined by $Q = (x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$ has $b_1(F) = 7$ while $b_1(B) = 5$. Thus the Milnor fiber is a more subtle invariant of a non-generic arrangement.*

Arrangements of Subspaces

In a their book [76] Goresky and MacPherson considered real arrangements of affine subspaces of possibly various dimensions as an example of their general results in stratified Morse theory. We call these **subspace** arrangements. In [76, pp.237-244] they computed the groups $H_*(M(\mathcal{A}); \mathbb{Z})$ using the order complex of Definition 17.2. They partially order $L(\mathcal{A})$ by inclusion. Our statement of their theorem is adjusted to agree with our conventions. Here complex arrangements are also considered as real arrangements.

Definition 24.13 *Let $(\mathcal{A}, V)$ be a real arrangement of subspaces. Let $L = L(\mathcal{A})$ be the set of all intersections partially ordered by reverse inclusion with rank function $r(X) = \operatorname{codim} X$. Given $X, Y \in L$ with $X < Y$ recall the segment $[X,Y) = \{Z \in L \mid X \le Z < Y\}$ from* Definition 4.10. *Define the segment*

$$(X,Y) = \{Z \in L \mid X < Z < Y\},$$

Write $\mathsf{K}_* = \mathsf{K}(*)$ for the corresponding order complexes.

Thus if $\mathcal{A}$ is a complex arrangement then the definition of the poset $L(\mathcal{A})$ is the same as in Definition 4.1, but the rank function in Definition 4.2 is complex codimension while here it is real codimension.

Theorem 24.14 *The homology of the complement $M = M(\mathcal{A})$ is given by*

$$H_i(M; \mathbb{Z}) = \bigoplus_{X \in L} H^{r(X)-i-1}(\mathsf{K}_{[V,X)}, \mathsf{K}_{(V,X)}; \mathbb{Z})$$

with the convention that $H^{-1}(\emptyset, \emptyset) = \mathbb{Z}$. Thus V contributes a copy of $\mathbb{Z}$ to $H_0(M)$. □

Example 24.15 *Let $\mathcal{A}$ consist of a 3-plane and two 2-planes in $\mathbb{R}^5$.*

$$A_1: \; x_1 = x_5 = 0$$
$$A_2: \; x_1 = x_2 = x_3 = 0$$
$$A_3: \; x_3 = x_4 = x_5 = 0.$$

These subspaces are not in general position. The intersection poset contains in addition to $V = \mathbb{R}^5$ and the elements of $\mathcal{A}$ the subspaces:

$$B_1: \; x_1 = x_2 = x_3 = x_5 = 0$$
$$B_2: \; x_1 = x_3 = x_4 = x_5 = 0$$
$$O: \; x_1 = x_2 = x_3 = x_4 = x_5 = 0.$$

The table below indicates their contributions.

X	$r(X)$	$\mathsf{K}_{[V,X)}$	$\mathsf{K}_{(V,X)}$	b_0	b_1	b_2
V	0	$\emptyset$	$\emptyset$	1		
A_1	2	D^0	$\emptyset$		1	
A_2	3	D^0	$\emptyset$			1
A_3	3	D^0	$\emptyset$			1
B_1	4	D^1	∂D^1			1
B_2	4	D^1	∂D^1			1
O	5	D^2	C			

Here D^k is a k-disk with boundary ∂D^k, and C is an arc in ∂D^2. It follows that $\operatorname{Poin}(M(\mathcal{A}), t) = 1 + t + 4t^2$.

It is natural to ask which properties of hyperplane arrangements hold for subspace arrangements. The calculation of the fundamental group of the complement has not been carried out. Suppose $A \in \mathcal{A}$ has codimension d. We may form $\mathcal{A}'$ and $\mathcal{A}''$ as before. Let $M' = M(\mathcal{A}')$ and $M'' = M(\mathcal{A}'')$. Then M'' has a tubular neighborhood in M' which is a trivial d-plane bundle. The Thom isomorphism arguments of section 22 hold and we get the long exact sequence:

$$\cdots \to H^k(M') \xrightarrow{i^*} H^k(M) \xrightarrow{\delta} H^{k+1-d}(M'') \xrightarrow{\phi} H^{k+1}(M') \to \cdots$$

We showed in Theorem 22.11 that for arrangements of complex hyperplanes this sequence splits into short exact sequences. A similar splitting for arbitrary subspace arrangements would result in the formula

(1) $$\operatorname{Poin}(M,t) = \operatorname{Poin}(M',t) + t^{d-1}\operatorname{Poin}(M'',t).$$

Example 24.15 illustrates that not all choices of $A \in \mathcal{A}$ result in a similar splitting. The choice of $A_1 \in \mathcal{A}$ as the distinguished element gives $d = 3$, $\operatorname{Poin}(M',t) = 1 + 2t^2 + t^3$ and $\operatorname{Poin}(M'',t) = 1 + 3t$. Thus (1) does not hold in this case. It is interesting to note that (1) holds with either A_2 or A_3 as distinguished element.

Chapter VI

25 Reflection Arrangements

In the Introduction we defined reflections, reflecting hyperplanes, reflection groups and reflection arrangements. In this section we discuss some of their special properties.

Algebraic and Combinatorial Properties

We introduced the B_3 arrangement in Example 2.7. The corresponding reflection group is called the Coxeter group of type B_3. It is generated by reflections about the symmetry planes of the cube. The braid arrangement of Example 2.9 is also a reflection arrangement corresponding to the symmetric group. It is generated by the transpositions (i,j) with fixed set $H_{i,j} = \ker(x_i - x_j)$. These are examples of real reflection groups. They are also called Coxeter groups because finite irreducible real reflection groups were classified by Coxeter [42]. Shephard and Todd [176] classified finite irreducible complex reflection groups. Every real reflection group may be viewed as a complex reflection group. We give two examples of complex reflection groups which are not complexified real reflection groups.

Example 25.1 *The monomial groups* $G(r,1,\ell)$.

Let $r \geq 2$ be an integer and let $C(r)$ be the cyclic group of order r generated by $\xi = e^{2\pi i/r}$. The group $G(r,1,\ell)$ is the wreath product of $C(r)$ and $Sym(\ell)$. It consists of all $\ell \times \ell$ monomial matrices with entries in $C(r)$. Its reflection arrangement is defined by

$$Q = x_1 \dots x_\ell \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r).$$

Its lattice is the Dowling lattice $Q_\ell(\mathbb{Z}_r)$, see [55]. In particular $G(r,1,2)$ is generated by the matrices:

$$s_1 = \begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 25.2 *The Hessian configuration.*

Every nonsingular cubic in $\mathbb{C}P^2$ is projectively equivalent to a nonsingular cubic defined by $f(x,y,z) = x^3 + y^3 + z^3 - 3axyz$ where $a^3 \neq 1$ and $a \neq \infty$. The inflection points of a cubic are the solutions of $f = 0 = H(f)$, where $H(f)$ is the Hessian determinant of second partials. Direct calculation shows that for this family of curves the nine inflection points are independent of a. Set $\omega = e^{2\pi i/3}$. The projective coordinates of the nine inflection points are:

$$\begin{array}{ccc} (0,1,-1) & (0,1,-\omega) & (0,1,-\omega^2) \\ (1,0,-1) & (1,0,-\omega) & (1,0,-\omega^2) \\ (1,-1,0) & (1,-\omega,0) & (1,-\omega^2,0) \end{array}$$

These 9 points lie on 12 projective lines, which are the four degenerate cubics corresponding to the parameter values $a = \infty$ and $a^3 = 1$:

$$(1) \qquad x = 0,\ y = 0,\ z = 0,\ x + \omega^i y + \omega^j z = 0$$

where $i,j = 0,1,2$. These 12 projective lines meet in 12 additional points. This configuration of 12 lines and 21 points is called the Hessian configuration. Each of the first nine points is contained in four lines, so we refer to them as quadruple points. Each of the second twelve points is contained in two lines, so we refer to them as double points. Each line contains three quadruple points and two double points. Figure 40 illustrates the configuration.

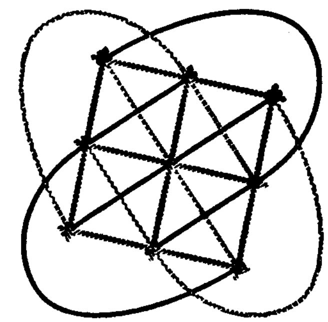


Figure 40: The Hessian configuration

The Hessian configuration has a distinguished history. For a complete account see Brieskorn and Knörrer's book [34, pp.289–305]. It appeared recently in the work of Hirzebruch [96]. Its group of symmetries was determined by Jordan in 1878 as a subgroup of $PGL(2,\mathbb{C})$ of order 216. It is generated by projective transformations of order 3 which leave one of the 12 projective lines pointwise fixed. In addition, the group contains 9 projective transformations of order 2 which fix projective lines. This group gives rise to two complex reflection groups. One is called G_{25} in the classification of Shephard and Todd [176]. It has order 648. It is generated by reflections of order 3. Its reflection arrangement is defined by the 12 lines of (1):

$$Q_{25} = xyz \prod_{i,j=0,1,2} (x + \omega^i y + \omega^j z).$$

The other is called G_{26}. It has order 1296 and contains additional reflections of order 2. Its reflection arrangement is defined by:

$$Q_{26} = Q_{25}(x^3 - y^3)(x^3 - z^3)(y^3 - z^3).$$

Let $G \subset GL(V)$ be any finite group. Let $\langle x, v\rangle = x(v)$ denote the usual pairing $V^* \times V \to \mathbb{K}$. Recall the S-module of derivations $\mathrm{Der}_S = \mathrm{Der}_{\mathbb{K}}(S)$ from Definition 2.18 and the S-module of differential 1-forms $\Omega_S = \Omega^1[V]$. If $v \in V$ let $D_v \in \mathrm{Der}_S$ be the derivation defined by $D_v(x) = \langle x, v\rangle$ for $x \in V^*$. The spaces S, Der_S, and Ω_S have G module structures.

Definition 25.3 *Let $g \in G$, $v \in V$, $a \in S$, $\theta \in \mathrm{Der}_S$, and $\omega \in \Omega_S$. Define the G module structure*

(1) in S by $(ga)(v) = a(g^{-1}v)$,
(2) in Der_S by $(g\theta)(a) = g(\theta(g^{-1}a))$,
(3) in Ω_S by $(g\omega)(\theta) = g(\omega(g^{-1}\theta))$.

Proposition 25.4 *The following transformation formulas hold:*

(1) $g(D_v a) = D_{gv}(ga)$, $d(ga) = g(da)$,
(2) $g(D_v) = D_{gv}$, $g(a\theta) = (ga)(g\theta)$,
(3) $g(a\omega) = (ga)(g\omega)$, $\langle g\omega, g\theta\rangle = g\langle\omega, \theta\rangle$. □

Let $R = S^G$ be the ring of G-invariant polynomials. Let Der_S^G be the R-module of G-invariant derivations, and let Ω_S^G be the R-module of G-invariant differential forms. If $G \subset GL(V)$ is a reflection group then Chevalley's theorem [39] describes R, and [148, Lemma 2.21] describes Der_S^G and Ω_S^G.

Theorem 25.5 *Let $G \subset GL(V)$ be a finite reflection group.*

(1) There exist homogeneous polynomials $f_1, \ldots, f_\ell$ such that $R = \mathbb{K}[f_1, \ldots, f_\ell]$.
(2) The R-module Ω_S^G is free of rank ℓ with basis $df_1, \ldots, df_\ell$.
(3) The R-module Der_S^G is free of rank ℓ with a basis of homogeneous elements. □

A set $\mathcal{F} = \{f_1, \ldots, f_\ell\}$ which satisfies (1) is called a set of **basic invariants** for G. The polynomials f_i are not unique but their degrees $d_i = \deg f_i$ are determined uniquely by G. They are called the basic degrees. The integers $m_i = d_i - 1$ are the **exponents** of G. It is customary to label them in increasing order:

$$m_1 \leq \ldots \leq m_\ell.$$

A homogeneous basis for Der_S^G is called a set of **basic derivations** $\Theta = \{\theta_1, \ldots, \theta_\ell\}$. Let $n_i = \mathrm{pdeg}\theta_i$. The integers n_i are called the **coexponents** of G. It is customary to label them in increasing order:

$$n_1 \leq \ldots \leq n_\ell.$$

If G is a Coxeter group then $m_i = n_i$. The following was shown in [187].

Theorem 25.6 *If $\mathcal{A}$ is a reflection arrangement then $D(\mathcal{A}) = S \otimes_R \mathrm{Der}_S^G$.* □

Thus we get from Theorems 25.5 and 25.6.

Theorem 25.7 *If $G \subset GL(V)$ is a finite reflection group then its reflection arrangement $\mathcal{A} = \mathcal{A}(G)$ is free with $\exp \mathcal{A} = \{n_1, \ldots, n_\ell\}$.* □

The group G acts on the lattice $L(\mathcal{A})$. The orbits of this action were computed for irreducible reflection groups in [145] and [146].

Proposition 25.8 *Let $G \subset GL(V)$ be a finite reflection group with reflection arrangement $\mathcal{A} = \mathcal{A}(G)$. For each $X \in L(\mathcal{A})$ with $r(X) = p$ there exist integers $b_1^X, \ldots, b_p^X$ such that*

$$\pi(\mathcal{A}^X, t) = \prod_{i=1}^{p}(1 + b_i^X t).$$ □

We demonstrate such a calculation for the group G_{25} defined in Example 25.2. For $v \in V$ let $G_v = \{g \in G \mid gv = v\}$ be the fixer of v. For $X \in L$ let $G_X = \cap_{v\in X} G_v$ be the fixer of X. The group G_{25} has five orbits on $L(\mathcal{A})$:

(i) V has fixer the identity group, called A_0,
(ii) the 12 hyperplanes form a single orbit with fixer $C(3)$,
(iii) the 12 lines which are in only two planes form an orbit with fixer $C(3) \times C(3)$,
(iv) the 9 lines which are in four planes form an orbit with fixer the group called G_4 in the classification,
(v) the origin is an orbit with fixer G_{25}.

The table below summarizes what we know about the lattice. The orbits are labeled by their fixers. The entry in row i column j is the number of elements in orbit j contained in an element in orbit i.

	A_0	$C(3)$	$C(3)^2$	G_4	G_{25}	b_1^X	b_2^X	b_3^X
A_0	1	12	12	9	1	1	4	7
$C(3)$		1	2	3	1	1	4	
$C(3)^2$			1	0	1	1		
G_4				1	1	1		
G_{25}					1			

It is clear how to compute b_1^X if X is a line in the orbit $C(3)^2$ or G_4. If X is a plane in the orbit $C(3)$ then $\mathcal{A}^X$ is a 2-arrangement with $|\mathcal{A}^X| = 5$ and hence $\pi(\mathcal{A}^X, t) = 1 + 5t + 4t^2 = (1+t)(1+4t)$. Finally, to compute $\pi(\mathcal{A}, t)$ note that if X is in the $C(3)$ orbit, then $\mu(X) = -1$, if X is in the $C(3)^2$ orbit then it is in two planes so $\mu(X) = 1$ and if X is in the G_4 orbit then it is in four planes so $\mu(X) = 3$. This allows calculation of $\mu(\{0\}) = -28$. Thus

$$\pi(\mathcal{A}, t) = 1 + 12t + 12t^2 + 27t^2 + 28t^3 = (1+t)(1+4t)(1+7t).$$

Proposition 25.8 motivated the conjecture that the restriction of a free arrangement is free. We would need to know not only that $\mathcal{A}^X$ is free but also that $\exp \mathcal{A}^X = \{b_1^X, \ldots, b_p^X\}$. In all the examples we have computed this is true, but we have been able to prove it in only one case [151]:

Theorem 25.9 *Let G be a finite Coxeter group with exponents $m_1 \le \ldots \le m_\ell$. Let $\mathcal{A} = \mathcal{A}(G)$ be its reflection arrangement. For $H \in \mathcal{A}$ the restriction $\mathcal{A}^H$ is free with*

$$\exp \mathcal{A}^H = \{m_1, \ldots, m_{\ell-1}\}. \quad \square$$

Unfortunately, induction may not be used in combination with Theorem 25.9 because in general $\mathcal{A}^H$ is not the arrangement of any Coxeter group. An even stronger conjecture says that every restriction of a reflection arrangement is inductively free. We have succeeded in showing this in many cases. The induction table below proves that the arrangement of the group G_{25} defined in Example 25.2 is inductively free. The reader willing to experiment with this example will soon discover how sensitive it is to the order of the hyperplanes.

$\exp \mathcal{A}'$	α_H	$\exp \mathcal{A}''$
0,0,0	$x+y+z$	0,0
0,0,1	$x+y+\omega z$	0,1
0,1,1	$x+y+\omega^2 z$	0,1
0,1,2	z	0,1
0,1,3	x	1,3
1,1,3	$x+\omega y+z$	1,3
1,2,3	$x+\omega y+\omega z$	1,3
1,3,3	$x+\omega y+\omega^2 z$	1,3
1,3,4	y	1,4
1,4,4	$x+\omega^2 y+z$	1,4
1,4,5	$x+\omega^2 y+\omega z$	1,4
1,4,6	$x+\omega^2 y+\omega^2 z$	1,4
1,4,7		

The $K(\pi,1)$ Problem

We close this section with a discussion of the topology of the complements of reflection arrangements. Brieskorn [33] generalized the constructions given in Remark 19.8 for the braid space. Let W be a finite Coxeter group and let $(\mathcal{A}, V)$ be its complexified reflection arrangement. Let $M = M(\mathcal{A})$. Let $p : V \to V/W$ be the orbit map. Note that W acts freely in M. Let $B = M/W$. The branch locus of p is called the discriminant locus. There is a natural embedding of B into $\mathbb{C}^\ell$ as follows. Let $\mathcal{F} = \{f_1, \ldots, f_\ell\}$ be a set of basic invariants for W. The map $\tau : V/W \to \mathbb{C}^\ell$ defined by

$$\tau(Wv) = (f_1(v), \ldots, f_\ell(v))$$

is a bijection. Since every reflection has order two, the polynomial $Q^2 = \prod_{H\in\mathcal{A}} \alpha_H^2$ is an invariant. We may define a polynomial $\Delta(T_1, \ldots, T_\ell; \mathcal{F})$, called the **discriminant**, in the indeterminates $T_1, \ldots, T_\ell$ and depending on $\mathcal{F}$ by:

$$\Delta(f_1, \ldots, f_\ell; \mathcal{F}) = \prod \alpha_H^2.$$

The discriminant locus

$$D = \{(z_1, \ldots, z_\ell) \in \mathbb{C}^\ell \mid \Delta(z_1, \ldots, z_\ell; \mathcal{F}) = 0\}$$

is the image under τ of N/W. Let $\pi = \tau p$. Then we have the commutative diagram:

$$\begin{array}{ccc} V & \supset & M = V \setminus N \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{C}^\ell & \supset & \tau B = \mathbb{C}^\ell \setminus D. \end{array}$$

Since W is a Coxeter group, it has a presentation with generators $s_1, \ldots, s_\ell$ and relations

(1) $\qquad s_i^2 = 1 \qquad 1 \le i \le \ell,$

(2) $\qquad s_i s_j s_i \ldots = s_j s_i s_j \ldots \qquad i \ne j$

where there are a certain number $q_{i,j} = q_{j,i} \ge 2$ terms on both sides of equation (2). Let A be the Artin group with generators $a_1, \ldots, a_\ell$ and relations

(3) $\qquad a_i a_j a_i \ldots = a_j a_i a_j \ldots \qquad i \ne j$

where there are $q_{i,j} = q_{j,i}$ terms on both sides of the equation. It is understood that the $q_{i,j}$ in (2) and (3) are the same. The natural surjection $A \to W$ gives rise to an exact sequence

(4) $\qquad 1 \to N_W \to A \to W \to 1.$

Brieskorn [32] proved that $\pi_1(B) = A$. It follows from Deligne's Theorem 19.15 that M is a $K(\pi,1)$ space. Thus (4) is the nontrivial part of the homotopy exact sequence of the fibration $p : M \to B$.

If G is a complex reflection group with reflection arrangement $(\mathcal{A}, V)$ the construction is similar. For each $H \in \mathcal{A}$ let e_H be the order of the cyclic subgroup fixing H. Then $\prod \alpha_H^{e_H}$ is a G-invariant polynomial. Given a set of basic invariants $\mathcal{F}$, we define the discriminant $\Delta(T_1, \ldots, T_\ell; \mathcal{F})$ by

$$\Delta(f_1, \ldots, f_\ell; \mathcal{F}) = \prod_{H\in\mathcal{A}} \alpha_H^{e_H}.$$

We define the orbit map $p : V \to V/G$, the bijection $\tau : V/G \to \mathbb{C}^\ell$, and the discriminant locus as before, and let $\pi = \tau p$ to get the commutative diagram:

$$\begin{array}{ccc} V & \supset & M = V \setminus N \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{C}^\ell & \supset & \tau B = \mathbb{C}^\ell \setminus D. \end{array}$$

It is conjectured that M and B are $K(\pi,1)$ spaces for all complex reflection groups. We observed in Proposition 19.6 that every 2-arrangement is $K(\pi,1)$. Nakamura [135] showed that if G is an imprimitive complex reflection group then its arrangement is $K(\pi,1)$. This holds in particular for the groups defined in Example 25.1. The fundamental groups $\pi_1(B)$ are not known for all G. They were computed for $\ell = 2$ by Bannai [15].

Regular complex polytopes were introduced by Shephard [175]. Their symmetry groups are irreducible complex reflection groups and we call them **Shephard groups**. There are finite irreducible complex reflection groups which are not Shephard groups. There are also Coxeter groups which are not Shephard groups. Coxeter [44] showed that every Shephard group admits a presentation with generating reflections $s_1, \ldots, s_\ell$ and relations

$$(5) \qquad s_i^{p_i} = 1 \qquad 1 \leq i \leq \ell,$$

$$(6) \qquad s_i s_j s_i \ldots = s_j s_i s_j \ldots \qquad i \neq j$$

where $p_i \geq 2$ are integers and there are a certain number $q_{i,j} = q_{j,i} \geq 2$ terms on both sides of equation (6). Not all complex reflection groups admit such presentations.

To each Shephard group G we associate a Coxeter group W by replacing p_i by 2 in (5). The group W is uniquely determined by G up to isomorphism. Thus G and W are both finite quotients of the same Artin group A. We call them an associated pair of groups and write (G, W). In general G and W are neither subgroups nor quotient groups of each other. When both groups are in consideration we use notation like M_G, M_W, d_i^G, d_i^W, etc. The following is a consequence of the main result of [149]:

Theorem 25.10 *Let G be a Shephard group and let W be the associated Coxeter group. There exist basic sets $\mathcal{F}_G = \{f_1^G, \ldots, f_\ell^G\}$ and $\mathcal{F}_W = \{f_1^W, \ldots, f_\ell^W\}$ such that their discriminant polynomials are equal:*

$$\Delta_G(T_1, \ldots, T_\ell; \mathcal{F}_G) = \Delta_W(T_1, \ldots, T_\ell; \mathcal{F}_W). \quad \square$$

Thus with this choice of coordinates G and W have the same discriminant loci, $D_G = D_W$ and hence $B_G = B_W$. Since it follows from Theorem 19.15 of Deligne that B_W is a $K(\pi, 1)$ space, we get:

Corollary 25.11 *If G is a Shephard group then $\mathcal{A}(G)$ is $K(\pi, 1)$.* $\square$

The group $G = G_{25}$ introduced in Example 25.2 is a Shephard group. The associated Coxeter group W is of type $A_3 = D_3$. We illustrate Theorem 25.10 with the pair (G_{25}, D_3). Maschke [125, p.326] constructed certain homogeneous polynomials C_6, C_9, C_{12}, C_{12} of degrees 6,9,12 and 12, where $C_{12} = Q_1$ defines $\mathcal{A}(G)$. Shephard and Todd [176, p.286] remarked that we may choose $\mathcal{F}_G = \{C_6, C_9, C_{12}\}$ as basic invariants for G. It follows from Maschke's work [125, p.326] that

$$C_{12}^3 \approx (432C_9^2 - C_6^3 + 3C_6C_{12})^2 - 4C_{12}^3.$$

Here $\approx$ means that the polynomials are equal up to a constant multiple. Since every reflection in G has order 3, $\prod \alpha_H^{e_H} = C_{12}^3$. Thus we have:

$$(7) \qquad \Delta_G(T_1, T_2, T_3; \mathcal{F}_G) = (432T_2^2 - T_1^3 + 3T_1T_3)^2 - 4T_3^3.$$

Next we compute the discriminant for $W = D_3$. Since D_3 is a subgroup of B_3 of index 2, we may think of W as the group of symmetries of one of the two tetrahedra inscribed in a cube, see Figure 41. Thus we may choose a basis x, y, z for V^* such that

$$\prod_{H \in \mathcal{A}(W)} \alpha_H = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2).$$

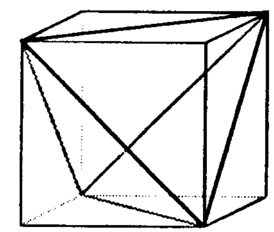


Figure 41: A tetrahedron in the cube

In this coordinate system $p_1 = x^2 + y^2 + z^2$, $p_2 = xyz$, and $p_3 = x^2y^2 + x^2z^2 + y^2z^2$ is a set of basic invariants for W. Consider a cubic polynomial with roots x^2, y^2, z^2. The formula for the discriminant of this cubic, expressed in terms of the elementary symmetric functions of the roots, gives the identity

$$\prod_{H \in \mathcal{A}(W)} \alpha_H^2 = (2p_1^3 - 9p_1p_3 + 27p_2^2)^2 - 4(p_1^2 - 3p_3)^3.$$

Let $f_1 = p_1$, $f_2 = p_2/4$ and $f_3 = p_1^2 - 3p_3$. Then $\mathcal{F}_W = \{f_1, f_2, f_3\}$ is also a set of basic invariants for W and we have

$$(8) \qquad \Delta_W(T_1, T_2, T_3; \mathcal{F}_W) = (432T_2^2 - T_1^3 + 3T_1T_3)^2 - 4T_3^3.$$

Comparing (7) and (8) illustrates Theorem 25.10 for the pair (G_{25}, D_3).

Appendix

26 Some Commutative Algebra

In this section we collect some definitions and facts in commutative algebra. These facts concern free modules, Krull dimension of rings and dimension of modules, and graded rings and modules.

Free Modules

Let R be a commutative Noetherian ring with unit and let M, N be R-modules. We assume that **all our modules are finitely generated**. The following lemma is one of the most fundamental results in linear algebra when R is a field.

Lemma 26.1 *Let* A *be an* $m \times n$*-matrix with entries in* R. *Suppose that all* n*-minors are zero (or* $m < n$*). Then there exists a non-zero vector* $\mathbf{a}$ *such that* $\mathsf{A}\mathbf{a} = 0$.

Proof. This is obvious when $\mathsf{A} = 0$. Let $r > 0$ be the number such that there is a non-zero r-minor of A and that all the $(r+1)$-minors are zero. (In linear algebra r is the rank of A.) We can assume without loss of generality that $n = r + 1$ and that the principal r-minor of A is not zero. Let $\hat{\mathsf{A}}$ be a square matrix consisting of the first $r + 1$ rows of A. The cofactor expansion formula gives

$$\mathsf{A}\ \mathrm{ad}(\hat{\mathsf{A}}) = 0,$$

where ad stands for the adjoint matrix. Since the $(r+1, r+1)$-entry of $\mathrm{ad}(\hat{\mathsf{A}})$ is not zero, take the $(r+1)$-column of $\mathrm{ad}(\hat{\mathsf{A}})$ as $\mathbf{a}$. □

Definition 26.2 *An* R*-module* M *is* **free** *if* M *has a linearly independent set of generators over* R. *We call such a set a* **basis**.

Proposition 26.3 *Let* M *be a free* R*-module with a basis* $\{f_1, \ldots, f_\ell\}$. *Let* $G = \{m_1, \ldots, m_k\}$ *be a finite subset of* M.
(1) If G *is linearly independent then* $k \le \ell$.
(2) If G *generates* M *then* $k \ge \ell$. *Moreover, if equality holds then* G *is a basis for* M.

Proof. There exists an $\ell \times k$-matrix A such that

$$[m_1, \ldots, m_k] = [f_1, \ldots, f_\ell]\mathsf{A}.$$

(1) If $k > \ell$ then by Lemma 26.1 there exists a non-zero vector $\mathbf{a}$ with $\mathsf{A}\mathbf{a} = \mathbf{0}$. Thus $[m_1, \ldots, m_k]\mathbf{a} = 0$, which implies that G is linearly dependent.
(2) Since G generates M, there exists a $k \times \ell$-matrix B such that

$$[f_1, \ldots, f_\ell] = [m_1, \ldots, m_k]\mathsf{B}.$$

If $k < \ell$ then there exists a non-zero vector $\mathbf{b}$ with $\mathsf{B}\mathbf{b} = \mathbf{0}$ by Lemma 26.1. Thus $[f_1, \ldots, f_\ell]\mathbf{b} = 0$. This contradicts the fact that $f_1, \ldots, f_\ell$ are linearly independent. Finally, assume that $k = \ell$. Then we get $\mathsf{AB} = \mathsf{I}_\ell$. Thus B is an invertible matrix and G is a basis. □

Definition 26.4 *Suppose* M *is a free* R*-module. It follows from* Proposition 26.3 *that the number of elements in a basis* $\{m_1, \ldots, m_\ell\}$ *for* M *over* R *is independent of the choice of basis. We call this number the* **rank** *of* M *and write* $\mathrm{rank}_R M = \ell$.

Krull Dimension

If $m \in M$ let

$$\mathrm{Ann}\ m = \mathrm{Ann}_R\ m = \{r \in R \mid rm = 0\}$$

be the **annihilator** of m. Let

$$\mathrm{Ann} M = \mathrm{Ann}_R M = \{r \in R \mid rM = 0\}$$

be the **annihilator** of M. Let $\mathrm{Spec} R$ be the set of all prime ideals of R. If I is an ideal of R let

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec} R \mid \mathfrak{p} \supseteq I\}.$$

If $\mathfrak{p} \in \mathrm{Spec} R$ let $R_\mathfrak{p}$ be the **localization** of R at $\mathfrak{p}$ and let $M_\mathfrak{p}$ be the **localization** of M at $\mathfrak{p}$ [126, 1.G]. Note that $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module.

Lemma 26.5 *Let* $\mathfrak{p} \in \mathrm{Spec} R$. *If* $m \in M$ *then* $m/1 \neq 0$ *in* $M_\mathfrak{p}$ *if and only if* $\mathfrak{p} \supseteq \mathrm{Ann}\ m$.

Proof. By definition $m/1 = 0$ if and only if there exists $s \in R \setminus \mathfrak{p}$ with $sm = 0$. □

Proposition 26.6 *Let* $\mathfrak{p} \in \mathrm{Spec} R$. *Then* $M_\mathfrak{p} \neq 0$ *if and only if* $\mathfrak{p} \supseteq \mathrm{Ann} M$.

Proof. Since M is finitely generated we may write $M = Rm_1 + \ldots + Rm_k$. Lemma 26.5 gives

$$\begin{aligned} M_\mathfrak{p} \neq 0 &\iff \text{there exists } i \text{ such that } m_i/1 \neq 0 \\ &\iff \text{there exists } i \text{ such that } \mathrm{Ann}\ m_i \subseteq \mathfrak{p} \\ &\iff \mathrm{Ann} M = \textstyle\bigcap_{i=1}^k \mathrm{Ann}\ m_i \subseteq \mathfrak{p}. \ \square \end{aligned}$$

Definition 26.7 *The* **support** *of* M *is defined by*

$$\mathrm{Supp} M = \mathrm{Supp}_R M = \{\mathfrak{p} \in \mathrm{Spec} R \mid M_\mathfrak{p} \neq 0\}.$$

The next result follows from Proposition 26.6 and Definition 26.7.

Proposition 26.8 $\mathrm{Supp} M = V(\mathrm{Ann} M)$. □

For the next result see [126, 3.D].

Theorem 26.9 *If $\wp \in \operatorname{Spec} R$ and $M' \to M \to M''$ is an exact sequence of R modules then $M'_\wp \to M_\wp \to M''_\wp$ is an exact sequence of $R_\wp$ modules.* □

Definition 26.10 *If $\wp \in \operatorname{Spec} R$ the* **height** *of $\wp$, $\operatorname{ht} \wp$, is the largest integer n for which there exists a chain $\wp = \wp_0 \supset \wp_1 \supset \ldots \supset \wp_n$ with $\wp_i \in \operatorname{Spec} R$ for $0 \le i \le n$.*

Definition 26.11 *The* **Krull dimension** *of R is defined by*

$$\operatorname{Krull} \dim R = \sup\{\operatorname{ht} \wp \mid \wp \in \operatorname{Spec} R\}.$$

Thus Krull dim$R$ is the largest integer n for which there exists a chain $\wp_0 \supset \wp_1 \supset \ldots \supset \wp_n$ with $\wp_i \in \operatorname{Spec} R$ for $0 \le i \le n$. For the next result see [126].

Proposition 26.12 *If $\varphi : R \to S$ is an epimorphism of rings then the correspondence $\wp \in \operatorname{Spec} S \mapsto \varphi^{-1}(\wp) \in \operatorname{Spec} R$ gives a bijection from Spec S to $\operatorname{Spec} R \cap V(\ker(\varphi))$.* □

Proposition 26.13 *Let I be an ideal of R. Then*

$$\operatorname{Krull} \dim(R/I) = \sup_{\wp \in V(I)} \operatorname{Krull} \dim(R/\wp).$$

Proof. This is clear from Definition 26.11 and Proposition 26.12. □

Definition 26.14 *If $M \neq 0$ then the* **dimension** *of M is defined by*

$$\dim M = \dim_R M = \operatorname{Krull} \dim(R/\operatorname{Ann} M).$$

Proposition 26.15 *If $M \neq 0$ then*

$$\dim M = \sup_{\wp \in \operatorname{Supp} M} \operatorname{Krull} \dim(R/\wp).$$

Proof. This is clear from Definition 26.14, Proposition 26.13, and Proposition 26.8. □

In (26.16)-(26.18) let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x_1, \ldots, x_\ell]$ where $x_1, \ldots, x_\ell$ are indeterminates. The next results are in [126, 14.A, Cor. and 14.H, Cor.3].

Proposition 26.16 *The ideal $(x_1, \ldots, x_k)$ is a prime ideal of height k for $1 \le k \le \ell$ and* Krull dim $R = \ell$. □

Proposition 26.17 *If $\wp \in \operatorname{Spec} R$ then* $\operatorname{Krull} \dim(R/\wp) = \ell - \operatorname{ht} \wp$. □

Proposition 26.18

$$\dim M = \ell - \min_{\wp \in \operatorname{Supp}(M)} \operatorname{ht} \wp.$$

Proof. This follows from Proposition 26.15 and Proposition 26.17. □

Graded Modules

For the rest of this section we assume that $R = \oplus_{p \in \mathbb{Z}} R_p$ is a **finitely generated $\mathbb{K}$-algebra with $R_p = 0$** for $p < 0$ **and $R_0 = \mathbb{K}$. Assume that M is a graded R-module,** $M = \oplus_{p \in \mathbb{Z}} M_p$. Let $R_+ = \oplus_{p > 0} R_p$. Then R_+ is a maximal ideal of R.

Theorem 26.19 *Let M be a free graded R-module of rank ℓ. A finite subset of M consisting of homogeneous elements is a basis if and only if it is a minimal set of generators.*

Proof. Suppose that $\{m_1, \ldots, m_k\}$ is a minimal set of generators. Let $\bar{M} = M/R_+M$. Then $\bar{M}$ is naturally a $\mathbb{K} = R_0 \simeq R/R_+$-vector space. Denote the class of x in $\bar{M}$ by $\bar{x}$. By Proposition 26.3 it is sufficient to show that $k \le \ell$. Assume $k > \ell$. Since ℓ elements (the classes of members of a basis) generate $\bar{M}$, $\bar{m}_1, \ldots, \bar{m}_k$ are linearly dependent over $\mathbb{K}$. Thus there exist $c_1, \ldots, c_k \in \mathbb{K}$, not all zero, such that

$$c_1\bar{m}_1 + \ldots + c_k\bar{m}_k = 0.$$

This implies that

$$c_1 m_1 + \ldots + c_k m_k = R_+ M.$$

There exist $r_1, \ldots r_k \in R_+$ such that

$$c_1 m_1 + \ldots + c_k m_k = r_1 m_1 + \ldots + r_k m_k.$$

Suppose $c_1 \neq 0$ and consider the homogeneous component of degree $\deg m_1$ in the last equation. Since $r_1 \in R_+$, $\deg r_1 m_1 > \deg c_1 m_1$. Thus

$$c_1 m_1 \in Rm_2 + \ldots + Rm_k.$$

This contradicts the minimality of $m_1, \ldots, m_k$. The converse is obvious. □

Theorem 26.20 *If a graded R-module M is free of rank ℓ then it has a homogeneous basis $\{m_1, \ldots, m_\ell\}$.*

Proof. Consider all homogeneous components of members of a basis. They form a set of homogeneous generators. Choose a minimal set of generators among them. By Theorem 26.19 it is a basis. □

Recall the Poincaré series of a finitely generated graded module from Definition 18.1.

Theorem 26.21 *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of finitely generated graded R-modules and S-homomorphisms of degree zero. Then

$$\operatorname{Poin}(M_1, t) - \operatorname{Poin}(M_2, t) + \operatorname{Poin}(M_3, t) = 0. \quad \square$$

Theorem 26.22 *Suppose $M \neq 0$. Then* Poin (M,t) *has a pole at $t = 1$ of order at most* $\dim_{\mathbb{K}} M$.

Proof. It is proved in [137, pp.346, Th.19] that $\dim M_n$ is a polynomial in n of degree $\dim M - 1$ for sufficiently large n. The weaker result here follows from the fact that every polynomial of degree $d-1$ in n is a linear combination of $\{n^{[i]}\}$, where

$$n^{[i]} = (n+1)(n+2)\dots(n+i) \quad 0 \le i \le d-1,$$

and

$$\frac{1}{(1-t)^i} = (-1)^{i-1}\sum_{n\ge 0} n^{[i-1]}t^n \quad 0 \le i \le d-1. \ \square$$

Example 26.23 *(1) Regard the polynomial ring $S = \mathbb{K}[x_1,\dots,x_\ell]$ as a graded $\mathbb{K}$-algebra by defining $\deg x_i = 1$ for $1 \le i \le \ell$. Then*

$$\text{Poin}\,(S,t) = (1-t)^{-\ell}.$$

(2) Let d be an integer. The graded R module $R(-d)$ is defined by shifting the grading of R by d so that $R(-d)_m = R_{m-d}$. Then

$$\text{Poin}\,(R(-d),t) = t^d\text{Poin}\,(R,t).$$

(3) Let d_i be integers for $1 \le i \le \ell$. If a graded R-module M is isomorphic to $\bigoplus_{i=1}^{\ell} R(-d_i)$ then by Theorem 26.21

$$\text{Poin}\,(M,t) = (t^{d_1} + \ldots + t^{d_\ell})\text{Poin}\,(R,t).$$

(4) If the set $m_1,\dots,m_\ell$ is a basis for the free graded R-module M with $m_i \in M_{d_i}$ then $M \cong \bigoplus_{i=1}^{\ell} R(-d_i)$ and thus by (3)

$$\text{Poin}\,(M,t) = (t^{d_1} + \cdots + t^{d_\ell})\text{Poin}\,(R,t).$$

Proposition 26.24 *Assume that the graded R-module M is free with a homogeneous basis $\{m_1,\dots,m_\ell\}$. Then the degrees $\{\deg m_1,\dots,\deg m_\ell\}$ (with multiplicity but neglecting the order) depend only on M.*

Proof. This follows from Example 26.23.4. $\square$

Proposition 26.25 *The following three conditions are equivalent:*

(1) M is finite dimensional over $\mathbb{K}$.

(2) $R_+ \subseteq \sqrt{\text{Ann} M}$.

(3) there exists a positive integer μ such that $(R_+)^\mu M = 0$.

Proof. Since R is Noetherian, (2) and (3) are equivalent.

(1) $\Rightarrow$ (2) There exists a positive integer μ such that $M_n = 0$ for $n \ge \mu$. Thus we have $(R_+)^\mu M \subseteq \oplus_{n\ge\mu} M_n = 0$.

(3) $\Rightarrow$ (1) Let $x_1,\cdots,x_m$ be a finite set of generators for M over R. Let $r_1,\dots,r_n$ be a basis for $\oplus_{p=0}^{\mu} R_p$ over $\mathbb{K}$. Then $\{r_i x_j \mid 1 \le i \le m, 1 \le j \le n\}$ generates M over $\mathbb{K}$. $\square$

References

[1] M. Aigner, "Combinatorial Theory," Grundlehren der Math. Wiss. **234**, Springer Verlag, Berlin/Heidelberg/New York, 1979.

[2] K. Aomoto, Un théorème du type de Matsushima–Murakami concernant l'intégrale des fonctions multiformes, *J. Math. Pures Appl.* **52** (1973), 1–11.

[3] K. Aomoto, Configurations and invariant theory of Gauss–Manin systems, in "Group representations and systems of differential equations," Adv. Studies in Pure Math. **4**, North Holland, 1984, pp. 165–179.

[4] K. Aomoto, Gauss–Manin connection of integral of difference products, *J. Math. Soc. Japan* **39** (1987), 191–208.

[5] V. I. Arnold, Braids of algebraic functions and cohomologies of swallowtails, *Uspehi Mat. Nauk* **23**(4) (1968), 247–248.

[6] V. I. Arnold, The cohomology ring of the colored braid group, *Mat. Zametki* **5** (1969), 227–231 : *Math. Notes* **5** (1969), 138–140.

[7] V. I. Arnold, Wave front evolution and equivariant Morse lemma, *Comm. Pure Appl. Math.* **29** (1976), 557–582.

[8] V. I. Arnold, Index of a singular point of a vector field, Petrovski–Oleinik inequality and mixed Hodge structures, *Funct. Anal. Appl.* **12** (1978), 1–14.

[9] E. Artal–Bartolo, Sur le premier nombre de Betti de la fibre de Milnor du cône sur une courbe projective plane et son rapport avec la position des points singuliers, preprint.

[10] E. Artin, Theorie der Zöpfe, *Hamb. Abh.* **4** (1925), 47–72.

[11] W. Arvola, The fundamental group of the complement of an arrangement of complex hyperplanes, PhD Thesis, University of Wisconsin, Madison, 1990.

[12] W. Arvola, The fundamental group of the complement of an arrangement of complex hyperplanes, *Topology*, to appear.

[13] W. Arvola, Complexified real arrangements of hyperplanes, preprint.

[14] K. Baclawski, Whitney numbers of geometric lattices, *Advances in Math.* **16** (1975), 125–138.

[15] E. Bannai, Fundamental groups of the spaces of regular orbits of the finite unitary reflection groups of dimension 2, *J. Math. Soc. Japan* **28** (1976), 447–454.

[16] H. Barcelo, On the action of the symmetric group on the free Lie algebra and the partition lattice, preprint.

[17] H. Barcelo and N. Bergeron, The Orlik–Solomon algebra on the partition lattice and the free Lie algebra, preprint.

[18] M. Barnabei, A. Brini, and G.-C. Rota, The theory of Möbius functions, *Russian Math. Surveys* **41**(3) (1986), 135–188.

[19] G. Barthel, F. Hirzebruch, and T. Höfer, "Geradenkonfigurationen und Algebraische Flächen," Vieweg Publishing, Wiesbaden, 1987.

[20] M. Bayer and B. Sturmfels, Lawrence polytopes, preprint.

[21] G. Birkhoff, "Lattice Theory," 3rd ed. Colloq. Publ. **25**, Amer. Math. Soc., 1967.

[22] G. Birkhoff, A determinant formula for the number of ways of coloring a map, *Ann. of Math.* **14** (1913), 42–46.

[23] J. Birman, "Braids, links, and mapping classes," Annals of Math. Studies **87** Princeton Univ. Press, 1974.

[24] A. Björner, On the homology of geometric lattices, *Algebra Universalis* **14** (1982), 107–128.

[25] A. Björner, Homotopy type of posets and lattice complementation, *J. Comb. Theory (A)* **30** (1981), 90–100.

[26] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G.M. Ziegler, "Oriented Matroids," Cambridge Univ. Press, to appear.

[27] A. Björner and J. W. Walker, A homotopy complementation formula for partially ordered sets, *Europ. J. Combinatorics* **4** (1983), 11–19.

[28] A. Björner, P. Edelman and G. Ziegler, Hyperplane arrangements with a lattice of regions, *Disc. and Comp. Geometry* **5** (1990), 263–288.

[29] A. Björner and G. M. Ziegler, Broken circuit complexes: factorizations and generalizations, preprint.

[30] R. Bott and L. W. Tu, "Differential forms in algebraic topology," GTM **82**, Springer Verlag, Berlin/Heidelberg/New York, 1982.

[31] N. Bourbaki, "Groupes et Algèbres de Lie," Chapitres 4,5 et 6, Hermann, Paris, 1968.

[32] E. Brieskorn, Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe, *Invent. math.* **12** (1971), 57–61.

[33] E. Brieskorn, Sur les groupes de tresses, in "Séminaire Bourbaki 1971/72," Lecture Notes in Math. **317**, Springer Verlag, Berlin/Heidelberg/New York, 1973, pp.21–44.

[34] E. Brieskorn and H. Knörrer, "Plane Algebraic Curves," Birkhauser, Boston, 1986.

[35] T. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972), 235–282.

[36] T. Brylawski, The broken circuit algebra, *Trans. Amer. Math. Soc.* **234** (1977), 417–433.

[37] R. C. Buck, Partition of space, *Amer. Math. Monthly* **50** (1943), 541–544.

[38] P. Cartier, Les arrangements d'hyperplans: un chapitre de géometrie combinatoire, in "Séminaire Bourbaki 1980/81," Lecture Notes in Math. **901**, Springer Verlag, Berlin/Heidelberg/New York, 1981, pp.1–22.

[39] C. Chevalley, Invariants of finite groups generated by reflections, *Amer. J. Math.* **77** (1955), 778–782.

[40] M. Cohen, Simplicial structures and transverse cellularity, *Annals of Math.* **85** (1967), 218–245.

[41] J. H. Conway and N. J. A. Sloane, "Sphere Packings, Lattices and Groups," Grundlehren der Math. Wiss. **290**, Springer Verlag, Berlin/Heidelberg/New York, 1988.

[42] H. S. M. Coxeter, Discrete groups generated by reflections, *Annals of Math.* **35** (1934), 588–621.

[43] H. S. M. Coxeter, The product of the generators of a finite group generated by reflections, *Duke Math. J.* **18** (1951), 765–782.

[44] H. S. M. Coxeter, "Regular Polytopes," 3rd ed., Dover, New York, 1973.

[45] H. S. M. Coxeter, "Regular Complex Polytopes," Cambridge Univ. Press, 1974.

[46] H. Crapo, The Möbius function of a lattice, *J. Comb. Theory* **1** (1966), 120–131.

[47] H. Crapo and G.-C. Rota, "Combinatorial Geometries," MIT Press, Cambridge, MA, 1971.

[48] R. Deheuvels, Homologie des ensembles ordonnés et des espaces topologiques, *Bull.Soc. Math. France* **90** (1962), 261–321.

[49] P. Deligne, Les immeubles des groupes de tresses généralisés, *Invent. math.* **17** (1972), 273–302.

[50] P. Deligne, Théorie de Hodge, *Publ. Math. IHES* **40** (1972), 5–57.

[51] P. Deligne and G. D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, *Publ. Math. IHES* **63** (1986), 5–89.

[52] T. tom Dieck and T. Petrie, Arrangements of lines with tree resolution, *Arch. Math.*, to appear.

[53] T. tom Dieck and T. Petrie, Homology planes and algebraic curves I, preprint.

[54] A. Dold, "Lectures on Algebraic Topology," Springer Verlag, Berlin/Heidelberg/New York, 1972.

[55] T. A. Dowling, A class of geometric lattices based on finite groups, *J. Comb. Theory (B)* **14** (1973), 61–86. Erratum, *ibid* **15** (1973), 211.

[56] P. Edelman, A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes, *Trans. Amer. Math. Soc.* **283** (1984), 617–631.

[57] H. Esnault, Fibre de Milnor d'un cône sur une courbe plane singulière, *Invent math.* **68** (1982), 477–496.

[58] E. Fadell and L. Neuwirth, Configuration spaces, *Math. Scand.* **10** (1962), 111–118.

[59] M. Falk, Geometry and topology of hyperplane arrangements, Ph.D. Thesis, University of Wisconsin-Madison, 1983.

[60] M. Falk, Combinatorics and the singularity defined by a product of linear forms, preprint.

[61] M. Falk, The minimal model of the complement of an arrangement of hyperplanes, *Trans. Amer. Math. Soc.* **309** (1988), 543–556.

[62] M. Falk, The cohomology and fundamental group of a hyperplane complement, in "Singularities," Contemporary Math., **90** Amer. Math. Soc. 1989, pp.55–72.

[63] M. Falk, On the algebra associated with a geometric lattice, *Advances in Math.*, to appear.

[64] M. Falk, A geometric duality for order complexes and hyperplane arrangements, preprint.

[65] M. Falk and R. Randell, The lower central series of a fiber-type arrangement, *Invent. math.* **82** (1985), 77–88.

[66] M. Falk and R. Randell, On the homotopy theory of arrangements, in "Complex Analytic Singularities," Advanced Studies in Pure Math. **8**, North Holland, 1987, pp. 101–124.

[67] M. Falk and R. Randell, The lower central series of generalized pure braid groups, in "Geometry and Topology," Lect. Notes in Pure and Appl. Math. **105**, Marcel Decker, New York, 1986, pp.103–108.

[68] M. Falk and R. Randell, Pure braid groups and products of free groups, in "Braids," Contemporary Math. **78**, Amer. Math. Soc., 1988, pp. 217–228.

[69] E. E. Floyd, On periodic maps and the Euler characteristic of associated spaces, *Trans. Amer. Math. Soc.* **72** (1952), 138–147.

[70] J. Folkman, The homology groups of a lattice, *J. Math. and Mech.* **15** (1966), 631–636.

[71] R. H. Fox and L. Neuwirth, The braid groups, *Math. Scand.* **10** (1962), 119–126.

[72] I. M. Gelfand, General theory of hypergeometric functions, *Soviet Math. Doklady* **33** (1986), 573–577.

[73] I. M. Gelfand and G. L. Rybnikov, Algebraic and topological invariants of oriented matroids, *Soviet Math. Doklady* **40** (1990), 148–152.

[74] I. M. Gelfand and V. V. Serganova, Combinatorial geometries and torus strata on homogeneous compact manifolds, *Russian Math. Surveys* **42**(2) (1987), 133–168.

[75] I. M. Gelfand and A. V. Zelevinsky, Algebraic and combinatorial aspects of the general theory of hypergeometric functions, *Funct. Anal. and Appl.* **20** (1986), 183–197.

[76] M. Goresky and R. MacPherson, "Stratified Morse Theory," Springer Verlag, Berlin/Heidelberg/New York, 1988.

[77] C. Greene, On the Möbius algebra of a partially ordered set, *Advances in Math.* **10** (1973), 177–187.

[78] C. Greene, An inequality for the Möbius function of geometric lattices, *Studies in Appl. Math.* **54** (1975), 71–74.

[79] C. Greene, The Möbius function of a partially ordered set, in "Ordered Sets," D. Reidel, 1982.

[80] C. Greene and T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs, *Trans. Amer. Math. Soc.* **280** (1983), 97–126.

[81] P. Griffiths and J. Morgan, "Rational homotopy theory and differential forms," Birkhauser, Boston, 1981.

[82] B. Grünbaum, "Convex polytopes," Interscience, New York, 1967.

[83] B. Grünbaum, Arrangements of hyperplanes, in "Proc. Second Lousiana Conf. on Combinatorics and Graph Theory," Baton Rouge 1971, pp. 41–106.

[84] B. Grünbaum, "Arrangements and spreads," CBMS Lecture Notes **10**, Amer. Math. Soc., 1972.

[85] B. Grünbaum and G. C. Shephard, Simplicial arrangements in projective 3-space, *Mitt. Math. Sem. Univ. Giessen* **166** (1984), 49–101.

[86] D. A. Gudkov, Topology of real projective algebraic manifolds, *Russian Math. Surveys* **29** (1974), 1–79.

[87] P. Hall, A contribution to the theory of groups of prime power order, *Proc. London Math. Soc. II. Ser.* **36** (1932), 39–95.

[88] P. Hall, The Eulerian functions of a group, *Quart. J. Math. Oxford* **7** (1936), 134–151.

[89] H. Hamm and Lê D. T., Un théorème de Zariski du type de Lefschetz, *Ann. sci. Éc. Norm. Sup.* **6** (1973), 317–366.

[90] A. Harnack, Ueber die Vieltheiligkeit der ebenen algebraischen Curven, *Math. Ann.* **10** (1876), 189–198.

[91] A. Hattori, Topology of $\mathbb{C}^n$ minus a finite number of affine hyperplanes in general position, *J. Fac. Sci. Univ. Tokyo* **22** (1975), 205–219.

[92] A. Hattori and T. Kimura, On the Euler integral representations of hypergeometric functions in several variables, *J. Math. Soc. Japan* **26** (1974), 1–16.

[93] H. Hendriks, Hyperplane arrangements of large type, *Invent. math.* **79** (1985), 375–381.

[94] D. Hilbert, Ueber die reeelen Züge algebraischer Curven, *Math. Ann.* **38** (1891), 115–138.

[95] M. Hirsch, "Differential Topology," Springer Verlag, Berlin/Heidelberg/New York, 1976.

[96] F. Hirzebruch, Arrangements of lines and algebraic surfaces, in "Arithmetic and Geometry," Vol. II, Progress in Math. **36**, Birkhauser, Boston, 1983, pp. 113–140.

[97] B. Hunt, Coverings and ball quotients with special emphasis on the 3-dimensional case, *Bonner Math. Schriften* **174**, 1986.

[98] M. Jambu, Algèbre d'holonomie de Lie et certaines fibrations topologiques, *C. R. Acad. Sci. Paris* **306** (1988), 479–482.

[99] M. Jambu, Sur l'algèbre d'Orlik–Solomon, *C.R. Acad. Sci. Paris* **307 Série I** (1988), 125–128.

[100] M. Jambu, Fiber-type arrangements and factorization properties, *Advances in Math.* **80** (1990), 1–21.

[101] M. Jambu and L. Leborgne, Fonction de Möbius et arrangements d'hyperplans, *C. R. Acad. Sc. Paris* **303** (1986), 311–314.

[102] M. Jambu and H. Terao, Arrangements libres d'hyperplans et treillis hyperrésolubles, *C. R. Acad. Sc. Paris* **296** (1983), 623–624.

[103] M. Jambu and H. Terao, Free arrangements of hyperplanes and supersolvable lattices, *Advances in Math.* **52** (1984), 248–258.

[104] M. Jambu and H. Terao, Arrangements of hyperplanes and broken circuits, in "Singularities," Contemporary Math., **90** Amer. Math. Soc. 1989, pp.147–162.

[105] R. Jozsa and J. Rice, On the cohomology of hyperplane complements, *Proc. Amer. Math. Soc.*, to appear.

[106] L. M. Kelly, A resolution of the Sylvester–Gallai problem of J. -P. Serre, *Discrete Comp. Geom.* **1** (1986), 101–104.

[107] T. Kohno, On the minimal algebra and $K(\pi, 1)$-property of affine algebraic varieties, preprint.

[108] T. Kohno, Differential forms and the fundamental group of the complement of hypersurfaces, in "Singularities," Proc. Symp. Pure Math. **40** Part 1, Amer. Math. Soc. 1983, pp.655–662.

[109] T. Kohno, On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces, *Nagoya Math. J.* **92** (1983), 21–37.

[110] T. Kohno, Série de Poincaré-Koszul associée aux groupes de tresses pures, *Invent. math.* **82** (1985), 57–75.

[111] T. Kohno, Homology of a local system on the complement of hyperplanes, *Proc. Japan Acad. Ser. A* **62** (1986), 144–147.

[112] T. KOHNO, Hecke algebra representations of braid groups and classical Yang–Baxter equations, in "Conformal field theory and solvable lattice models," Adv. Studies in Pure Math. **16** 1988, pp.255–269.

[113] T. KOHNO, Poincaré series of the Malcev completion of generalized pure braid groups, preprint.

[114] T. KOHNO, Rational $K(\pi,1)$ arrangements satisfy the LCS formula, preprint.

[115] T. KOHNO, Holonomy Lie algebras, logarithmic connections, and the lower central series of fundamental groups, in "Singularities," Contemporary Math., **90** Amer. Math. Soc. 1989, pp.171–182.

[116] V. S. KULIKOV, On the fundamental group of the complement of a hypersurface in $\mathbf{C}^n$, preprint.

[117] M. LAS VERGNAS, Convexity in oriented matroids, *J. Comb. Theory (B)* **29** (1980), 231–243.

[118] G. I. LEHRER, On the Poincaré series associated with Coxeter group actions on complements of hyperplanes, preprint.

[119] G. I. LEHRER, On hyperoctahedral hyperplane complements, preprint.

[120] G. I. LEHRER AND L. SOLOMON, On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes, *J. Algebra*, **104**(2) (1986), 410–424.

[121] F. LOESER, Arrangements d'hyperplans et sommes de Gauss, preprint.

[122] S. MACLANE, "Homology," Springer Verlag, Berlin/Heidelberg/New York, 1963.

[123] YU. I. MANIN AND V. V. SCHECHTMAN, Higher Bruhat orders related to the symmetric group, *Funct. Anal. and Appl.* **20** (1986), 148–150.

[124] YU. I. MANIN AND V. V. SCHECHTMAN, Arrangements of hyperplanes, higher braid groups and higher Bruhat orders, preprint.

[125] H. MASCHKE, Aufstellung des vollen Formensystems einer quaternären Gruppe von 51840 linearen substitutionen, *Math. Ann.* **33** (1888), 317–344.

[126] H. MATSUMURA, "Commutative Algebra," Benjamin/Cummings, Second Edition, 1980.

[127] E. MELCHIOR, Uber Vielseite der projektiven Ebene, *Deutsche Mathematik* **5** (1940), 461–475.

[128] J. MILNOR, "Singular points of complex hypersurfaces," Annals of Math. Studies **61**, Princeton University Press, 1968.

[129] J. MILNOR AND P. ORLIK, Isolated singularities defined by weighted homogeneous polynomials, *Topology* **9** (1970), 385–393.

[130] N. E. MNËV, On manifolds of combinatorial types of configurations and convex polyhedra, *Soviet Math. Dokl.* **32** (1985), 335–337.

[131] J. MORGAN, The algebraic topology of smooth algebraic varieties, *Publ. Math. IHES* **48** (1978), 137–204.

[132] A. F. MÖBIUS, Ueber eine besondere Art von Umkehrung der Reihen, *J. reine angew. Math.* **9** (1832), 105–123.

[133] J. MUNKRES, "Elements of Algebraic Topology," Addison–Wesley, 1984.

[134] I. NARUKI, The fundamental group of the complement of Klein's arrangement of twenty-one lines, *Topology and Appl.* **34** (1990), 167–181.

[135] T. NAKAMURA, A note on the $K(\pi,1)$-property of the orbit space of the unitary reflection group $G(m,\ell,n)$, *Sci. Papers College of Arts and Sciences, Univ. Tokyo* **33** (1983), 1–6.

[136] NGUYÊÑ VIÊT DŨNG, The fundamental group of the space of regular orbits of the affine Weyl groups, *Topology* **22** (1983), 425–435.

[137] G. D. NORTHCOTT, "Lessons on Rings, Modules and Multiplicities," Cambridge University Press, London/New York, 1968.

[138] P. ORLIK, Basic derivations for unitary reflection groups, in "Singularities," Contemporary Math., **90** Amer. Math. Soc. 1989, pp.211–228.

[139] P. ORLIK, "Introduction to arrangements," CBMS Lecture Notes **72**, Amer. Math. Soc. 1989.

[140] P. ORLIK, Complements of subspace arrangements, *J. Alg. Geom. and Sing.*, to appear.

[141] P. ORLIK AND R. RANDELL, The Milnor fiber of a generic arrangement, preprint.

[142] P. ORLIK AND L. SOLOMON, Combinatorics and topology of complements of hyperplanes, *Invent. math.* **56** (1980), 167–189.

[143] P. ORLIK AND L. SOLOMON, Unitary reflection groups and cohomology, *Invent. math.* **59** (1980), 77–94.

[144] P. ORLIK AND L. SOLOMON, Complexes for reflection groups, in "Algebraic Geometry," Lecture Notes in Math. **862**, Springer Verlag, Berlin/Heidelberg/New York, 1981, pp.193–207.

[145] P. ORLIK AND L. SOLOMON, Coxeter arrangements, in "Singularities," Proc. Symp. Pure Math. **40** Part 2, Amer. Math. Soc. 1983, pp.269–292.

[146] P. ORLIK AND L. SOLOMON, Arrangements defined by unitary reflection groups, *Math. Annalen* **261** (1982), 339–357.

[147] P. ORLIK AND L. SOLOMON, Arrangements in unitary and orthogonal geometry over finite fields, *J. Combinatorial Theory Ser.A* **38**(2) (1985), 217–229.

[148] P. ORLIK AND L. SOLOMON, The Hessian map in the invariant theory of reflection groups, *Nagoya Math. J.* **109** (1988), 1–21.

[149] P. ORLIK AND L. SOLOMON, Discriminants in the invariant theory of reflection groups, *Nagoya Math. J.* **109** (1988), 23–45.

[150] P. ORLIK, L. SOLOMON AND H. TERAO, Arrangements of hyperplanes and differential forms, in "Combinatorics and Algebra," Contemporary Math. **34**, Amer Math. Soc. 1984, pp.29–65.

[151] P. ORLIK, L. SOLOMON AND H. TERAO, On Coxeter arrangements and the Coxeter number, in "Complex Analytic Singularities", Advanced Studies in Pure Math. **8** North Holland, 1987, pp.461–477.

[152] I. PETROWSKY, On the topology of real plane algebraic curves, *Ann. of Math.* **39** (1938), 189–209.

[153] F. PHAM, "Introduction a l'Étude Topologique des Singularités de Landau," Mémorial des Sci. Math. **164** Gauthier-Villars, Paris, 1967.

[154] D. QUILLEN, Homotopy properties of the poset of non-trivial p-subgroups of a group, *Advances in Math.* **28** (1978), 101–128.

[155] R. RANDELL, On the topology of non-isolated singularities, in "Geometric Topology," Academic Press, New York, 1979, pp.445–473.

[156] R. RANDELL, On the fundamental group of the complement of a singular plane curve, *Quart. J. Math. Oxford* (2) **31** (1980), 71–79.

[157] R. RANDELL, The fundamental group of the complement of a union of complex hyperplanes, *Invent. math.* **69** (1982), 103–108. Correction *Invent. math.* **80** (1985), 467–468.

[158] R. RANDELL, Lattice-isotopic arrangements are topologically isomorphic, *Proc. Amer. Math. Soc.* **107** (1989), 555–559.

[159] S. ROBERTS, On the figures formed by the intercepts of a system of straight lines in a plane, and on analogous relations in space of three dimensions, *Proc. London Math. Soc.* **19** (1889), 405–422.

[160] V. A. ROKHLIN, Congruence modulo 16 in Hilbert's sixteenth problem, *Funct. Anal. Appl.* **6** (1972), 301–306.

[161] L. ROSE AND H. TERAO, Hilbert polynomials and geometric lattices, *Advances in Math.*, **84** (1990), 209–225.

[162] L. ROSE AND H. TERAO, A free resolution of the module of logarithmic forms of a generic arrangement, *J. Algebra* **136** (1991), 376–400.

[163] G.-C. ROTA, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitsrechnung* **2** (1964), 340–368.

[164] G.-C. ROTA, On the combinatorics of the Euler characteristic, in "Studies in Pure Math." presented to Richard Rado (L. Mirsky, ed.) Academic Press, London 1971, pp.221–233.

[165] K. SAITO, Regularity of Gauss-Manin connection of flat family of isolated singularities, in "Conference Notes," Centre de Mathématiques de l'Ecole Polytechnique, 1973.

[166] K. SAITO, On the uniformization of complements of discriminant loci, in "Conference Notes," AMS Summer Institute, Williamstown, 1975.

[167] K. SAITO, On a linear structure of a quotient variety by a finite reflexion group, *RIMS Kyoto preprint* **288**, 1979.

[168] K. SAITO, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect.IA Math.* **27** (1981), 265–291.

[169] K. SAITO, T. YANO AND J. SEKIGUCHI, On a certain generator system of the ring of invariants of a finite reflection group, *Communications in Algebra* **8**(4) (1980), 373–408.

[170] M. SALVETTI, Topology of the complement of real hyperplanes in $\mathbf{C}^N$, *Invent. math.* **88** (1987), 603–618.

[171] M. SALVETTI, Arrangements of lines and monodromy of plane curves, in "Algebraic Geometry," Lecture Notes in Math. **862**, Springer Verlag, Berlin/Heidelberg?New York, 1981, pp.107–192.

[172] M. SALVETTI, Arrangements of lines and monodromy of plane curves, *Compositio Math.* **68** (1988), 103–122.

[173] M. Salvetti, Generalized braid groups and self-energy Feynmann integrals, in "Braids," Contemporary Math. **78**, Amer. Math. Soc. 1988, pp.675–686.

[174] V. V. Schechtman and A. N. Varchenko, Arrangements of hyperplanes and Lie algebra homology, preprint.

[175] G. C. Shephard, Regular complex polytopes, *Proc. London Math. Soc.* (3)**2** (1952), 82–97.

[176] G. C. Shephard and J. A. Todd, Finite unitary reflection groups, *Canad. J. Math.* **6** (1954), 274–304.

[177] L. Solomon and H. Terao, A formula for the characteristic polynomial of an arrangement, *Advances in Math.* **64** (1987), 305–325.

[178] A. Sommese, On the density of ratios of Chern numbers of algebraic surfaces, *Math. Ann.* **268** (1984), 207–221.

[179] E. Spanier, "Algebraic Topology," McGraw-Hill, New York, 1966.

[180] R. P. Stanley, Modular elements of geometric lattices, *Algebra Universalis* **1** (1971), 214–217.

[181] R. P. Stanley, Supersolvable lattices, *Algebra Universalis* **2** (1972), 214–217.

[182] R. P. Stanley, Acyclic orientations of graphs, *Discrete Math.* **5** (1973), 171–178.

[183] R. P. Stanley, T-free arrangements of hyperplanes, in "Progress in Graph Theory," Academic Press, 1984, p.539.

[184] R. P. Stanley, "Enumerative Combinatorics," Vol. I, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[185] D. Sullivan, Infinitesimal computations in topology, *Publ. Math. IHES* **47** (1977), 269–331.

[186] H. Terao, Arrangements of hyperplanes and their freeness I, II *J. Fac. Sci. Univ. Tokyo* **27** (1980), 293–320.

[187] H. Terao, Free arrangements of hyperplanes and unitary reflection groups, *Proc. Japan Acad. Ser. A* **56** (1980), 389–392.

[188] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shephard–Todd–Brieskorn formula, *Invent. math.* **63** (1981), 159–179.

[189] H. Terao, On Betti numbers of complements of hyperplanes, *Publ. RIMS Kyoto Univ.* **17** (1981), 567–663.

[190] H. Terao, The exponents of a free hypersurface, in "Singularities" Proc. Symp. Pure Math. **40** Part 2, Amer. Math. Soc. 1983, pp.561–566.

[191] H. Terao, Discriminant of a holomorphic map and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **30** (1983), 379–391.

[192] H. Terao, Free arrangements of hyperplanes over an arbitrary field, *Proc. Japan Acad. Ser.A* **59** (1983), 301–303.

[193] H. Terao, The bifurcation set and logarithmic vector fields, *Math. Ann.* **263** (1983), 313–321.

[194] H. Terao, Modular elements of lattices and topological fibration, *Advances in Math.* **62** (1986), 135–154.

[195] H. Terao, Factorizations of Orlik-Solomon algebras, *Advances in Math.*, to appear.

[196] H. Terao and T. Yano, The duality of the exponents of free deformations associated with unitary reflection groups, in "Algebraic Groups and Related Topics," Advanced Studies in Pure Math. **6**, North Holland, 1985, pp.339–348.

[197] A. N. Varchenko, Combinatorics and topology of the disposition of affine hyperplanes in real space, *Funct. Anal. and Appl.* **21** (1987), 9–19.

[198] A. N. Varchenko, Morse theory on configurations of hyperplanes and periods of hypergeometric functions, preprint (in Russian).

[199] A. N. Varchenko, Multidimensional hypergeometric functions in conformal field theory, algebraic K-theory, algebraic geometry, preprint.

[200] A. N. Varchenko and I. M. Gelfand, Heaviside functions of configurations of hyperplanes, *Funct. Anal. Appl.* **21** (1987), 255–270.

[201] O. Ja. Viro, Progress in the topology of real algebraic varieties over the last six years, *Russian Math. Surveys* **41** (1986), 55–82.

[202] B. L. van der Waerden, "Modern Algebra," Ungar, New York, 1950.

[203] M. Ward, The algebra of lattice functions, *Duke Math. J.* **5** (1939), 357–371

[204] L. Weisner, Abstract theory of inversion of finite series, *Trans. Amer. Math. Soc.* **38** (1935), 474–484.

[205] J. E. Wetzel, On the division of the plane by lines, *Amer. Math. Monthly* **85** (1978), 648–656.

[206] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

[207] T. YANO AND J. SEKIGUCHI, The microlocal structure of weighted homogeneous polynomials associated with Coxeter systems, I *Tokyo J. Math.* **2**(2) (1979), 193–219; II *Tokyo J. Math.* **4**(1) (1981), 1–34.

[208] S. YUZVINSKY, Cohen-Macaulay seminormalizations of unions of linear subspaces, preprint.

[209] S. YUZVINSKY, Cohomology of local sheaves on arrangement lattices, *Proc. Amer. Math. Soc.*, to appear.

[210] S. YUZVINSKY, First two obstructions to the freeness of arrangements, preprint.

[211] T. ZASLAVSKY, "Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes," Memoirs Amer. Math. Soc. No. 154, 1975.

[212] T. ZASLAVSKY, Counting the faces of cut-up spaces, *Bull. Amer. Math. Soc.* **81** (1975), 916–918.

[213] T. ZASLAVSKY, Maximal dissections of a simplex, *J. Comb. Theory (A)* **20** (1976), 244–257.

[214] T. ZASLAVSKY, A combinatorial analysis of topological dissections, *Advances in Math.* **25** (1977), 267–285.

[215] T. ZASLAVSKY, Arrangements of hyperplanes; matroids and graphs, in "Proc. Tenth Southeastern Conf. on Combinatorics, Graph Theory and Computing (Boca Raton, 1979)," Vol. II pp. 895–911.

[216] T. ZASLAVSKY, The slimmest arrangements of hyperplanes: I. Geometric lattices and projective arrangements, *Geometriae Dedicata* **14** (1983), 243–259; II. Basepointed geometric lattices and Euclidean arrangements, *Mathematika* **28** (1981), 169–190.

[217] A. V. ZELEVINSKY, Geometry and combinatorics related to vector partition functions, preprint.

[218] G. M. ZIEGLER, Algebraic combinatorics of hyperplane arrangements, Ph.D. Thesis, MIT, 1987.

[219] G. M. ZIEGLER, The face lattice of hyperplane arrangements, *Discrete Math.* **73** (1989), 233–238.

[220] G. M. ZIEGLER, Multiarrangements of hyperplanes and their freeness, in "Singularities," Contemporary Math., **90** Amer. Math. Soc. 1989, pp.345–359.

[221] G. M. ZIEGLER, Matroid representations and free arrangements, *Trans. Amer. Math. Soc.*, to appear.

[222] G. M. ZIEGLER, Combinatorial construction of logarithmic differential forms, *Advances in Math.* **76** (1989), 116–154.

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