ASYMPTOTICS OF MULTIVARIATE SEQUENCES, PART I: SMOOTH POINTS OF THE SINGULAR VARIETY

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ABSTRACT. Given a multivariate generating function $F(z_1, \ldots, z_d) = \sum a_{r_1,\ldots,r_d} z_1^{r_1} \cdots z_d^{r_d}$, we determine asymptotics for the coefficients. Our approach is to use Cauchy's integral formula near singular points of F, resulting in a tractable oscillating integral. This paper treats the case where the singular point of F is a smooth point of a surface of poles. Companion papers G treat singular points of F where the local geometry is more complicated, and for which other methods of analysis are not known.

1. INTRODUCTION

The generating function $F(z) := \sum_{r=0}^{\infty} a_r z^r$ for the sequence $a_0, a_1, a_2, \ldots$ is one of the most useful constructions in combinatorics. If the function F has a simple description, it is usually not too hard to obtain F as a formal power series once one understands a recursive or combinatorial description of the numbers $\{a_r\}$. One may then analyze the analytic properties of F in order to obtain asymptotic information about the sequence $\{a_r\}$. While still part art and part science, this latter analytic step has become quite systematized. Stanley (1997) in his introduction to enumerative combinatorics gives the example of the function $F(z) = \exp(z + \frac{z^2}{2})$, from which he says "it is routine (for someone sufficiently versed in complex variable theory) to obtain the asymptotic formula $a_r = 2^{-1/2} r^{r/2} e^{-r/2+\sqrt{r}-1/4}$." Routine, in this case, means a single application of the saddle point method. When F has singularities in the complex plane, the analysis is often more direct: the location of the singularities and the behavior of F near these determine almost algorithmically the asymptotic behavior of the sequence $\{a_r\}$. For those not sufficiently versed in complex variable theory, two useful sources are Henrici (1977) and Odlyzko (1995). The transfer theorems of Flajolet & Odlyzko (1990) encapsulate much of this knowledge in a very useful way; see also Wilf (1994) for an elementary introduction.

When the sequence $a_0, a_1, a_2, \ldots$ is replaced by a multidimensional array $\{a_{r_1,\ldots,r_d}\}$, things become much more hit and miss. Let us use boldface to denote vectors in $\mathbb{C}^d$ or $\mathbb{N}^d$, and use multi-index notation, so that $a_{\mathbf{r}}$ denotes the multi-index $a_{r_1,\ldots,r_d}$ and $\mathbf{z}^{\mathbf{r}}$ denotes the product $z_1^{r_1} \cdots z_d^{r_d}$ which we shall sometimes write in expanded form for clarity. The generating function $F : \mathbb{C}^d \to \mathbb{C}$ is defined analogously to the one-dimensional generating function by

$$F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}.$$

Surprisingly, techniques for extracting asymptotics of $\{a_{\mathbf{r}}\}$ from the analytic properties of F were, until recently, almost entirely missing. In a survey of asymptotic methods, Bender (1974) says:

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> Practically nothing is known about asymptotics for recursions in two variables even when a generating function is available. Techniques for obtaining asymptotics from bivariate generating functions would be quite useful.

In the intervening 25 years, some results have appeared, addressing chiefly the case where the array $\{a_{\mathbf{r}}\}$ obeys a central limit theorem. Common to all of these is the following method. Treat $\{a_{\mathbf{r}}\}$ as a sequence of $(d-1)$-dimensional arrays indexed by r_d; show that the n^{th} $(d-1)$-dimensional generating function is roughly the n^{th} power of a given function; use this approximation to invert the characteristic function and obtain a Central Limit Theorem. We refer to these methods as GF-sequence methods. The other body of work on multivariate sequences, which we will call the diagonal method, is based on algebraic extraction of the diagonal, as found in Hautus & Klarner (1971) (see also Furstenberg (1967) and later Lipshitz (1988) for an algebraic description of the scope of this method; variants are described in Stanley (1999) and Pippenger (2000)).

The most fundamental GF-sequence result is probably Bender & Richmond (1983), with extensions appearing in later work of the same authors. Flajolet & Sedgewick (1997) present a version of the same idea which holds in much greater generality. Gao & Richmond (1992) go beyond the central limit case, using the transfer theorems of Flajolet & Odlyzko (1990) to handle functions that are products of powers with powers of logs. Recent work of Bender and Richmond (Bender & Richmond 1996, Bender & Richmond 1999) extends the applicability of the central limit results to many problems of combinatorial interest; see also (Hwang 1995, Hwang 1998*b*), where more precise asymptotics are given, and Hwang (1998*a*), which extends some results to the combinatorial schemes of Flajolet & Soria (1993). This does not exhaust the recent work on the problem of multivariable coefficient extraction, but does circumscribe it.

The present paper, together with forthcoming companion papers, takes aim at a large class of multivariable coefficient extraction problems, for which a fair amount of information can be read off in a systematic way. An ultimate goal (not our only goal) is to systematize the extraction of multivariate asymptotics sufficiently that it may be automated, say in Maple. Everything we do, we do with complex contour integration. In this regard, our methods are most similar to those of Bertozzi & McKenna (1993), who, as we do, provide a general framework for harnessing the multivariable theory of residues for exact and series computation of coefficients. A more detailed description of our method will be given in Section 3, but here is an outline.

(1) Use the multidimensional Cauchy integral formula to represent $a_{\mathbf{r}}$ as an integral over a d-dimensional torus inside $\mathbb{C}^d$.

(2) Expand the surface of integration across a point $\mathbf{z}$ where F is singular, and use the residue theorem to represent $a_{\mathbf{r}}$ as a $(d-1)$-dimensional integral of one-variable residues. The choice of $\mathbf{z}$ determines the directions in which asymptotics may be computed.

(3) Put this in the form of an integral $\int \exp(\lambda f(\mathbf{z}))\psi(\mathbf{z})\, d\mathbf{z}$ for which the large-λ asymptotics can be read off from the theory of oscillating integrals.

In the rest of this introductory section, we describe the scope of our methods. Figure 1 depicts a classification of generating functions and illustrates the remainder of this paragraph. If a formal power series is nowhere convergent, analytic methods are useless. Among those power series converging in some neighborhood of the origin, there are three possibilities: a function may be entire, may have singularities around which analytic continuations exist, or it may be defined only on some bounded subset of $\mathbb{C}^d$. Our methods are tailored to the second class. The third class, although in some sense generic, seldom arises in any problem for which the generating function may be effectively described. Incomplete asymptotic information is available via Darboux' method; details of this method in the univariate case are given in

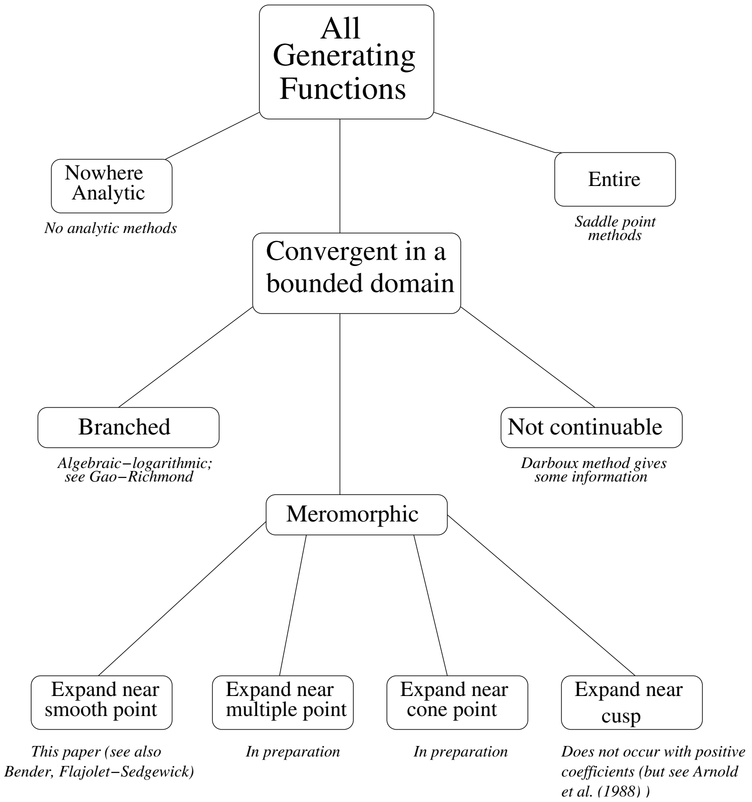

FIGURE 1. Classification of generating functions

Henrici (1977) and Odlyzko (1995). The first class can and does arise frequently. Our methods are simply not equipped to handle entire functions, and systematizing the asymptotic analysis of coefficients of entire generating functions remains an important open problem.

For the remainder of this paper, we will assume that the formal power series F converges in a neighborhood of the origin and may be analytically continued everywhere except a set $\mathcal{V}$ of complex dimension $d-1$ which we call the *singular variety*. The point $\mathbf{z}$ in step 2 is an element of $\mathcal{V}$, and the behavior of $\mathcal{V}$ near $\mathbf{z}$ greatly affects the subsequent analysis in step 3. This paper addresses the case where $\mathbf{z}$ is a smooth point of $\mathcal{V}$ at which F has a pole. The forthcoming companion papers will address cases where $\mathbf{z}$ is a multiple point or a cone point. We do not know whether cases where $\mathbf{z}$ is a cusp of $\mathcal{V}$ arise (in certain degenerate situations) but if so, one might hope to find methods of analysis in Arnol'd, Guseĭn-Zade & Varchenko (1988).

The chief purpose of this study is to give a solution to the problem of asymptotic evaluation of coefficients that is as general as possible. An important part of this is re-derivation in a general setting of results obtainable via GF-sequence or *ad hoc* methods. We show in Section 6 how unifying these results allows us to show that our method successfully finds asymptotics for every function in a certain large class. Familiar examples from this class include: lattice path counting, various known generating functions for polyominos and stacked balls, enumeration of Catalan trees by number of components or surjections by image cardinality (see Flajolet & Sedgewick (1997)), stopping times for certain random walks (see Larsen & Lyons (1999)), as well as the examples given in the GF-sequence papers of Bender (1973) and Bender & Richmond (1983): ordered set partitions enumerated by number of blocks, permutations enumerated by rises, and Tutte polynomials of recursive sequences of graphs.

Nevertheless, our pursuit of this problem was also motivated by some specific applications which we mention briefly now and discuss more thoroughly later. These are cases where known methods do not suffice to obtain complete asymptotic information. There is a class of tiling enumeration problems for which an explicit three variable rational generating function may be obtained. This class includes the Aztec Diamond domino tilings of Cohn, Elkies & Propp (1996). Asymptotics in the so-called *region of fixation* are obtained from analysis of the smooth points of $\mathcal{V}$ (Theorem 3.5 below), while asymptotics in the region of positive entropy are derived from analysis of the cone point. Cohn& Pemantle (2001) applies a cone point analysis to a tiling enumeration problem for which the only previous results are some pictures via simulation (http://www.math.harvard.edu/˜cohn/picture.gif). Another motivation has been to solve the general multivariable linear recursion. Depending on whether one allows forward recursion in some of the variables, one typically obtains either rational or algebraic generating functions (less well-behaved fucntions can also result — see Bousquet-Mélou & Petkovšek (2000) for more details). The general rational function may have any of the types of singularities mentioned above: smooth points, nodes, cones, cusps, branchpoints, etc. Even the simple rational generating function $1/(3 - 3z - w + z^2)$ of Example 3.4 requires two separate analyses in order to get asymptotics in all directions. We shall see that Theorem 3.1 gives asymptotics in one region, while Theorem 3.3 is required for other directions.

Asymptotics derived near smooth pole points nearly always exhibit central limit behavior. Smooth pole points are the topic of this first paper, and are exactly the case to which existing methods may apply. While one function of this paper is to lay foundations for the cases in which the singularity is more complicated, there are several ways in which it improves upon available analyses of the smooth case.

First, most of the existing results assume that the singular point $\mathbf{z} \in \mathcal{V}$ has positive real coordinates, and that it is strictly minimal in a sense defined in the next section. This assumption often holds when the coefficients $\{a_\mathbf{r}\}$ are nonnegative reals, though it will fail if, for example, there is any periodicity. The assumption always fails when the coefficients $\{a_\mathbf{r}\}$ have mixed signs, as is the case for example with the generating functions $(1 - zw)/(1 - 2zw + w^2)$ and $1/(1 - 2zw + w^2)$ for the Chebyshev polynomials of the first and second kinds (Comtet 1974, page 50). GF-sequence methods may be adapted to some of these situations. Indeed, the presentation of these methods by Flajolet & Sedgewick (1997, Theorem 9.7) accomplishes this adaptation in great generality. But certainly there are cases such as the rational generating function $1/(1 - z - w + \beta zw)$, where the points $\mathbf{z}$ with given moduli form a continuum and standard GF-sequence methods are not sufficient.

Second, our methods obtain automatically a full asymptotic expansion of $a_{r_1,\ldots,r_d}$ in decreasing powers of the indices r_j. This is certainly not inherent in the existing results, whose relatively short proofs involve inversion of the characteristic function (see however Hwang (1995) and Hwang (1996) for something in this direction). The expansion to n terms is completely effective in terms of the first n partial derivatives of $1/F$ at $\mathbf{z}$, as is the error bound.

Third, these results explicitly cover the case where the pole at $\mathbf{z}$ has order greater than 1. The behavior in this case is not according to the central limit theorem. The only existing work addressing this case is Gao & Richmond (1992), and they require nonnegativity assumptions, as mentioned above. In the case where $F = G^k$ is an exact power, one could attempt first to solve the problem for G and then to take the k-fold convolution. This is much harder than the present approach, as may be seen by the rather involved computation in Cohn et al. (1996).

Fourth, the potential for increasing the scope to new applications seems greater for contour methods than for GF-sequence methods. The contour method reduces the asymptotic problem to the problem of an oscillating integral near a singularity, which can almost certainly be done. By contrast, the GF-sequence method requires first an understanding of the sequence of $(d-1)$-dimensional generating functions arising from the given d-dimensional generating function, and then another result in order to transfer this information to asymptotics of the coefficients $a_\mathbf{r}$.

Fifth, although our results in the case of smooth pole points are often similar to those obtained by GF-sequence methods, our hypotheses are quite different. In Section 6 we show how our hypotheses may be universally established for functions that generate nonnegative values and are meromorphic through their domain of convergence.

Finally, we compare our method to recent results from the diagonal method. It is known (Lipshitz 1988) that the diagonal sequence $a_{n,n,\ldots,n}$ of a multivariate sequence with rational generating function has a generating function satisfying a linear differential equation over rational functions. Much is known about how to compute this equation (see for example Chyzak & Salvy (1998)). If one wants asymptotics on the diagonal, or in any direction where the coordinate ratios are rational numbers with small denominators, then these methods give results that are in theory at least as good as ours. The method, however, is inherently non-uniform in the direction, so there is no hope of extending it to larger sets of directions, which is what we accomplish in the present work.

The remainder of the paper is organized as follows. In the next section we set forth notation and define the terms necessary to state the main results of the paper. The main results are stated in Section 3, and examples are given. The next section contains a proof of these results, modulo the computation of some oscillating integrals. This computation is carried out in Section 5. Section 6 outlines some details of taxonomy and discusses universality of the method of complex contour integration. The final section discusses some open problems.

2. Notation and Preliminaries

The main results of this paper give asymptotics valid under certain geometric assumptions on $\mathcal{V}$ and computable from some quantities that are in turn effectively computable from the generating function F. Thus in addition to setting out basic notation, we need to define some terms related to the geometry of $\mathcal{V}$ and some quantities associated with F.

2.1. **Notation.** Throughout the paper, F will denote a function on $\mathbb{C}^d$ analytic in a neighborhood of the origin, with Taylor series representation $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. We also suppose that there are analytic functions G, H such that $F = G/H$ wherever F is defined. The (open) domain of convergence of the power series will be denoted $\mathcal{D}$. For $\mathbf{z} \in \mathbb{C}^d$, let $T(\mathbf{z})$ denote the torus consisting of points $\mathbf{w}$ with $|w_j| = |z_j|$ for $1 \leq j \leq d$ and let $D(\mathbf{z})$ denote the closed polydisk of points $\mathbf{w}$ with $|w_j| \leq |z_j|$ for $1 \leq j \leq d$. Recall (see Hörmander (1990)) that the domain $\mathcal{D}$ is a union of tori $T(\mathbf{z})$ and is logarithmically convex, that is, the set

$$\log\mathcal{D} := \{\mathbf{x} \in \mathbb{R}^d : (e^{x_1}, \ldots, e^{x_d}) \in \mathcal{D}\}$$

is a convex subset of $\mathbb{R}^d$ and is an order ideal (a subset closed under $\leq$ in the coordinatewise partial order).

Many of our examples will be in dimension 2, in which case we will often use z and w in place of z_1 and z_2, use (z, w) in place of $\mathbf{z}$, and use (r, s) in place of (r_1, r_2). We sometimes need to treat $\mathbb{C}^d$ as $\mathbb{C}^{d-1} \times \mathbb{C}$ (although symmetry of the coordinates is preserved most of the time). Accordingly, when the dimension is greater than 2, we use $\widehat{\mathbf{z}}$ to denote $(z_1, \ldots, z_{d-1})$. Partial derivatives will be denoted H_1 for $\frac{\partial H}{\partial z_1}$ and so forth; in dimension 2 we will also use H_z and H_w.

As is usual for asymptotic analyses, we let $f \sim g$ denote $f/g \to 1$, with the limit taken at infinity unless otherwise specified. The function f is said to be *rapidly decreasing* if $f(x) = O(x^{-N})$ for every N, and is said to be *exponentially decreasing* if $f(x) = O(e^{-cx})$ for some $c > 0$. We also use the symbol "$\sim$" to denote asymptotic expansion. Thus

$$f \sim \sum b_n g_n$$

is normally taken to mean that $f - \sum_{n=0}^N b_n g_n = o(b_N g_N)$, where $b_n \in \mathbb{C}$ and $\{g_n\}$ is a fixed sequence of functions such that $g_{n+1} = o(g_n)$ for each n. We broaden this to allow $b_n = 0$ when $n \neq 0$, so that the remainder term need only be $o(g_n)$ and not $o(b_n g_n)$. In particular, if

$$f(x) \sim g(x) \cdot \sum_{n=0}^{\infty} c_n x^{-n}$$

with $c_0 = 1$, then we say we have obtained a full asymptotic expansion for f in decreasing powers of x with leading term g.

2.2. **Geometry of $\mathcal{V}$.** As in the one-dimensional case, the points of $\mathcal{V}$ nearest the origin are the most important. Accordingly we define a point $\mathbf{z} \in \mathcal{V}$ to be *minimal* if $\mathcal{V} \cap D(\mathbf{z}) \subseteq T(\mathbf{z})$ and, in addition, G and H are analytic on a neighborhood of $D(\mathbf{z})$ [1]. We say that $\mathbf{z}$ is *locally minimal* if the analogous relation holds with $\mathcal{V}$ replaced by a neighborhood of $\mathbf{z}$ in $\mathcal{V}$. Divide the minimal points of $\mathcal{V}$ into three types. Say that $\mathbf{z}$ is *strictly minimal*, *finitely minimal* or *toral*, according to whether the cardinality of $\mathcal{V} \cap D(\mathbf{z})$ is 1, finite, or infinite. When infinite, the intersection must be uncountable. If $\mathbf{z}$ is a minimal point of $\mathcal{V}$ then the interior of $D(\mathbf{z})$ is contained in $\mathcal{D}$, so the assumption that G and H are analytic on a neighborhood of $D(\mathbf{z})$ is just a little stronger than what is true automatically.

A *simple pole* of F is a point $\mathbf{z} \in \mathcal{V}$ where H vanishes to order 1. Equivalently, the gradient ∇H does not vanish. Let $\mathbf{z}$ be a simple pole of F and assume for specificity that H_d is nonzero at $\mathbf{z}$. By the implicit function theorem, there is a neighborhood of $\mathbf{z}$ where $\mathcal{V}$ may be parametrized by $z_d = g(z_1, \ldots, z_{d-1})$ for some analytic function g. We will always use g to denote this parametrization.

We shall see later (in the proof of Theorem 6.3) that under some hypotheses on F, minimal points of $\mathcal{V}$ are always found in the positive real orthant. A relation true in complete generality is the following.

Lemma 2.1. *Let $\mathbf{z}$ be a simple pole of F and suppose that $z_d H_d$ does not vanish there. If $\mathbf{z}$ is locally minimal then for all $j < d$, the quantity $z_j H_j / (z_d H_d)$ is real and nonnegative.*

Proof. Given θ and j, let $\mathbf{z}^{(\theta)}$ be the result of varying $\mathbf{z}$ by multiplying the j^{th} coordinate by $e^{i\theta}$ and adjusting the last coordinate so as to remain on $\mathcal{V}$ (that is, $z_d^{(\theta)} = g(z_1, \ldots, z_{j-1}, z_j e^{i\theta}, z_{j+1}, \ldots, z_{d-1})$). Differentiating the relation $H(\mathbf{z}^{(\theta)}) = 0$ implicitly with respect to θ at 0 yields

$$iz_j H_j + H_d \frac{dz_d^{(\theta)}}{d\theta} = 0. \tag{2.1}$$

By minimality of $\mathbf{z}$, we know that the modulus of $z_d^{(\theta)}$ has a minimum at $\theta = 0$, hence $(dz_d^{(\theta)}/d\theta)/z_d$ is purely imaginary. Plugging this into (2.1) proves that $z_j H_j/(z_d H_d)$ is real. If $z_j H_j/(z_d H_d) = -\beta < 0$ then $\mathcal{V}$ has a tangent vector at $\mathbf{z}$ in the direction $-z_j e_j - \beta z_d e_d$, where e_j is the j^{th} coordinate vector. This contradicts minimality. Hence $z_j H_j/(z_d H_d) \geq 0$. □

Definition 2.2. Suppose that $\mathbf{z}$ has all coordinates nonzero. Define $\mathbf{dir}(\mathbf{z})$ to be the equivalence class of (complex) scalar multiples of the vector $(z_1 H_1, \ldots, z_d H_d)$.

The previous lemma shows that when $\mathbf{z}$ is a minimal pole of F with nonzero coordinates, $\mathbf{dir}(\mathbf{z})$ can be considered as a well defined element of $\mathbb{RP}^{d-1}$.

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[1] This last condition is satisfied by all globally meromorphic F; the greater generality allows us to cover examples such as the generating function for self-avoiding random walks (Chayes & Chayes 1986) or percolation paths in the subcritical regime (Campanino, Chayes & Chayes 1991). In these cases, all the work is in showing the function is meromorphic in a neighborhood of $D(\mathbf{z})$. Without further knowledge, the authors then conclude central limit behavior.

TABLE 1. Reserved notation in remainder of this article

Given information:

the function F in the form G/H

First level:

g parametrizes the zero set, $\mathcal{V}$ of H

$\mathbf{dir}(\mathbf{z})$ is the coordinatewise product $(\nabla H) \cdot (\mathbf{z})$ in projective space

Second level:

ψ is the residue in z_d of F/z_d at points $(\widehat{\mathbf{z}}, g(\widehat{\mathbf{z}}))$

f is $\log g$, plus a term linear in $\log z_j$ and depending on $\mathbf{r}$.

Third level:

$\widetilde{\psi}, \tilde{g}$ and $\tilde{f}$ are ψ, g and f expressed in terms of θ

The importance of $\mathbf{dir}$ is that analysis of F near $\mathbf{z}$ yields asymptotic information about $a_{\mathbf{r}}$ with $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$ (of course, since $\mathbf{r}$ has integer coordinates, for a given $\mathbf{r}$ this latter inclusion will fail to hold for most values of $\mathbf{z}$, so that $\mathbf{z}$ must be chosen appropriately as we shall see later). The function $\mathbf{dir}$ appears in GF-sequence method literature as $\mathbf{m}$. When $\mathbf{z} \in \partial\mathcal{D}$ is on the boundary of the domain of convergence, $\mathbf{dir}(\mathbf{z})$ is the normal to the support hyperplane of the convex set $\log\mathcal{D}$ at the point $(\log|z_1|, \ldots, \log|z_d|)$.

We now define a few more quantities associated with F and g. Again, we shall reserve the names of these functions, so as not to burden the notation with subscripts and arguments. If $\mathbf{z}$ is a simple pole of F with $z_d H_d$ not vanishing there, define a function ψ on a neighborhood of $\widehat{\mathbf{z}}$ by

$$\psi(\widehat{\mathbf{w}}) = - \lim_{w \to g(\widehat{\mathbf{w}})} (w - g(\widehat{\mathbf{w}})) \frac{F(\widehat{\mathbf{w}}, w)}{w}. \tag{2.2}$$

Suppose now that $\widehat{\mathbf{w}} \in T(\widehat{\mathbf{z}})$ and write $w_j = z_j e^{i\theta_j}$. For fixed $\mathbf{r}$ with $r_d \neq 0$, define a function f on a neighborhood of $\widehat{\mathbf{z}}$ in $T(\widehat{\mathbf{z}})$ by

$$f(\widehat{\mathbf{w}}) = \log\left(\frac{g(\widehat{\mathbf{w}})}{g(\widehat{\mathbf{z}})}\right) + i \sum_{j=1}^{d-1} \frac{r_j}{r_d} \theta_j. \tag{2.3}$$

We shall be parametrizing integrals over $T(\widehat{\mathbf{z}})$ by θ, so we shall want the above function expressed in terms of $\widehat{\theta}$. We therefore compose with the map M taking $\widehat{\theta}$ to $\widehat{\mathbf{w}}$ defined by $M(\theta_1, \ldots, \theta_{d-1}) = (z_1 e^{i\theta_1}, \ldots, z_{d-1} e^{i\theta_{d-1}})$, and define the functions $\tilde{g} := g \circ M, \tilde{f} := f \circ M, \widetilde{\psi} := \psi \circ M$.

Although it is not obvious yet, $\tilde{f}$ will always vanish at $\mathbf{0}$ to at least two orders (Lemma 4.2 below), and the hypothesis $Q \neq 0$ in Theorem 3.1 is equivalent to $\tilde{f}$ having nonvanishing quadratic term. For ease of reference, Table 1 summarizes the foregoing definitions, stratified by how many times the given data G and H have been manipulated.

3. STATEMENT OF RESULTS, WITH EXAMPLES

Before going on, we pause to state a prototype of our results in the simplest possible setting, namely where the number of variables is 2, the functions G and g are as nondegenerate as possible, and only the leading term asymptotic is given. The proof is in Section 4.

Theorem 3.1. *Let $F = G/H$ be a meromorphic function of two variables, not singular at the origin. Define*

$$Q(z,w) := -w^2 H_w^2 z H_z - w H_w z^2 H_z^2 - w^2 z^2 \left(H_w^2 H_{zz} + H_z^2 H_{ww} - 2 H_z H_w H_{zw} \right).$$

Then

$$a_{r,s} \sim \frac{G(z,w)}{\sqrt{2\pi}} z^{-r} w^{-s} \sqrt{\frac{-wH_w}{sQ}}$$

uniformly as (z,w) varies over a compact set of strictly minimal, simple poles of F on which Q and G are nonvanishing, and $(r,s) \in \mathbf{dir}(z,w)$.

Remark. Usually the expression in the radical will be positive real, as will the coefficients a_{rs}. The result is true in general, though, as long as the square root is taken to be $-(wH_w)^{-1}$ times the principal root of $(-wH_w^3)/(sQ)$. Also note that when $(r,s) \in \mathbf{dir}(z,w)$ then the expression wH_w/s is coordinate-invariant, that is, equal to zH_z/r. Thus the given expression for $a_{r,s}$ has the expected symmetry.

Example 3.2 (Lattice paths)**.** Let $a_{r,s}$ be the number of nearest-neighbor paths from the origin to (r,s) moving only north, east and northeast; these are sometimes called *Delannoy numbers* (Stanley 1999, page 185). The generating function is $F(z,w) = 1/(1-z-w-zw)$. The zero set $\mathcal{V}$ of $H = 1-z-w-zw$ is given by $w = (1-z)/(1+z)$, and the minimal points of $\mathcal{V}$ are those where $w \in [0,1]$. With the help of relations that hold when $\mathbf{z} \in \mathcal{V}$ we may compute as follows.

$$\begin{aligned} H_z &= -1-w \\ -zH_z &= 1-w \\ Q &= (1-z)(1-w)(1-zw) \\ \frac{zH_z}{wH_w} &= \frac{1-w}{1-z} = \frac{1-w^2}{2w} \end{aligned}$$

with H_w and $-wH_w$ given by reversing z and w. As z varies over $[\varepsilon, 1-\varepsilon]$, the functions Q and $G := 1$ do not vanish. The minimal pair (z,w) that solves $(r,s) \in \mathbf{dir}(z,w)$ is given by $z = (\sqrt{r^2+s^2}-s)/r$ and $w = (\sqrt{r^2+s^2}-r)/s$. Theorem 3.1 then gives

$$\begin{aligned} a_{rs} &\sim \left(\frac{\sqrt{r^2+s^2}-s}{r} \right)^{-r} \left(\frac{\sqrt{r^2+s^2}-r}{s} \right)^{-s} \sqrt{\frac{1}{2\pi}} \sqrt{\frac{1-z}{s} \frac{1}{1-zw}} \\ &= \left(\frac{\sqrt{r^2+s^2}-s}{r} \right)^{-r} \left(\frac{\sqrt{r^2+s^2}-r}{s} \right)^{-s} \sqrt{\frac{1}{2\pi}} \sqrt{\frac{rs}{(r+s-\sqrt{r^2+s^2})^2 \sqrt{r^2+s^2}}}, \end{aligned}$$

uniformly when r/s and s/r remain bounded. In particular, when $r = s = n$, this gives the following formula for the n^{th} diagonal coefficient (which may alternatively be obtained by computing the diagonal generating function $(1-6s+s^2)^{-1/2}$ according to the method given in Stanley (1999, Section 6.3):

$$(\sqrt{2}-1)^{-2n} \sqrt{\frac{1}{2\pi} \frac{2^{-1/4}}{2-\sqrt{2}}}.$$

The computations in Theorem 3.1 in terms of the values and derivatives of G and H are explicit. As we state more general theorems, it becomes cumbersome and in fact obfuscating to give formulae for the expansion coefficients directly in terms of derivatives of G and H. This is one reason we have already introduced the functions in Table 1. It should be emphasized, however, that while we use higher level quantities in the statements of subsequent theorems, each expansion coefficient can be computed from finitely many derivatives of G and H. We begin with a relatively explicit computation for the general two-variable case.

For k at least 2, we define constants

$$A_+(k,l) := \frac{1}{k}\Gamma\left(\frac{l+1}{k}\right) \tag{3.1}$$

$$A(k,l) := \frac{1}{k}\Gamma\left(\frac{l+1}{k}\right)\left(1+e^{\operatorname{sgn}\operatorname{Arg}(c_k)i\pi(\frac{l+1}{k})}\right) \text{ if } k \text{ is odd,} \tag{3.2}$$

$$A(k,l) := \frac{2}{k}\Gamma\left(\frac{l+1}{k}\right) \text{ if } k,l \text{ are even,} \tag{3.3}$$

$$A(k,l) := 0 \text{ if } k \text{ is even and } l \text{ is odd.}$$

In situations of interest to us, there is a neighborhood of 0 in which $\operatorname{Re} f \geq 0$. In that case let

$$y(x) = f(x)^{1/k} = c_k^{1/k} x\left(1+\frac{f(x)-c_kx^k}{c_kx^k}\right)^{1/k}, \tag{3.4}$$

where c_k is the first nonvanishing Taylor coefficient of $f(x) = \sum_{j=k}^\infty c_j x^j$ and the argument of $c_k^{1/k}$ is taken between $-\pi/(2k)$ and $\pi/(2k)$. We also let η denote the inverse function to y, and let $\{b_j\}$ denote the Taylor coefficients of $(\widetilde{\psi}\circ\eta)\cdot\eta'$. Clearly each $\{b_j\}$ is determined by finitely many partial derivatives of G and H, and the index l_0 of the first nonvanishing b_l is the same as the order of vanishing of $\widetilde{\psi}$ at 0. The coefficients b_l are easily computed from the coefficients $\tilde{b}_j := \widetilde{\psi}^{(j)}(0)/j!$ and $c_j := \tilde{f}^{(j)}(0)/j!$; in particular, if $\tilde{f} \sim c_k x^k$ near 0 then

$$b_{l_0} = \tilde{b}_{l_0} c_k^{-1/k}. \tag{3.5}$$

Theorem 3.3. *Let $F = G/H = \sum a_{rs} z^r w^s$ have a strictly minimal, simple pole at (z,w). Let k be the order of vanishing of $\tilde{f}$ at 0. Let l_0 be the order to which G vanishes near (z,w) on $\mathcal{V}$, that is, the largest l such that $G(z',w') = O(|z-z'|^l + |w-w'|^l)$ as $(z',w') \to (z,w)$ in $\mathcal{V}$. Then there is a full asymptotic expansion*

$$a_{r,s} \sim \frac{1}{2\pi} z^{-r} w^{-s} \sum_{l\geq l_0} \mathcal{A}(k,l) b_l s^{-(l+1)/k}, \tag{3.6}$$

where $\mathcal{A}(k,l)$ denotes $A(k,l)$ if $\operatorname{Im}\{c_k\} \geq 0$ and $\overline{A(k,l)}$ otherwise. The expansion is uniform as (z,w) varies over a compact set of strictly minimal poles with $(r,s) \in \mathbf{dir}(z,w)$ and k and l_0 not changing.

Example 3.4 (Cube root asymptotics). Let $F(z,w) = 1/(3-3z-w+z^2)$. The set $\mathcal{V}$ is the set $\{w = z^2-3z+3\}$ and $g(z) = z^2-3z+3$. The point $(1,1)$ is in $\mathcal{V}$, indicating that the maximal exponential growth rate will be zero. Indeed, for directions above the diagonal, Theorem 3.1 or 3.3 may be used at the minimal points $\{(z,g(z)) : 0 < z < 1\}$, while each direction below the diagonal corresponds to a pair of complex minimal points fitting the hypotheses of Corollary 3.7; the result is that the coefficients decay exponentially at a rate that is uniform over compact subsets of directions not containing the diagonal.

The interesting behavior is near the diagonal. The relevant minimal point is $(1,1)$, where $z^r w^s \equiv 1$ and the decay is sub-exponential. Computing $\tilde{f}''(0)$ via equation (4.8) below gives

$$\tilde{f}''(z) = -3\frac{z(z^2-4z+3)}{(z^2-3z+3)^2}.$$

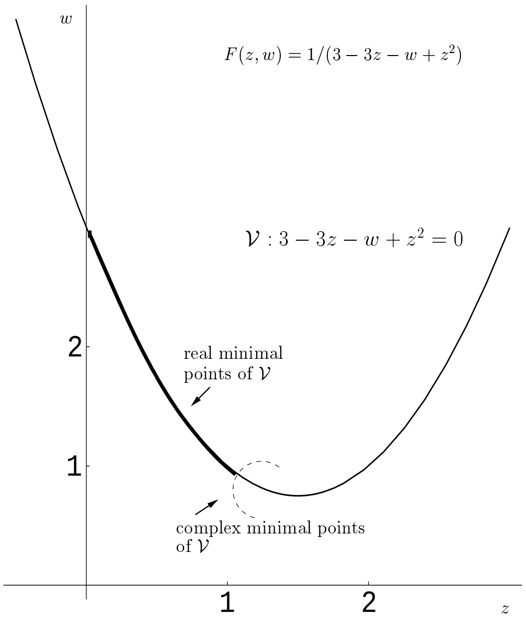

FIGURE 2. $\mathcal{V}$ for Example 3.4

This vanishes when $z = 1$, and computing further, we find that $\tilde{f}$ vanishes to order exactly 3 here, with $c_3 := \tilde{f}'''(0)/3! = i$. Along with $\widetilde{\psi}(0) = 1$, this then results in an asymptotic expansion whose leading term is given by

$$a_{r,r} \sim \frac{1}{2\pi} A(3,0) i^{-1/3} (1 + e^{-i\pi/3}) r^{-1/3} = \frac{\Gamma(2/3)}{6\sqrt{3}\pi} r^{-1/3} .$$

Combinatorially natural examples of such cubic singularities have been discussed by Banderier, Flajolet, Schaeffer & Soria (2000) under the name of "Airy phenomena". In Section 7 we discuss the question of computing asymptotics "in the gaps" so as to be able to conclude that $\limsup \log a_{\mathbf{r}} / \log |\mathbf{r}| = -1/3$ or even $\limsup |\mathbf{r}|^{1/3} a_{\mathbf{r}} = \frac{\Gamma(2/3)}{6\sqrt{3}\pi}$.

For more than two variables a result holds similar to the two-variable result.

Theorem 3.5. *Let $F = G/H = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ have a strictly minimal, simple pole at $\mathbf{z}$. Suppose $z_d H_d$ does not vanish. If the Hessian of $\tilde{f}$ at $\mathbf{z}$ is nonsingular, then there is an expansion*

$$a_{\mathbf{r}} \sim \mathbf{z}^{-\mathbf{r}} \sum_{l \geq l_0} C_l r_d^{(1-d-l)/2}$$

where l_0 is the degree to which G vanishes on $\mathcal{V}$ near the point $\mathbf{z}$. When G does not vanish at $\mathbf{z}$ then $l_0 = 0$ and

$$C_0 = (2\pi)^{(1-d)/2} \, \mathcal{H}^{-1/2} \, \frac{G(\mathbf{z})}{z_d H_d}$$

where $\mathcal{H}$ is the determinant of the Hessian at $\mathbf{z}$.

Example 3.6 (Domino tilings)**.** Random perfect tilings of planar regions by dominos have been a subject of some interest, since the analysis by Fisher (1961) of this model for dimer packing uncovered an exact expression for the partition function of the ensemble. A generating function is given in Cohn et al. (1996)

which allowed the authors to determine, after some cumbersome analysis, which parts of a diamond-shaped region (a union of lattice squares approximating the region $|x| + |y| \leq k$) were asymptotically deterministic and which contained randomness in the limit as the edge size of the diamond grew.

An easier analysis in the region of non-randomness is available via Theorem 3.5 together with a slightly more informative generating function than was used by Cohn et al. (1996). In particular, let

$$F(x, y, z) = \sum_{t=0}^{\infty} \sum_{|r|+|s| \leq t} a_{r,s,t} x^r y^s z^t$$

be the generating function for the probability $a_{r,s,t}$ that the tile covering position (r, s) of a random diamond of size t will be horizontal. For brevity, we omit formal descriptions of the diamond and its indexing. We remark that the use of negative indices (for each fixed t, the sum $\sum_{|r|+|s|\leq t}$ is a polynomial in x, x^{-1}, y and y^{-1}) does not require any alterations in the theory (see Cohn & Pemantle (2001) for justification), and that the natural way to parametrize directions is by the pair $(r/t, s/t)$ which varies over the diamond $|r/t| + |s/t| = 1$. From Cohn et al. (1996) or from the generation algorithm in Gessel, Ionescu & Propp (1995), one finds

$$F(x, y, z) = \frac{z/2}{(1 - yz)(1 - (x + x^{-1} + y + y^{-1})z/2 + z^2)} .$$

Cohn & Pemantle (2001) show that whenever (r, s, t) satisfy

$$t = \sqrt{r^2 + s^2 + 2\sqrt{r^2 + 1}\sqrt{s^2 + 1} - s}\,,$$

then there is a smooth minimal point (x, y, z) on the pole manifold of F for which $(r, s, t) \in \mathbf{dir}(x, y, z)$, yielding exponential decay in the direction (r, s, t). The set of directions so parametrized turns out to be the region between the diamond $|r/t| + |s/t| = 1$ and the inscribed circle $(r/t)^2 + (s/t)^2 = 1/2$. Thus they recover the description of the region of non-randomness as the complement of the inscribed circle. They also obtain descriptions of the region of fixation for related tiling problems in which no other analysis has been carried out.

The extension of all of the above results to finitely minimal points is routine.

Corollary 3.7. *Suppose* $\mathbf{z}$ *is a finitely minimal point of* $\mathcal{V}$ *with* $\mathcal{V} \cap T(\mathbf{z}) = \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$. *Then*

$$a_{\mathbf{r}} \sim \sum_{j=1}^{n} E_j(\mathbf{r})$$

where $E_j(\mathbf{r})$ *is the asymptotic expression given by the previous theorems with* $\mathbf{z} = \mathbf{z}_j$. *In other words, if there are finitely many points on* $\mathcal{V} \cap T(\mathbf{z})$, *then sum the contributions as if each were strictly minimal.* □

Example 3.8 (Chebyshev polynomials)**.** Let $F(z, w) = 1/(1 - 2zw + w^2)$ be the generating function for Chebyshev polynomials of the second kind (Comtet 1974); of course asymptotics for these are well known and easy to derive by other means. To use Corollary 3.7, first find the minimal points for the direction (r, s), which are $(i(\beta - \beta^{-1})/2, i\beta)$ for $\beta = \pm\sqrt{\frac{s-r}{s+r}}$. Computing $Q = 4a^2(1 - a^2)$ and summing the two contributions then gives

$$a_{rs} \sim \sqrt{\frac{2}{\pi}}(-1)^{(s-r)/2} \left(\frac{2r}{\sqrt{s^2 - r^2}}\right)^{-r} \left(\sqrt{\frac{s-r}{s+r}}\right)^{-s} \sqrt{\frac{s+r}{r(s-r)}}$$

when $r + s$ is even and zero otherwise, uniformly as r/s varies over compact subsets of $(0, 1)$.

4. Proofs of main results

In this section we prove Theorems 3.1–3.5 modulo some technical results on oscillatory integrals, which are stated and proved in Section 5.

In each case it is straightforward to see that if $\mathbf{r}$ is bounded away from $\mathbf{dir}(\mathbf{z})$, then $|\mathbf{z}^{\mathbf{r}}\mathbf{a}_{\mathbf{r}}|$ decreases exponentially. Indeed, consider Cauchy's formula

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{w}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{w})\, d\mathbf{w} \tag{4.1}$$

where the multi-exponent $\mathbf{r}-\mathbf{1}$ means $(r_1-1,\dots,r_d-1)$. If $\mathbf{z}$ is a minimal point of $\mathcal{V}$, then letting T approach $T(\mathbf{z})$ from the inside, we see that $|\mathbf{z}^{\mathbf{r}}|a_{\mathbf{r}}$ does not increase exponentially. If, furthermore, the hyperplane through $(\log|z_1|,\dots,\log|z_d|)$ normal to $\mathbf{r}$ is not a support hyperplane for $\log\mathcal{D}$, then some $\mathbf{x}\in\log\mathcal{D}$ has $\mathbf{x}\cdot\mathbf{r} > (\log|z_1|,\dots,\log|z_d|)\cdot\mathbf{r}$, and integrating on the torus $T(e^{\mathbf{x}})$ shows that $|\mathbf{z}^{\mathbf{r}}|a_{\mathbf{r}}$ decreases exponentially.

Our theorems demonstrate the converse, namely that when the hyperplane normal to $\mathbf{r}$ is a support hyperplane, then $\mathbf{z}^{-\mathbf{r}}$ does give the right exponential order for $a_{\mathbf{r}}$. This is done by evaluating $a_{\mathbf{r}}$.

Theorems 3.1–3.5 all begin with the reduction of an iterated Cauchy integral to an oscillating integral in one fewer dimension.

Lemma 4.1. *Let $\mathbf{z}$ be a strictly minimal simple pole of $F = G/H$. Assume that $z_d H_d \neq 0$. For a neighborhood $\widetilde{\mathcal{N}}$ of $\mathbf{0}$ in $\mathbb{R}^{d-1}$ define a quantity*

$$\Xi := (2\pi)^{1-d}\mathbf{z}^{-\mathbf{r}} \int_{\widetilde{\mathcal{N}}} \exp(-r_d\tilde{f}(\widehat{\theta}))\widetilde{\psi}(\widehat{\theta})\, d\widehat{\theta}. \tag{4.2}$$

Then the quantity

$$|\mathbf{z}^{\mathbf{r}}|\,|a_{\mathbf{r}} - \Xi|$$

decreases exponentially as $\widetilde{\mathcal{N}}$ remains fixed and $\mathbf{r}\to\infty$.

Proof. For $\varepsilon\in(0,|z_d|)$, let T be the torus $T(\mathbf{z})$ shrunk in the last coordinate by ε, that is, the set of $\mathbf{w}$ for which $|w_j| = |z_j|$, $j<d$ and $|w_d| = |z_d|-\varepsilon$. Write Cauchy's formula as an iterated integral

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{T(\widehat{\mathbf{z}})} \widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}-\mathbf{1}} \left[\int_{\mathcal{C}_1} w_d^{-r_d} F(\mathbf{w})\,\frac{dw_d}{w_d}\right] d\widehat{\mathbf{w}}\,. \tag{4.3}$$

Here $\mathcal{C}_1$ is the circle of radius $|z_d|-\varepsilon$. Let $K\subseteq T(\widehat{\mathbf{z}})$ be a compact set not containing $\widehat{\mathbf{z}}$. For each fixed $\widehat{\mathbf{w}}\in K$, the function $F(\widehat{\mathbf{w}},\cdot)$ has radius of convergence greater than $|z_d|$. Hence the inner integral in equation (4.3) is $O(|z_d|+\delta)^{-r_d}$ for some $\delta>0$. By continuity of the radius of convergence,we may integrate over K to see that

$$|\mathbf{z}^{\mathbf{r}}|\int_{K\times\mathcal{C}_1} \mathbf{w}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{w})\,d\mathbf{w}$$

decreases exponentially. Thus if $\mathcal{N}$ is any neighborhood of $\widehat{\mathbf{z}}$ in $T(\widehat{\mathbf{z}})$, the quantity

$$|\mathbf{z}^{\mathbf{r}}|\left|a_{\mathbf{r}} - \left(\frac{1}{2\pi i}\right)^d \int_{\mathcal{N}} \widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}-\mathbf{1}} \left[\int_{\mathcal{C}_1} \frac{F(\mathbf{w})}{w_d^{r_d+1}}\,dw_d\right] d\widehat{\mathbf{w}}\right|$$

decreases exponentially. Thus we have reduced the problem to an integral over a neighborhood of $\widehat{\mathbf{z}}$.

Near $\mathbf{z}$ there is a parametrization $w_d = g(\widehat{\mathbf{w}})$ of $\mathcal{V}$. Let $\mathcal{C}_2$ be the circle of radius $|z_d|+\varepsilon$. Then when $\mathcal{N}$ is sufficiently small compared to ε, the image of $\mathcal{N}$ under g is disjoint from $\mathcal{C}_2$. Fix such a neighborhood

$\mathcal{N}$. For any $\widehat{\mathbf{w}} \in \mathcal{N}$, the function $F(\widehat{\mathbf{w}}, \cdot)$ has a single simple pole in the annulus bounded by $\mathcal{C}_1$ and $\mathcal{C}_2$, occurring at $g(\widehat{\mathbf{w}})$. The residue in the last variable of F at $g(\widehat{\mathbf{w}})$ is equal to

$$R(\widehat{\mathbf{w}}) := -\psi(\widehat{\mathbf{w}}) g(\widehat{\mathbf{w}})^{-r_d} \tag{4.4}$$

where ψ is defined in (2.2). Therefore, for each fixed $\widehat{\mathbf{w}} \in \mathcal{N}$,

$$\int_{\mathcal{C}_1} \frac{F(\mathbf{w})}{w_d^{r_d+1}}\, dw_d = \int_{\mathcal{C}_2} \frac{F(\mathbf{w})}{w_d^{r_d+1}}\, dw_d - 2\pi i R(\widehat{\mathbf{w}}).$$

But $|\mathbf{z}^{\mathbf{r}} \int_{\mathcal{C}_2} F(\mathbf{w}) dw_d / \mathbf{w}^{\mathbf{r}+\mathbf{1}}|$ is bounded by a constant multiple of $(1 + \varepsilon/|z_d|)^{-r_d}$ (the constant depending on the maximum of F on $\mathcal{C}_2$) and hence $|\mathbf{z}^{\mathbf{r}}||a_{\mathbf{r}} - X|$ is exponentially decreasing, where

$$\begin{aligned} X &= (2\pi i)^{1-d} \int_{\mathcal{N}} (\widehat{\mathbf{w}})^{-\widehat{\mathbf{r}}-1} g(\widehat{\mathbf{w}})^{-r_d} \psi(\widehat{\mathbf{w}})\, d\widehat{\mathbf{w}} \\ &= (2\pi i)^{1-d} \mathbf{z}^{-\mathbf{r}} \int_{\mathcal{N}} \frac{\widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}}}{\widehat{\mathbf{z}}^{-\widehat{\mathbf{r}}}} \frac{d\widehat{\mathbf{w}}}{\prod_{j=1}^{d-1} w_j} \left(\frac{g(\widehat{\mathbf{w}})}{g(z_d)} \right)^{-r_d} \psi(\widehat{\mathbf{w}}) \end{aligned} \tag{4.5}$$

Changing variables to $w_j = z_j e^{i\theta_j}$ and $dw_j = i w_j d\theta_j$ turns the quantity X into

$$(2\pi)^{1-d} \mathbf{z}^{-\mathbf{r}} \int_{\widetilde{\mathcal{N}}} \prod_{j=1}^{d-1} e^{-i r_j \theta_j} \widetilde{\psi}(\widehat{\theta}) \left(\frac{g(\widehat{\mathbf{w}})}{g(\widehat{\mathbf{z}})} \right)^{-r_d} d\widehat{\theta}$$

and plugging in the definitions of f and $\tilde{f}$ at (2.3) above yields

$$(2\pi)^{1-d} \mathbf{z}^{-\mathbf{r}} \int_{\widetilde{\mathcal{N}}} \exp(-r_d \tilde{f}(\widehat{\theta})) \widetilde{\psi}(\widehat{\theta})\, d\widehat{\theta}$$

which is none other than Ξ. $\square$

Remark. It is possible to compute from Cauchy's integral formula in a more coordinate-free way as follows. There is a unique holomorphic $(d-1)$-form ω_F on $\mathcal{V}$ for which $\omega \wedge dH = G\, dz_1 \wedge \cdots \wedge dz_d$. Let Ω be a $(d+1)$-manifold that is a homotopy from a small torus to a torus at infinity. Then $M := \Omega \cap \mathcal{V}$ is a $(d-1)$-manifold and $a_{\mathbf{r}} = (2\pi i)^{-d} \int_M \mathbf{w}^{-\mathbf{r}-\mathbf{1}} dF$ in the sense of currents, which is none other than $\int_M \mathbf{w}^{\mathbf{r}-\mathbf{1}} \omega_F$. See Kenyon & Pemantle (2001) for a more thorough discussion of the foregoing. The manifold M is any member of a certain homology class in $\mathcal{V}$ with the coordinate axes removed, and choosing M to pass through the stationary phase point for the integrand replicates the selection of $\mathbf{z}$ with $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. Although more canonical, the coordinate-free method is less suitable for explicit computation, so we do not pursue it further here. Suffice it to point out that the conclusion of Theorem 3.5 may of course be written in terms more evidently symmetric, as was done in Theorem 3.1.

Equation (4.2) is easily recognized as the standard form for an oscillating integral. The only unusual feature is that the phase is neither real nor purely imaginary. This presents no difficulties, but it does necessitate the statement of a result in Section 5 that is a little different from the usual results on purely oscillating integrals, found in, for example, Stein (1993) or Bleistein & Handelsman (1986). We first establish that $\widehat{\theta} = \mathbf{0}$ is a stationary phase point for the function $\tilde{f}$ when $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$.

Lemma 4.2. *The quantity $\tilde{f}(\mathbf{0})$ always vanishes. If $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$ then $\nabla \tilde{f}(\mathbf{0}) = \mathbf{0}$ and the real part of $\tilde{f}$ has a strict minimum at $\mathbf{0}$.*

Proof. The first statement is immediate. To prove the second, let $j \leq d-1$ and see from the definition of f that

$$r_d f_j(\widehat{\mathbf{z}}) = \frac{r_d g_j(\widehat{\mathbf{z}})}{g(\widehat{\mathbf{z}})} + \frac{r_j}{z_j}.$$

By definition of **dir**, the ratio $r_j/(z_jH_j)$ is some constant c independent of j, hence

$$c^{-1}r_d f(\mathbf{z}) = g_j(\mathbf{z})H_d(\mathbf{z}) + H_j(\mathbf{z}).$$

The right hand side of this is the derivative of $H(w_1, \ldots, w_{d-1}, g(\widehat{\mathbf{w}}))$ with respect to w_j at $\widehat{\mathbf{z}}$. By definition of g this vanishes, and hence $f_j(\widehat{\mathbf{z}}) = 0$. But $\tilde{f}_j(\mathbf{0}) = iz_jf_j(\mathbf{z})$, so the gradient of $\tilde{f}$ must vanish at $\mathbf{0}$. Finally, observe that $\mathrm{Re}\{\tilde{f}(\widehat{\theta})\} = -\log|\tilde{g}(\widehat{\theta})/z_d|$. By strict minimality of $\mathbf{z}$, the modulus of $g(\widehat{\mathbf{w}}) = \tilde{g}(\widehat{\theta})$ is greater than $|z_d|$ for any $\widehat{\mathbf{w}} \in T(\widehat{\mathbf{z}})$. □

We now prove Theorems 3.1, 3.3 and 3.5 in reverse order. We see from Lemma 4.1 that proving any of these theorems amounts to evaluating the quantity Ξ in equation (4.2). From Lemma 4.2 we see that $\mathbf{0}$ is a stationary point for the function $\tilde{f}$ as long as $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. The function $\tilde{f}$ is in general complex valued, but we shall see in Theorem 5.4 that it may be treated as if it were real valued, given the strict minimality of the zero guaranteed by Lemma 4.2 and the nonsingularity hypothesis. In particular the leading term of the integral in (4.2) is $(2\pi)^{(d-1)/2}\widetilde{\psi}(\mathbf{0})r_d^{(1-d)/2}$ divided by the product of the square roots of the eigenvalues of the Hessian. Once we have identified $\widetilde{\psi}(\mathbf{0}) = \psi(\mathbf{0})$ as $G(\mathbf{0})/(z_dH_d)$, Theorem 3.5 follows directly from Theorem 5.4.

Theorem 3.3 follows from the more explicit asymptotic development given in Corollary 5.3. Finally, to prove Theorem 3.1, it remains to compute the quantity $\tilde{f}''(0)$ in terms of the partial derivatives of H. We make use of the notation and computations immediately preceding the statement of Theorem 3.3.

First we compute the derivatives of g in the 2-variable case.

Lemma 4.3. *In a neighborhood of (z, w), ψ and the derivatives of g are as follows.*

$$g'(z) = -\frac{H_z}{H_w} \tag{4.6}$$

$$g''(z) = -\frac{1}{H_w}\left[H_{zz} - 2\frac{H_z}{H_w}H_{zw} + \frac{H_z^2}{H_w^2}H_{ww}\right]. \tag{4.7}$$

$$\psi(z) = \frac{G(z,w)}{-wH_w(z,w)}.$$

Proof. Differentiate the equation $H(z, g(z)) = 0$ to get $H_z + g'(z)H_w = 0$ which is the same as (4.6). Differentiate again to get

$$H_{zz} + 2g'H_{zw} + g''H_w + (g')^2H_{ww} = 0$$

and use (4.6) to eliminate g', giving (4.7). The formula for ψ follows from the definitions of ψ and of the partial derivative. □

Now, making use of the notation and computations immediately preceding the statement of Theorem 3.3, we can prove Theorem 3.1.

Proof of Theorem 3.1 via direct computation: We know from Lemma 4.2 that $\tilde{f}$ vanishes to order at least two at 0. To compute $\tilde{f}''(0)$, observe first that $\tilde{f}'' - \log\tilde{g}$ is linear in θ, so $\tilde{f}'' = (\log\tilde{g})''$. When $Z = ze^{i\theta}$, we have $(d/d\theta) = iZ(d/dZ)$, so

$$\tilde{f}'' = iZ\frac{d}{dZ}\left(iZ\frac{d\log g}{dZ}\right) = -Z\frac{d}{dZ}\left(\frac{Zg'}{g}\right).$$

Expanding this yields

$$\tilde{f}'' = -Z\frac{g' + Zg''}{g} + \frac{Z^2(g')^2}{g^2}. \tag{4.8}$$

By our assumption, G does not vanish at (z, w), so as long as $\tilde{f}''(0) \neq 0$, we may use Theorem 3.3 to conclude that the leading term asymptotic for $a_{r,s}$ is the $k = 2, l = 0$ term of (3.6). The term b_0 there is equal to

$$\widetilde{\psi}(0)\eta'(0) = \psi(z)\sqrt{2/\tilde{f}''(0)} = \frac{G(z,w)}{-wH_w(z,w)}\sqrt{\frac{2}{\tilde{f}''(0)}}.$$

Thus from Theorem 3.3,

$$a_{r,s} \sim \frac{A(2,0)}{2\pi} z^{-r} w^{-s} \frac{G(z,w)}{-wH_w(z,w)}\sqrt{\frac{2}{s\tilde{f}''(0)}}.$$

Now evaluate this using the value $A(2, 0) = \sqrt{\pi}$ and equation (4.8) along with (4.6) and (4.7) to obtain

$$a_{r,s} \sim \frac{1}{\sqrt{2\pi}} z^{-r} w^{-s} \frac{G(z,w)}{-wH_w(z,w)}\sqrt{\frac{(-wH_w(z,w))^3}{sQ}}$$

where

$$Q = (-wH_w(z,w))^3 \tilde{f}''(0) = (-wH_w(z,w))^3 z \frac{-g'(z) - zg''(z)}{g(z)} + \frac{z^2(g'(z))^2}{(g(z))^2}.$$

With the help of Lemma 4.3 we see (using $g(z) = w$) that

$$Q = (-wH_w)^3 \left[-z\frac{H_z}{-wH_w} - z^2 \frac{1}{-wH_w}\left(H_{zz} - 2\frac{H_z}{H_w}H_{zw} + \frac{H_z^2}{H_w^2}H_{ww}\right) + \frac{z^2H_z^2}{w^2H_w^2}\right],$$

evaluated at (z, w), which simplifies to the expression in Theorem 3.1. We see also that the nonvanishing hypotheses on Q is enough to guarantee $\tilde{f}''(0) \neq 0$, which finishes the proof of Theorem 3.1. □

5. Some oscillating integrals

The oscillating integrals we require are integrals over a neighborhood of zero in $\mathbb{R}^d$ of the complex-valued integrand:

$$\int_{\mathcal{N}} \exp(-\lambda f(\mathbf{x}))\psi(\mathbf{x})\, d\mathbf{x}$$

where f is analytic, $f(\mathbf{0}) = 0, \nabla f(\mathbf{0}) = \mathbf{0}$ and $\operatorname{Re} f \geq 0$. They are not difficult to compute, but since the standard references assume f is either real or purely imaginary, we sketch the development of these results. We mostly follow the exposition of Stein (1993), adapting it to complex-valued phase functions and simplifying it to take advantage of the decay of the magnitude of the integrand in this case.

We begin with one-dimensional results. Let C_0^∞ denote the class of smooth functions with compact support. The following proposition is a well known consequence of Watson's Lemma (see, for example, Wong (1989, Ch. 2, Theorem 1)).

Proposition 5.1. *Let $\psi \in C_0^\infty(\mathbb{R})$ and let $b_j = \psi^{(j)}(0)/j!$. Then as $\lambda \to \infty$, there is an asymptotic development*

$$\int_0^\infty \exp(-\lambda x^k)\psi(x)\, dx \sim \sum_{l=0}^\infty A_+(k,l) b_l \lambda^{-(l+1)/k},$$

where, as in (3.1),

$$A_+(k,l) := k^{-1}\Gamma\left(\frac{l+1}{k}\right).$$

□

We extend this to more general one-sided integrals by a complex change of variables. Given any f as above defined on an interval $[0, B]$, suppose also that $f' \neq 0$ on $(0, B]$, and let $k \geq 2$ be minimal so that $f^{(k)}(0) \neq 0$. Let $\psi \in C_0^\infty$ vanish to order $l \geq 0$ at 0. Let $c_j = f^{(j)}(0)/j!$ and $b_j = \psi^{(j)}(0)/j!$.

The real part of c_k is necessarily nonnegative. Define a function y on $[0, B]$ by

$$y(x) = f(x)^{1/k} = c_k^{1/k} x \left(1 + \frac{f(x) - c_k x^k}{c_k x^k} \right)^{1/k} ,$$

where the argument of $c_k^{1/k}$ is between $-\pi/(2k)$ and $\pi/(2k)$. The quantity $f(x) - c_k x^k$ is $O(x^{k+1})$ near zero, so y is analytic near 0, and, in particular, is a diffeomorphism between $[0, B]$ and a contour γ from 0 to some B^*. Let η invert y. The derivatives of η at 0 are easy to compute formally and the first $j + 1$ starting from the k^{th} depend only on the first j coefficients of f starting at c_k. Define

$$\begin{aligned} \psi^* &= (\psi \circ \eta) \cdot \eta' \; ; \\ b_j^* &= \psi^{*\,(j)}(0)/j! \, . \end{aligned} \tag{5.1}$$

Theorem 5.2. *Let f be analytic and complex-valued on an interval $[0, B]$ and suppose that $k \geq 2$ is minimal such that $f^{(k)}(0) \neq 0$ (so in particular $f(0) = f'(0) = 0$). Assume that $f' \neq 0$ on $(0, B]$, and that $\operatorname{Re} f$ has a strict minimum at 0.*

Let m be minimal so that the real part of $f^{(m)}(0)$ does not vanish. Let $\psi \in C_0^\infty$, let l be minimal such that $\psi^{(l)}(0) \neq 0$, and denote $c_j := f^{(j)}(0)/j!$, $b_j := \psi^{(j)}(0)/j!$. Define b_j^* as in (5.1).

Then there is an asymptotic development

$$\int_0^B \exp(-\lambda f(x)) \psi(x) \, dx \sim \sum_{j=l}^\infty A_+(k, j) b_j^* \lambda^{-(j+1)/k}. \tag{5.2}$$

The constant in the $O(\lambda^{-(N+1)/k})$ term depends continuously (only) on the derivatives of f and ψ up to $(N + 1)m/k - 1$.

Proof. Changing variables to $y = f(x)^{1/k}$, the integral becomes

$$\int_\gamma \exp(-\lambda y^k) \psi^*(y) \, dy;$$

the curve γ is the image of $[0, B]$ under y, so $\gamma'(0) = c_k^{1/k}$ and γ remains in the right half plane, strictly except at 0. For $0 < N < M$ write ψ^* as $P_M + y^{M+1} R_M$, where P_M is a polynomial of degree M and R_M is bounded; this can be done since ψ^* may be approximated by a degree M polynomial to within $O(y^{M+1})$ at 0.

First, evaluate

$$\int_\gamma \exp(-\lambda y^k) P_M(y) \, dy$$

by moving the contour. Replace γ by two line segments, the first of which goes some distance ε along the positive real axis and the second of which is strictly in the right half plane (we assumed $\operatorname{Re}\{f\} > 0$ except at 0). The integral along the second segment is exponentially small since the integrand is. Hence the combined contribution is the series (5.2) out to the $j = M$ term.

Next, bound

$$\left| \int_\gamma \exp(-\lambda y^k) y^{M+1} R_M(y) \, dy \right| .$$

With C representing different constants in different lines, we now observe that on γ we have $\mathrm{Re}\{-y^k\} < -C|y|^m$. Thus, parametrizing γ by arc-length, an upper bound is given by

$$\int_0^\infty \exp(-\lambda C t^m) t^{M+1} C |R_M(\gamma(t))| \, dt.$$

This is easily seen to be bounded above by $C\lambda^{-(M+2)/m}$ where C depends on the first M derivatives of f and ψ. Choosing $M \geq m(N+1)k - 1$ we have a remainder term that is $O(\lambda^{-(N+1)/k})$, proving the theorem. □

The value of a two-sided integral follows as a corollary.

Corollary 5.3. *Assume the hypotheses of Theorem 5.2, with f now defined on an interval $[-B, B]$. Then there is an asymptotic development*

$$\int_{-B}^{B} \exp(-\lambda f(x)) \psi(x) \, dx \sim \sum_{j=l}^{\infty} A(k,j) b_j^* \lambda^{-(j+1)/k}. \tag{5.3}$$

with $A(k,j)$ given by (3.2) and (3.3). The bounds on the remainder terms each depend continuously on finitely many derivatives of f and ψ on $[-B, B]$.

Proof. The two-sided integral is the sum of two one-sided integrals on intervals $[0, B]$ and $[-B, 0]$. The integral over $[-B, 0]$ may be written as an integral over $[0, B]$ of the function $\exp(-\lambda f(-x))\psi(-x)\, dx$. With b_l^* still denoting the coefficients resulting form the application of Theorem 5.2 to the first integral, let $\check{b}_l^*$ denote the coefficients when Theorem 5.2 is applied to the second integral. In order to add the two integrals, we write $\check{b}_l^*$ in terms of b_l^* by means of the following routine computation.

Let $c_k := c^k e^{i\alpha}$ with $c > 0$ and $|\alpha| \leq \pi/2$ and define the analytic quantity R so that

$$y(x) = \left[c_k x^k (1 + R(x))^k\right]^{1/k} = c e^{i\alpha/k} x (1 + R(x)).$$

If k is odd, then then the hypothesis $\mathrm{Re}\{f\} \geq 0$ implies that c_k is purely imaginary. We have

$$\check{y}(x) = \left[-c_k x^k (1 + R(-x))^k\right]^{1/k} = c e^{-i\alpha/k} x(1 + R(-x)) = -y(-x) e^{-2i\alpha/k}.$$

Writing η for the inverse function to y and $\check{\eta}$ for the inverse function to $\check{y}$ we then have

$$\check{\eta}(x) = -\eta(-e^{2i\alpha/k} x) \, .$$

Hence, letting $\mathcal{C}_l[\cdot]$ denote the coefficient of y^l,

$$\begin{aligned} \check{b}_l^* &= \mathcal{C}_l \left[\psi(-\check{\eta}(x)) \cdot \check{\eta}'(x)\right] \\ &= \mathcal{C}_l \left[\psi(\eta(-e^{2i\alpha/k} x)) \cdot e^{2i\alpha/k} \cdot \eta'(-e^{2i\alpha/k} x)\right] \end{aligned}$$

and thus

$$\check{b}_l^* = (-1)^l e^{2i\alpha(l+1)/k} b_l^* \, .$$

When k is even, the computation is similar but easier, resulting in

$$\check{b}_l^* = (-1)^l b_l^* \, .$$

Now observe that if k is odd, hence c_k is purely imaginary, then $e^{2i\alpha(l+1)/k} = e^{\pm i\pi(l+1)/k}$ according to the sign of the argument of c_k. Setting $A(k,l) = (1 + (-1)^l) A_+(k,l)$ if k is even and $(1 + e^{\mathrm{sgn}\,\mathrm{Arg}(c_k) i\pi(l+1)/k}) A_+(k,l)$ if k is odd, we recover the definition in (3.2) and (3.3) and prove the Corollary. □

Theorem 5.4. *Let f be a smooth complex-valued function on a neighborhood of $\mathbf{0}$ in $\mathbb{R}^d$ such that $\mathrm{Re}\{f\} \geq 0$ with equality only at $\mathbf{0}$. Suppose further that $\nabla f(\mathbf{0}) = 0$, and that the Hessian (matrix of second partials) of f has eigenvalues with positive real parts. Let $\mathcal{H}$ denote the Hessian determinant at $\mathbf{0}$. Then for $\psi \in C_0^\infty$, there is an asymptotic expansion*

$$\int \exp(-\lambda f(\mathbf{x}))\psi(\mathbf{x})\, d\mathbf{x} \sim \sum_{j \geq l} C_j \lambda^{-(j+d)/2}$$

where l is the degree of vanishing of ψ at $\mathbf{0}$. If $l = 0$ then $C_0 = \psi(\mathbf{0})(2\pi)^{d/2}\,\mathcal{H}^{-1/2}$. The choice of square root is determined by $\mathcal{H}^{-1/2} = \prod_{j=1}^d \mu_j^{-1/2}$ where μ_j are the eigenvalues of the Hessian and the principal square root is taken in each case.

Proof. Let $Q = \sum_{i,j=1}^d q_{i,j} z_i z_j$ be the quadratic form determined by the Hessian at the origin. Denote the eigenvalues of Q by $\{\mu_j : 1 \leq j \leq d\}$ and note that each μ_j has nonnegative real part.

Step 1: change coordinates to make f exactly equal to the quadratic form Q. Indeed since $f(\mathbf{x}) = Q(\mathbf{x})/2 + O(|\mathbf{x}|^3)$, and the Hessian is nondegenerate, there is a locally smooth change of variables $\{x_j(\mathbf{z}) : 1 \leq j \leq d\}$ such that $f(\mathbf{z}) = Q(\mathbf{x}(\mathbf{z}))/2$ and the Jacobian at the origin is 1.

Step 2: normalize by $\mathcal{H}^{1/2}$. For any quadratic form Q there is a linear change of variables $\mathbf{y}(\mathbf{x})$ such that $Q(\mathbf{x}) = \sum_{j=1}^d y_j^2$. The change of variables matrix P satisfies $PP^T = M(Q)$, the symmetric matrix representing Q. Changing variables to $\mathbf{y}$ introduces an integrating factor of $\det P$ which is a square root of $\mathcal{H}$ since $M(Q)$ is just the Hessian. Let $\mathcal{N}'$ be the region of integration over which $\mathbf{y}$ varies when $\mathbf{z}$ varies over an appropriately small neighborhood of $\mathbf{0}$.

Step 3: Expand $\widetilde{\psi}$ into monomials. The function ψ has now become $\widetilde{\psi}$, where $\widetilde{\psi}(\mathbf{0}) = \mathcal{H}^{-1/2}\,\psi(\mathbf{0})$ and the sign of the square root will be chosen later. We may expand $\widetilde{\psi}$ into monomials, using the same argument as in the proof of Theorem 5.2 to show the remainder term can be made $O(|\mathbf{y}|^N)$ for any N. It remains to evaluate the integral over the region of integration, $\mathcal{N}'$ of

$$\int_{\mathcal{N}'} \exp\left(-\lambda \sum_{j=1}^d y_j^2\right) \widetilde{\psi}(\mathbf{y})\, d\mathbf{y}$$

when $\widetilde{\psi}$ is a monomial.

Step 4: move the region of integration to the real d-space. Let $\mathcal{N}''$ be the projection of $\mathcal{N}'$ onto $\mathbb{R}^d$ by setting the imaginary part to zero. We claim that changing the region of integration from $\mathcal{N}'$ to $\mathcal{N}''$ alters the integral by an amount rapidly decreasing in λ. To show this, let Ω be the region $\{\mathrm{Re}\{\mathbf{x}\} + it\,\mathrm{Im}\{\mathbf{x}\} : \mathbf{x} \in \mathcal{N}', t \in [0,1]\}$. The boundary of Ω (as a manifold) is composed of $\mathcal{N}', \mathcal{N}''$ (with opposite signs) together with $S := \{\mathrm{Re}\{\mathbf{x}\} + it\,\mathrm{Im}\{\mathbf{x}\} : \mathbf{x} \in \partial\mathcal{N}', t \in [0,1]\}$. For any d-form ω, $\int_\Omega d\omega = \int_{\partial\Omega} \omega$. When $\omega = \exp(-\lambda \sum_{j=1}^d \mu_j y_j^2)\mathbf{y}^{\mathbf{r}}\, dy_1 \wedge \cdots \wedge dy_d$ is a holomorphic d-form, we see that $d\omega$ vanishes (being the sum of $\partial/\partial\overline{z_j}$ terms) so that

$$\int_{\mathcal{N}'} \omega = \int_{\mathcal{N}''} \omega + \int_S \omega.$$

We know that $\mathrm{Re}\{\sum_j \mu_j y_j^2\}$ is bounded away from 0 on $\partial\mathcal{N}'$, and its minimal value on S lies on $\partial\mathcal{N}'$, hence the integral over S decays exponentially.

Step 5: evaluate the integral. Factoring $\int_{\mathcal{N}''} \mathbf{y}^{\mathbf{r}} \exp(-\lambda \sum_{j=1}^d y_j^2)$ into one-dimensional integrals and plugging into Proposition 5.1 yields an asymptotic expansion whose leading term (when $l = 0$) is equal to $(2\pi)^{d/2}\psi(\mathbf{0})\,\mathcal{H}^{-1/2}$. When $f(\mathbf{z})$ is the function $\sum_{j=1}^d z_j^2$, then the positive square root is taken. The choice of square root must be continuous in the analytic topology on functions having nondegenerate Hessians and having eigenvalues with positive real parts, and the only such choice is the product of the principal square roots of the eigenvalues of the Hessian. □

6. Classification of cases

For purposes of classification some natural questions are:

(i) what are all possible local geometries of minimal points of $\mathcal{V}$?
(ii) which of these can be handled by variants of the methods in this article?
(iii) are these sufficient to yield a good approximation to $a_{\mathbf{r}}$ no matter what the direction, $\mathbf{r}/|\mathbf{r}|$, and no matter which generating function in the class, say, of functions meromorphic in a neighborhood of their domain of convergence?

To make the last question more concrete, consider the simplest possible example, namely binomial coefficients, where $F = 1/(1 - z - w)$ and $\mathcal{V}$ is a complex line. There are no nonsmooth points here, but how do we know that as (z, w) varies over minimal points of $\mathcal{V}$, the direction $\mathbf{dir}(z, w)$ will cover all of $\mathbb{RP}^1$?

This question will be answered by Theorem 6.3, but first we need to add some detail to the geometric discussion begun in Section 2.2. It will be evident that quite a few cases need to be considered, some of which require new tools and some of which require only minor modifications. Accordingly, the results will appear in several papers, currently under preparation. In other words, a discussion of taxonomy will necessarily refer to results not yet published, and we shall indicate to the best of our knowledge which ones are expected to be routine.

Given a point $\mathbf{z} \in \mathcal{V}$, we extend the definition of $\mathbf{dir}(\mathbf{z})$ to mean the set of limits of $\mathbf{dir}(\mathbf{y})$ as $\mathbf{y} \to \mathbf{z}$ along smooth points. When $\mathbf{z}$ is minimal, this is just the set of normals to support hyperplanes of $\log\mathcal{D}$ at the point $(\log|z_1|, \ldots, \log|z_d|)$, so this is consistent with the old definition. As we shall see shortly, $\mathbf{dir}(\mathbf{z})$ may be a $(d-1)$-dimensional subset of $\mathbb{RP}^{d-1}$ when $\mathbf{z}$ is a nonsmooth point of $\mathcal{V}$.

When H has a repeated factor, the residue computation in equation (4.4) must be replaced by one involving the derivative. The remainder of the computation proceeds without a hitch as before. Details are given in Pemantle & Wilson (2001b). For the remainder of the taxonomy, we assume H to be square-free. Toral smooth points may be handled by methods exactly the same as strictly minimal points. The inner integrand in (4.3) will in this case have its maximal modulus on a set of dimension larger than zero. A modification of the necessary oscillating integral computation that works in this case is also given in Pemantle & Wilson (2001b).

If $\mathbf{z} \in \mathcal{V}$ is not smooth, all the first partials vanish. The expansion of $H(\mathbf{x})$ near $\mathbf{z}$ is then a sum of terms of degrees 2 and higher. We call $\mathbf{z}$ a *homogeneous point* of degree k if this expansion contains terms $(x_j - z_j)^k$ for each $j = 1, \ldots, d$, and contains no terms of total degree less than k.

Lemma 6.1. *If $\mathbf{z}$ is a locally minimal point of $\mathcal{V}$ with all coordinates nonzero and F is meromorphic in a neighborhood of $\mathbf{z}$, then $\mathbf{z}$ is homogeneous.*

Proof. Passing to $F(z_1 x_1, \ldots, z_d x_d)$ if necessary, we may assume $\mathbf{z} = \mathbf{1}$. Setting $x_j = 1$ for all but one index j, we cannot obtain the zero function (by minimality), and so some term in the expansion around $\mathbf{1}$ is a pure power of $(x_j - 1)$, and we denote the minimal degree such term by $c_j(x_j - 1)^{k_j}$. If $\mathbf{z}$ is not a homogeneous point, then there is some j for which some monomial has total degree lower than k_j. Assume without loss of generality that $j = d$. The function $F(x, x, \ldots, x, y)$ then has a minimal degree pure $y - 1$ term $c_0(y-1)^k$, $k := k_d$, and some term $c'(x-1)^a(y-1)^b$ with $a + b < k$. In other words, the Newton Polygon of $F(x, \ldots, x, y)$ around $(1,1)$ has a support line passing through $(0, k)$ with slope $-p/q$ in lowest terms, and $p > q$. It is well known that we may describe the solutions $y(x)$ of the equation

$$F(1 + x, \ldots, 1 + x, 1 + y) = 0$$

as follows. Write

$$H := (y-1)^k(c_0 + c_1(y-1)^{-p}(x-1)^q + c_2(y-1)^{-2p}(x-1)^{2q} + \cdots + c_s(y-1)^{-sp}(x-1)^{sq})$$

for the polynomial collecting all the terms on this support line. Then for each q^{th} root of unity, ω, and each root λ of $\sum c_{s-j}\lambda^j = 0$, there is a solution $y = \lambda^{1/p}x^{q/p}(\omega + o(1))$ as $x \to 0$. A proof may be found in Brieskorn & Knörrer (1986).

Varying x over the set $|\pi - \arg(x)| \le \pi/4$, we see that the solutions $y(x)$ must sometimes be in this set as well. For those x, the points $(1+x, \ldots, 1+x, 1+y)$ will be in $\mathcal{V} \cap D(\mathbf{1}) \setminus T(\mathbf{1})$, violating minimality of $\mathbf{1}$. By contradiction, we have shown that no monomial in the expansion around $\mathbf{1}$ has lower total degree than any pure power term, hence $\mathbf{1}$ is minimal. $\square$

Continuing the taxonomy, suppose that $\mathbf{z}$ is a homogeneous point of $\mathcal{V}$ of degree $k \ge 2$. We say that $\mathbf{z}$ is a *multiple point* if $\mathcal{V}$ is locally the union of k analytic surfaces. Algebraically, this means that the leading (order k) terms in the expansion of H near $\mathbf{z}$ factors into linear pieces. If the homogeneous point $\mathbf{z}$ is not a multiple point, we say it is a *cone point*. When $d = 2$ there are no cone points, since any homogeneous polynomial in 2 variables factors completely over $\mathbb{C}$.

Our understanding of cone points is not yet complete, but an analysis involving cone points is underway in Cohn & Pemantle (2001). For multiple points, most of the story is given in Pemantle & Wilson (2001a). In particular, the following theorem is proved there.

Theorem 6.2 (Pemantle & Wilson (2001a))**.** *Let $\mathbf{z}$ be an isolated, minimal, multiple point of $\mathcal{V}$ with multiplicity k. Let $S \subseteq \mathbb{RP}^{d-1}$ be the set of outward normals to support hyperplanes to* $\log\mathcal{D}$ *at the point $(\log|z_1|, \ldots, \log|z_d|)$. Then there is an integer $p \ge 0$ and a polynomial function $\phi : S \to \mathbb{R}$ such that the asymptotic expansion*

$$a_{\mathbf{r}} \sim \mathbf{z}^{-\mathbf{r}}\phi(\mathbf{r})\sum_j C_j(r_d)^{k-p/2-j/2} \tag{6.1}$$

holds uniformly as $\mathbf{r}$ varies over compact subsets of the interior of S. $\square$

The extension to toral multiple points is given in Pemantle & Wilson (2001b). If a multiple point is not isolated or toral, then the degree of multiplicity, k, must be less than the dimension, d. This cannot happen of course when $d = 2$, but does happen when $d \ge 3$. The method for handling this case, toral or otherwise, is given in Pemantle & Wilson (2001b). That paper will also contain some subcases of the isolated multiple point case, namely when the sheets of $\mathcal{V}$ intersect non-transversely.

Having more or less completed the taxonomy, we now discuss when we can guarantee that our methods yield asymptotics in all directions.

Theorem 6.3. *Let $F = G/H = \sum a_{r,s}z^r w^s$ be the quotient of analytic functions $G, H : \mathbb{C}^2 \to \mathbb{C}$. Suppose that the coefficients $a_{r,s}$ are all nonnegative, and that $F(z,0)$ and $F(0,w)$ are not entire. Then for every direction $\alpha \in \mathbb{RP}^1$ there is a minimal $\mathbf{z} \in \mathcal{V}$ with $\alpha \in \mathbf{dir}(\mathbf{z})$.*

Proof. Let (x,y) be any point on the boundary of $\log\mathcal{D}$. For $u < e^x$ and $v < e^y$ the power series for F is convergent at (u,v). As $u \uparrow e^x$ and $v \uparrow e^y$ therefore, $F(u,v)$ is finite and increasing. On the other hand, the power series for F is not absolutely convergent on $T(e^x, e^y)$, since we know F to have some singularity on this torus. Hence $F(u,v) \uparrow \infty$ as $(u,v) \uparrow (e^x, e^y)$. Since F is meromorphic, it must have a pole at (e^x, e^y), hence $(e^x, e^y) \in \mathcal{V}$ and is a minimal point of $\mathcal{V}$. As (x,y) varies over the boundary of $\log\mathcal{D}$, we let $\gamma \subseteq \mathcal{V}$ denote the curve traced out by this minimal point.

Pick any $\alpha \in \mathbb{RP}^1$. The convex set $\log\mathcal{D}$ has horizontal and vertical support hyperplanes (by non-entirety of $F(z,0)$ and $F(0,w)$), and therefore has a support hyperplane normal to α; let (x,y) be a point of intersection of this support plane with $\log\mathcal{D}$. We have just seen that $\mathbf{z}(\alpha) := (e^x, e^y)$ is a minimal point of $\mathcal{V}$. If $\mathbf{z}$ is a smooth point of $\mathcal{V}$ then $\alpha \in \mathbf{dir}(\mathbf{z})$: either $\mathbf{z}$ is finitely minimal, in which case Theorem 3.3

applies, or it is toral, in which case the toral version of this theorem from Pemantle & Wilson (2001b) applies.

Assume now that $\mathbf{z}$ is not a smooth point. By Lemma 6.1, $\mathbf{z}$ is a homogeneous point, and since $d = 2$, $\mathbf{z}$ is a multiple point. Theorem 6.2 then shows that $\alpha \in \mathbf{dir}(\mathbf{z})$ in this case as well. This finishes the proof. $\square$

7. Further details and open questions

The theorems in this and subsequent papers give estimates that are uniform away from the boundary of the domain in which they are valid. In order for all of these to be patched together so as to give estimates valid now matter how $\mathbf{r} \to \infty$, one must determine the bandwidth around the boundary for which the boundary estimates on either side hold. For instance, suppose (z, w) is a multiple point of degree 2 and that $\mathbf{dir}(z, w)$ is the set of slopes between 1/2 and 2. It appears that the asymptotic estimate in Pemantle & Wilson (2001a) holding near the line $\{s = 2r\}$ can be written so it is valid out to $s = 2r + c\sqrt{r}$. If the estimate for the region $s/r > 2 + \varepsilon$ can be widened so it holds to $s = 2r + c\sqrt{r}$ and a description given that is valid in the regime $(s - 2r)/\sqrt{r} \to c$, then the estimates will patch together completely.

Another natural question is the universality of the method when the coefficients have mixed signs. We conjecture that Theorem 6.3 still holds, in the sense that for every direction there is point $\mathbf{z} \in \mathcal{V}$ for which integration near $\mathbf{z}$ yields correct asymptotics. What we know is that $\mathbf{z}$ may no longer be minimal. For example, if $G = 1$ and

$$H = (1 - (2/3)w - (1/3)z)(1 + (1/3)w - (2/3)z)$$

then the point $(3/2, 3/4)$ is not minimal but yields asymptotics in the diagonal direction; one sees this by integrating along a deformed torus rather than along $T(3/2, 3/4)$. In fact we conjecture that such a deformation always exists, but the topology seems not transparent enough to yield an easy proof.

The class of algebraic functions is in some ways almost as nice as the set of rational functions, and nicer than the meromorphic functions. For one thing, an algebraic function is determined by a finite amount of data, and may thus easily be input into a symbolic math package. Gao & Richmond (1992) give an analysis of algebraic and logarithmic singularities, but sometimes the relevant singularities for algebraic functions are poles. For example, in Larsen and Lyons' analysis (Larsen & Lyons 1999) of merge times for coalescing particles, they find an algebraic function of the form

$$F(z, w) = \frac{\chi(z, w)}{w - 1 - \sqrt{1 - z}}$$

with χ analytic. The branch of the square root is chosen so that at the origin the denominator is -2, not 0. There is a branchline at $z = 1$, but for all directions in $\mathbb{RP}^1$, there is a smooth pole on the curve $w = 1 + \sqrt{1 - z}$ yielding asymptotics in the desired direction. It is natural to ask when this will happen, and how one can tell effectively. Some questions of effectiveness are addressed in Pemantle & Wilson (2001a) and Pemantle & Wilson (2001b), but there is probably substantial room for improvements on an algorithmic level.

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