P.L. HOMEOMORPHIC MANIFOLDS ARE EQUIVALENT BY ELEMENTARY SHELLINGS

Udo Pachner

Dedicated to Professor Günter Ewald to his 60th birthday

Bericht Nr.126 März 1989

Preprint Nr. 126/1989

Ruhr-Universität Bochum, Institut für Mathematik
Universitätsstraße 150, 4630 Bochum 1

Abstract

Shellability of simplicial complexes has been a powerful concept in polyhydral theory, in p.l. topology and recently in connection with Cohen-Macaulay rings and toric varieties. It is well known that all 2-spheres and all boundary complexes of convex polytopes are shellable, but the analogous theorem fails for general simplicial balls and spheres. In this paper we study transformations of simplicial p.l. manifolds by elementary boundary operations (shellings and inverse shellings). As the main result we shall show that a simplicial p.l. manifold $\mathcal{M}$ can be transformed in any other simplicial p.l. manifold $\mathcal{M}'$ hommeomorphic to $\mathcal{M}$ using these elementary operations. The tools we need (which were partly not published in English) and related results are summarized. In the last part we study generalized shellings of totally strongly connected simplicial complexes and the effect on the face numbers of the complex.

1. Introduction

The concept of stellar subdivision and shellability has an interesting history going back to the 19th century. The early "proofs" of the Euler relation for convex polytopes (Schläfli, 1852) were based on the then unproved assumption that boundary complexes of convex polytopes are shellable (see Grünbaum [20]). This incompleteness was rectified 120 years later by Bruggesser and Mani [8]. Shellability then played a key role in the first complete proof of the upper-bound conjecture (Motzkin, 1957) by McMullen [29], which provides a tight upper bound on the number of faces of a convex d-polytope with n vertices.

The study of convex polytopes and polyhydral sets was stimulated since the early 1950's by many problems arising from linear programming. In the past 15 years the interest in convex polytopes and simplicial manifolds has advanced greatly by the developement of strong connections to Cohen-Maccaulay rings and toric varieties. The proof of McMullen's g-conjecture - the complete characterization of the face numbers of simplicial polytopes - is one of the fundamental results based on this theory (Billera/Lee [5], Stanley [43]). More detailed background and motivation is presented in the following sections.

2. Basic concepts

Let P be a convex polytope. The boundary complex of P is denoted by $\mathcal{B}(P)$ and $\mathcal{F}(P) := \mathcal{B}(P) \cup \{P\}$. For a single point p we write $\mathcal{F}(\{p\}) =: \bar{p}$. For more informations about polytopes the reader is referred to [20]. In the sequel T^d always denotes a d-dimensional simplex.

A finite simplicial complex $\mathcal{C}$ is defined in the usual way in an abstract sense. Nevertheless we also use notations and constructions arising from geometrical realizations of simplicial complexes. The members of $\mathcal{C}$ are the *faces* of $\mathcal{C}$ and dim A denotes the dimension of a face A of $\mathcal{C}$. $\mathcal{C}$ is a simplicial n-complex if n is the maximum dimension of its faces. We use the following notations:

$$\mathrm{st}(A;\mathcal{C}) := \{B \in \mathcal{C}\colon\ A \subseteq B\}$$ *"(open) star"*

$$\mathrm{clst}(A;\mathcal{C}) := \bigcup\{\mathcal{F}(B)\colon\ B \in \mathrm{st}(A;\mathcal{C})\}$$ *"(closed) star"*

$$\mathrm{ast}(A;\mathcal{C}) := \{B \in \mathcal{C}\colon\ B \cap A = \emptyset\}$$ *"antistar"*

$$\mathrm{link}(A;\mathcal{C}) := \mathrm{ast}(A;\mathcal{C}) \cap \mathrm{clst}(A;\mathcal{C})$$

$$\Delta_k(\mathcal{C}) := \{A \in \mathcal{C}\colon\ \dim A = k\}$$

$$\mathrm{skel}_k(\mathcal{C}) := \{A \in \mathcal{C}\colon\ \dim A \le k\}$$ *"k-skeleton"*

$$\mathrm{vert}(\mathcal{C}) := \Delta_1(\mathcal{C})$$ *"vertices"*

$$|\mathcal{C}| := \bigcup \mathcal{C}$$ *"underlying polyhydron (topological space)"*

The maximal faces of $\mathcal{C}$ are the *facets* of $\mathcal{C}$. $\mathcal{C}$ is *pure* provided all facets have the same dimension. A pure simplicial complex $\mathcal{C}$ is *strongly connected* provided every two facets F, F' of $\mathcal{C}$ can be linked together by a *path of facets* $F=F_0,\dots,F_r=F'$, that means $F_{i-1} \cap F_i$ is a common facet of F_{i-1}, F_i for $i=1,\dots,r$. A *missing face* of $\mathcal{C}$ is a simplex $D \notin \mathcal{C}$ with $\mathcal{B}(D) \subseteq \mathcal{C}$, $\dim D \ge 1$. A subcomplex $\mathcal{C}'$ of $\mathcal{C}$ is *full* in $\mathcal{C}$ provided $A \in \mathcal{C}$, $\mathrm{vert}(A) \subseteq \mathcal{C}'$ implies $A \in \mathcal{C}'$. A simplicial n-complex $\mathcal{M}$ is called a *simplicial n-ball, sphere or manifold* if $|\mathcal{M}|$ is a ball, a sphere or a manifold, respectively.

All balls, spheres, manifolds and homeomorphisms to be considered are piecewise linear.

Bd$(\mathcal{C})$ denotes the *boundary complex* of a pure simplicial n-complex $\mathcal{C}$. This is the subcomplex of $\mathcal{C}$ which has as facets those (n-1)-faces of $\mathcal{C}$ which are contained in only one facet of $\mathcal{C}$. The set of the *interior faces* of $\mathcal{C}$ is denoted by $\mathit{Int}(\mathcal{C}) := \mathcal{C} \setminus \mathit{Bd}(\mathcal{C})$. We use "$\cong$" for homeomorphic polyhedrons and "$\approx$" for isomorphic complexes. But, because additional isomorphisms are always allowed (and often necessary) we shall mostly write "=" instead of "$\approx$".

The *join* of simplicial complexes $\mathcal{C}, \mathcal{C}'$ is defined by $\mathcal{C}\cdot\mathcal{C}' := \{A\cdot A'\colon\ A \in \mathcal{C}, A' \in \mathcal{C}'\}$ where $A\cdot A' := A \cup A'$ if the complexes are considered as abstract complexes. A realization in euclidean space is given by the convex hull $A\cdot A' := \mathrm{conv}\,(A \cup A')$. Here it is always assumed that $|\mathcal{C}|, |\mathcal{C}'|$ are joinable (see [19,22]). This is, for instance, the case if $|\mathcal{C}|, |\mathcal{C}'|$ are embedded into disjoint affine subspaces containing no parallel lines. The join of subsets of joinable complexes is defined in the obvious way. We shortly write

$\mathcal{C}\cdot A$ instead of $\mathcal{C}\cdot\langle A\rangle$. But realize that one has to distinguish between the join of $\mathcal{C}$ with the empty simplex ($\mathcal{C}\cdot\langle\emptyset\rangle = \mathcal{C}$) and the join with the empty complex ($\mathcal{C}\cdot\emptyset = \emptyset$).

(2.1) DEFINITION. (1) Let $\mathcal{M}$ be a simplicial n-manifold, and let $F = A\cdot B$ be a facet of $\mathcal{M}$ such that $A \in \mathit{Int}(\mathcal{M})$, $\mathcal{B}(A)\cdot B \subseteq \mathit{Bd}(\mathcal{M})$ and $\dim A, \dim B \geq 0$. Then we call

$$\mathcal{M}' := \rho_{-F}\mathcal{M} := \mathcal{M} \setminus \mathcal{F}(A)\cdot B$$

an *(elementary) k-shelling* of $\mathcal{M}$ where $k := \dim B$.
The inverse operation is denoted by $\rho_{+F}\mathcal{M} := \rho_{-F}^{-1}\mathcal{M}$ and $\rho^{\pm}$ stands for an *elementary boundary operation* which is a shelling or an inverse shelling.
(2) For simplicial n-manifolds $\mathcal{M}, \mathcal{M}'$ we define:

$$\mathcal{M} \xrightarrow{sh} \mathcal{M}' :\Leftrightarrow \mathcal{M}' = \rho_r \;\ldots\; \rho_1\mathcal{M}$$

$$\mathcal{M} \approx_{sh} \mathcal{M}' :\Leftrightarrow \mathcal{M}' = \rho_r^{\pm} \;\ldots\; \rho_1^{\pm}\,\mathcal{M}$$

(3) For a simplicial n-ball $\mathcal{K}$ we say:

$$\mathcal{K} \text{ is } \textit{shellable} :\Leftrightarrow \mathcal{K} \xrightarrow{sh} \mathcal{F}(T^n)$$

A simplicial n-sphere $\mathcal{S}$ is called shellable if there exists a facet F of $\mathcal{S}$ such that $\mathcal{S} \setminus \{F\}$ is a shellable n-ball.

Remarks and additional notations. (1) It can happen that there exists a face $A \in \mathit{Int}(\mathcal{M})$ and different faces B_1, B_2 such that $\mathcal{B}(A)\cdot B_1, \mathcal{B}(A)\cdot B_2 \subseteq \mathit{Bd}(\mathcal{M})$ and $A\cdot B_1, A\cdot B_2$ are both facets of $\mathcal{M}$. But for every $B \in \mathit{Bd}(\mathcal{M})$ there exists at most one $A \in \mathit{Int}(\mathcal{M})$ such that $A\cdot B$ is a facet of $\mathcal{M}$ with $\mathcal{B}(A)\cdot B \subseteq \mathit{Bd}(\mathcal{M})$. Thus ρ_{-F} is uniquely determined by B and we write $\rho_{-F} =: \rho_B^-$. Conversely we write ρ_A^+ for an inverse elementary shelling. This implies that $A \in \mathit{Bd}(\mathcal{M})$ and $\mathrm{link}(A;\mathit{Bd}(\mathcal{M})) = \mathcal{B}(B)$ for a missing face B of $\mathcal{M}$.
(2) $\mathcal{M} \xrightarrow{sh} \mathcal{M}'$ as well as $\mathcal{M}' \approx_{sh} \mathcal{M}'$ imply $|\mathcal{M}| \cong |\mathcal{M}'|$.
obviously "$\approx_{sh}$" is an equivalence relation.

There is a strong connection between shellings and certain stellar operations.

(2.2) DEFINITION. Let $\mathcal{M}$ be a simplicial n-manifold and let $\emptyset \neq A \in \mathcal{M}$ such that $\mathrm{link}(A;\mathcal{M}) = \mathcal{B}(B)\cdot\mathcal{L}$, where $B \neq \emptyset$ is a simplex not contained in $\mathcal{M}$. Then we call

$$\varkappa_{(A,B)}\mathcal{M} := (\mathcal{M} \setminus A\cdot\mathcal{B}(B)\cdot\mathcal{L}) \cup \mathcal{B}(A)\cdot B\cdot\mathcal{L}$$

a ***stellar exchange***.

Remarks, examples and additional notations. (1) Clearly $\varkappa_{(A,B)}\mathcal{M}$ is again a simplicial n-manifold with $|\varkappa_{(A,B)}\mathcal{M}| \cong |\mathcal{M}|$.

Obviously $\varkappa^{-1}_{(A,B)} = \varkappa_{(B,A)}$ holds.

The equivalence of simplicial manifolds by stellar exchanges is denoted by "$\approx_{stex}$".

(2) In the case of dim B = 0, i.e. B = {b} is a (new) vertex, the operations $\varkappa_{(A,B)} =: \sigma_{(A,b)} =: \sigma_A$ are well known as *stellar subdivisions* (see [19,22]). Here it is $A \in Bd(\mathcal{M})$ or $A \in Int(\mathcal{M})$ respectively, depending on whether $\mathcal{L}$ is a ball or a sphere. Conversely $\varkappa^{-1}_{(A,B)} = \sigma^{-1}_B$ is an *inverse stellar subdivision* in the case of dim A = 0.

Clearly the definition of stellar subdivisions and their inverses are still applicable to arbitrary simplicial complexes (and even to more general complexes). Conform with the former notations $\mathcal{C} \xrightarrow{st} \mathcal{C}'$ means that $\mathcal{C}'$ is obtainable from $\mathcal{C}$ by stellar subdivisions and "$\approx_{st}$" denotes the stellar equivalence using both stellar and inverse stellar subdivisions.

(3) $\varkappa(A,B) = \sigma^{-1}_B\sigma_A$ holds.

(4) If dim A + dim B = n (i.e. $\mathcal{L} = \{\emptyset\}$) then $\varkappa_{(A,B)} =: \chi_{(A,B)}$ is called a *bistellar k-operation* if dim A = k.

Obviosly we have $\chi^{-1}_{(A,B)} = \chi_{(B,A)}$. The related equivalence relation is denoted by "$\approx_{bst}$".

If dim B ≥ 1, $B = p\cdot B'$, then $\chi_{(A,B)}$ is uniquely determined by p and the facet $F := A\cdot B'$ of $\mathcal{M}$. We then say that F *is visible* from p and we write $\chi(A,B) =: \chi_{p/F}$ (for motivation see part 5).

(5) $\mathcal{M} \xrightarrow{sh,bst} \mathcal{M}'$, $\mathcal{M} \approx_{sh,bst} \mathcal{M}'$ is defined in the obvious way.

Note that these notations do not imply any order for the perfor-

In conclusion: A bist... a stellar ... of full dimension.

You all... STELLAR SUBD... EXCHANGE
YES, THAT'S WHAT ... BFS ...

mance of the involved types of operations. An *elementary operation* is an elementary boundary operation or a bistellar operation.

3. STELLAR EQUIVALENCE

The concept of stellar subdivision belongs to the standard tools in the theory of simplicial complexes and has an old and rich tradition. For more informations the reader can consult any book about p.l. topology [19,22]. Later on we need the following fundamental theorem.

(3.1) THEOREM. For arbitrary simplicial complexes the following holds:

$$|\mathcal{C}'| \cong |\mathcal{C}| \Leftrightarrow \mathcal{C}' \approx_{st} \mathcal{C}$$

Remark. From remarks (2) and (3) for (2.2) follows that the same holds for stellar exchanges.

A complete proof of the above theorem can be found in the book of Glaser [19]. For earlier results see [1,31]. There exist many theorems of the above type. Ewald and Shephard proved a convex version of (3.1). Indeed they showed the bistellar equivalence of boundary complexes of simplicial polytopes, but they did not emphasize this.

(3.2) THEOREM (Ewald/Shephard, 1974 [18]). Boundary complexes of (simplicial) polytopes are stellar (bistellar) equivalent in a geometrical sense. This means this can be done in such a way that all the spheres appearing in the equivalence are polytopal.

In the next sections we shall prove some generalizations of this theorem. There are many interesting unsolved problems concerning stellar equivalence. We only mention here the following long outstanding problem which is not solved even for polytopal spheres (see in [22]).

(3.3) PROBLEM. Let $\mathcal{C}_1, \mathcal{C}_2$ be stellar equivalent simplicial complexes. Does there exists a common stellar subdivision

$$\mathcal{C}_1 \xrightarrow{st} \mathcal{C} \xleftarrow{st} \mathcal{C}_2 \;?$$

How do they prove theorem 3.1

In dimension 2 the answer is known to be yes.

(3.4) THEOREM (Ewald, 1984 [14]). Let $\mathcal{E}_1, \mathcal{E}_2$ be simplicial 2-complexes, $|\mathcal{E}_1| = |\mathcal{E}_2|$. Then there exists a common stellar subdivision $\mathcal{E}$.

Since about 1970 an interesting connection between the theory of convex bodies and algebraic geometry has developed. Let $P \subseteq \mathbb{Q}^d$ be a full dimensional polytope with $0 \in \text{int } P$ and let then Σ be the fan of convex cones spanned by the faces of P. With every cone is assoziated an affin variety, namely the spectrum of the ring of all Laurent polynomials with support in the dual of the cone. These affin varieties can be glued together in a natural way by using the combinatorial structure of Σ. The resulting variety is a projective toric variety. This far-reaching result leads to a complete characterization of the face numbers of simplicial and simple polytopes [43].

Stellar subdivions of fans correspond to blow-ups of the assoziated varieties. Thus (3.2) and (3.4) respectively yield transforms of projective toric varieties and complete toric 3-varieties into projective space by composite of blow-ups and blow-downs (Ewald [15]).

4. Basic construction theorems

At the beginning of this section we shall enumerate some basic construction methods which may be of intrinsic interest. The first Lemmas deal with permutations of elementary operations.

(4.1) LEMMA. Let $\mathcal{M}_1, \mathcal{M}_1', \mathcal{M}_2$ be simplicial n-manifolds such that

(a) $\mathcal{M}_1' \xrightarrow{sh} \mathcal{M}_1$

(b) $\mathcal{M}_1'$ contains no missing face of $\mathcal{M}_1$

(c) $\mathcal{M}_2 = \chi_{(A,B)} \mathcal{M}_1$

Then the following holds:

$\mathcal{M}_2' := \chi_{(A,B)} \mathcal{M}_1' \xrightarrow{sh} \mathcal{M}_2$

The elementary shellings correspond to something hanging this can be removed!!

deletion and contraction???

Proof. $\mathcal{M}_1' \supseteq \mathcal{M}_1$ and (b) imply $B \notin \mathcal{M}_1'$. Hence $\mathcal{M}_2'$ is well defined. On the other hand every $\rho_C^- = \rho_{-C \cdot D}$ appearing in the process inverse to

(a) is applicable to $\mathcal{M}_2$, because (b) guarantees $D \neq B$.

(4.2) LEMMA. Let $\mathcal{M}_1, \mathcal{M}_1', \mathcal{M}_2$ be simplicial manifolds. Then we have:

$$\mathcal{M}_2 = \chi_{(A,B)}\mathcal{M}_1,\ \mathcal{M}_1' = \rho_{-F}\mathcal{M}_1,\ A \not\subseteq F \Rightarrow \mathcal{M}_2' := \rho_{-F}\mathcal{M}_2 = \chi_{(A,B)}\mathcal{M}_1'$$

Proof. $A \not\subseteq F$ implies $A\cdot\mathcal{B}(B) \cap \mathcal{F}(F) = \emptyset$. Furthermore the applicability of $\chi_{(A,B)}$ on $\mathcal{M}_1$ implies $B \notin \mathcal{M}_1$ and therefore we have $B \not\subseteq F$. Hence $\mathcal{B}(A)\cdot A \cap \mathcal{F}(F)$ holds too. From this it is easy to verify the applicability of the operations and the validity of the identity on the right hand side.

(4.3) LEMMA ([35]). Let $\mathcal{M}$ be a simplicial n-manifold and
$\varkappa_{(A,B)}\mathcal{M} = (\mathcal{M} \setminus A\cdot\mathcal{B}(B)\cdot\mathcal{L}) \cup \mathcal{B}(A)\cdot B\cdot\mathcal{L}$ and
$\varkappa_{(C,D)}\mathcal{L} = (\mathcal{L} \setminus C\cdot\mathcal{B}(D)\cdot\mathcal{L}') \cup \mathcal{B}(C)\cdot D\cdot\mathcal{L}'$.
Then the following holds

(1) $\varkappa_{(B\cdot C,D)}\varkappa_{(A,B)}\mathcal{M} = \varkappa_{(A,B)}\varkappa_{(A\cdot C,D)}\mathcal{M}$

(2) $\mathrm{link}(B\cdot C;\varkappa_{(A,B)}\mathcal{M}) = \mathcal{B}(A)\cdot\mathcal{B}(D)\cdot\mathcal{L}'$

(3) $\mathrm{link}(A\cdot C;\mathcal{M}) = \mathcal{B}(B)\cdot\mathcal{B}(D)\cdot\mathcal{L}'$

(4) $\mathrm{link}(A;\varkappa_{(A\cdot C,D)}\mathcal{M}) = \mathcal{B}(B)\cdot\varkappa_{(C,D)}\mathcal{L}$

Proof (Details are left to the reader). First one have to establish (2) which shows that the left hand side of (1) is well defined. The validity of (3) guarantees the existence of $\varkappa_{(A\cdot C,D)}\mathcal{M}$. Then (4) can be proved to show that the right hand side of (1) is well defined. Finish with the proof of identity (1).

Next we study how to replace certain constructions by elementary operations.

(4.4) LEMMA ([37]). Let $\mathcal{M}$ be a simplicial n-manifold and $\mathcal{K} \subseteq \mathcal{M}$ a shellable n-ball. Then the following holds.

$$\mathcal{M} \approx_{bst} (\mathcal{M} \setminus \mathit{Int}(\mathcal{K})) \cup p\cdot \mathit{Bd}(\mathcal{K})$$

Proof. By our assumption follows the existence of an inverse shelling $\mathcal{K} = \rho^+_{A_m} \ldots \rho^+_{A_1} \mathcal{F}(F_0)$.

From this we obtain by induction on m:

$$(\mathcal{M} \setminus \mathit{Int}(\mathcal{K})) \cup p\cdot \mathit{Bd}(\mathcal{K}) = \chi_{A_m} \ldots \chi_{A_1} \chi_{(F_0,p)}\mathcal{M}.$$

(4.5) LEMMA ([32]). Let $\mathcal{M}$ be a simplicial n-manifold, $A \in Int(\mathcal{M})$ and $p \in link(A;\mathcal{M})$ such that

(a) $ast(p;link(A;\mathcal{M}))$ is shellable

(b) $link(p;\mathcal{M}) \cap Int(ast(p;link(A;\mathcal{M}))) = \{\emptyset\}$

Then we have

$$\mathcal{M} \approx_{bst} (\mathcal{M} \setminus st(A;\mathcal{M})) \cup p\cdot ast(p;link(A;\mathcal{M})\cdot \mathcal{B}(A)) =: \mathcal{M}'$$

Proof. The proof works with induction on the number r of facets of $ast(p;link(A;\mathcal{M}))$. In the case of r=1 we clearly have $\mathcal{M}' = \chi_A \mathcal{M}$. Otherwise let $\rho_{+S_r} \cdots \rho_{+S_2} \mathcal{F}(S_1) = \rho^+_{B_r} \cdots \rho^+_{B_2} \mathcal{F}(S_1) = ast(p;link(A;\mathcal{M}))$

be an inverse shelling. Then case 1 can be applied on $\mathcal{M}, A\cdot B_r, p$ to get $\mathcal{M}'' := \chi_{A\cdot B_r} \mathcal{M} = \chi_{p/A\cdot S_r} \mathcal{M}$.

Now the conditions (a),(b) hold for $\mathcal{M}''$,A,p and furthermore we have $\rho_{+S_{r-1}} \cdots \rho_{+S_2} \mathcal{F}(S_1) = \rho^+_{B_{r-1}} \cdots \rho^+_{B_2} \mathcal{F}(S_1) = ast(p;link(A;\mathcal{M}''))$ which completes the proof.

(4.6) LEMMA ([37]). Let $\mathcal{M}$ be a simplicial n-manifold and let $\mathcal{K} \subseteq Bd(\mathcal{M})$ be a shellable (n-1)-ball. Then we have

$$\mathcal{M} \cup p\cdot\mathcal{K} \xrightarrow{sh} \mathcal{M}$$

Proof. Given a shelling $\rho_{-F_1} \cdots \rho_{-F_m} \mathcal{K} = \mathcal{F}(F_0)$ of $\mathcal{K}$, we get

$\mathcal{M} = \rho_{-p\cdot F_0}\, \rho_{-p\cdot F_1} \cdots \rho_{-p\cdot F_m} (\mathcal{M} \cup p\cdot\mathcal{K})$.

(4.7) LEMMA ([37]). Let M be a simplicial n-manifold, $\mathcal{K} \subseteq Bd(\mathcal{M})$ a shellable (n-1)-ball and $\mathcal{K} \subseteq link(p;\mathcal{M})$ for a vertex $p \in Int(\mathcal{M})$. Then

$$\mathcal{M} \xrightarrow{sh} \mathcal{M} \setminus p\cdot Int(\mathcal{K}) =: \mathcal{M}'$$

Proof. Let $\rho_{-F_1} \cdots \rho_{-F_m} \mathcal{K} = \mathcal{F}(F_1)$ be a shelling of $\mathcal{K}$.

By induction on m one obtains:

$\rho_{-p\cdot F_1} \cdots \rho_{-p\cdot F_m} \mathcal{M} = \mathcal{M} \setminus p\cdot Int(\mathcal{M}) =: \mathcal{M}'$ and

$Bd(\mathcal{M}') = (Bd(\mathcal{M}) \setminus Int(\mathcal{K})) \cup p\cdot Bd(\mathcal{K})$.

Now we are able to replace, under certain niceness conditions, stellar subdivisions by elementary operations.

(4.8) LEMMA ([32]). Let $\mathcal{M}$ be a simplicial n-manifold and $A \in Int(\mathcal{M})$. If link(A;$\mathcal{M}$) is shellable then

$$\mathcal{M} \approx_{bst} \sigma_A \mathcal{M}$$

Proof. This follows immediately from Lemma (4.5).

(4.9) LEMMA ([37]). Let $\mathcal{M}$ be a simplicial n-manifold and $A \in Bd(\mathcal{M})$. If both link(A;$\mathcal{M}$) and link(A;$Bd(\mathcal{M})$) are shellable then

$$\sigma_A \mathcal{M} \xrightarrow{sh,\, bst} \mathcal{M}.$$

Proof. Following Lemma (4.6) the shellability of clst(A;$Bd(\mathcal{M})$) implies $\mathcal{M}' := \mathcal{M} \cup p\cdot \text{clst}(A;Bd(\mathcal{M})) \xrightarrow{sh} \mathcal{M}$.
Furthermore the shellability of link(A;$\mathcal{M}$) implies the shellability of ast(p;link(A;$\mathcal{M}'$)) = $\mathcal{B}(A)\cdot$link(A;$\mathcal{M}$) and it is easy to see that (b) of Lemma (4.5) holds too. So we get

$$\begin{aligned} \mathcal{M}' &\approx_{bst} (\mathcal{M}' \setminus \text{st}(A;\mathcal{M}')) \cup p\cdot\mathcal{B}(A)\cdot \text{link}(A;\mathcal{M}) \\ &= (\mathcal{M} \setminus \text{st}(A;\mathcal{M})) \cup p\cdot\mathcal{B}(A)\cdot \text{link}(A;\mathcal{M}) \\ &\approx \sigma_A \mathcal{M} \end{aligned}$$

(4.10) LEMMA ([37]). Let $\mathcal{M}$ be a simplicial n-manifold and $A \in Bd(\mathcal{M})$. If both lin\` A;$\mathcal{M}$) and link(A;$Bd(\mathcal{M})$) are shellable then

$$\mathcal{M} \xrightarrow{sh,\, bst} \sigma_A \mathcal{M}.$$

Proof. Following Lemma (4.5) the shellability of clst(A;$\mathcal{M}$) implies $\mathcal{M} \approx_{bst} (\mathcal{M} \setminus \text{st}(A;\mathcal{M})) \cup p\cdot Bd(\text{clst}(A;\mathcal{M})) =: \mathcal{M}'$.
Now clst(A;$Bd(\mathcal{M}')$) = clst(A;$Bd(\mathcal{M})$) is a shellable n-ball which is contained in link(p;$\mathcal{M}$). So we obtain by Lemma (4.7)
$\mathcal{M}' \xrightarrow{sh} \mathcal{M}' \setminus p\cdot \text{st}(A;Bd(\mathcal{M}')) \approx \sigma_A \mathcal{M}$.

(4.11) LEMMA. Let $\mathcal{M}, \mathcal{M}'$ be simplicial manifolds. Then it holds:
$\mathcal{M}' = \chi_{(A,B)} \mathcal{M}$ and clst(A,$\mathcal{M}$) $\cap$ $Bd(\mathcal{M})$ = $\mathcal{F}$(F), F a facet of $Bd(\mathcal{M})$

$$\Rightarrow \mathcal{M}' \approx_{sh} \mathcal{M}$$

Proof. As bistellar operations do not affect the boundary we clearly have $\mathcal{M}'' := \mathcal{M} \setminus (A\cdot\mathcal{B}(B) \cup \langle F\rangle) = \mathcal{M}' \setminus (\mathcal{B}(A)\cdot B \cup \langle F\rangle)$.
$F \in$ clst(A,$\mathcal{M}$) implies $F = A'\cdot B'$ where $A = A'\cdot a$, $B = B'\cdot b$.
Let dim A = k and $A_k, \ldots, A_0$ and $B_{n-k}, \ldots, B_0$ be any ordering of the facets of A and B respectivly such that $A_0 = A'$ and $B_0 = B'$. Then the

following holds

$\mathcal{M}'' = \rho_{-A\cdot B_{n-k}} \ldots \rho_{-A\cdot B_0}\mathcal{M}$ and

$\mathcal{M}'' = \rho_{-B\cdot A_k} \ldots \rho_{-B\cdot A_0}\mathcal{M}'$.

(4.12) LEMMA ([35]). For every simplicial n-ball $\mathcal{K}$ holds

$$\mathcal{K} \text{ shellable} \Rightarrow \sigma_A\mathcal{K} \text{ shellable}$$

Sketch of the Proof. Let be $\sigma_A = \sigma_{(A,b)}$ and let be given an inverse shelling of $\mathcal{K}$

$$(*) \qquad \rho^+_{A_r} \ldots \rho^+_{A_2}\mathcal{F}(F_1) = \rho_{+F_r} \ldots \rho_{+F_2}\mathcal{F}(F_1) = \mathcal{K}.$$

Let us consider one of the facets $F_i \in st(A;\mathcal{K})$, say $F_i = A\cdot S$ and let be $A' := A \cap A_i$ $(A_1 := \emptyset)$. Then we have to replace in (*) ρ_{F_i} by the sequence $\rho_{+a\cdot A_k\cdot S} \ldots \rho_{+a\cdot A_1\cdot S}$, where $A_1,\ldots,A_k$ is an order of the facets of A starting with the facets of $st(A',\mathcal{B}(A))$ if $A' \in \mathcal{B}(A)$ and any arbitrary order else.

The following decomposition lemma plays and importent role for the inductive ar‹ument in the proof of the main theorem of this section.

(4.13) LEMMA ([34]). Let $\mathcal{C}$ be a simplicial complex. Then there exists a unique decomposition $\mathcal{C} = \mathcal{B}(P)\cdot\mathcal{C}'$ such that P is a simplexoid (i.e. $\mathcal{B}(P) = \mathcal{B}(T_1)\cdot\ldots\cdot\mathcal{B}(T_r)$, T_k-s simplices) and P is maximal with this property.

Idea for the Proof. Let $\mathcal{D}$ be the simplicial complex which has as facets the missing faces of $\mathcal{C}$. The connected components of $\mathcal{D}$ yield the desired decomposition.

Remark. Clearly, if P_1, P_2 are simplexoids then $\mathcal{B}(P_1)\cdot\mathcal{B}(P_2)$ is again isomorphic to the boundary complex of a simplexoid.

Now we are able to replace stellar subdivisions by elementary operations without any niceness assumptions.

(4.14) THEOREM ([37]). Let $\mathcal{M},\mathcal{M}'$ be simlicial n-manifolds. Then

$$|\mathcal{M}'| \cong |\mathcal{M}| \Leftrightarrow \mathcal{M}' \xrightarrow{sh,\,bst} \mathcal{M}.$$

Especially we have $\mathcal{F}(T^n) \xrightarrow{sh,bst} \mathcal{K}$ and $\mathcal{K} \xrightarrow{sh,bst} \mathcal{F}(T^n)$ for every simplicial n-ball $\mathcal{K}$.

Sketch of the Proof. The suffiency follows at once from remark (2) for (2.1) and remark (1) for (2.2). In order to prove the existence of our transformation we can assume $\mathcal{M}' = \varkappa_{(A,B)}\mathcal{M}$ (apply remark for Theorem (3.1)).

Now let $\text{link}(A;\mathcal{M}) = \mathcal{B}(B)\cdot\mathcal{L}$ and let $\mathcal{L} = \mathcal{B}(P)\cdot\mathcal{L}'$ be the unique decomposition of $\mathcal{L}$, according to Lemma (4.13), and let be given an equivalence $\varkappa_r \ldots \varkappa_1 \mathcal{L}' = \mathcal{B}(T^{m+1})$ or $\mathcal{F}(T^m)$, according to (3.1).
If $m := \dim \mathcal{L}' \leq 2$ or $r \leq 2$, then $\mathcal{L}'$ is polytopal and hence shellable (see Steinitz Theorem in [20]) which implies the shellability of $\mathcal{L}$. From this and $\varkappa_{(A,B)} = \sigma_B^{-1}\sigma_A$ we conclude our assertion immediately with the help of Lemma (4,8),(4.9) and (4.10).
We proceed by induction on m and r. Let $\varkappa_1 = \varkappa_{(C,D)}$. Then we may apply $\varkappa_1$ to $\mathcal{B}(P)\cdot\mathcal{L}'$ and get $\varkappa_{(C,D)}(\mathcal{B}(P)\cdot\mathcal{L}') = \mathcal{B}(P)\cdot\varkappa_1\mathcal{L}'$. Two cases arise.

<u>Case 1.</u> $D \notin \mathcal{M}$.
Following (1) c_ Lemma (4.3) can construct $\mathcal{M}'$ by steps:

$$\mathcal{M} \xrightarrow{\varkappa_{(A\cdot C,D)}} \mathcal{M}_1 \xrightarrow{\varkappa_{(A,B)}} \mathcal{M}_2 \xleftarrow{\varkappa_{(B\cdot C,D)}} \mathcal{M}'.$$

(2),(3),(3) of Lemma (3.4) then enable to show that in each step the inductive assumption concerning m respective r is applicable. We remark that $D \notin \mathcal{M}$ if $\dim D = 0$.

<u>Case 2</u>. $D \in \mathcal{M}$.
This case can be reduced to case 1. As mentioned above we may assume $\dim D \geq 1$. Let $D = p\cdot E$, p a vertex of D. Then we subdivide $\mathcal{L}'$ in the 0-face p (which clearly yields an isomorphic complex), $\varkappa_{(p,q)}\mathcal{L}' = (\mathcal{L}' \setminus p\cdot\text{link}(p;\mathcal{L}')) \cup q\cdot\text{link}(p;\mathcal{L}')$, where q is a new vertex not contained in $\mathcal{M}$.
From Lemma (4.3) we then derive

$$\mathcal{M} \xrightarrow{\varkappa_{(B\cdot p,q)}} \mathcal{M}_1 \xrightarrow{\varkappa_{(A,B)}} \mathcal{M}_2 \xleftarrow{\varkappa_{(A\cdot p,q)}} \mathcal{M}',$$

from which we obtain our assertion by the inductive argument or by applying Case 1 in the second step respectively.

5. Transformations of Closed Manifolds and Spheres

In 1978 Ewald realized the connection between bistellar operations and shellings. Moving along a suitable ray starting from a vertex of a simplicial polytope P "one can see" (having realized P in a suitable manner) all the facets of P in a certain ordering. This implies both a shelling of the boundary complex of P and a bistellar equivalence between the boundary complex of P and that of a simplex.

(5.1) THEOREM (Ewald, 1978 [13]). Let P be a simplicial d-polytope and let p be a vertex of P. Then there exists a (geometrical) bistellar equivalence

$$\chi_{p/F_r} \cdots \chi_{p/F_1} \mathcal{B}(P) = \mathcal{B}(T^d)$$

Remarks. (1) An alternative proof works with the help of Gale-diagrams (see [26]).
(2) Kleinschmidt [24] has generalized this process on non-simplicial polytopes.
(3) So called regular bistellar operations of fans were used to prove that a complete smooth toric 3-variety can be transformed into a projective one by blow-ups with non-singular centers (Ewald [16], Danilov [11]).

(5.2) THEOREM (Bruggesser/Mani, 1972 [8]). Boundary complexes of polytopes are shellable (this can be done starting with the facets of the star of an arbitrary vertex of the polytope).

This deep result has produced many applications. As already mentioned, (5.2) was the basis for McMullen's proof of the upper bound conjecture. With the help of shellings of simplicial polytopes Blind and Mani [7] proved in 1986 the conjecture of Perles that simple polytopes have isomorphic boundary complexes provided their 1-skeletons are isomorphic.

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring with the natural grading by degree, where the variables are interpretet as the vertices of a (d-1)-dimensional simplicial complex $\mathcal{C}$. Let then be I the ideal generated by the missing faces of $\mathcal{C}$. Factoring out I from

R yields the so called *Stanley-Reisner ring* A of $\mathscr{C}$ (see[38, 42]). A result of Reisner [38] in 1976 states that the Stanley-Reisner rings of homological spheres are Cohen-Macaulay rings. One consequence of this result was Stanley's new proof of the upper bound theorem for convex polytopes and its extension to homological spheres [41]. There was spend much effort to get combinatorial proofs of the results of Reisner and Stanley. In 1979 Kind and Kleinschmidt proved that shellable simplicial complexes are Cohen-Macaulay [23]. Another method was used by Stanley [40].

Combining the global construction in (5.1) with similar local processes enabled us to prove:

(5.3) THEOREM (Pachner, 1981 [33]). Let P,P' be simplicial d-polytopes with the same number of vertices. Then there exists a (geometrical) bistellar equivalence

$$\chi_r \ \dots \ \chi_1 \mathscr{B}(P) = \mathscr{B}(P')$$

such that all the polytopes appearing in the equivalence have the same number of vertices (especially one can choose P' to be a stacked polytope (see [4,20])).

It has tuned out that shellability is not a property which holds for general spheres.

(5.4) THEOREM (Edwards, 1975 [12]). There exist non-shellable triangulated (topological!) 5-spheres.

For this reason it was surprising that Theorem (5.1) could be generalized to simlicial spheres.

(5.5) THOREM (Pachner [35]). Let $\mathcal{M}, \mathcal{M}'$ be closed simplicial manifolds. Then we have

$$|\mathcal{M}'| \cong |\mathcal{M}| \leftrightarrow \mathcal{M}' \approx_{bst} \mathcal{M}$$

Proof. Replace the corresponding keywords in the proof of Theorem (4.14).

(5.6) COROLLARY. Every simplicial n-sphere is bistellar equivalent to the boundary complex of the (n+1)-Simplex.

For simplicial 3-spheres with up to nine vertices this was proved by computational constructing (Altshuler/Bokowski/Steinberg, 1980 [3]) all these spheres. Using the ideas of Kind/Kleinschmidt and based on (5.6) Lee recently has found a new proof of the Cohen-Macaulay property for simplicial spheres [27].

We do not know, whether Theorem (5.3) can be generalized to simplicial spheres. It is only known:

(5.7) THEOREM (Pachner [32]). Let $\mathcal{S}$ be a simplicial n-sphere. If $\mathcal{S}$ can be transformed into $\mathcal{B}(T^{n+1})$ by bistellar operations without bistellar n-operations then $\mathcal{S}$ can be transformed into the boundary complex of a stacked polytope by bistellar operations without changing the number of vertices during the process.

For the construction of special collars in part 6 we need the following strengthening of a theorem in [35].

(5.8) THEOREM. Every simplicial n-sphere $\mathcal{S}$ is the boundary complex of a shellable simplicial (n+1)-ball $\mathcal{K}$. $\mathcal{K}$ can be choosen such that $\mathcal{S}$ is full in $\mathcal{K}$.

Proof. Following Theorem (5.6) it is sufficient to prove our assertion for $\mathcal{S}' = \chi_{(A,B)}\mathcal{S}$ assuming that the assertion holds for $\mathcal{S}$.

Case 1 $B \notin \mathcal{K}$

Then $\mathcal{K}' := \rho_A^+ \mathcal{K}$ is a shellable ball with boundary complex $\mathcal{S}'$.

Case 2 $B \in \mathcal{K}$

Indeed we than have $B \in Int(\mathcal{K})$ and following Lemma (4.12) $\sigma_B\mathcal{K}$ is again a shellable ball with boundary complex $\mathcal{S}$. As $B \notin \sigma_B\mathcal{K}$ we then can apply case 1 to get a shellable ball $\mathcal{K}'$ with boundary complex $\mathcal{S}'$.

Stellar subdivions applied in all faces $C \in Int(\mathcal{K}')$ which are missing faces of $\mathcal{S}'$ then yields the desired ball (Lemma (4.12)).

For further informations about bistellar equivalence and related problems the reader can consult [13,25,33,35].

6. Transformations of manifolds with boundary and balls

A survey about shelling can be found in [10]. In addition to the facts presented till now we mention the following important

Otro ejemplito!!

result.

THEOREM (6.1) (Rudin, 1958 [39], Grünbaum, 1972 [21]). There exist non-shellable simplicial balls.

As we have seen in Theorem (4.14) one succeeds with additional bistellar operations. Certainly, it would be more convenient to deal with boundary operations alone. In order to replace bistellar operations by shellings and inverse shelling we need special partial collars.

(6.2) LEMMA. Let $\mathcal{M}$ be a simplicial manifold an F be a facet of $\mathcal{M}$. Then there exists a simplicial manifold $\mathcal{M}'$ such that

(1) $\mathcal{M}' \xrightarrow{sh} \mathcal{M}$

(2) $\mathcal{M}$ is full in $\mathcal{M}'$

(3) $Bd(\mathcal{M}') \cap Bd(\mathcal{M}) = \mathcal{F}(F)$

Proof. Let be $A \in Bd(\mathcal{M})$. Then link($A;Bd(\mathcal{M})$) is a simplicial sphere and Theorem (5.5) asserts the existence of a ball $\mathcal{K}$ such that

(a) $Bd(\mathcal{K})$ = link($A;Bd(\mathcal{M})$)

(b) $Bd(\mathcal{K})$ is full in $\mathcal{K}$

(c) $\mathcal{K}$ is shellable

From (a),(b) follows, that $\mathcal{M}' := \mathcal{M} \cup A\cdot\mathcal{K}$ is again a simplicial manifold with $Bd(\mathcal{M}') = (\mathcal{M} \setminus st(A;Bd(\mathcal{M})) \cup \mathcal{B}(A)\cdot\mathcal{K}$. Furthermore (c) implies the shellability of $\mathcal{B}(A)\cdot\mathcal{K}$ and $\mathcal{B}(A)\cdot\mathcal{K}$ is containted in link($a;\mathcal{M}'$) for every vertex a of A. Hence we obtain $\mathcal{M}' \xrightarrow{sh} \mathcal{M}$ directly from Lemma (4.7). Further (b) implies that $\mathcal{M}'$ contains no missing face of $\mathcal{M}$.

Applying the above process to all the faces of $\mathcal{M} \setminus \mathcal{F}(F)$ after having them ordered by decreasing dimension, yields the desired manifold.

Remark. Generalizations of (6.2) are obvious.

Now we are able to prove the main theorem of this paper.

(6.3) THEOREM. Let $\mathcal{M}',\mathcal{M}$ be simplicial manifolds with boundary. Then the following holds:

$$|\mathcal{M}'| \cong |\mathcal{M}| \leftrightarrow \mathcal{M}' \approx_{sh} \mathcal{M}$$

Proof. Following Theorem (4.14) it is sufficient to proof $\mathcal{M}'$: =

$\chi_{(A,B)}\mathcal{M} \approx_{sh} \mathcal{M}$. We may assume that $\mathcal{M}$ is connected. From this follows that $\mathcal{M}$ is strongly connected (see [2]) and hence there exists a sequence $F_0,\ldots,F_r$ of facets of $\mathcal{M}$ such that $F_0 \in st(A,\mathcal{K})$, $F := F_r$ is a facet having one of its facets in the boundary of $\mathcal{K}$ and F_{i-1}, F_i have a facet in common. We choose such a sequence with minimum r and then prove by induction on r:

There exists a simplicial n-manifold $\mathcal{M}_1$ such that

$$\mathcal{M}_1 \xrightarrow{sh} \mathcal{M},\ \mathcal{M}_1' := \chi_{(A,B)}\mathcal{M}_1 \xrightarrow{sh} \mathcal{M}' \text{ and } \mathcal{M}_1' \approx_{sh} \mathcal{M}_1$$

Let S be the facet of F contained in $Bd(\mathcal{M})$ and let then $\mathcal{M}_2$ be the simplicial n-manifold constructed in Lemma (6.2) with respect to $\mathcal{M}$, S. From (2) of Lemma (6.2) follows that we can apply $\chi_{(A,B)}$ on $\mathcal{M}_2$ and Lemma (4.1) yields $\mathcal{M}_2' := \chi_{(A,B)}\mathcal{M}_2 \xrightarrow{sh} \mathcal{M}'$.

In case of r=0 (3) of Lemma (6.2) enables us to apply Lemma (4.11) which yields $\mathcal{M}_2' \xrightarrow{sh} \mathcal{M}_2$. That means our assertion holds for $\mathcal{M}_1 := \mathcal{M}_2$.

Otherwise (1) and (3) of Lemma (6.2) allows to apply ρ_{-F} on $\mathcal{M}_2$. The minimality of r implies $F \notin st(A;\mathcal{M}_2)$. Hence $\chi_{(A,B)}$ can be applied on $\mathcal{M}_3 := \rho_{-F}\mathcal{M}_2$ and Lemma (4.2) yields $\rho_{-F}\mathcal{M}_3' = \mathcal{M}_2'$, where $\mathcal{M}_3' := \chi_{(A,B)}\mathcal{M}_3$. Now the inductiv assumption applied on $\mathcal{M}_3$ proves our assertion.

(6.4) COROLLARY (Pachner [37]). For every simplicial n-ball $\mathcal{K}$ holds:

$$\mathcal{K} \approx_{sh} \mathcal{F}(T^n)$$

(6.5) COROLLARY (Pachner [37]). Let $\mathcal{S}$ be a simplicial n-sphere and $p \in vert(\mathcal{S}_1)$.

Then there exists a transformation

$$\chi^{\pm}_{p/F_r} \cdots \chi^{\pm}_{p/F_1} \mathcal{S} = \mathcal{B}(T^{n+1}).$$

Proof. Apply Theorem (6.3) on $ast(p;\mathcal{S})$.

7. Pseudoshellings and Facenumbers

Let $f_i(\mathscr{C})$ be the number of i-dimensional faces of a simplicial (d-1)-complex. The vector $f(\mathscr{C}) := (f_0,\dots,f_{d-1})$ will be called the *f-vector* of $\mathscr{C}$. For many purposes the *h-vector* $h(\mathscr{C})$ defined by

$$\textbf{(7.1)}\quad h_i(\mathscr{C}) := \sum_{j=0}^{i} (-1)^{i-j}\begin{bmatrix} d-j \\ d-i \end{bmatrix} f_{j-1}(\mathscr{C}),\ i = 0,\dots,d.$$

is easier to deal with. For the algebraic interpretation of $h(\mathscr{C})$ in Stanley-Reisner rings the reader is referred to [42].

Different from stellar subdivisions elementary operations allow an explicit computation of the numbers of faces. Following Corollary (6.4) $\mathscr{K} \approx_{sh} \mathscr{F}(T^{d-1})$ holds for every simplicial (d-1)-ball $\mathscr{K}$. Let λ_i^- and λ_i^+ respectively denote the numbers of elementary k-shellings and inverse elementary k-shellings if $\mathscr{K}$ is constructed from $\mathscr{F}(T^{d-1})$ in this way. It is well known and easy to calculate with the help of (7.1) that $h(\rho_A^- \mathscr{K}) = h(\mathscr{K}) - e_k$ holds, where e_k denotes the k-th unit vector (k=0,...,d) and dim A = k-1. Hence we get:

$$\textbf{(7.2)}\quad h_i(\mathscr{K}) = \lambda^+_{d-1-i} - \lambda^-_{i-1},\ i = 1,\dots,d-1$$

Obviously these equations remain true if we use generalized shellings which change the f-vector in the same way. Many authors have already considered simplicial complexes which can be constructed from a single simplex by inverse generalized shellings (compare [9,23]). These complexes are in general no manifolds, but they are of the following type:

(7.3) DEFINITION. A *totally strongly connected* complex $\mathscr{C}$ is a pure simplicial complex with the property that link$(A,\mathscr{C})$ is strongly connected for every $A \in \mathscr{C}$ ($A = \emptyset$ included!).

We want to preserve this property when allowing additional generalized shellings.

(7.4) DEFINITION. Let $\mathscr{C}$, $\mathscr{C}'$ be totally strongly connected (d-1)-complexes, $A \in \mathscr{C}$, dim A = k. Then we call $\hat{\rho}_{+F}\ \mathscr{C} := \hat{\rho}^+_{(A,B)}\ \mathscr{C}$

:= $\mathscr{C}'$ an *inverse (elementary) k-pshelling* (= *pseudoshelling*) provided $\mathscr{C}' = \mathscr{C} \cup A\cdot\mathscr{B}(B)$, where $F = A\cdot B$ is a (d-1)-simplex such that $\mathscr{F}(F) \cap \mathscr{C} = A\cdot\mathscr{B}(B)$ holds. The inverse operation is called an *elementary (d-1-k)-pshelling* and is denoted by $\hat{\rho}_{-F} = \hat{\rho}^{-}_{(B,A)}$.

Remarks and examples. (1) Note that a pseudoshellings is not determined by A or B alone. The applicabillity of $\hat{\rho}^{-}_{(B,A)}\mathscr{C}'$ implies $B\cdot\mathscr{B}(A) \subseteq Bd(\mathscr{C}')$. The notations "$\xrightarrow{psh}$", "$\approx_{psh}$" are defined as usual.

(2) We allow inverse (d-1)-pshellings $\rho_{(F,\emptyset)}\mathscr{C}$, F a (d-1)-simplex. This implies $\mathscr{F}(F) \cap \mathscr{C} = \langle F\rangle\cdot\emptyset = \emptyset$. From this follows $\emptyset \notin \mathscr{C}$ which implies $\mathscr{C} = \emptyset$ and $\emptyset \xrightarrow{psh} \mathscr{F}(T^{d-1})$. Thus given an equivalence $\emptyset \approx_{psh} \mathscr{C}$ we always may assume that there appears precisely one inverse (d-1)-pseudoshelling and no (-1)-pshelling (which is the inverse operation).

(3) Let $\mathscr{S}$ be a simplicial (d-1)-sphere and F a facet of $\mathscr{S}$. Then $\hat{\rho}_{-F}\mathscr{S}$ is a simplicial ball.

From remark (3) of Definition (7.4) and Corollary (6.4) we get:

(7.5) $\mathscr{C}$ simplicial ball or sphere $\Rightarrow \emptyset \approx_{psh} \mathscr{C}$

Let be $\emptyset \approx_{psh} \mathscr{C}$. Analogously as for elementary operations let then $\lambda^{-}_{k}(\mathscr{C})$ respective $\lambda^{+}_{k}(\mathscr{C})$ denote the number of k-pshellings and inverse k-pshellings in this process (starting with the empty complex $\emptyset$). Clearly these numbers do not depend only on $\mathscr{C}$, but also on the present equivalence. (7.2) generalizes to:

(7.6) $(\lambda^{+}_{d-1-i} - \lambda^{-}_{i-1})(\mathscr{C})$ depends only on $\mathscr{C}$ as

$h_i(\mathscr{C}) = (\lambda^{+}_{d-1-i} - \lambda^{-}_{i-1})(\mathscr{C})$ holds for $i=0,\dots,d$. It is $h_0 = 1$.

Examples. $h(\mathscr{F}(T^{d-1})) = (1,0,\dots,0)$, $h(\mathscr{B}(T^{d})) = (1,\dots,1)$.

The number of facets increases by one or decreases by one respectively if an inverse elementary pshelling or an elementary pshelling is applied. Obviously the number of vertices calculates as follows (compare remark (2) for (7.4)):

$f_0 = d(\lambda^+_{d-1} - \lambda^-_{-1})(\mathscr{C}) + (\lambda^+_{d-2} - \lambda^-_0)(\mathscr{C}) = d + (\lambda^+_{d-2} - \lambda^-_0)(\mathscr{C})$

Hence we get from (7.6):

(7.7) $f_0(\mathscr{C}) = d + h_1(\mathscr{C})$ and $f_{d-1}(\mathscr{C}) = \sum_{i=0}^{d} h_i(\mathscr{C})$

Obviously every inverse k-pshelling of $\mathscr{C}$ induces one inverse j-pshelling in $skel_{d-2}(\mathscr{C})$ for $j = k-1,\ldots,-1$. Thus we obtain:

$\lambda^+_{k-1}(skel_{d-2}(\mathscr{C})) = \sum_{j=k}^{d-1} \lambda^+_j(\mathscr{C})$ and analogously it is

$\lambda^-_k(skel_{d-2}(\mathscr{C})) = \sum_{j=0}^{i} \lambda^-_j(\mathscr{C})$. Consequently we get:

(7.8) Let be $\emptyset \approx_{psh} \mathscr{C}$, $\dim \mathscr{C} = d-1$. Then it holds:

(1) $\emptyset \approx_{psh} skel_{d-2}(\mathscr{C})$

(2) $(h_i - h_{i-1})(skel_{d-2}(\mathscr{C})) = h_i(\mathscr{C})$, $i = 0,\ldots,d-1$, and

$h_i(skel_{d-2}(\mathscr{C})) = \sum_{j=0}^{i} h_j(\mathscr{C})$

Together with (7.7) this leads to:

(7.9) $f_j(\mathscr{C}) := \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_{i-1}(\mathscr{C})$, $j=-1,\ldots,d-1$

Let us consider now bistellar operations. Let $\mu_i(\mathcal{M},\mathcal{M}')$ denote the number of bistellar k-operations in an equivalence $\mathcal{M}' \approx_{bst} \mathcal{M}$ (transforming $\mathcal{M}$ into $\mathcal{M}'$). For a simplicial (d-1)-sphere we write $\mu_i(\mathscr{S}) := \mu_i(\mathscr{B}(T^d),\mathscr{S})$. From (7.1) follows easily that a bistellar k-operation decreases h_i for $k+1 \le i \le d-1-k$ by one if $2k \le d-1$, and increases h_i for $d-k \le i \le k$ by one if $2k \ge d-1$. Hence we get:

(7.10) Let be $\mathcal{M} \approx_{bst} \mathcal{M}'$. Then

(1) $(\mu_{d-1-i} - \mu_i)(\mathcal{M},\mathcal{M}') = (h_{i+1} - h_i)(\mathcal{M}') - (h_{i+1} - h_i)(\mathcal{M})$, $0 \le i \le d-1$.

That means $(\mu_{d-1-i} - \mu_i)(\mathcal{M},\mathcal{M}')$ depends only on $\mathcal{M},\mathcal{M}'$.

Especially for spheres we have $(\mu_{d-1-i} - \mu_i)(\mathcal{S}) = (h_{i+1} - h_i)(\mathcal{S})$.

$$(2)\quad h_i(\mathcal{M}') - h_i(\mathcal{M}) = \sum_{j=0}^{i-1} (\mu_{d-1-j} - \mu_j)(\mathcal{M},\mathcal{M}')$$

This leads to an easy proof of the *Dehn-Sommerville equations*.

(7.11) For closed simplicial (d-1)-manifolds $(h_{d-i} - h_i)(\mathcal{M})$ are topological invariants, $0 \le i \le d$. Especially for a sphere $\mathcal{S}$ it holds $(h_{d-i} - h_i)(\mathcal{S}) = 0$.

Proof. $|\mathcal{M}'| \cong |\mathcal{M}|$ implies $\mathcal{M}' \approx_{bst} \mathcal{M}$ (Theorem (5.5)) and following (7.10) we then get:

$$(h_{d-i} - h_i)(\mathcal{M}') - (h_{d-i} - h_i)(\mathcal{M})$$

$$= \sum_{j=0}^{d-i-1} (\mu_{d-1-j} - \mu_j)(\mathcal{M},\mathcal{M}') - \sum_{j=0}^{i-1} (\mu_{d-1-j} - \mu_j)(\mathcal{M},\mathcal{M}')$$

$$= \sum_{j=0}^{d-1} \mu_{d-1-j}(\mathcal{M},\mathcal{M}') - \sum_{j=0}^{d-1} \mu_j(\mathcal{M},\mathcal{M}') = 0$$

Especially for spheres we get

$(h_{d-i} - h_i)(\mathcal{S}) = (h_{d-i} - h_i)(\mathcal{B}(T^d)) = 0$.

Remark. For i=0 this yields the Euler equation (use (7.8)).

Pseudoshellings make it possible to construct other manifolds than spheres or balls.

(7.12) THEOREM. For every orientable closed simplicial 2-manifold $\mathcal{M}$ holds $\emptyset \approx_{psh} \mathcal{M}$. It is $(h_2 - h_1)(\mathcal{M}) = -6g(\mathcal{M})$, $g(\mathcal{M})$ the genus of $\mathcal{M}$.

Proof. Let $F = x\cdot y\cdot z$ and $F' = x'\cdot y'\cdot z'$ be two disjoint triangles of $\mathcal{M}$. $\mathrm{Skel}_1(\mathcal{M})$ contains a path $x = x_0, x_1, \ldots, x_r = x'$. Using stellar subdivisions and Theorem (6.3) we may assume $x_0 \cdot x_i \notin \mathcal{M}$ for $i = 1,\ldots,r$. Now we are able to construct a surface $\mathcal{M}'$ of genus $g(\mathcal{M}') = g(\mathcal{M}) + 1$ by elementary pshellings and its inverses by the following steps:

$$\mathcal{E}_1 := \hat{\rho}^{-}_{(F',\emptyset)} \hat{\rho}^{-}_{(F,\emptyset)} \mathcal{M},$$

$$\mathcal{E}_2 := \hat{\rho}^{+}_{(x_{r-1},\, x_0\cdot x_r)} \hat{\rho}^{+}_{(x_{r-2},\, x_0\cdot x_{r-1})} \cdots \hat{\rho}^{+}_{(x_1,\, x_0\cdot x_2)} \mathcal{E}_1,$$

$$\mathcal{C}_3 := \hat{\rho}^{+}_{(\emptyset, x\cdot x'\cdot z')}\hat{\rho}^{+}_{(z, x\cdot z')}\hat{\rho}^{+}_{(y', z\cdot z')}\hat{\rho}^{+}_{(y, z\cdot y')}\hat{\rho}^{+}_{(x', y\cdot y')}\hat{\rho}^{+}_{(x, y\cdot x')}\mathcal{C}_2,$$

$$\mathcal{M}' := \hat{\rho}^{-}_{(x_0\cdot x_2, x_1)} \cdots \hat{\rho}^{-}_{(x_0\cdot x_{r-1}, x_{r-2})} \hat{\rho}^{-}_{(x_0\cdot x_{r-1}\cdot x_r, \emptyset)}\mathcal{C}_3 .$$

From this follows

$h(\mathcal{M}') - h(\mathcal{M}) = -2e_2 + re_1 + (5e_1 + e_2) - (e_2 + (r-1)e_1) = 4e_1 - 2e_2$.

Together with (7.5),(7.11) this implies our assertion.

(7.13) CONJECTURE. $\emptyset \approx_{psh} \mathcal{M}$ holds for simplicial manifolds.

Remark. We believe that a general proof is possible with the help of *handle-body theorems* (see [36]).

Last we shall present a combinatorial interpretation for some known consequences of the Dehn-Sommerville equations (see [30]). Let $\mathcal{S}$ be a simplicial (d-1)-sphere, $p \in vert(\mathcal{S})$ and $\mathcal{K} := ast(p;\mathcal{S})$, $\mathcal{L} := link(p;\mathcal{S})$. From Corollary (6.4) we get an equivalence $\mathcal{F}(T^{d-1}) \approx_{sh} \mathcal{K}$ which yield bistellar equivalences $\mathcal{B}(T^d) \approx_{bst} \mathcal{S}$ and $\mathcal{B}(T^{d-1}) \approx_{bst} \mathcal{L}$ (compare Corollary (6.5)). Obviously we have:

Every elementary k-shelling of $\mathcal{K}$ induces a bistellar k-operation of $\mathcal{S}$ and $\mathcal{L}$. Every inverse elementary k-shelling of $\mathcal{K}$ induces a bistellar k-operation of $\mathcal{L}$ and a bistellar (k+1)-operation of $\mathcal{S}$ respectivly. Hence we obtain:

$\mu_j(\mathcal{S}) = \lambda^{+}_{j-1}(\mathcal{K}) + \lambda^{-}_{j}(\mathcal{K})$, $j = 0,\dots,d$, and

$\mu_j(\mathcal{L}) = \lambda^{+}_{j}(\mathcal{K}) + \lambda^{-}_{j}(\mathcal{K})$, $j = 0,\dots,d-1$.

From (7.6) and (7.10) then easily follows:

(7.14) (1) $(h_{i+1} - h_i)(\mathcal{S}) = (h_{i+1} - h_{d-i})(\mathcal{K})$, $i = 0,\dots,d-1$

$h_i(\mathcal{S}) = (h_0 + \dots + h_i)(\mathcal{K}) - (h_{d+1-i} + \dots + h_d)(\mathcal{K})$, $i = 0,\dots,d$

(2) $(h_i - h_{i-1})(\mathcal{L}) = (h_i - h_{d-i})(\mathcal{K})$, $i = 0,\dots,d-2$

$h_i(\mathcal{L}) = (h_0 + \dots + h_i)(\mathcal{K}) - (h_{d-i} + \dots + h_d)(\mathcal{K})$, $i = 0,\dots,d-1$

Certainly this implies:

(7.15) $h_i(\mathcal{S}) = h_i(\mathcal{K}) + h_{i-1}(\mathcal{L}) = h_{d-i}(\mathcal{K}) + h_i(\mathcal{L})$, $i = 0,\dots,d$.

We hope that additional ideas will solve the following problem:

(7.16) PROBLEM. Find combinatorial proofs of:
For every simplicial ball $\mathcal{K}$ holds
(1) $h(\mathcal{K}) \geq 0$
(2) $(h_{i+1} - h_{d-i})(\mathcal{K}) \geq 0$, $2i \leq d-1$

Acknowledgment

I would like to thank Andreas Krecht who has proved the details of a part of the theorems.

References

1. Alexander, J.W.: The combinatorial theory of complexes. Ann. of Math. 31, 292 - 320 (1930).
2. Alexandroff, P.S., Hopf, H.: Topologie I, Springer Verlag Berlin (1935).
3. Altshuler, A., Bokowski, J., Steinberg, L.: The classification of simplicial 3-spheres with nine vertices into polytopes and non polytopes. Discrete Math. 31 (1980), 115 - 124.
4. Barnette, D.: A proof of the lower bound conjecture for convex polytopes. Pacific J. Math. 46, 349 -354 (1973).
5. Billera, L.J., Lee, C.W.: A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes. J. Comb. Theory Ser. A 31 (1981), 237 - 255.
6. Björner, A.: Shellable and Cohen-Macaulay partially ordered sets. Trans. Amer. Math. Soc. 260 (1980), 159 - 183.
7. Blind, R., Mani-Levitska, P.: Puzzles And Polytope Isomorphisms. Aequationes Math. (to appear).
8. Bruggesser, H., Mani, P.: Shellable decompositions of cells and spheres. Math. Scand. 29 (1972), 197 - 205.
9. Danaraj, G., Klee, V.: Shellings of spheres and polytopes. Duke Math. J. 41, 443 - 451 (1974).
10. Danaraj, G., Klee, V.: Which spheres are shellable? Ann. of Discrete Math. 2 (1978), 33 - 52.
11. Danilov, V.I.: Birational Geometry Of Toric 3-Folds. Math. USSR Izvestiya, Vol. 21 (1983), No. 2., 269 -280.
12. Edwards, R.D.: The double suspension of a certain homology 3-sphere is S 65 1. A.M.S. Notices 22 (1975), A - 334.

13. Ewald, G.: Über die stellare Äquivalenz konvexer Polytope. Resultate d. Math. 1 (1978), 54 - 64.
14. Ewald, G.: Über stellare Unterteilung von Simplizialkomplexen. Archiv Math. 46, 153 -158 (1986).
15. Ewald, G.: Torische Varietäten und konvexe Polytope. Studia Scient. Math. Hung. (to appear).
16. Ewald, G.: Blow-ups of Smooth Toric-Varieties. Abh. Math. Sem. Hamb. 57 (1987), 193 - 201.
17. Ewald, G., Kleinschmidt, P., Pachner, U., Schulz, Chr.: Neuere Entwicklungen in der kombinatorischen Konvexgeometrie. In: Contributions to Geometry. Proceedings of the Geometry Symposium in Siegen 1978. Birkhäuser Verlag Basel, 1979.
18. Ewald, G., Shephard, G.C.: Stellar subdivisions of boundary complexes of convex polytopes. Math. Ann. 210 (1974), 7 - 16.
19. Glaser, L.C.: Geometrical Combinatorial Topology. Vol. 1. Van Nostrand Reinhold Company, New York 1970.
20. Grünbaum, B.: Convex Polytopes. Interscience Publishers, New York 1967.
21. Grünbaum, B.: Two non-shellable triangulations of the 3-cell. Manuscript (1972).
22. Hudson, J.F.P.: Piecewise Linear Topology. Math. Lecture Note Ser. New York: Benjamin 1969.
23. Kind, B., Kleinschmidt, P.: Schälbare Cohen-Macaulay Komplexe und ihre Parametrisierung. Math. Z. 167 (1979), 173 - 179.
24. Kleinschmidt, P.: Stellare Abänderungen und Schälbarkeit von Komplexen und Polytopen. J. of Geom. 11/2 (1978), 161 - 176.
25. Lee, C.W.: Two combinatorial properties of a class of simplicial polytopes. Isr. J. Math. 47 (1984), no. 4, 261 - 269.
26. Lee, C.W.: Gale triangulations of convex polytopes. In preparation.
27. Lee, C.W.: PL-spheres and convex polytopes. In preparation.
28. Mandel, A.: Topology or Oriented Matroids. Ph. D. Thesis. University of Waterloo, Ontario, Canada 1982.
29. McMullen, P.: The maximum numbers of faces of a convex polytope. Mathematika 17 (1970), 179 - 184.
30. McMullen, P, Walkup, D.W.: A generalized lower-bound conjecture for simplicial polytopes. Mathematika 18 (1971), 264 - 273.

31. Newman, M.H.A.: Combinatorial topology of convex regions. Proc. Nat. Acad. Sci. 16 (3), 240 - 242 (1930).

32. Pachner, U.: Bistellare Äquivalenz kombinatorischer Mannigfaltigkeiten. Archiv f. Math. 30 (1978), 89 - 98.

33. Pachner, U.: Über die bistellare Äquivalenz simplizialer Sphären und Polytope. Math. Z. 176 (1981), 565 - 576.

34. Pachner, U.: Diagonalen in Simplizialkomplexen. Geom. Dedicata 24 (1987), 1 - 28.

35. Pachner, U.: Konstruktionsmethoden und das kombinatorische Homöomorphieproblem für Triangulationen kompakter semilinearer Mannigfaltigkeiten. Abh. Math. Sem. Hamb. 57, 69 -85 (1987).

36. Pachner, U.: Ein Henkeltheorem für geschlossene semilineare Mannigfaltigkeiten. Resultate d. Math. 12 (1987), 386 - 394.

37. Pachner, U.: Shellings of simplicial balls and p.l. manifolds with boundary. Discrete Math. (to appear).

38. Reisner, G.: Cohen-Macaulay quotients of polynomial rings. Advances in Math. 21 (1976), 30 - 49.

39. Rudin, M.E.: An unshellable triangulation of a tetrahedron. Bull. Amer. Math. Soc. 64 (1958), 90 - 91.

40. Stanley, R.P.: Cohen-Macaulay rings and constructible polytopes. Bull. Amer. Math. Soc. 81 (1975), 133 - 135.

41. Stanley, R.P.: The upper bound conjecture and Cohen-Macauley rings. Stud. Appl. Math. 54 (1975), 135 - 142.

42. Stanley, R.P.: Cohen-Macaulay complexes, Higher Combinatorics, 51 - 62, NATO Adv. Study Inst. Ser. C: Math. and Phys. Sci. 31, Reidel, Dordrecht, 1977.

43. Stanley, R.P.: The number of faces of a simplicial convex polytope. Adv. in Math. 35 (1980), 236 - 238.