ASYMPTOTICS OF MULTIVARIATE SEQUENCES II. MULTIPLE POINTS OF THE SINGULAR VARIETY.

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Abstract. Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ be a multivariate generating function which is meromorphic in some neighborhood of the origin of $\mathbb{C}^d$, and let $\mathcal{V}$ be its set of singularities. Effective asymptotic expansions for the coefficients can be obtained by complex contour integration near points of $\mathcal{V}$.

In the first article in this series, we treated the case of smooth points of $\mathcal{V}$. In this article we deal with multiple points of $\mathcal{V}$.

1. Introduction

In [PW99], we began a series of articles addressing the general problem of computing asymptotic expansions for a multivariate sequence whose generating function is known. Such problems are encountered frequently in combinatorics and probability; see for instance Examples 2 – 8 in Section 1 of [Pem], which collects examples from various sources including [LL99, FM77, Wil94, Com74]. Our aim is to present methods which are as general as possible, and lead to effective computation. Our apparatus may be applied to any function whose dominant singularities are poles. Among other things, we showed in [PW99] that for nonnegative bivariate sequences of this type, our method is always applicable. (Further work may be required in the presence of the divisor $c - z_j$.)

Our approach is analytic. For simplicity, we restrict to the two-variable case in this introduction, though our methods work for any number of variables. Given a sequence a_{rs} indexed by the nonnegative integer lattice, we seek asymptotics as $r, s \to \infty$. Form the generating function $F(z,w) = \sum_{r,s} a_{rs} z^r w^s$ of the sequence; we assume that F is analytic in some neighborhood of the origin.

The iterated Cauchy integral formula yields

$$a_{rs} = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}'} \int_{\mathcal{C}} \frac{F(z,w)}{z^{r+1} w^{s+1}} \, dw \, dz,$$

where $\mathcal{C}$ and $\mathcal{C}'$ are circles centered at 0 and F is analytic on a polydisk containing the torus $\mathcal{C} \times \mathcal{C}'$.

Expand the torus, by expanding (say) $\mathcal{C}$, slightly beyond a *minimal singularity* of F (that is, a point (z_0, w_0) at which the expanding torus first touches the singular set $\mathcal{V}$ of F). The difference between the corresponding inner integrals is then computed via residue theory. Thus a_{rs} is represented as a sum of a residue and an integral on a large torus. One hopes that the residue term is dominant, and gives a good approximation to a_{rs}. The residue term is itself an integral (since

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the residue is taken in the inner integral only). A stationary phase analysis of the residue integral shows when our hope is realized, and yields an asymptotic expansion for a_{rs}.

In [PW99] we considered the case when the minimal singularity in question is a smooth point of $\mathcal{V}$. The present article deals with the more general case where the minimal singularity is a *multiple point*: locally, the singular set is a union of finitely many graphs of analytic functions. (This case is more general, since Theorem 3.9 with n set to 0 captures the previously analyzed case of a smooth point.) In the two-variable case, we know from Lemma 6.1 of [PW99] that every minimal singularity must have this form (see also [Tsi93, Lemma 3.1]), though more complicated singularities may arise in higher dimensions.

The rest of this article is organized as follows. In the remainder of this section we describe in more detail the program begun in [PW99] and continued here and in future articles in this series. Section 2 deals with notation and preliminaries required for the statement of our main results. Those results, along with illustrative examples, are listed in Section 3. In Section 4 we give a separate proof of the easiest special case of our main results. Although not strictly necessary from a logical point of view, this section is highly recommended to the reader. Our more general results are proved in Section 5. We discuss some further details and outline future work in Section 6.

Details of the Program. Our notation is similar to that in [PW99]. For clarity we shall reserve the names of several objects throughout. We use boldface to indicate a (row or column) vector. The number of variables will be denoted $d+1$. The usual multi-index notation is in use: $\mathbf{z}$ denotes a vector $(z_1, \ldots, z_{d+1})^T \in \mathbb{C}^{d+1}$, and if $\mathbf{r}$ is an integer vector then $\mathbf{z}^{\mathbf{r}} = \prod_j z_j^{r_j}$. We also use the convention that a function, ostensibly of 1 variable, applied to an element of $\mathbb{C}^{d+1}$ acts on each coordinate separately — for example $e^{\mathbf{x}} = (e^{x_1}, \ldots, e^{x_{d+1}})$. Throughout, G and H denote functions analytic in some polydisk about $\mathbf{0}$ and $F = G/H = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. The set where H vanishes will be called the *singular variety* of F and denoted by $\mathcal{V}$. Let $\mathbf{D}(\mathbf{z}), \mathbf{T}(\mathbf{z})$ denote respectively the polydisk and torus (both centered at the origin) on which $\mathbf{z}$ lies.

A crude preliminary step in approximating $a_{\mathbf{r}}$ is to determine its exponential rate; in other words, to estimate $\log |a_{\mathbf{r}}|$ up to a factor of $1 + o(1)$. Let $\mathcal{D}$ denote the (open) domain of convergence of F and let $\log\mathcal{D}$ denote the logarithmic domain in $\mathbb{R}^{d+1}$, that is, the set of $\mathbf{x} \in \mathbb{R}^{d+1}$ such that $e^{\mathbf{x}} \in \mathcal{D}$. If $\mathbf{z} \in \mathcal{D}$ then Cauchy's integral formula

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^{d+1} \int_{\mathbf{T}(\mathbf{z})} \frac{F(\mathbf{w})}{\mathbf{w}^{\mathbf{r}+\mathbf{1}}}\, d\mathbf{w} \tag{1.1}$$

shows that $a_{\mathbf{r}} = O(|\mathbf{z}|^{-\mathbf{r}})$. Letting $\mathbf{z} \to \partial\mathcal{D}$ gives

$$\log |a_{\mathbf{r}}| \leq -\mathbf{r} \cdot \log |\mathbf{z}| + o(|\mathbf{r}|),$$

and optimizing in $\mathbf{z}$ gives $\log |a_{\mathbf{r}}| \leq \gamma(\mathbf{r}) + o(|\mathbf{r}|)$ where

$$\gamma(\mathbf{r}) := - \sup_{\mathbf{x} \in \log\mathcal{D}} \mathbf{r} \cdot \mathbf{x}\,. \tag{1.2}$$

The cases in which the most is known about $a_{\mathbf{r}}$ are those in which this upper bound is correct, that is, $\log |a_{\mathbf{r}}| = \gamma(\mathbf{r}) + o(|\mathbf{r}|)$. To explain this, note first that the supremum in (1.2) is equal to $\mathbf{r} \cdot \mathbf{x}$ for some $\mathbf{x} \in \partial\log\mathcal{D}$. The torus $\mathbf{T}(\mathbf{x})$ must contain some minimal singularity $\mathbf{z} \in \mathcal{V} \cap \partial\mathcal{D}$. Asking that $\log |a_{\mathbf{r}}| \sim -\mathbf{r} \cdot \mathbf{z}$ is then precisely the same as requiring the Cauchy integral (1.1) — or the residue integral of Section 1 — to be of roughly the same order as its integrand. This is the situation in which it easiest to estimate the integral.

Our program may now be summarized as follows. Associated to each minimal singularity $\mathbf{z}$ is a cone $\mathbf{K}(\mathbf{z}) \subseteq (\mathbb{R}^+)^{d+1}$. Given $\mathbf{r}$, we find one or more $\mathbf{z} = \mathbf{z}(\mathbf{r}) \in \mathcal{V} \cap \partial\mathcal{D}$ where the upper bound is least. We then attempt to compute a residue integral there. This works only if $\mathbf{r} \in \mathbf{K}(\mathbf{z})$ and if the residue computation is of a type we can handle. Our program is guaranteed to succeed in some cases, and conjectured to succeed in others. It is known to fail only in some cases where the $a_\mathbf{r}$ are not nonnegative reals (not an important case in combinatorial or probabilistic applications) and even then a variant seems to work.

To amplify on this, define a point $\mathbf{z} \in \mathcal{V}$ to be *minimal* if $\mathbf{z} \in \partial\mathcal{D}$ and each z_j is nonzero. There are only three possible types of minimal singularities [PW99, Lemma 6.1]), namely smooth points of $\mathcal{V}$, multiple points and cone points (all defined below). It is conjectured that for all three types of points, and any $\mathbf{r} \in \mathbf{K}(\mathbf{z})$, we indeed have

$$\log|a_\mathbf{r}| = \gamma(\mathbf{r}) + o(|\mathbf{r}|) = -\mathbf{r} \cdot \log|\mathbf{z}| + o(|\mathbf{r}|)\,.$$

This is proved for smooth points in [PW99] via residue integration, and the complete asymptotic series obtained. It is proved in the present work for multiple points under most nondegeneracy assumptions; the fact that these do not cover all cases seems due more to taxonomical problems rather than the inapplicability of the method. The problem remains open for cone points, along with the problem of computing asymptotics. It is also shown in [PW99] that when $a_\mathbf{r}$ are all nonnegative, then $\mathbf{r} \in \mathbf{K}(\mathbf{z}(\mathbf{r}))$, and therefore that a resolution of the above problem yields a complete analysis of nonnegative sequences with pole singularities.

When the hypothesis that $a_\mathbf{r}$ be real and nonnegative is removed, it is not always true that $\mathbf{r} \in \mathbf{K}(\mathbf{z}(\mathbf{r}))$. In all examples we have worked, there have been points $\mathbf{z} \notin \overline{\mathcal{D}}$ for which $\mathbf{r} \in \mathbf{K}(\mathbf{z})$ and a residue integral near $\mathbf{z}$ may be proved a good approximation to $a_\mathbf{r}$; the method is to contract the torus of integration to the origin in some way other than simply iterating the contraction of each coordinate circle to a point. Thus a second open question is to settle whether there is always such a point $\mathbf{z}$ in the case of mixed signs (see the discussion in Section 7 of [PW99]).

The scope of the present article is as follows. We define multiple points and carry out the residue integral arising near a generic, strictly minimal, multiple point. Unlike the case for smooth points, the integral is not readily recognizable as a standard multivariate oscillatory integral, and a key result of this article is a more manageable representation of the residue to be integrated (Lemmas 5.1 and 4.3). The analysis of the resulting integrals is somewhat technical, and for expositional clarity, we give a self-contained proof of the simplest case ($d+1 = 2$ variables, multiplicity 2) before handling general d and multiplicity.

The asymptotics arising from multiple point singularities differ substantially from asymptotics in the smooth case. In the remainder of this introduction, we give examples to illustrate this.

1.1. **Example** (simplest possible multiple point). Let $F(z,w)$ be a two-variable generating function and suppose that the point $(1,1)$ is a double pole of F (thus $F = G/H$) with $G(\mathbf{z}) \neq 0$ and $H(\mathbf{z})$ vanishing to order 2). If F has no other poles (z,w) with $|z|, |w| \leq 1$, and if the two branches of the singular variety $\mathcal{V}$ meet transversely at $(1,1)$ as in Figure 1, then for some positive constants c and C,

$$a_{rs} = C + O(e^{-c|(r,s)|}) \tag{1.3}$$

for all (r,s) in a certain cone, $\mathbf{K}$ in the positive integer quadrant. This is proved in Theorem 3.1 below, and the constant C computed. Exact statement of the transversality hypothesis requires some discussion of the geometry of $\mathcal{V}$. The constant C is computed in terms of some algebraic quantities

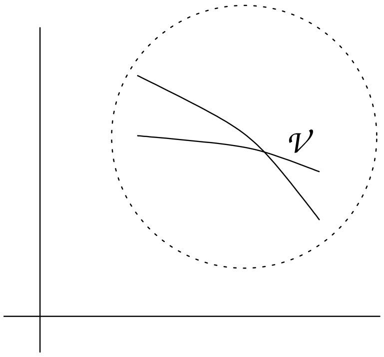

FIGURE 1. Local picture of a double pole

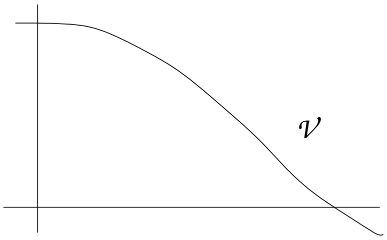

FIGURE 2. When $\mathcal{V}$ has only smooth points

derived from F. The cone $\mathbf{K}$ in which this holds is easily described in terms of the tangents to the branches of the double pole. The need for some preliminary algebraic and geometric analysis to define transversality and to compute C and $\mathbf{K}$ motivates our somewhat lengthy Section 2.

Example 3.2 below shows the details of this computation for a particular F. The exponential bound on the error follows from results in [Pem00] which are cited in Section 5. If the multiple pole is moved to a point (z, w) other than $(1, 1)$, a factor of $z^{-r}w^{-s}$ is introduced.

Compare this to the case where $\mathcal{V}$ is smooth, intersecting the positive real quadrant as shown in Figure 2. In this case one has Ornstein-Zernike (central limit) behavior: as $(r, s) \to \infty$ with r/s fixed, a_{rs} is rapidly decreasing for all but one value of r/s; for that distinguished direction, $a_{rs} \sim C|(r, s)|^{-1/2}$ for some constant, C, and the error terms may be developed in a series of decreasing powers of $|(r, s)|$. The two main differences between the double and single pole cases are thus the existence of a plateau in the double pole case, and the flatness up to an exponentially small correction versus a correction of order $|(r, s)|^{-3/2}$ for a single pole.

1.2. **Example** (pole of greater order). Alter the previous example so that $\mathcal{V}$ has a pole of some order $n + 1$ at $(1, 1)$, as in Figure 3. Then the formula (1.3) becomes instead

$$a_{rs} = P(r, s) + O(e^{-c|(r,s)|})$$

where P is a polynomial of degree $n - 1$. In Example 3.7 below, the polynomial P is explicitly computed in the case $n + 1 = 3$. Again, we see the chief differences from the smooth case being a

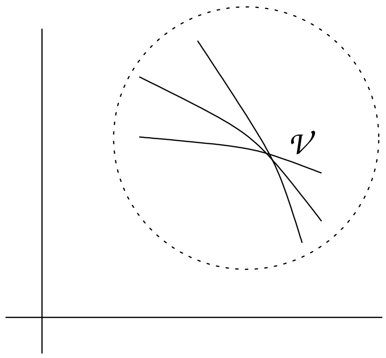

FIGURE 3. A pole of order 3

cone of non-exponential decay, and an exponentially small correction to the polynomial $P(r,s)$ in the interior of the cone.

In higher dimensions there are more possible behaviors, but the same sorts of results hold. There is a cone on which the exponential rate has a plateau; under some conditions the correction terms within this cone are exponentially small. The following table summarizes the results proved in this paper; transversality assumptions have been omitted.

Theorem number	**Hypotheses**	**Result**
Theorem 3.1	two curves in 2-space	$C +$ exp. small
Theorem 3.3	$d+1$ sheets in $(d+1)$-space	$C +$ exp. small
Theorem 3.6	more sheets than the dimension	polynomial $+$ exp. small
Theorem 3.9	fewer sheets than the dimension	asymptotics start with $\lvert\mathbf{r}\rvert^{n/2-d/2}$
Theorem 3.11	all sheets tangent	asymptotics start with $\lvert\mathbf{r}\rvert^{n-d/2}$

We conclude the introduction by giving a combinatorial application of Example 1.1.

1.3. **Example** (combinatorial application). An independent sequence of random numbers uniform on $[0,1]$ is used to generate biased coin-flips: if p is the probability of HEADS then a number $x \le p$ means HEADS and $x > p$ means TAILS. The coins will be biased 2/3 heads to 1/3 tails for the first n flips, and 2/3 TAILS to 1/3 HEADS thereafter. A player desires to get r heads and s tails and is allowed to choose n. On average, how many choices of $n \le r+s$ will be winning choices?

The probability that n is a winning choice for the player is the sum over $a+b=n$ of

$$\binom{n}{a}(2/3)^a(1/3)^b\binom{r+s-n}{r-a}(1/3)^{r-a}(2/3)^{s-b}\,.$$

Let a_{rs} be this expression summed over n. The array $\{a_{rs}\}_{r,s\ge 0}$ is just the convolution of the arrays $\binom{r+s}{r}(2/3)^r(1/3)^s$ and $\binom{r+s}{r}(1/3)^r(2/3)^s$, so the generating function $F(z,w) := \sum a_{rs}z^r w^s$ is the

product
$$F(z,w) = \frac{1}{(1-\frac{1}{3}z-\frac{2}{3}w)(1-\frac{2}{3}z-\frac{1}{3}w)}\,.$$
Applying Theorem 3.1 with $G \equiv 1$ and $\det \mathbf{H} = -1/9$, we see that $a_{rs} = 3$ plus a correction which is exponentially small as $r, s \to \infty$ with $r/(r+s)$ staying in any subinterval of $(1/3, 2/3)$. A purely combinatorial analysis of the sum may be carried out to yield the leading term, 3, but says nothing about the correction terms. The diagonal extraction method of [HK71] yields very precise information for $r = s$ but nothing more general in the region $1/3 < r/(r+s) < 2/3$.

2. Preliminary definitions and notation

For each $\mathbf{w} \in \mathbb{C}^{d+1}$, the truncation $(w_1, \ldots, w_d)$ will be denoted $\widehat{\mathbf{w}}$, and the last coordinate w_{d+1} simply by w. We do not specify the size for constant vectors — for example $\mathbf{1}$ denotes the vector $(1, \ldots, 1)^T$ of whatever size is appropriate. Thus we write $\widehat{\mathbf{1}} = \mathbf{1}$. A minimal singularity $\mathbf{z}$ is *strictly minimal* if $\mathcal{V} \cap \mathbf{D}(\mathbf{z}) = \mathcal{V} \cap \mathbf{T}(\mathbf{z})$ is just the singleton $\{\mathbf{z}\}$. When a minimal point is not strictly minimal, one must add (or integrate) contributions from all points of $\mathcal{V} \cap \mathbf{T}(\mathbf{z})$. This step is routine and will be carried out in a future article; we streamline the exposition here by assuming strict minimality.

Geometry of $\mathcal{V}$. This article deals entirely with minimal points $\mathbf{z}$ of $\mathcal{V}$ near which $\mathcal{V}$ decomposes as a union of sheets $\mathcal{V}_j$, each of which is a graph of an analytic function $w = u_j(\widehat{\mathbf{w}})$. The algebraic description of this situation is as follows. By the Weierstrass Preparation Theorem [GH78] there is a neighborhood of $\mathbf{z}$ in which we may write $H(\mathbf{w}) = \chi(\mathbf{w}) W(\mathbf{w})$ where χ is analytic and nonvanishing and W is a *Weierstrass polynomial.* This means that
$$W(\mathbf{w}) = w^{n+1} + \sum_{j=0}^{n} \chi_j(\widehat{\mathbf{w}}) w^j$$
where the multiplicity $n+1$ is at least 2 if $\mathbf{z}$ is not a smooth point, and the analytic functions χ_j vanish at $\widehat{\mathbf{z}}$.

Now suppose that $\mathbf{z}$ has all coordinates nonzero. Recalling from Lemma 6.1 of [PW99] that any such minimal point of $\mathcal{V}$ is locally homogeneous, we see in fact that χ_j vanishes to homogeneous degree $n+1-j$ at $\widehat{\mathbf{z}}$ and that χ_0 has nonvanishing pure w_i^{n+1} terms for each $1 \leq i \leq d$. Then $\mathcal{V}$ is locally the union of smooth sheets if the degree $n+1$ homogeneous part of W (the leading term) factors completely into linear terms. This is equivalent to the local factorization
$$H(\mathbf{w}) = \chi(\mathbf{w}) \prod_{j=0}^{n} [w - u_j(\widehat{\mathbf{w}})] \tag{2.1}$$
for analytic (not necessarily distinct) functions u_j mapping a neighborhood of $\widehat{\mathbf{z}}$ to a neighborhood of z_{d+1}.

Remark. If the expansion of H near $\mathbf{z}$ vanishes to order $n+1$ in z_{d+1}, then there are always $n+1$ solutions (counting multiplicity) to $H(\widehat{\mathbf{w}}, w) = 0$ for $\widehat{\mathbf{w}}$ near $\widehat{\mathbf{z}}$. These vary analytically, but may be parametrized by $n+1$ analytic functions u_j only if there is no monodromy, that is, if one stays on the same branch moving around a cycle in the complement of the singular set. Thus a multiple point singularity is one whose monodromy group is trivial.

It turns out to be more convenient to deal with the reciprocals $v_j = 1/u_j$. The basic setup throughout the rest of this article is as follows.

2.1. **Definition.** The point $\mathbf{z}$ of $\mathcal{V}$ is a *multiple point* if there are analytic functions $v_0, \ldots, v_n, \phi$ and a local factorization

$$F(\mathbf{w}) = \frac{\phi(\mathbf{w})}{\prod_{j=0}^{n}(1 - wv_j(\widehat{\mathbf{w}}))}, \tag{2.2}$$

which we call the *factored form* of F, such that

- $z_{d+1}v_j(\widehat{\mathbf{z}}) = 1$ for all j;
- each $\dfrac{\partial v_j}{\partial w_i}(\mathbf{z}) \neq 0$;
- $w_{d+1}v_j(\widehat{\mathbf{w}}) = 1$ for some j if and only if $\mathbf{w} \in \mathcal{V}$.

Remark. Let $\mathcal{V}_j$ denote the local hypersurface parametrized by $w = u_j(\widehat{\mathbf{w}})$. Then the first of the conditions says that each $\mathcal{V}_j$ passes through $\mathbf{z}$: there are no extraneous factors in the denominator representing surface elements not passing through $\mathbf{z}$. The last condition says that the zeros of the denominator are exactly the poles of F: there are no extraneous factors in the denominator vanishing at $\mathbf{z}$ and cancelling a similar divisor in the numerator.

It may appear that some generality has been lost in imposing the second condition, since we are assuming that each sheet of $\mathcal{V}$ projects diffeomorphically onto any coordinate hyperplane. This latter property is in fact guaranteed by the non-vanishing of the pure w_i^{n+1} terms of W. Thus, in particular, in the two-variable case every minimal singularity with nonzero coordinates is a multiple point .

To compute ϕ directly from G and H, differentiate (2.1) $n+1$ times in the $w := w_{d+1}$ coordinate at the point $\mathbf{z}$ to write

$$\left(\frac{\partial}{\partial w}\right)^{n+1} H(\mathbf{z}) = (n+1)!\chi(\mathbf{z}).$$

We may then write

$$\frac{\phi}{\prod_{j=0}^{n}(1 - wv_j\widehat{\mathbf{w}})} = \frac{G}{H} = \frac{G}{\chi(\prod_{j=0}^{n}(-u_j))\prod_{j=0}^{n}(1 - wv_j(\widehat{\mathbf{w}}))}$$

and solve for ϕ at $\mathbf{w} = \mathbf{z}$ to obtain

$$\phi(\mathbf{z}) = \frac{(n+1)!}{(-z_{d+1})^{n+1}} \frac{G(\mathbf{z})}{\left(\frac{\partial}{\partial w_{d+1}}\right)^{n+1} H(\mathbf{z})}. \tag{2.3}$$

Since we will be integrating over a torus containing $\mathbf{z}$, it will be helpful to understand how $\log(z_{d+1}v_j(\widehat{\mathbf{z}}e^{i\widehat{\boldsymbol{\theta}}}))$ varies with $\widehat{\boldsymbol{\theta}}$.

2.2. **Proposition.** *Suppose that F is in factored form with respect to the strictly minimal point $\mathbf{z}$. Then for each j, and k, the Mclaurin expansion of $\log[z_{d+1}v_j(\widehat{\mathbf{z}}e^{i\widehat{\boldsymbol{\theta}}})]$ with respect to θ_k has the following properties:*

- *the constant term is 0;*
- *the degree 1 term is ic_j where $c_j \geq 0$;*

- *the first coefficient with nonzero real part occurs for an even index m_j, and this real part is negative (we write the coefficient as $-d_j$ where* $\text{Re}\{d_j\} > 0$*).*

Proof. By strict minimality, for $r \geq 0$ the function $r \mapsto |v_j((1-r)\widehat{\mathbf{z}})|$ has a strict maximum at $r = 0$. Also the function $\theta_k \mapsto |v_j(\widehat{\mathbf{z}}e^{i\widehat{\boldsymbol{\theta}}})|$ has a strict maximum at $\theta_k = 0$. Thus so does its logarithm $\theta_k \mapsto \text{Re}\{\log v_j(\widehat{\mathbf{z}}e^{i\widehat{\boldsymbol{\theta}}})\}$. This shows the degree 1 term is purely imaginary. Varying θ_k in the positive imaginary direction reduces the modulus of z_k, hence by minimality must not reduce the modulus of z_{d+1}; thus if e_k is the k^{th} elementary basis vector, then $|v_j(\widehat{\mathbf{z}}e^{i(\widehat{\boldsymbol{\theta}}+ite_k)})|$ has a maximum at $t = 0$ on $[0, \varepsilon]$, implying that $c_j \geq 0$. The result follows from elementary calculus. □

For the remaining definitions, fix a strictly minimal element $\mathbf{z} \in \mathcal{V}$ which is a multiple point, and let $v_0, \ldots, v_n$ be as above.

2.3. **Definition.** For each sheet j, $0 \leq j \leq n$, let $\boldsymbol{\delta}_j(\mathbf{z})$ be the row vector whose kth component is $\frac{w_k}{w_{d+1}}\partial v_j/\partial w_k$ (evaluated at $\mathbf{z}$) for $1 \leq k \leq d$, and whose $d+1$ component is 1.

Let $\mathbf{C}(\mathbf{z})$ be the matrix whose jth row is $\boldsymbol{\delta}_j(\mathbf{z})$. Say that $\mathbf{z}$ is *nondegenerate* if the rank of $\mathbf{C}$ is $d+1$, *transverse* if the rank is $n+1$, and *completely nondegenerate* if it is both transverse and nondegenerate; in this case necessarily $n = d$ and the multiplicity of each sheet is 1.

We denote by $\mathbf{K}(\mathbf{z})$ the positive hull of all the $\boldsymbol{\delta}_j(\mathbf{z})$, a cone in $(\mathbb{R}^+)^{d+1}$. Its intersection with the hyperplane $z_{d+1} = 1$ is denoted $\mathbf{K}_0$. A geometric description of $\mathbf{K}$ is that it is the collection of outward normal vectors to support hyperplanes of the logarithmic domain of convergence of F at the point $(\log|z_1|, \ldots, \log|z_{d+1}|)$; see [PW99] for details.

Remark. The importance of $\mathbf{K}$ is that analysis near $\mathbf{z}$ will yield asymptotics for $\mathbf{r}$ in precisely the directions in $\mathbf{K}(\mathbf{z})$. In the smooth case, $n = 0$ and so $\mathbf{K}(\mathbf{z})$ reduces to a single ray. The point $\mathbf{z}$ is transverse if and only if the normals there to the surfaces $\mathcal{V}_j$ span a space of dimension $n+1$. Alternatively, the $n+1$ tangent hyperplanes intersect transversally. When $n > d$, transversality must be violated; nondegeneracy means that there is as little violation as possible.

Asymptotic expansions. Our notation is standard. We consider only expansions of a complex-valued function $f(\lambda)$, as $\lambda \to \infty$ in some specified way in $\mathbb{R}$ or $\mathbb{R}^k$. We allow generalized expansions of the form $f(\lambda) \sim \sum_j b_j \phi_j(\lambda)$ where $b_j \in \mathbb{C}$ and $\phi_{N+1} = o(\phi_N)$ for each N. This means that $f - \sum_{j=0}^N b_j\phi_j = O(\phi_{N+1})$ for each N as $\lambda \to \infty$; this is equivalent to the apparently weaker $f - \sum_{j=0}^N b_j\phi_j = o(\phi_N)$.

Sometimes an asymptotic series will depend on a parameter, y. In this case, we say the asymptotic series is *uniform* in y, if the implied constant $K(y)$ for which

$$\left| f(\lambda, y) - \sum_{j=0}^N b_j(y)\phi_j(\lambda) \right| \leq K(y)\phi_{N+1}(\lambda)$$

may be chosen independently of y. A weaker condition is for an asymptotic series is *integrable* in y, by which we mean that each b_j is integrable in y and

$$\left| f(\lambda, y) - \sum_{j=0}^N b_j(y)\phi_j(\lambda) \right| \leq K(y)\phi_N(\lambda)$$

for some integrable function $K(y)$.

A useful observation is that integrable asymptotic expansions may be integrated term-by-term, and thus uniform expansions may be integrated term-by-term with respect to a finite measure.

2.4. **Proposition.** *Let μ be a positive measure on a set A. If $f(\lambda, y)$ has an asymptotic expansion $\sum_j b_j(y)\phi_j(\lambda)$ which is integrable in y, then*

$$\int_A f(\lambda, y) d\mu(y) \sim \sum_j \left[\int_A b_j(y)\, d\mu(y)\right] \phi_j(\lambda).$$

Proof. By the definition of asymptotic expansion, for each fixed y there is a function $K_y(\lambda)$ going to zero as λ goes to infinity, such that

$$\left| f(\lambda, y) - \sum_{j=0}^{N} b_j(y)\phi_j(\lambda) \right| \le K_y(\lambda)\phi_N(\lambda)\,.$$

When the expansion is integrable, we then have

$$\int_A \left| f(\lambda, y) - \sum_{j=0}^{N} b_j(y)\phi_j(\lambda) \right| \, d\mu(y) \le \phi_N(\lambda) \int_A \left(\min\{K_y(\lambda), K(y)\}\right)\, d\mu(y)\,.$$

By Lebesgue dominated convergence, the integral on the right hand side goes to zero as λ goes to infinity, so we conclude that

$$\left| \int_A f\, d\mu - \sum_{j=0}^{N} \int_A b_j\phi_j\, d\mu \right| \le \int_A \left| f - \sum_{j=0}^{N} b_j\phi_j\, d\mu \right| = o(\phi_N)$$

which establishes the asymptotic expansion. □

Oscillatory integrals. We will require one almost standard asymptotic integration result. Let $\mathcal{E} \subseteq \mathbb{R}^m$ be a product of simplices and open intervals. Suppose smooth functions f and ψ are defined on $\mathcal{E}$ and let $I(\lambda; f, \psi)$ denote the integral $\int_{\mathcal{E}} \exp(-f(\mathbf{x}))\psi(\mathbf{x})\, d\mathbf{x}$. When f is real and nonnegative (the Laplace case), or when f is purely imaginary and ψ is compactly supported (the Fourier case), then an asymptotic series expansion for $I(\lambda)$ as $\lambda \to \infty$ is well known. We require the following analogue, whose proof may be found in [PW00, Theorem 2.4 and Corollary 2.5].

2.5. **Lemma** (Complex Fourier-Laplace asymptotics). *Let $\mathcal{E}$ be a d-dimensional product of simplices and intervals inside $\mathbb{R}^m$, and denote by $\mathcal{E}^\circ$ the product of the interiors of these, that is, the set of points whose neighborhood in $\mathcal{E}$ is diffeomorphic to $\mathbb{R}^d$. Let $f, \psi : \mathcal{E} \to \mathbb{C}$ be smooth functions with the real part of f nonnegative on $\mathcal{E}$. Suppose there is a unique point $\mathbf{0} \in \mathcal{E}$ for which both (i) the real part of f vanishes, and (ii) the gradient of f is orthogonal to every curve in $\mathcal{E}$ through $\mathbf{0}$. Suppose further that the Hessian $\mathbf{H}$ of f is nonsingular at $\mathbf{0}$.*

If $\mathbf{0} \in \mathcal{E}^\circ$, then there is an asymptotic expansion

$$I(\lambda; f, \psi) \sim \sum_{j=0}^{\infty} a_j \lambda^{-d/2-j}.$$

In particular, if $\psi(\mathbf{0}) \neq 0$, then

$$a_0 = \psi(\mathbf{0}) \frac{(2\pi)^{d/2}}{\sqrt{\det(\mathbf{H})}}$$

where the square root is defined to be the product of the principal square roots of the eigenvalues of $\mathbf{H}$.

If instead, $\mathbf{0}$ *has a neighborhood in* $\mathcal{E}$ *diffeomorphic to a d-dimensional half-space, then the same result holds but with the value of* a_0 *halved. As* $\mathbf{0}$ *ranges over any compact set where* $\det \mathbf{H}$ *is nonvanishing, there is a bound* $|I(\lambda)| \leq C\lambda^{-d/2}$. □

3. Main theorems and illustrative examples

All of the results in this section are ultimately proved by reducing the problem to the computation of asymptotics for an oscillatory integral (the proofs are presented in later sections). Owing to the large number of cases arising in this analysis, constructing a taxonomy of cases is rather challenging and the number of possible theorems is ridiculous.

We have chosen to present a series of theorems of varying complexity and generality. Taken together, they completely describe asymptotics associated with nondegenerate multiple points. Our analysis of other types of multiple points requires additional (mild) hypotheses, which will almost always be satisfied in applications. The most important case is that of transversal points, but we also treat various types of tangencies and degeneracies. More cases will be dealt with in future articles. However, it is always possible that a practical problem involving a meromorphic generating function may not fit neatly into our classification scheme. We hope to convince the reader that in such a situation our basic method will yield oscillatory integrals from which asymptotics can almost certainly be extracted in a systematic way.

In two variables, every strictly minimal point of $\mathcal{V}$ is a multiple point, so that locally $\mathcal{V}$ is the union of $n+1$ curves (counting multiplicity). Despite the fact that many possible tangencies can occur, it turns out that there are essentially only two cases: as long as at least two of the $n+1$ tangents are distinct, nondegeneracy is assured. This case is treated in Theorem 3.3. The degenerate situation, where all tangents coincide, is more complicated and we require some extra hypotheses; this falls under Theorem 3.11.

We begin with the simplest case, in which the result admits a relatively self-contained statement, with as little extra notation as possible.

3.1. **Theorem** (2 curves meeting transversally in 2-space). *Let* F *be a meromorphic function of two variables, not singular at the origin, with* $F(z,w) = G(z,w)/H(z,w) = \sum_{r,s} a_{rs} z^r w^s$.

Suppose that (z,w) *is a strictly minimal, double point of* $\mathcal{V}$. *Let* $\mathbf{H}(z,w)$ *denote the Hessian of* H *at* (z,w). *If* $\det \mathbf{H}(z,w) \neq 0$, *then*

$$a_{rs} = z^{-r} w^{-s} \left(\frac{G(z,w)}{\sqrt{-z^2 w^2 \det \mathbf{H}(z,w)}} + O(e^{-c|(z,w)|}) \right),$$

uniformly as r/s *varies over a compact subset of the interior of* $\mathbf{K}_0(z,w)$.

Remark. The geometric significance of the nonsingularity of $\mathbf{H}(z,w)$ is that this is equivalent to the curves $\mathcal{V}_j$ intersecting transversally. If, on the other hand, $\det \mathbf{H}(z,w) = 0$ (equivalently the curves are tangent), then higher order information is required in order to compute the relevant asymptotics. We treat this situation in Theorem 3.11.

Note that in the situation of the above theorem, the asymptotic exponential rate of a_{rs} is constant on the interior of the cone $\mathbf{K}(z,w)$, that is, $a_{rs} \sim \exp(-(r,s)\cdot\mathbf{v})$ where $\mathbf{v} = (\log z, \log w)$ is

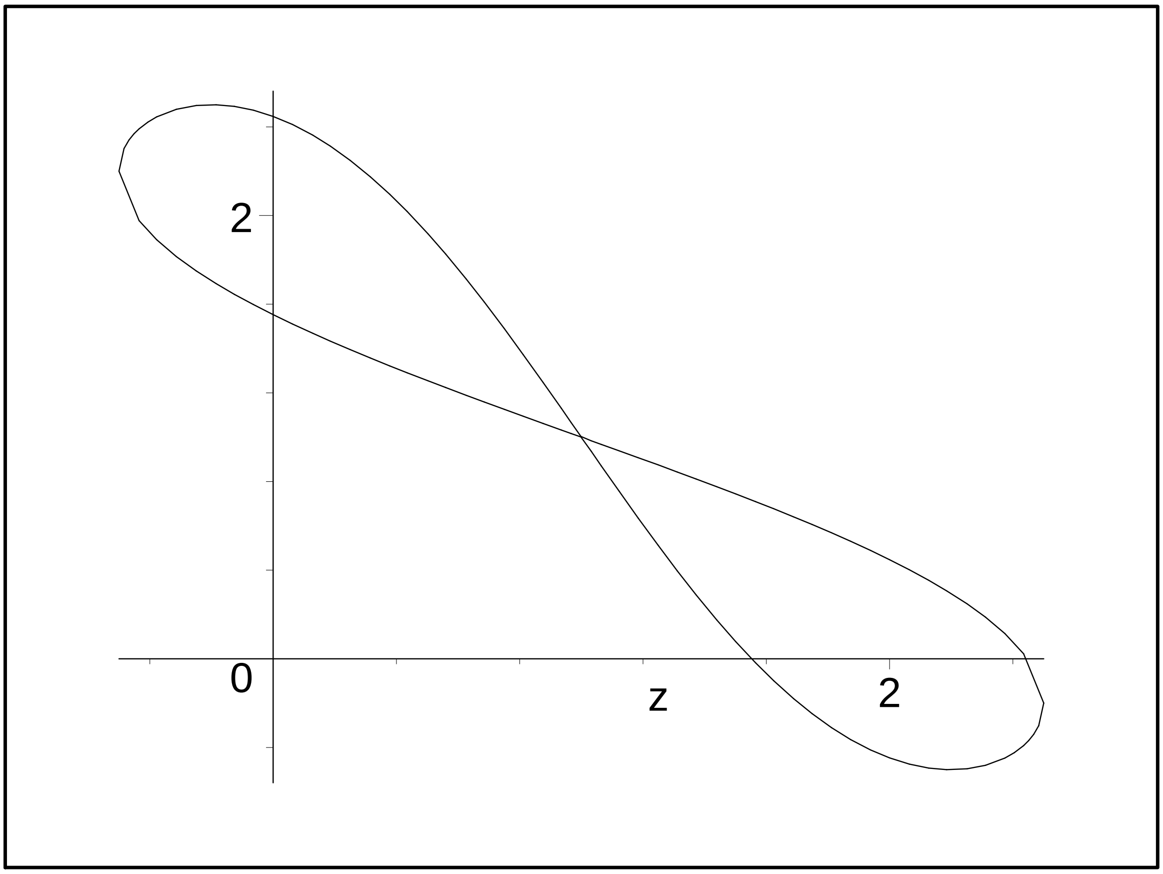

FIGURE 4. Figure-eight shape

constant on $\mathbf{K}(z, w)$; this differs considerably from the asymptotics previously derived for smooth points [PW99].

In more than one complex variable, it is possible to have a pole where F is still continuous; one simply needs G to vanish to as high an order as H. In the two variable case, under an assumption of Hölder-1/2 continuity, [Tsi93, Theorem 6.1] shows that subexponential decay occurs only at directions in the boundary of $\mathbf{K}_0$, and that the $|a_{rs}|$ are in fact summable. This phenomenon in any number of variables is explained in [Pem00].

3.2. **Example** (the figure 8). Consider $F = 1/H$, where $H(z, w) = 19 - 20z - 20w + 5z^2 + 14zw + 5w^2 - 2z^2w - 2zw^2 + z^2w^2$.

The only common zero of H and ∇H is at $(1, 1)$, and this is the only candidate for a strictly minimal double point. All other strictly minimal elements of $\mathcal{V}$ must be smooth. The Taylor expansion of H around $(1, 1)$ is, to second order, $4(w - 1)^2 + 10(z - 1)(w - 1) + 4(z - 1)^2 = 4[w - (3 - 2z)][w - (3/2 - z/2)]$. The Hessian determinant at $(1, 1)$ is therefore -36. The two tangent lines have slopes $-1/2$ and -2, so that asymptotics in directions $r/s = \kappa \in [1/2, 2]$ cannot be obtained by smooth points. For slopes in the interior of this interval, the asymptotic $a_{rs} \sim 1/6$ holds, by Theorem 3.1.

Directions corresponding to slopes outside the interval $[1/2, 2]$ may be treated by the methods of [PW99]. On the boundary of the cone (slopes 1/2 or 2) it turns out that $a_{rs} \sim 1/12$. The justification for this is given in a remark following the proof of Theorem 3.3.

This example demonstrates that only the local behavior of H at $(1,1)$ is of relevance. In fact H is irreducible in $\mathbb{C}[z-1, w-1]$ and hence does not factor globally (in the power series ring $\mathbb{C}[[z-1, w-1]]$).

The previous theorem treated the simplest case $d = n = 1$, and our subsequent results will be labelled in a similar way. We begin with some theorems involving nondegenerate points. The simplest case is when the point is completely nondegenerate; the picture to keep in mind here is of $d+1$ complex hypersurfaces in $\mathbb{C}^{d+1}$ intersecting at a single point. We again obtain asymptotic constancy on the interior of the cone of allowable directions, provided $G(\mathbf{z}) \neq 0$. Recall from Definition 2.3 that $\mathbf{C}$ is the $n+1$ by $d+1$ matrix whose j,k-element is $(w_k/w_{d+1})(\partial v_j/\partial w_k)(\widehat{\mathbf{z}})$, defined to be 1 if $k = d+1$.

3.3. **Theorem** ($d = n$). *Let F be a function of $d+1$ variables, not singular at the origin, with $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. Suppose that $\mathbf{z}$ is a strictly minimal, completely nondegenerate multiple point of $\mathcal{V}$ and that F is meromorphic in a neighborhood of $\mathbf{T}(\mathbf{z})$. Then for each compact subset K of the interior of $\mathbf{K}(\mathbf{z})$,*

$$a_{\mathbf{r}} \sim \mathbf{z}^{-\mathbf{r}} \left(C + e^{-c|\mathbf{r}|} \right) \text{ uniformly for } \mathbf{r} \in K.$$

If $G(\mathbf{z}) \neq 0$, the constant C is given by

$$C = \sqrt{d+1} \frac{\phi(\mathbf{z})}{|\det \mathbf{C}(\mathbf{z})|}$$

where, as in (2.3),

$$\phi(\mathbf{z}) = \frac{(n+1)!}{(-z_{d+1})^{n+1}} \frac{G(\mathbf{z})}{\left(\frac{\partial}{\partial w_{d+1}} \right)^{n+1} H(\mathbf{z})}.$$

In case $d < n$, a modification of the last result is required, involving some linear algebra which we now introduce. Let $\Delta = \Delta_n$ denote the standard n-simplex

$$\Delta := \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=0}^{n} \alpha_j = 1 \text{ and } \alpha_j \geq 0 \text{ for all } j \right\}.$$

The hyperplane $\mathbf{1}^{\perp}$ parallel to Δ will be denoted by $\mathcal{H}$. Assuming nondegeneracy of a minimal point $\mathbf{z}$, the rows of $\mathbf{C}$ are $n+1$ vectors spanning $\mathbb{R}^{d+1}$, whence the space of solutions $\mathcal{A} := \{\boldsymbol{\alpha} : \boldsymbol{\alpha}\mathbf{C}(\mathbf{z}) = \boldsymbol{\delta}\}$ will always be $(n-d)$-dimensional. The set $\mathcal{A} \cap \Delta$ is just the space of coefficients of convex combinations of rows of $\mathbf{C}$ that yield the given vector $\boldsymbol{\delta}$. If $\boldsymbol{\delta} \in \mathbf{K}_0$ then $\mathcal{A}$ is a subset of the hyperplane in $\mathbb{R}^{n+1}$ containing Δ, thus if $\boldsymbol{\delta}$ is in the interior of $\mathbf{K}$ then $\mathcal{A} \cap \Delta$ is $(n-d)$-dimensional. Let $\mathcal{A}^{\perp}$ denote the orthogonal complement in $\mathcal{H}$ of $\mathcal{A}_0$, the set $\mathcal{A}$ translated to the origin. The dimension (later denoted ρ) of $\mathcal{A}^{\perp}$ is d. The space $\mathcal{A}^{\perp}$ does not depend on $\boldsymbol{\delta}$, since changing $\boldsymbol{\delta}$ only translates $\mathcal{A}$.

The role that $\mathcal{A}$ will play is this: there will be an integral over Δ whose stationary points are the points of $\mathcal{A}$, each contributing the same amount; thus the total contribution will be the measure of $\mathcal{A}$ in the the *Lebesgue-complementary measure* which we denote by σ. The proper normalization in the definition of σ is

$$\sigma(S) := \sigma(S \cap \mathcal{A} \cap \Delta) := \frac{\mu_{n-d}(S \cap \mathcal{A} \cap \Delta)}{\mu_n(\Delta)}$$

where μ_k denotes k-dimensional Lebesgue measure. One may think of this as: $\sigma \times \mu_d =$ the normalized Lebesgue measure, μ, on Δ.
Warning: When $\mathcal{A}$ is a single point, it is tempting to assume that $\sigma(\mathcal{A}) = 1$, but in fact in this case $\sigma(\mathcal{A}) = \mu(\Delta)^{-1}$.

3.4. **Proposition.** *Assume the columns of* $\mathbf{C}$ *are linearly independent. Then the projection of the column space of* $\mathbf{C}$ *onto* $\mathcal{A}^\perp$ *has dimension* d.

Proof. The matrix $\mathbf{C}$ is the sum of three columnwise projections, namely one onto $\mathcal{A}_0$, one onto $\mathcal{A}^\perp$ and one onto the span of $\mathbf{1}$. The sum of the first two projections annihilates the last column of $\mathbf{C}$, mapping the column space to a space of dimension d. By definition of $\mathcal{A}$, the space $\mathcal{A}\mathbf{C}$ is a single point, hence $\mathcal{A}_0\mathbf{C} = 0$, meaning that the first projection is null. Therefore, the second projection has rank d. □

3.5. **Definition.** When the columns of $\mathbf{C}$ are linearly independent, let $\overline{\mathbf{C}}$ denote the matrix representing the linear transformation $\mathbf{v} \mapsto \mathbf{v}\mathbf{C}$ on $\mathcal{A}^\perp$ with respect to some basis of $\mathcal{A}^\perp$. Note that $\overline{\mathbf{C}}$ is independent of $\boldsymbol{\delta}$ since $\mathcal{A}^\perp$ is, and that $|\overline{\mathbf{C}}|$ is independent of the choice of basis of $\mathcal{A}^\perp$. For general $\mathbf{C}$, we extend this definition so that $\overline{\mathbf{C}}$ is the projection of the first d columns of $\mathbf{C}$ onto the space $\mathcal{A}^\perp$, then represented in a (fixed but arbitrary) orthonormal basis of $\mathcal{A}^\perp$.

In stating subsequent results, we shall rely increasingly on derived data such as $\overline{\mathbf{C}}$. It is in principle possible to give formulae for these asymptotics in terms of the original data G and H, but such expressions rapidly become too cumbersome to be useful.

3.6. **Theorem** ($d \leq n$)**.** *Let* F *be a function of* $d+1$ *variables, not singular at the origin, with* $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. *Suppose that* $\mathbf{z}$ *is a strictly minimal, nondegenerate multiple point of degree* $n+1$ *of* $\mathcal{V}$ *and that* F *is meromorphic in a neighborhood of* $\mathbf{T}(\mathbf{z})$. *Then* $\mathbf{K}$ *is a finite union of cones* K_j *such that for each compact (in projective space) subcone* K *of the interior of some* K_j, *there is a polynomial* P *of degree at most* $n-d$ *such that*

$$a_{\mathbf{r}} = \mathbf{z}^{-\mathbf{r}} \left(P(\mathbf{r}) + O(e^{-c|\mathbf{r}|}) \right) \qquad \textit{uniformly for } \mathbf{r} \in K.$$

If $G(\mathbf{z}) \neq 0$, *the leading coefficient is given by*

$$P(\mathbf{r}) \sim \frac{\phi(\mathbf{z})\sigma(\mathcal{A} \cap \Delta)}{|\det \overline{\mathbf{C}}|} (r_{d+1})^{n-d} ,$$

where $\mathcal{A}$ *is the solution set to* $\boldsymbol{\alpha}\mathbf{C} = \boldsymbol{\delta}$ *with* $\boldsymbol{\delta} = \mathbf{r}/r_{d+1} \in \mathbf{K}_0$, *and* σ *is the complementary measure.*

Remark. The approximation to $a_{\mathbf{r}}$ is actually piecewise polynomial and is asymptotically valid throughout the interior of $\mathbf{K}$. The correction term may, however, fail to be exponentially small on some lower-dimensional surfaces in the interior of $\mathbf{K}$ where the piecewise polynomial is pieced together. Note also that we have broken the coordinate symmetry in the formula for the leading term by parametrizing $\mathbf{r}$ in terms of r_{d+1} and $\boldsymbol{\delta}$.

3.7. **Example** (3 curves in 2-space). The simplest interesting case to which Theorem 3.6 applies is that where $\mathcal{V}$ is the union of 3 curves in 2-space which intersect at a strictly minimal point. Suppose that the strictly minimal singularity in question is at $(1,1)$. In this case Theorem 3.6 shows that

$$a_{rs} \sim f(r/s) s^{n-1}$$

for some f, whenever r/s lies in the interior of the interval formed by the slopes $c_j = v_j'(1)$. We shall obtain a more explicit expression for f.

We have $d = 1$ and $n = 2$. Without loss of generality, we suppose that $c_0 \leq c_1 \leq c_2$, at least one of the two inequalities being strict. Assume for now that both inequalities are strict: $c_0 < c_1 < c_2$. Let $\mathcal{A}_\delta = \{(\alpha_0, \alpha_1, \alpha_2) \in \Delta \mid \alpha_0 c_0 + \alpha_1 c_1 + \alpha_2 c_2 = \delta\}$. When δ belongs to the convex hull of c_0, c_1 and c_2, the affine set $\mathcal{A}_\delta$ is a line segment whose endpoints are on the boundary of the simplex Δ. Since $\mathcal{A}$ is orthogonal to $\mathbf{c}$, the set $\mathcal{A}^\perp$ is parallel to the projection $\bar{\mathbf{c}}$ of $\mathbf{c}$ onto $\mathcal{H}$. Letting m denote the mean of the c_j, $\Sigma/\sqrt{3}$ the standard deviation and $\mathbf{m} = m\mathbf{1}$, we see that the projection is $\mathbf{c} - \mathbf{m}$ and its euclidean length is Σ.

If $c_0 < \delta < c_1$, then one endpoint of the line segment is on the face $\alpha_2 = 0$, with $\alpha_0 = \frac{c_1 - \delta}{c_1 - c_0}, \alpha_1 = \frac{\delta - c_0}{c_1 - c_0}$. The other endpoint is when $\alpha_1 = 0$, with $\alpha_0 = \frac{c_2 - \delta}{c_2 - c_0}, \alpha_2 = \frac{\delta - c_0}{c_2 - c_0}$. The squared euclidean length of this line segment simplifies to

$$3 \frac{(\delta - c_0)^2 \Sigma^2}{(c_1 - c_0)^2 (c_2 - c_0)^2}.$$

A similar argument gives the answer when $c_1 < \delta < c_2$. The squared euclidean length of the line segment is then

$$3 \frac{(\delta - c_2)^2 \Sigma^2}{(c_2 - c_1)^2 (c_2 - c_0)^2}.$$

Both answers agree, and by continuity give the correct answer, when $\delta = c_1$. Since the area of Δ_2 is $\sqrt{3}/2$, the complementary measure σ of $\mathcal{A}_\delta$ is $2/\sqrt{3}$ times its euclidean length. Thus we obtain

$$f(\delta) = \begin{cases} 2 \frac{|\delta - c_0|}{|c_1 - c_0||c_2 - c_0|}, & c_0 \leq \delta \leq c_1 \\ 2 \frac{|\delta - c_2|}{|c_2 - c_1||c_2 - c_0|}, & c_1 \leq \delta \leq c_2. \end{cases}$$

Note that $f = 0$ on the boundary of the cone, namely when $\delta = c_0$ or $\delta = c_2$. The above formula extends to the case $c_0 = c_1$ or $c_1 = c_2$ in the obvious way. This example illustrates the piecewise polynomial nature of the asymptotics in $\mathbf{K}$. The approximation $a_{rs} \sim P(r,s)$ is exponentially close for δ in compact subintervals of $(c_0, c_1) \cup (c_1, c_2)$, but requires polynomial correction when $\delta \to c_1$ as well as at the endpoints c_0 and c_2.

More elegant formulas for the polynomial P can no doubt be derived in greater generality, but we shall not focus on that aspect. Instead, we turn to cases which are degenerate according to our terminology. We begin with the simplest, namely the transverse case. Note that in this case, the set $\mathcal{A}$ of linear combinations of the rows of $\mathbf{C}$ that yield $\boldsymbol{\delta}$ is always a single point since the rows of $\mathbf{C}$ are linearly independent.

3.8. **Definition.** If $\mathbf{z}$ is a strictly minimal multiple point and $\boldsymbol{\alpha}$ is any element of Δ, we define a matrix $Q = Q(\mathbf{z}, \boldsymbol{\alpha})$ to be the Hessian (matrix of second partial derivatives in the variables

$(\theta_1, \ldots, \theta_d)$ of the function

$$\widehat{\boldsymbol{\theta}} \mapsto -\log \sum_{j=0}^{n} \alpha_j v_j(\widehat{\mathbf{z}} e^{i\widehat{\boldsymbol{\theta}}}) .$$

at the point $\widehat{\boldsymbol{\theta}} = \mathbf{0}$. Then define a matrix M by

$$M := M(\mathbf{z}, \boldsymbol{\alpha}) := \begin{pmatrix} 0 & -i\overline{\mathbf{C}} \\ -i\overline{\mathbf{C}}^T & Q(\boldsymbol{\alpha}) \end{pmatrix} . \tag{3.1}$$

3.9. **Theorem** ($\rho = n \leq d$). *Let F be a function of $d+1$ variables, not singular at the origin, with $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. Suppose that $\mathbf{z}$ is a strictly minimal, transverse multiple point of degree $n+1$ of $\mathcal{V}$ and that F is meromorphic in a neighborhood of $\mathbf{T}(\mathbf{z})$. Given the direction $\boldsymbol{\delta}$, suppose also that* $\det M(\mathbf{z}, \boldsymbol{\alpha}) \neq 0$, *where $\boldsymbol{\alpha}(\boldsymbol{\delta})$ is the unique point in $\mathcal{A} := \mathcal{A}(\boldsymbol{\delta})$.*

The set $\mathbf{K}_0(\mathbf{z})$ has dimension n, and we refer to its interior as the set of convex combinations of the rows of $\mathbf{C}$ with no vanishing coefficient. Let K be a compact subset of the interior of $\mathbf{K}_0(\mathbf{z})$ such that $\det M(\mathbf{z}, \boldsymbol{\alpha}(\boldsymbol{\delta}))$ *is nonvanishing when $\boldsymbol{\delta} \in K$. Then there is a complete asymptotic expansion*

$$a_{\mathbf{r}} \sim \mathbf{z}^{-\mathbf{r}} \sum_{k \geq 0} b_k (r_{d+1})^{\frac{n-d}{2} - k} \quad \textit{uniformly for } \mathbf{r} \in K .$$

If $G(\mathbf{z}) \neq 0$, the leading coefficient is given by

$$b_0 = \frac{n!}{\sqrt{n+1}} \frac{(2\pi)^{\frac{n-d}{2}} \phi(\mathbf{z})}{\det M(\mathbf{z})^{1/2}} .$$

3.10. **Example** (2 planes in 3-space). Consider the trivariate sequence given recursively for $r, s, t \geq 0$ by $a_{0,0,0} = 1$ and

$$\begin{aligned} 16 a_{r,s,t} &= 12 a_{r-1,s,t} + 12 a_{r,s-1,t} + 8 a_{r,s,t-1} - 5 a_{r-1,s-1,t} - 3 a_{r-1,s,t-1} - 3 a_{r,s-1,t-1} \\ &\quad - 2 a_{r-2,s,t} - 2 a_{r,s-2,t} - a_{r,s,t-2} . \end{aligned}$$

Let $F(x,y,z) = \sum_{r,s,t} a_{r,s,t} x^r y^s z^t$ be the associated generating function. Then we obtain directly the explicit form

$$F(x,y,z) = \frac{16}{(4 - 2x - y - z)(4 - x - 2y - z)} .$$

The point $(1,1,1)$ is a multiple point of $\mathcal{V}$ which is readily seen to be strictly minimal. Near this point, $\mathcal{V}$ is parametrized by $z = 4 - 2x - y$ and $z = 4 - x - 2y$. The cone $\mathbf{K}$ is the positive hull of $(2,1,1)$ and $(1,2,1)$, with $\mathbf{K}_0$ being the line segment between these. Thus the multiple point computes asymptotics for a_{rst} with $r + s = 3t$ and $r, s \geq t$. Given $\boldsymbol{\delta} = (2 - \alpha, 1 + \alpha, 1) \in \mathbf{K}_0$, the set $\mathcal{A}$ is the single point $(1 - \alpha, \alpha)$. The complement $\mathcal{A}^\perp$ is always all of $\mathcal{H}$, which is the span of $\mathbf{x} := (\sqrt{1/2}, -\sqrt{1/2})$ (we have chosen the constant so this is an orthonormal basis).

The matrix $\overline{\mathbf{C}}$ is equal to $((2,1) \cdot \mathbf{x}, (1,2) \cdot \mathbf{x}) = \mathbf{x}$. This leads to $\det M = A + B + C + D$, where A, B, C and D are the entries of the restricted Hessian, Q. A routine computation shows that $A + B + C + D = 12$, leading to $\det M = 6$. Computing $\phi(\mathbf{1})$ from (2.3) then gives

$$\phi(\mathbf{1}) = \frac{2!}{(-1)^2} \frac{16}{2} = 16 .$$

Plugging this into Theorem 3.9 we find (to first order, as $t \to \infty$) that

$$a_{rst} \sim \frac{16}{\sqrt{24\pi t}} \qquad \text{if } r+s=3t \text{ and } r/s \in (1/2,2).$$

The above first order approximation to $a_{90,90,60}$ is accurate to within 0.3%.

As mentioned in the introduction, instead of producing enough variants of these theorems for a complete taxonomy, we will stop with just one more result. The case we describe is the most degenerate, namely where all the sheets $\mathcal{V}_j$ are tangent at $\mathbf{z}$. In this case M varies with α and the resulting formula is an integral over α. The methods used here may be adapted to prove results when the degeneracy is somewhere between this extreme and the nondegenerate cases.

3.11. **Theorem** ($\rho = 0$). *Let F be a function of $d+1$ variables, not singular at the origin, with $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. Suppose that $\mathbf{z}$ is a strictly minimal, multiple point of degree $n+1$ of $\mathcal{V}$ and that F is meromorphic in a neighborhood of $\mathbf{T}(\mathbf{z})$. Further suppose that all tangent hyperplanes to the sheets $\mathcal{V}_j$ at $\mathbf{z}$ coincide. Then $\mathbf{K}(\mathbf{z})$ is a single point. If $\det M(\mathbf{z};\boldsymbol{\alpha}) \neq 0$ on Δ, then in this unique direction we have*

$$a_{\mathbf{r}} = \mathbf{z}^{-\mathbf{r}} b_0 (r_{d+1})^{n-d/2} + O(r_{d+1})^{n-d/2-1}\,.$$

The value of b_0 is given by

$$b_0 = \frac{\phi(\mathbf{z})}{(2\pi)^{d/2}} \int_\Delta \det M(\mathbf{z};\boldsymbol{\alpha})^{-1/2}\, d\mu(\boldsymbol{\alpha}).$$

Remark. This theorem covers the situation where all the sheets coincide (as opposed to merely being tangent). This can also be analyzed in a different way by a slight modification of the proof of the smooth case, as discussed in [PW99]. The methods of [PW99] yield a complete asymptotic expansion.

3.12. **Example.** The simplest case illustrating Theorem 3.11 is that of two curves $w = u_j(z)$ in two-space, intersecting tangentially at the strictly minimal point $\mathbf{z} = (1,1)$. Suppose that $\mathrm{Re}\{\log v_j(e^{i\theta})\} = -d_j\theta^2 + \dots$ with $d_j > 0$. The simplex in question is one-dimensional and we identify it with the interval $0 \le t \le 1$. Then for $0 \le t \le 1$ the matrix $M(\mathbf{z};t)$ is just the 1×1 matrix $d_t = (1-t)d_0 + td_1$. Hence when $d_0 \neq d_1$ we obtain asymptotics in the unique direction $\boldsymbol{\delta}$ of the theorem:

$$\begin{aligned} a_{rs} &\sim \frac{\phi(1,1)}{\sqrt{2\pi}} \int_0^1 (d_t)^{-1/2}\, dt \\ &= \frac{\phi(1,1)}{\sqrt{2\pi}} \frac{1}{d_1 - d_0} \int_{d_0}^{d_1} y^{-1/2}\, dy \\ &= \frac{2\phi(1,1)}{\sqrt{2\pi}} \frac{\sqrt{d_1} - \sqrt{d_0}}{d_1 - d_0} \\ &= \frac{\phi(1,1)}{\sqrt{2\pi}} \frac{2}{\sqrt{d_0} + \sqrt{d_1}}. \end{aligned}$$

This final formula is easily seen to hold also in the case $d_0 = d_1$, in which case the formula $\phi(1,1)/\sqrt{2\pi d_0}$ agrees with the formula in [PW99].

Remark. Since analysis in 1 variable is considerably easier than any other case, it is possible to derive a result for plane curves even when the quadratic term of v_j vanishes. We omit the details here, but note that such a derivation is possible because we must have $\log wv_j(ze^{i\theta}) = ic_j\theta - d\theta^m + \ldots$, where $c_j \geq 0$, m is even and $\mathrm{Re}\{d\} > 0$.

If all tangents are equal at (z, w), then $\mathbf{K}_0(z, w)$ is a singleton $\{\boldsymbol{\delta}\}$. If all $\mathrm{Re}\{\log wv_j(z^* e^{i\theta})\}$ vanish to the same exact order m and $(r, s) \in \mathbf{K}$, then

$$a_{rs} = z^{-r}w^{-s}b_0 s^{n-1/m} + O(s^{n-2/m})$$

with b_0 given by

$$b_0 = \frac{\phi(z,w)\Gamma(1/m)}{2\pi(1-1/m)} \frac{d_1^{1-1/m} - d_0^{1-1/m}}{d_1 - d_0}.$$

The above formula for b_0 also holds in the limit when $d_0 = d_1$, when it becomes $b_0 = \dfrac{\phi(z,w)\Gamma(1/m)}{2\pi(d_0)^{1/m}}$.

4. Proof of Theorem 3.1

In this section we give a separate proof of Theorem 3.1. That result is a corollary of the more general proofs in the next section, and hence can be logically omitted. However, we include it because the notation in the next section is necessarily rather complicated and may obscure the essential idea.

The outline of the proof is as follows. We first use apply the residue theorem to show that a_{rs} is asymptotically approximated by a one-dimensional integral of a sum of two residues. Each of these residues fails to be integrable on its own, so we cannot distribute the integral over the sum. Instead, our second step is to write the sum of residues as a one-dimensional integral. The third step is to convert this to two-dimensional oscillatory integral by means of the substitution $z = e^{i\theta}$. This integral is not quite of standard form, but is covered by the analysis in [PW00]. Once we have recovered the leading term, the last step is to quote a result implying exponential decay of the correction term.

We recall our standard notation in this case; since the number of variables is $d + 1 = 2$, the d-dimensional vector $\widehat{\mathbf{z}}$ becomes a scalar, and in addition we will write (z, w) in place of $\mathbf{z}$. Our generating function is $F(z, w) = G(z, w)/H(z, w) = \sum_{r,s} a_{rs} z^r w^s$. Suppose that we have a strictly minimal double point of $\mathcal{V}$. We assume that in some neighborhood of this point, we have $H(z, w) = \chi(z, w)(1 - wv_0(z))(1 - wv_1(z))$ where the v_j are analytic. We write $v_j = 1/u_j$ and $\phi = G/\chi$. We shall first prove the result in the case where the minimal point is $(1, 1)$, and then derive the general result from that case.

Copying the proof of [PW99, Lemma 4.1] with the sum of all residues in place of the single residue leads immediately to a proposition valid for a strictly minimal multiple point in any dimension and of any multiplicity. The proof is simply to write the integral as an iterated integral over $\widehat{\mathbf{w}}$ and w_{d+1}, represent this last integral as a residue, then notice that the magnitude of the integrand is small away from $\widehat{\mathbf{z}} = \widehat{\mathbf{w}}$. A version of this argument in two variables appears in [Tsi93, Theorem 1.1].

4.1. **Proposition** (Local residue formula). *Let $R(\widehat{\mathbf{z}})$ denote the sum of the residues of $w^{-r_{d+1}-1}F(\widehat{\mathbf{w}}, w)$ at $w = u_j(\widehat{\mathbf{w}})$. Then for a sufficiently small neighborhood $\mathcal{N}$ of $\mathbf{1}$ in $\mathbf{T}(\mathbf{1})$,*

$$|\mathbf{z}|^{-\mathbf{r}} \left| a_{\mathbf{r}} - (2\pi i)^{-1} \int_{\mathcal{N}} -\widehat{\mathbf{z}}^{-\widehat{\mathbf{r}}-1} R(\widehat{\mathbf{z}})\, d\widehat{\mathbf{z}} \right| \text{ is exponentially decreasing in } s. \tag{4.1}$$

□

The second step is to to express the residue sum $R(z)$ in a way which is suitable for asymptotic analysis. The following integral formula does the trick.

4.2. **Lemma.** *For each $t \in [0,1]$, define $v_t := (1-t)v_0 + tv_1$. Also define a function h depending on s and z by $h(w) := h_{s,z}(w) := w^{s+1}\phi(z, 1/w)$. Then*

$$-R(z) = \int_0^1 h'(v_t)\, dt.$$

Proof. First suppose that $v_0 \neq v_1$. Then each pole at $w = 1/v_j(z)$ is simple. The corresponding residues are

$$R_0(z) = v_0^{s+1}\frac{\phi(z, 1/v_0)}{v_1 - v_0}; \qquad R_1(z) = v_1^{s+1}\frac{\phi(z, 1/v_1)}{v_0 - v_1}.$$

The form of $R(z)$ as a difference quotient now encourages us to write $R(z)$ as an integral: first write $v_t = (1-t)v_0 + tv_1$. Then

$$-R(z) = \int_{[v_0, v_1]} h'(w)\, \frac{dw}{v_1 - v_0} = \int_0^1 h'(v_t)\, dt.$$

Now we note that this latter formula also works when $v_0 = v_1$, by continuity. □

It follows from the above that

$$a_{rs} \sim (2\pi i)^{-1} \int_{\mathcal{N}} z^{-r-1} \int_0^1 h'_{s,z}(v_t)\, dt\, dz, \tag{4.2}$$

We now proceed to analyze this integral.

4.3. **Lemma.** *Define functions f, ψ_1 and ψ_2 by*

$$f(\theta, t) = \frac{ir\theta}{s} - \log v_t(e^{i\theta}) \tag{4.3}$$

$$\psi_k(\theta, t) = \left(\frac{\partial}{\partial w}\right)^k \phi(e^{i\theta}, v_t(e^{i\theta})) \tag{4.4}$$

where the partial derivative is with respect to the second coordinate. There is a rectangle $\mathcal{E} := \mathcal{N} \times [0,1] \subseteq \mathbb{R}^2$,

$$a_{rs} \sim (2\pi)^{-1} \int_{\mathcal{E}} e^{-sf(\mathbf{x})}\, [(s+1)\psi_0(\mathbf{x}) + \psi_1(\mathbf{x})]\; d\nu(\mathbf{x}), \tag{4.5}$$

where ν is Lebesgue measure.

Proof. Substituting $z = e^{i\theta}$ for $-\pi \le \theta < \pi$ into (4.2), we obtain

$$a_{rs} \sim (2\pi)^{-1} \int_{\mathcal{E}} \exp(-ir\theta) h'_{s,e^{i\theta}}(v_t(e^{i\theta}))\, d\nu(\theta, t).$$

Expand $h'_{s,z}(y)$ by the product rule, to obtain

$$(s+1)y^s\phi(z, 1/y) + y^{s+1}(\partial/\partial y)\phi(z, 1/y) = y^s((s+1)\phi^{(0)} + y\phi^{(1)})$$

and substitute $z = e^{i\theta}$ and $y = v_t(e^{i\theta})$ to obtain the result. □

Proof of Theorem 3.1. First assume the double point is at $(1,1)$. Define

$$I(s; f, \psi_k) := \int_{\mathcal{N}} \exp(-sf(\mathbf{x}))\psi_k(\mathbf{x})\, d\mathbf{x}.$$

The real part of f is a convex combination of the logs of the moduli of $v_j(e^{i\theta})$, $j = 1, 2$; by minimality, $|v_j(e^{i\theta})| \leq 1$ with equality only when $\theta = 0$, so we see that $\mathrm{Re}\{f\} \geq 0$ with equality if and only if $\theta = 0$.

For $j \in \{0,1\}$, write $v_j(e^{i\theta}) = ic_j\theta + O(\theta^2)$, where $c_j > 0$. Thus $v_t(e^{i\theta}) = ic_t\theta + O(\theta^2)$, where $c_t = (1-t)c_0 + tc_1$. It follows that

$$f(\theta, t) = i(r/s)\theta - \log v_t(e^{i\theta}) = i(r/s - c_t)\theta + O(\theta^2). \tag{4.6}$$

Say that a point (θ, t) is a *critical point* for f if the real part of f and the partial derivatives of f vanish there. We see that (θ, t) is critical if and only if $\theta = 0$ and $c_t = r/s$. which occurs only when r/s lies between c_0 and c_1, and then for a unique value t^* of t. The generalized Fourier-Laplace expansion in Lemma 2.5 now shows that

$$I(s; f, \psi_k) \sim (2\pi)s^{-1}\psi_k(0, t^*)(\det \mathbf{H})^{-1/2}$$

where $\mathbf{H}$ is the Hessian of f. The hypotheses to check are that $\det \mathbf{H} \neq 0$, and that at $(0,0)$ and $(0,1)$, the gradient of f has a nonvanishing θ component.

Since f is linear in t, the second derivative with respect to t vanishes, so the Hessian of f is $-(\partial^2 f/\partial t \partial\theta)^2$. From (4.6) we see that this is $(c_0 - c_1)^2$. Putting this together with the Fourier-Laplace result and plugging into Lemma 4.3 gives

$$a_{rs} \sim \frac{\psi_0(0, t^*)}{|c_0 - c_1|}$$

with the contribution from ψ_1 being of smaller order since $\psi_0(0, t^*) = \phi(1,1) \neq 0$.

This expression for the constant is quite handy, but to recover the statement of the theorem, we also express it in terms of the partial derivatives of H. Recall that $H = \chi Q$ where $Q(z,w) = (1 - v_0 w)(1 - v_1 w)$. Then at $(1,1)$, $Q = Q_z = Q_w = 0$ and so $H_{zz} = \chi Q_{zz} = 2\chi c_0 c_1$. Similarly, $H_{wz} = \chi(c_0 + c_1)$ and $H_{ww} = 2\chi$. Thus c_0, c_1 are the roots of the quadratic $(H_{ww})x^2 - (2H_{wz})x + H_{zz} = 0$. Solving via the quadratic formula gives $|c_0 - c_1| = 2\sqrt{-\det \mathbf{H}}/H_w w(1,1)$ and using formula (2.3) then yields

$$a_{rs} \sim \frac{\phi(1,1)}{|c_0 - c_1|} = \frac{2G(1,1)}{H_{ww}(1,1)|c_0 - c_1|} = \frac{G(1,1)}{\sqrt{-\det \mathbf{H}}}.$$

To finish the proof of Theorem 3.1 in the case where the double point is at $(1,1)$, we quote [Pem00, Corollary 3.2] which says that the only possible asymptotics in the region $c_0 < r/s < c_1$ are a constant (possibly zero, but we know not in this case) plus an exponentially small correction.

The general result follows by a change of scale mapping the multiple point (z^*, w^*) to $(1,1)$. Details of the computation are as follows. Make the change of variables $\widetilde{z} = z/z^*, \widetilde{w} = w/w^*$, and write $\widetilde{G}$ for G considered as a function of $\widetilde{z}, \widetilde{w}$. In an obvious notation, $\widetilde{a}_{rs} = (z^*)^r (w^*)^s a_{rs}$. Then, evaluating at $(\widetilde{z}, \widetilde{w}) = (1,1)$, we obtain $H_{\widetilde{z}} = z^* H_z$, $H_{\widetilde{w}} = w^* H_w$. Repeating this yields $H_{\widetilde{z}\widetilde{z}} = (z^*)^2 H_{zz}$, $H_{\widetilde{w}\widetilde{z}} = z^* w^* H_{wz}$ and $H_{\widetilde{w}\widetilde{w}} = (w^*)^2 H_{ww}$.

The special case above now yields

$$\widetilde{a}_{rs} \sim \frac{\widetilde{G}(1,1)}{\sqrt{-\widetilde{D}(1,1)}} = \frac{G(z^*,w^*)}{\sqrt{-(z^*)^2(w^*)^2 \det \mathbf{H}(z^*,w^*)}}.$$

□

5. Proofs of other theorems

In this section we adopt the same basic approach as in Section 4, but more work is required. Throughout, we assume that $\mathbf{z}$ is a strictly minimal multiple point of $\mathcal{V}$, of multiplicity $n+1$, with Weierstrass representation $F(\mathbf{w}) = \phi(\mathbf{w})/\prod_{j=0}^n (1 - wv_j(\widehat{\mathbf{w}}))$ near $\mathbf{z}$ as in Definition 2.1. Additionally, we assume $\mathbf{z} = \mathbf{1}$, leaving the adjustments for general $\mathbf{z}$ to the reader this time.

Reduction to an oscillatory integral. The main new feature in this part of the analysis, over the analysis up through Lemma 4.3 in the previous section, is as follows. Since we are operating in $d+1$ dimensions, after using the residue theorem we must still perform a d-dimensional integration of the sum of residues. Allowing $n+1$ residue functions instead of just two necessitates the following replacement for Lemma 4.2 when writing the sum of these residues as an integral. The proof may be found in [DL93, p. 121, Eqs 7.7 & 7.12].

5.1. **Lemma.** *Let h be a function of one complex variable, analytic at 0. Then*

$$\sum_{j=0}^n \frac{h(v_j)}{\prod_{r\neq j} v_j - v_r} = \int_{\Delta_n} h^{(n)}(\boldsymbol{\alpha}\mathbf{v})\, d\mu(\boldsymbol{\alpha})$$

both as formal power series in $n+1$ variables $v_0, \ldots, v_n$ and in a neighborhood of the origin in $\mathbb{C}^{n+1}$. □

As a corollary, we obtain the residue sum that we need in order to plug into the local residue formula. The notation $\boldsymbol{\alpha}\mathbf{v}$ denotes the scalar product $\sum_{j=0}^n \alpha_j v_j$. To ease the notation from here on, denote r_{d+1} by s.

5.2. **Corollary.** *Let $R(s,\widehat{\mathbf{w}},\varepsilon)$ be the sum of the residues of the function $w \mapsto w^{-s-1}F(\mathbf{w})$ inside the ball $|w-z| < \varepsilon$. Define $h_{s,\widehat{\mathbf{w}}}(y) = y^{s+n}\phi(\widehat{\mathbf{w}}; 1/y)$ and use the notation $h^{(n)}$ for the nth derivative of h with respect to y.*

Then for sufficiently small ε, there is $\delta > 0$ such that for $|\widehat{\mathbf{w}} - \widehat{\mathbf{z}}| < \delta$,

$$R(s,\widehat{\mathbf{w}},\varepsilon) = \int_{\Delta} h_{s,\widehat{\mathbf{w}}}^{(n)}(\boldsymbol{\alpha}\mathbf{v})\, d\mu(\boldsymbol{\alpha}). \tag{5.1}$$

Proof. First suppose that the functions $v_0, \ldots, v_n$ are distinct. Choose ε sufficiently small and $\delta > 0$ such that f has exactly $n+1$ simple poles in $|w - z| < \varepsilon$ whenever $|\widehat{\mathbf{w}} - \widehat{\mathbf{z}}| < \delta$. Then there are $n+1$ residues, the jth one being

$$\frac{v_j(\widehat{\mathbf{w}})^{s+n}\phi(\widehat{\mathbf{w}}, 1/v_j(\widehat{\mathbf{w}}))}{\prod_{r\neq j}(v_r(\widehat{\mathbf{w}}) - v_j(\widehat{\mathbf{w}}))} = \frac{h_{s,\widehat{\mathbf{w}}}(v_j(\widehat{\mathbf{w}}))}{\prod_{r\neq j}(v_r(\widehat{\mathbf{w}}) - v_j(\widehat{\mathbf{w}}))}.$$

The result in this case now follows by summing over j and applying Lemma 5.1.

In the case when $v_0, \ldots, v_n$ are not distinct, let v_j^t be functions approaching v_j as $t \to 0$, and let $f^t(w) = w^{-s-1}\phi(\mathbf{w})/\prod_{j=0}^n (w - 1/v_j^t(\widehat{\mathbf{w}}))$, so that $f^t \to f$ as well. The sum of the residues of

f^t may be computed by integrating f^t around the circle $|w - z| = \varepsilon$, and since $f^t \to f$, this sum approaches the sum of the residues of f. Since the expression in (5.1) is continuous in the variables v_j, this proves the general case. □

Now plugging this expression for the residue sum into Proposition 4.1 leads to a multi-residue analogue of (4.2):

$$a_{\mathbf{r}} \sim (2\pi i)^{-d} \int_{T(\mathbf{1})} -\widehat{\mathbf{w}}^{-\widehat{\mathbf{r}}-\mathbf{1}} \beta(\widehat{\mathbf{w}}) \int_{\Delta} h^{(n)}_{s,\widehat{\mathbf{w}}}(\boldsymbol{\alpha}\mathbf{v})\, d\mu(\boldsymbol{\alpha})\, d\widehat{\mathbf{w}}, \tag{5.2}$$

where we have let s denote r_{d+1}.

With Step 1 (local residue formula) and Step 2 (expanding the residue sum) out of the way, we turn to the third step, namely putting the integral into a more familiar form. The appearance of the end result differs from formula (4.5) of Lemma 4.3 in that the n-fold differentiation of h introduces a sequence of $n+1$ polynomials $p_k(s)$.

5.3. **Lemma.** *Let $\mathcal{E}$ be the product of the ball $B := |\widehat{\mathbf{w}} - \widehat{\mathbf{z}}| < \varepsilon$ from Corollary 5.2 with the n-simplex, Δ. Define a measure $d\nu = d\rho \times d\mu$, where $d\mu$ is normalized Lebesgue measure on Δ and $d\rho$ is the image in B of $d\widehat{\boldsymbol{\theta}}$ under $\widehat{\boldsymbol{\theta}} \mapsto e^{i\widehat{\boldsymbol{\theta}}}$. For $0 \le k \le n$, define*

$$\begin{aligned} p_k(s) &:= \frac{n!(s+n)!}{k!(n-k)!(s+k)!} \\ f(\widehat{\boldsymbol{\theta}}, \boldsymbol{\alpha}) &:= \frac{i\widehat{\mathbf{r}}}{s} - \log(\boldsymbol{\alpha}\mathbf{v}(e^{i\widehat{\boldsymbol{\theta}}})) \\ \psi_k((\widehat{\boldsymbol{\theta}}, \boldsymbol{\alpha}) &:= \left(\frac{d}{dy}\right)^k \phi(\widehat{\mathbf{w}}, 1/y)\Big|_{y=\mathbf{v}(e^{i\widehat{\boldsymbol{\theta}}})} . \end{aligned}$$

Then

$$a_{\mathbf{r}} \sim (2\pi)^{-d} \sum_{k=0}^{n} p_k(s) \int_{\mathcal{E}} e^{-sf(\mathbf{x})} \psi_k(\mathbf{x})\, d\nu(\mathbf{x}) . \tag{5.3}$$

Proof. The proof is entirely analogous to that of Lemma 4.3. Applying Leibniz' rule for differentiating h yields

$$\begin{aligned} h^{(n)}_{s,\widehat{\mathbf{w}}}(y) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dy}\right)^{n-k} (y^{s+n}) \left(\frac{d}{dy}\right)^k \phi(\widehat{\mathbf{w}}, 1/y) \\ &= \sum_{k=0}^{n} \frac{n!(s+n)!}{k!(n-k)!(s+k)!} y^{s+k} \left(\frac{d}{dy}\right)^k \phi(\widehat{\mathbf{w}}, 1/y) \\ &= y^s \sum_{k=0}^{n} p_k(s) y^k \left(\frac{d}{dy}\right)^k \phi(\widehat{\mathbf{w}}, 1/y) . \end{aligned}$$

Substituting this into (5.3) with $\widehat{\mathbf{w}} = e^{i\widehat{\boldsymbol{\theta}}}$ completes the proof and the third step. □

Analysis of the oscillatory integral. The proofs of all the theorems in Section 3 proceed in parallel while we compute the asymptotics for (5.3) via the complex Fourier-Laplace expansion. Observe, for the application of Lemma 2.5, that f and ψ_k are smooth functions on $\mathcal{E}$, and that the real part of f is nonnegative on $\mathcal{E}$. Say that a point $\mathbf{x} \in \mathcal{E}$ is *critical* for f if $\text{Re}\{f(\mathbf{x})\} = 0$ and if ∇f is orthogonal to every curve through $\mathbf{x}$ in $\mathcal{E}$. Recall from Section 2 the matrix $\mathbf{C}$ whose rows are extreme rays in the cone of outward normals to support hyperplanes to the logarithmic domain of convergence of F. Recall also that we have assumed without loss of generality that $\mathbf{z} = \mathbf{1}$.

5.4. **Proposition.** *Write $\mathbf{C} = \mathbf{C}(\mathbf{1})$. For $\boldsymbol{\delta} := \widehat{\mathbf{r}}/s \in \mathbf{K}_0$, let $\mathcal{A}$ denote the solution set $\{\boldsymbol{\alpha} : \boldsymbol{\alpha}\mathbf{C} = \boldsymbol{\delta}\}$. Let $\mathcal{S}$ denote the set of critical points of f. Then $\mathcal{S}$ consists precisely of those points $(\mathbf{0}, \boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \in \mathcal{A} \cap \Delta$.*

Proof. By strict minimality of $\mathbf{z}$, for each j the modulus of $v_j(\widehat{\mathbf{z}})$ achieves its maximum of 1 only when $\widehat{\mathbf{z}} = \mathbf{1}$. Thus any convex combination of $v_j(\widehat{\mathbf{z}})$ with $\widehat{\mathbf{z}} \neq \mathbf{1}$ has modulus less than 1. In other words, when $\widehat{\boldsymbol{\theta}} \neq 0$, the real part of f is strictly positive, from which it follows that all critical points of f are of the form $(\mathbf{0}, \boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in \Delta$.

In fact f is somewhat degenerate: $f(\mathbf{0}, \boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha} \in \Delta$, so not only the real part of f vanishes everywhere $\widehat{\boldsymbol{\theta}} = 0$, but also the $\boldsymbol{\alpha}$-gradient of f vanishes there. We compute the $\widehat{\boldsymbol{\theta}}$-derivatives at $\widehat{\boldsymbol{\theta}} = \mathbf{0}$ as follows. For $1 \leq j \leq d$,

$$\frac{\partial f}{\partial \theta_j} = i \left[\frac{r_j}{s} - \frac{w_j}{w_{d+1}} \boldsymbol{\alpha} \left(\frac{\partial}{\partial w_j} \right) \bigg|_{\widehat{\mathbf{w}} = \mathbf{1}} \mathbf{v}(\widehat{\boldsymbol{w}}) \right] .$$

Recalling the definition of $\mathbf{C}$, we see that these vanish simultaneously if and only if $\boldsymbol{\delta} = \boldsymbol{\alpha}\mathbf{C}$, which finishes the proof. $\square$

With the critical points identified, the remainder of the program is as follows. Rewrite the integral over $\mathcal{E}$ as the integral over $\mathcal{S}$ of an integral along a complementary manifold indexed by $\mathcal{S}$. The inner integral has a single critical point, so its asymptotics are given by the Fourier-Laplace expansion in Lemma 2.5. We check that the asymptotics are uniform, then integrate them term by term over $\mathcal{S}$.

The complementary manifolds are not hard to define, since they may be taken to be linear. Recall that $\mathcal{A}^\perp$ is the orthogonal complement to $\mathcal{A}$ in $\mathcal{H}$; for each point y of $\mathcal{A} \cap \Delta$, let B_y be the product of the ball B (of Lemma 5.3) with the affine subspace of Δ containing y and parallel to $\mathcal{A}^\perp$. Then the integrals in Lemma 5.3 may be decomposed as

$$\int_{\mathcal{E}} e^{-sf(\mathbf{x})} \psi_k(\mathbf{x}) \, d\nu(\mathbf{x}) = \int_{\mathcal{S}} d\sigma(y) \int_{B_y} e^{-sf(\mathbf{x})} \psi_k(\mathbf{x}) \, d(\rho \times \lambda)(\mathbf{x}) \tag{5.4}$$

where $d\rho$ is the image of $d\widehat{\boldsymbol{\theta}}$ under the exponential map into B, and λ is Lebesgue measure.

According to Lemma 2.5, we may expand the inner integral into an asymptotic series long as the determinant of the Hessian of f restricted to B_y does not vanish. Although the entire expansion need not be integrable, we will give a reference in the proof of Theorem 3.9 below to the fact that the leading term is integrable as long as the determinant of the Hessian of f restricted to B_y remains bounded away from 0. By Proposition 2.4, the leading term of the integral on each B_y may then be integratedi in y, leading to the following expression as long as $\mathcal{S}$ intersects the interior of Δ.

$$a_{\mathbf{r}} \sim (2\pi)^{-d} p_0(s) s^{-\dim(B_y)/2} \int_{\mathcal{S}} \psi_0(\mathbf{0}, y) \frac{(2\pi)^{\dim(B_y)/2}}{\sqrt{\det(\mathbf{H}(\mathbf{0}, y))}} \, d\sigma(y) \tag{5.5}$$

where $\mathbf{H}(\mathbf{0}, y)$ is the Hessian of f restricted to B_y at $(\mathbf{0}, y)$. To finish from here, we describe the Hessian in more detail as follows.

5.5. **Proposition.** *The Hessian $M(y)$ of f restricted to B_y has the block form*

$$M(y) = \begin{pmatrix} 0 & -i\overline{\mathbf{C}} \\ -i\overline{\mathbf{C}}^T & Q(y) \end{pmatrix}. \tag{5.6}$$

In this decomposition:

- *the jth column of $\overline{\mathbf{C}}$ is the projection of the jth column of $\mathbf{C}$ onto $\mathcal{A}^\perp$ (expressed in some orthonormal basis)*
- *$Q(y)$ is the Hessian at y of the restriction of f to the $\widehat{\boldsymbol{\theta}}$-directions*
- *the zero block in $M(y)$ has dimensions $\rho \times \rho$, where $\rho = \dim \mathcal{A}^\perp = \operatorname{rank}(\overline{\mathbf{C}})$*

Proof. Constancy of f in the Δ directions at $\widehat{\boldsymbol{\theta}} = \mathbf{0}$ shows that the second partials in Δ directions vanish, giving the upper left block of zeros. Computing $(\partial/\partial\theta_j) f$ up to a constant gives

$$-\frac{i}{\boldsymbol{\alpha}\mathbf{v}(\mathbf{1})} \boldsymbol{\alpha} w_j \frac{\partial}{\partial w_j} \mathbf{v}(\mathbf{1})$$

and since $\boldsymbol{\alpha}\mathbf{v}(\mathbf{1})$ is constant when $\widehat{\boldsymbol{\theta}} = \mathbf{0}$, differentiating in the $\boldsymbol{\alpha}$ directions recovers the blocks $-i\overline{\mathbf{C}}$ and $-i\overline{\mathbf{C}}^T$. The second partials in the $\widehat{\boldsymbol{\theta}}$ directions are of course unchanged. To see that the dimension of $\mathcal{A}^\perp$ is the rank of $\overline{\mathbf{C}}$, observe that no nonzero element of $\mathcal{A}^\perp$ can be orthogonal to column space of $\mathbf{C}$, since then it would have been in $\mathcal{A}$; thus the projection of the columns of $\mathbf{C}$ onto $\mathcal{A}^\perp$ spans $\mathcal{A}^\perp$. □

Proof of Theorem 3.3. By assumption, $\operatorname{rank}(\mathbf{C}) = d + 1 = n + 1$. Hence $\mathcal{A}$ is a single point, $(\mathbf{0}, \boldsymbol{\alpha})$. This is in the interior of $\mathcal{E}$ as long as $\boldsymbol{\delta}$ is in the interior of $\mathbf{K}$. The function f in the representation (5.3) already has only one critical point, so the decomposition (5.4) is not necessary:

$$a_{\mathbf{r}} \sim (2\pi)^{-d} \sum_{k=0}^{n} p_k(s) \int_{\mathcal{E}} e^{-s\, f(\mathbf{x})} \psi_k(\mathbf{x})\, d\nu(\mathbf{x})\,. \tag{5.7}$$

The leading contribution is from $k = 0$, where $p_0(s) = s^n + O(s^{n-1})$. The value of ψ_0 at the critical point $\mathbf{0}$ is given by $\psi_0(\mathbf{0}, \boldsymbol{\alpha}) = \phi(\mathbf{1})$. The Hessian of f is computed by Proposition 5.5 as follows.

Since $\mathcal{A}^\perp$ is all of $\mathcal{H}$, and has dimension $d = \rho = \operatorname{rank}(\overline{\mathbf{C}})$. The block decomposition of $M(y)$ in (5.6) is into four blocks of dimensions $d \times d$, implying that $M(y)$ is nonsingular with determinant $(\det \overline{\mathbf{C}})^2$ (in performing column operations to bring $M(y)$ to block diagonal form, we pick up a factor of $(-1)^{d^2}$ which cancels the factor i^{2d}). The last column of $\mathbf{C}$ is $\mathbf{1}$, whose projection onto $\mathcal{A}^\perp$ is null; its projection onto the space spanned by $\mathbf{1}$ has length $\sqrt{d+1}$, whence $|\det \mathbf{C}| = \sqrt{d+1} |\det \overline{\mathbf{C}}|$. Putting this together and applying the Fourier-Laplace expansion to the integral yields

$$a_{\mathbf{r}} \sim (2\pi)^{-d} s^d a_0 s^{-(2d)/2-j}$$

where

$$a_0 = \frac{\phi(\mathbf{1})(2\pi)^{(2d)/2}}{|\det \overline{\mathbf{C}}|}\,.$$

To finish the proof, we quote again [Pem00, Corollary 3.2] to see that all but the leading terms are exponentially small. □

Remark. If $\boldsymbol{\delta}$ is on the boundary of $\mathbf{K}$, it may happen that the point $(\mathbf{0}, \boldsymbol{\alpha})$ has $\boldsymbol{\alpha}$ lying on a face of Δ of dimension $d-1$. In that case, the Fourier-Laplace expansion gives a leading term with exactly half the previous magnitude. In Example 3.2, this happens when $r/s \in \{1/2, 2\}$.

Proof of Theorem 3.6. Again, by the nondegeneracy assumption, $\text{rank}(\mathbf{C}) = d + 1$, hence $\rho := \text{rank}(\overline{\mathbf{C}}) = d$ and $\dim(B_y) = 2d$. This time, however, the set $\mathcal{S}$ has dimension $n - d$; we assume this to be strictly positive since the case $n = d$ has been dealt with already. At each stationary point $(\mathbf{0}, \boldsymbol{\alpha}) \in \mathcal{S}$ we find the Hessian of f restricted to B_y to be the constant value $(\det \overline{\mathbf{C}})^2$. If the first term of the Fourier-Laplace expansion is integrable, then (5.5) gives

$$\begin{aligned} a_{\mathbf{r}} &\sim (2\pi)^{-d} p_0(s) s^{-d} (2\pi)^d \phi(\mathbf{1}) \int_{\mathcal{S}} \det \mathbf{H}(\mathbf{0}, y)^{-1/2} \, d\sigma(y) \\ &\sim s^{n-d} \frac{\phi(\mathbf{1}) \sigma(\mathcal{S})}{|\det \overline{\mathbf{C}}|}, \end{aligned}$$

which establishes the formula for the leading term. That the expansion is a polynomial plus an exponentially smaller correction then follows from [Pem00, Theorem 3.1].

To see that the first term of the Fourier-Laplace expansion is integrable, we refer to its derivation in [PW00, Theorem 2.4]. We see there that as long as the determinant of the Hessian of f on B_y remains bounded away from zero, as does the size of a neighborhood of y in B_y on which a smooth change of variables can be defined mapping f to $\sum_{j=1}^{\dim B_y} x_j^2$, the leading term is uniform, hence integrable. Bounding the neighborhood size from below is done in [PW00, Theorem 2.6]. $\square$

Remark. The expansion via Leibniz' formula is not generally integrable past the leading term, and therefore does not give a method of computing the lower terms of the polynomial P; contrast this to the effective computation of coefficients in the upcoming proof of Theorem 3.9.

Proof of Theorem 3.9. The rank of $\mathbf{C}$ is $n + 1$. Independence of the rows of $\mathbf{C}$ implies that $\mathcal{A}$ is a single point, $\boldsymbol{\alpha}$. Hence $\mathcal{A}^\perp = \mathcal{H}$ and has dimension n. Since $\mathcal{A}$ is a singleton, we may use the formula (5.7) again. The expansion of $\int_{\mathcal{E}} e^{-sf} \psi_k$ will have terms $s^{-(n+d)/2-j}$ for all j, and when multiplied by p_k, will contribute to the $s^{(n-d)/2-k-j}$ terms for all j. Each of these terms is explicitly computable from the partial derivatives of G and H via ψ_k, but these become rapidly messy. The leading term of $s^{(n-d)/2}$ comes only from $k = 0$. Being careful to remember that $\sigma(\mathcal{A}) = n!/\sqrt{n+1}$ is the inverse of the Lebesgue measure of the n-simplex, we find that

$$a_{\mathbf{r}} \sim (2\pi)^{-d} s^n \frac{(2\pi)^{(n+d)/2} s^{-(n+d)/2} \phi(\mathbf{1})}{\sqrt{\det \mathbf{H}(\mathbf{0}, \boldsymbol{\alpha})}} \frac{n!}{\sqrt{n+1}}$$

which proves the theorem. In this case we find no simplification of the determinant of the Hessian. $\square$

Proof of Theorem 3.11. When all the tangent planes to the $\mathcal{V}_j$ at $\mathbf{z}$ coincide, the rows of $\mathbf{C}$ are all equal to a common value, $\boldsymbol{\delta}$. Then $\mathbf{K}_0 = \{\boldsymbol{\delta}\}$, and so the set $\mathcal{A}$ is all of Δ. The rank of $\mathcal{A}^\perp$ is zero, which means that the matrix $M(y)$ is just the $\widehat{\boldsymbol{\theta}}$-Hessian of f which we have denoted $Q(y)$ in Proposition 5.5. Use (5.5) once more with $\dim(B_y) = d$ to get

$$a_{\mathbf{r}} \sim (2\pi)^{-d} p_0(s) s^{-d/2} (2\pi)^{d/2} \int_{\Delta} \frac{\phi(\mathbf{1})}{\sqrt{\det \mathbf{H}(\mathbf{0}, y)}} \, d\mu(y)$$

which simplifies to the conclusion of the theorem. $\square$

6. Further discussion

The reader may have noticed that the proof of Theorem 3.11 did not establish an asymptotic series in decreasing powers of s for $a_{\mathbf{r}}$. Such a series does indeed exist, and may be found by mapping the integral on the left hand side of (5.4) to a Laplace integral with exponent $\tilde{f}$ and then computing the asymptotic volume series for the measure of the set $f = t$ near $t = 0$. Details may be found in [PW00].

The problem of determining asymptotics when $\widehat{\mathbf{r}}/s$ converges to the boundary of $\mathbf{K}_0$ is dual to the problem raised in [PW99] of letting $\widehat{\mathbf{r}}/s$ converge to $\partial\mathbf{K}_0$ from the outside. Solutions to both of these problems are necessary before we understand asymptotics "in the gaps", that is, in any region asymptotic to and containing a direction in the boundary of $\mathbf{K}_0$. For example, in the case Example 3.12 with $\boldsymbol{\delta}(1,1)$, what are the asymptotics for $a_{r,r+\sqrt{r}}$ as $r \to \infty$?

Many of our theorems rule out analysis of a minimal point $\mathbf{z}$ if one of the coordinates z_j is zero. The directions in $\mathbf{K}(\mathbf{z})$ will always have $r_j = 0$, in which case the analysis of coefficient asymptotics reduces to a case with one fewer variable. Thus it appears no generality is lost. If we are, however, able to solve the previous problem, wherein $\mathbf{r}/s$ converges to $\partial\mathbf{K}_0$, then we may choose to let $\mathbf{r}/s$ converge to something with a zero component. The problem of asymptotics when some $r_j = o(r_k)$ now makes sense and is not reducible to a previous case. Presumably these asymptotics are governed by the minimal point $\mathbf{z}$ still, but it must be sorted out which of our results persist when $z_j = 0$. Certainly the geometry near $\mathbf{z}$ has more possibilities, since it is easier to be a minimal point (it is easier to maintain $|z'_j| \geq |z_j|$ for $\mathbf{z}'$ near $\mathbf{z}$ when $z_j = 0$).

We have not made the greatest possible attempt to write the conclusions in a coordinate-free manner. This reflects the symmetry breaking in the parametrization of $\mathcal{V}$ by $\widehat{\mathbf{z}}$. Theorem 3.1 recasts the conclusion in a coordinate-free manner; while this seems to lead in general to messy expressions, formulae such as appear in Example 3.7 should be possible to generate methodically. A modest start would be to compute these for any number of curves in two dimensions.

The greatest obstacle to making all these computations completely effective lies in the location of the minimal point $\mathbf{z}$ given $\mathbf{r}$. Assuming the existence of a $\mathbf{z}(\mathbf{r})$ with $\mathbf{r} \in \mathbf{K}(\mathbf{z})$, how may we compute $\mathbf{z}(\mathbf{r})$ and test whether it is a minimal point? Since the moduli of the coordinates of $\mathbf{z}$ are involved in the definition of minimality, this is a problem in real rather than complex computational geometry and does not appear easy. For example, how easily can one prove that the double point in Example 3.2 is minimal?

The problem of completing some kind of classification for singularities of $\mathcal{V}$ that may arise as minimal points is still open. This becomes more interesting if we restrict to generating functions F with nonnegative coefficients. Of particular interest is whether a minimal point can have some kind of cusp. It was shown in [PW99] that this cannot happen without a vanishing coordinate. Even when a coordinate vanishes, we know of no such example. If there is such an example, a residue integral would most likely involve the sort of singularity analysis found in [AGZV88]. At this point, expansions are not known even for homogenous points that are not multiple points, though some work is underway on that.

Many of our results fail when the Hessian of f unexpectedly vanishes. Surprisingly, we do not know whether this ever happens in cases of interest. Specifically, we do not know of a generating function with nonnegative coefficients, for which the Hessian of f on B_y (notation of Section 5) ever vanishes. It goes without saying, therefore, that we do not know whether cases ever arise of degeneracies of mixed orders, such as may arise in the case of the remark following Example 3.12.

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