# Generating functions with high-order poles are nearly polynomial

Robin Pemantle [1,2]

**ABSTRACT:**

Consider the problem of obtaining asymptotic information about a multi-dimensional array of numbers $a_{\mathbf{r}}$, given the generating function $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. When $F$ is meromorphic, it is known how to obtain various asymptotic series for $a_{\mathbf{r}}$ in decreasing powers of $|\mathbf{r}|$. The purpose of this note is to show that, when the pole set of $F$ has singularities of a certain kind, then there can be only finitely many terms in such an asymptotic series. As a consequence, in the presence of a singularity of this kind, the whole asymptotic series for $a_{\mathbf{r}}$ is an effectively computable object.

Keywords: Asymptotic, automatic coefficient extraction, meromorphic, multiple point, Cauchy integral formula, convolution.

Subject classification: Primary: 05A16l, 05A15 ; Secondary 14Q20, 32A20, 13P10.

[1]Research supported in part by National Science Foundation grant # DMS 9803249

[2]Department of Mathematics, Ohio State University, 231 W. 18th Avenue, Columbus, OH 43210

# 1 Introduction

Given a generating function $F(z) = \sum_{r=0}^\infty a_r z^r$, analytic in a neighborhood of the origin, it is usually possible to obtain a good explicit approximation for $a_r$. The transfer method of Flajolet and Odlyzko [FlaOdl90], for example, translates information about $F$ near a singularity automatically into asymptotic information about $a_r$.

The corresponding problem in more dimensions, when $r$ is replaced by a multi-index, $\mathbf{r}$, is much harder. Even rational functions, whose approximation theory in one dimension is trivial, are not well understood. The paper [CohElkPro96], for example, spends many pages deriving asymptotics by hand for an array $\{a_{rst}\}$ whose generating function is, up to minor changes,

$$F(x, y, z) = \sum_{r,s,t} a_{rst} x^r y^s z^t = \frac{1}{(1 - yz)(1 - \frac{x+x^{-1}+y+y^{-1}}{2} z + z^2)} .$$

The body of literature dealing with the problem of multivariate coefficient extraction in a systematic way is quite small. The purpose of this note is to shed some light on coefficient approximation for a class of meromorphic generating functions to be defined shortly.

The problem of finding an asymptotic expression for $a_\mathbf{r}$ falls naturally into two steps. The first is to find the correct exponential rate, namely a homogeneous function $\gamma(\mathbf{r})$ of degree 1 for which

$$\log a_\mathbf{r} = (1 + o(1))\gamma(\mathbf{r}) . \tag{1.1}$$

This step is geometric and amounts to finding an appropriate point on the variety $\mathcal{V}$ of poles of $F$. This step, which is not the main concern of this paper, will be discussed in Section 2.

If the first step can be carried out, the next step is to find an asymptotically valid expression (or better yet, a complete asymptotic series) for $a_\mathbf{r}$. This step is analytic. All known methods involve complex variable methods, namely contour integration or Fourier transforms. When this step can be carried out, one typically finds something like

$$a_\mathbf{r} \sim \exp(\gamma(\mathbf{r})) \sum_{j=0}^\infty b_j(\mathbf{r})$$

where $\gamma$ is homogeneous of degree 1 and $\{b_j\}_{j\geq 0}$ is a sequence of homogeneous functions whose degrees decrease, typically as $(l-j)/2$ for some $l \in \mathbf{Z}$. An example of the leading term asymptotic (the $j = 0$ term) is given by $F(z, w) = 1/(1 - z - w - zw) = \sum a_{(r,s)} z^r w^s$ which generates the number of lattice paths from $(0, 0)$ to $(r, s)$ that go north, east or northeast. Here the leading term asymptotic is given by

$$a_{rs} \sim \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r} \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \sqrt{\frac{1}{2\pi}} \sqrt{\frac{rs}{(r+s-\sqrt{r^2+s^2})^2\sqrt{r^2+s^2}}}, \quad (1.2)$$

so $\gamma(r, s)$ is the logarithm of the first two terms and $b_0$ is the product of the last two terms, with $l = -1$.

Generally, if $F$ is expressed as the quotient of analytic functions $G/H$, the function $\gamma$ is determined by $H$, as is $l$ in nondegenerate cases. As $G$ varies, the space of possible asymptotic series will be quite large: even holding $\mathbf{r}$ fixed in projective space, any set of values for $b_0, \ldots, b_N$ will typically be possible, and the possible values of the sequence $\{b_j\}_{j\geq 0}$ will typically form an infinite-dimensional vector space. In some cases, however, the possible sequences $\{b_j\}_{j\geq 0}$ will form a finite vector space, and what is more, will consist of terminating sequences. Furthermore, each $b_j$ will then be a polynomial function of $\mathbf{r}$, whence the whole asymptotic expansion up to terms of exponentially smaller order is a finite object. This is the topic of the present note.

The methods herein are more algebraic than geometric or analytic, and are not useful for computing the coefficients $\{b_j\}_{j\geq 0}$. The point is to find out *a priori* how many coefficients one has to compute for a complete asymptotic expansion, thus enabling computation algorithms such as those in [PemWil00b] to terminate. This introductory section concludes with an imprecise statement of the main result of the paper. Section 2 gives some background on the determination of the correct exponential order. Section 3 sets forth the remaining notation and states the main theorem, Theorem 3.1, along with examples and corollaries. Section 4 shows how a local ring of analytic functions maybe be extended over a polydisk and characterizes when partial fraction expansions are available in the local and global rings. Section 5 finishes the proof of Theorem 3.1.

Some notation in use throughout the paper is as follows. Say that $g(\mathbf{r}) = o_{\exp} h(\mathbf{r})$ if $g(\mathbf{r})/h(\mathbf{r}) = O(e^{-\delta|\mathbf{r}|})$ for some $\delta > 0$. The function $F$ is always taken to be analytic on a neighborhood of the origin in $\mathbf{C}^d$. The formal power series $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ that represents $F$ will then converge near the origin and its domain of convergence is denoted $\mathcal{D}$; here and throughout, $\mathbf{z}^{\mathbf{r}} := z_1^{r_1} \cdots z_d^{r_d}$. In order to accommodate functions such as the generating functions for self-avoiding walks and percolation probabilities, which are meromorphic near the origin but not necessarily near infinity [ChaCha86, CamChaCha91], the natural hypothesis that $F$ be meromorphic will be weakened as follows. Let $\mathbf{z}$ be a point on the boundary of $\mathcal{D}$ where $F$ is singular and let $D(\mathbf{z})$ be the closed polydisk $\{\mathbf{w} : |w_j| \leq |z_j|, j = 1, \ldots, d\}$. Assume that $F$ is meromorphic in a neighborhood $\Omega$ of $D(\mathbf{z})$, and express $F = G/H$ in this neighborhood (such a global choice is always possible). In particular, $\mathbf{z}$ is assumed to be a pole of $F$.

Let $\mathcal{V}$ denote the analytic variety where $H$ vanishes. We say that $\mathbf{z}$ is a *multiple point* of $\mathcal{V}$ if near $\mathbf{z}$, $\mathcal{V}$ is locally the union of smooth manifolds. We say that the point $\mathbf{z}$ is a *complete multiple point* if in addition, the common intersection of these manifolds is locally the singleton $\{\mathbf{z}\}$. Define

$$\gamma(\mathbf{r}, \mathbf{z}) = -\sum_{j=1}^{d} r_j \log |z_j| \,. \tag{1.3}$$

As discussed later, when $\mathbf{z}$ is a multiple point of $\mathcal{V}$ on the boundary of the domain of convergence, $\mathcal{D}$, then

$$\frac{\log a_{\mathbf{r}}}{\gamma(\mathbf{r}, \mathbf{z})} \to 1$$

as $\mathbf{r}$ varies over a certain cone $\mathcal{C} \cap (\mathbf{Z}^+)^d$. The cone $\mathcal{C}$ depends on $H$ but not on $G$.

The main result of this paper is as follows. Assume $\mathcal{C} = \mathcal{C}(H)$ has non-empty interior. Then there is a finite dimensional vector space $\mathbf{W}$ of polynomials in $\mathbf{r}$ such that for any $G$ analytic in a neighborhood of $D(\mathbf{z})$ there is a $P \in \mathbf{W}$ with

$$a_{\mathbf{r}} = \exp(\gamma(\mathbf{r}, \mathbf{z}))\,(P(\mathbf{r}) + o_{\exp}(1)) \,. \tag{1.4}$$

Contrast this to (1.2), which is only the leading term, to see what complications are avoided when $P$ is a polynomial. This result will be stated more precisely as Theorem 3.1 after the appropriate terminology has been introduced. This theorem does not address the possibility that $P$ is always zero, but in fact this is ruled out by results of [PemWil00b, PemWil00c].

## 2 The exponential order of $a_{\mathbf{r}}$

The problem of determining the exponential order of $a_{\mathbf{r}}$ is completely solved only when $d \leq 2$ and the coefficients $a_{\mathbf{r}}$ are assumed to be nonnegative. This section summarizes most of what is known about determining the exponential order.

If one is interested only in those $a_{\mathbf{r}}$ with $\mathbf{r}$ on the diagonal, then relatively powerful results may be obtained. When $d = 2$, the generating function $\xi(z) = \sum_r a_{rr} z^r$ may be extracted analytically [HauKla71], reducing the problem to one dimension. A rational two-variable function has an algebraic diagonal [Fur67], so for nice two-variable functions, extraction of the diagonal is effective and asymptotics may then be obtained. In more than two dimensions, no analytic expression for the diagonal is available [Sta99], but the diagonal is still D-finite [Lip88] and a recursion for the diagonal may be effectively derived [ChySal96], which allows the derivation of asymptotics by solving difference equations with polynomial coefficients. This has in fact been implemented [LeyTsa00] and has no problem running on a standard laptop (circa 1999) when the inputs are reasonable. The methods used in these cases, though superficially analytic, are really algebraic and may be carried out over formal power series rings and modules over the Weyl algebra. The methods may, in theory, be applied to other rays such as $\{a_{rs} : s = 2r\}$, but unfortunately, they are inherently non-uniform in $s/r$, and may not therefore be applied when the direction of the ray is a changing parameter.

When the direction of $\mathbf{r}$ varies, all known results require analytic methods. To review what is known here, begin by defining a function $\mathbf{dir} = \mathbf{dir}_F$ on $\mathcal{V}$. The function $\mathbf{dir}$ takes values in $\mathbf{CP}^{d-1}$ and may be multi-valued.

If $\mathbf{z}$ is a simple pole and no $z_j$ vanishes, then $\mathbf{dir}(\mathbf{z}) = \mathbf{dir}_F(\mathbf{z})$ is the single value

$(z_1 \frac{\partial H}{\partial z_1}, \ldots, z_d \frac{\partial H}{\partial z_d})$, which is a nonzero element of $\mathbf{C}^d$ and thus defines an element of $\mathbf{CP}^{d-1}$. Under the additional assumption that $\mathbf{z}$ is on the boundary of $\mathcal{D}$, an equivalent definition of $\mathbf{dir}(\mathbf{z})$ is the normal to the support hyperplane at $(\log|z_1|, \ldots, \log|z_d|)$ of the (convex) logarithmic domain of convergence $\log\mathcal{D} := \{\mathbf{x} \in \mathbf{R}^d : (e^{x_1}, \ldots, e^{x_d}) \in \mathcal{D}\}$. If $\mathbf{z}$ is a manifold point of $\mathcal{V}$ but not a simple pole, then $H$ is not square-free and $\mathbf{dir}(\mathbf{z})$ may be defined by replacing $H$ with its radical. In the above cases, $\mathbf{dir}$ is single-valued. The remaining case is when $\mathbf{z}$ is not a manifold point of $\mathcal{V}$; in this case, define $\mathbf{dir}(\mathbf{z})$ as the set of limit points of $\mathbf{dir}(\mathbf{w})$ as $\mathbf{w} \to \mathbf{z}$ along manifold points of $\mathcal{V}$. When $\mathbf{z}$ is on the boundary of $\mathcal{D}$, this is again the set of normals to support hyperplanes $(\log|z_1|, \ldots, \log|z_d|)$ of the logarithmic domain of convergence, $\log\mathcal{D}$, but is now, in general, multi-valued.

An illustration will help to clarify the definition of $\mathbf{dir}$. Suppose $H = (1 - (2/3)z - (1/3)w)(1 - (1/3)z - (2/3)w)$ so that $\mathcal{V}$ is the union of two lines, as in figure 1.

`figure 1 goes here`

As $\mathbf{z}$ varies linearly from $(0, 3)$ to $(1, 1)$, not including $(1, 1)$, the quantity $\mathbf{dir}(\mathbf{z})$ is single-valued and goes from slope $\infty$ to slope 2. The value of $\mathbf{dir}(1, 1)$ is the cone of slopes between 2 and 1/2. As $\mathbf{z}$ varies linearly from $(1, 1)$ to $(3, 0)$, the quantity $\mathbf{dir}(\mathbf{z})$ is once more single-valued and goes from slope 1/2 to slope 0. The remaining points of $\mathcal{V}$ do not concern us, since they are not on the boundary of the domain of convergence of $F$.

For each $\mathbf{z} \in \mathcal{V}$, and each $\epsilon > 0$, the asymptotic inequality

$$a_{\mathbf{r}} = o\left((1+\epsilon)^{|\mathbf{r}|}\mathbf{z}^{-\mathbf{r}}\right) \tag{2.1}$$

is immediate from Cauchy's integral formula. In fact more is true:

**Lemma 2.1** *If $\mathbf{z} \in \mathcal{V}$ is on the boundary of $\mathcal{D}$ and $\mathbf{r}$ varies over a compact subset of the complement of $\mathbf{dir}(\mathbf{z})$, then $a_{\mathbf{r}} = o_{\exp}(\mathbf{z}^{-\mathbf{r}})$.*

PROOF: Let $\mathbf{r}$ be a fixed direction not in $\mathbf{dir}(\mathbf{z})$. Since the hyperplane through $\mathbf{x} := (\log|z_1|, \ldots, \log|z_d|)$ normal to $\mathbf{r}$ is not a support hyperplane to $\log\mathcal{D}$, there is a $\mathbf{y}$ in the interior of $\log \mathcal{D}$ with $\mathbf{y} \cdot \mathbf{r} > \mathbf{x} \cdot \mathbf{r}$. From (2.1) we see that $a_{\mathbf{r}} = O((1+\epsilon)^{|\mathbf{r}|} e^{-\mathbf{y} \cdot \mathbf{r}})$ for any $\epsilon > 0$. Choosing $\epsilon$ small enough, the conclusion follows for fixed directions. The rate of decay (the $\delta$ in the definition of $o_{\text{exp}}$) may be chosen continuously in $\mathbf{r}$, so the uniformity over compact sets follows. $\square$

In particular, the true exponential rate for $a_{\mathbf{r}}$ is at most $\gamma(\mathbf{r}, \mathbf{z}) = -\sum_{j=1}^{d} r_j \log |z_j|$ for any $\mathbf{z}$. This is minimized precisely when $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. One might hope that the reverse inequality holds in this case, namely, that $\log a_{\mathbf{r}} = (-1 + o(1)) \sum_{j=1}^{d} r_j \log |z_j|$ for $\mathbf{z} \in \mathcal{V} \cap \partial \mathcal{D}$ such that $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. Indeed this is conjectured always to hold when $F$ has nonnegative coefficients, which is the case of greatest combinatorial interest. Results in this direction are as follows.

Pemantle and Wilson [PemWil00a, Theorem 6.3] show that if $F$ has nonnegative coefficients, then for every $\mathbf{r}$ there is always a $\mathbf{z} \in \mathcal{V} \cap \partial \mathcal{D}$ with $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. The argument consists of generalizing the example in figure 1.

When $\mathbf{z} \in \partial \mathcal{D}$ is a smooth point of $\mathcal{V}$ and $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$, there are several known proofs that $\gamma(\mathbf{r}, \mathbf{z})$ is the correct exponential rate for $a_{\mathbf{r}}$. Bender and Richmond [BenRic83] proved this in 1983, under some additional hypotheses, and also derived the leading term asymptotic. A different proof in a more general framework is given in [PemWil00a]; see also the book [FlaSed00]. When $\mathbf{z}$ is a singular point of $\mathcal{V}$, less is known. The preprint [PemWil00b] shows that $\gamma(\mathbf{r}, \mathbf{z})$ is the correct exponential order when $\mathbf{z} \in \partial \mathcal{D}$ is a multiple point and $\mathbf{r} \in \mathbf{dir}(\mathbf{z})$. As a consequence, if one assumes $a_{\mathbf{r}} \geq 0$, one has complete knowledge of the exponential order. Another consequence is that Theorem 3.1 is non-trivial (meaning that $P$ is not identically zero). The case where $d \geq 3$ and $\mathbf{z}$ is some singularity other than a multiple point is addressed in a manuscript in preparation [CohPem00].

# 3 Statement of results

The definition of an isolated point may be made precise by the introduction of local rings. Let $\Omega$ be any open set containing a point $\mathbf{z}$ and let $\Re_{\mathbf{z}}$ be the complex algebra of germs of analytic functions in $d$ complex variables, $w_1, \ldots, w_d$ at $\mathbf{z}$. This is naturally identified with the set of power series in $(\mathbf{w} - \mathbf{z})$ that converge in some neighborhood of $\mathbf{z}$ [GunRos65, Theorem 1 of Ch. II]. The ring $\Re_{\mathbf{z}}$ is a noetherian UFD and is a local ring, with the maximal ideal $\mathcal{M}$ consisting of functions vanishing at $\mathbf{z}$. Any $h \in \mathcal{M}$ may be factored uniquely into powers of irreducible factors $\prod_{j=1}^k h_j^{n_j}$ with each $h_j \in \mathcal{M}$. This corresponds to the decomposition of the variety $V(h)$ locally as the union of $V_j$, where $\mathcal{V}_j$ is the zero set of $h_j$. Taking $h = H$, the denominator of $F$, the point $\mathbf{z} \in \mathcal{V} = V(H)$ is defined to be a multiple point if each $h_j$ has non-vanishing linear part. It is a complete multiple point if, in addition, the common intersection of the sets $\mathcal{V}_j$ (locally a hyperplane arrangement) is the singleton $\{\mathbf{z}\}$.

Given complete multiple point, $\mathbf{z} \in V(H)$, let $\mathbf{dir}_j(\mathbf{z})$ be the limit of $\mathbf{dir}(\mathbf{w})$ as $\mathbf{w} \to \mathbf{z}$ in $\mathcal{V}_j$. This is the normal to $\mathcal{V}_j$ in logarithmic coordinates. The set $\mathbf{dir}(\mathbf{z})$ is simply the convex hull of $\{\mathbf{dir}_j(\mathbf{z}) : 1 \leq j \leq k\}$. Let $\mathcal{C}$ denote the cone $\mathbf{dir}(\mathbf{z})$ and for $S \subseteq \{1, \ldots, k\}$, let $\mathcal{C}(S)$ denote the convex hull of $\{\mathbf{dir}_j(\mathbf{z}) : j \in S\}$. Define $\mathcal{S}$ to be the family of sets $S \subseteq \{1, \ldots, k\}$ for which $\bigcap_{j \in S} \mathcal{V}_j \neq \{\mathbf{z}\}$, that is, the intersection is a variety $\mathcal{V}_S$ of dimension at least 1. By convention, $\emptyset \in \mathcal{S}$. It follows that if $S \in \mathcal{S}$, then for $j \in S$, each $\mathcal{V}_j$ contains $\mathcal{V}_S$, so each $\mathbf{dir}_j(\mathbf{z})$ is normal to $\mathcal{V}_S$ in logarithmic coordinates, implying that $\mathcal{C}(S)$ is a subspace of codimension at least 1. Let $U = \bigcup_{S \in \mathcal{S}} \mathcal{C}(S)$. The main result may now be stated as follows.

**Theorem 3.1** *Let $H$ be analytic on $D(\mathbf{z})$ and have a complete multiple point at $\mathbf{z}$ which is the only zero of $H$ on the closed polydisk $D(\mathbf{z})$. Let $h_1^{n_1}, \ldots, h_k^{n_k}$, $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be the local factorization of $H$ at $\mathbf{z}$ and define $\mathbf{dir}_j$, $\mathcal{C}$, $\mathcal{C}(S)$ and $\mathcal{S}$ as above. Assume $\mathcal{C} = \mathbf{dir}(\mathbf{z})$ has non-empty interior. Then there exists a finite-dimensional complex vector space $\mathbf{W}$ of polynomials in $\mathbf{r}$ such that for every function $G$ analytic in a neighborhood of $D(\mathbf{z})$ and any* *compact subcone $\mathcal{C}_1$ of $\mathcal{C} \setminus U$, the coefficients of $F := G/H$ are given by*

$$a_{\mathbf{r}} = \mathbf{z}^{-\mathbf{r}} \left(P(\mathbf{r}) + E(\mathbf{r})\right)$$

*with $P \in \mathbf{W}$ and $E = o_{\exp}(1)$ uniformly on $\mathcal{C}_1$.*

*Remark:* The set $U$ contains the boundary of $\mathcal{C}$ but also possibly some hyperplanes in the interior of $\mathcal{C}$. Thus it is possible for $\mathcal{C} \setminus U$ to be disconnected. In this case, it can happen that $a_{\mathbf{r}}$ is approximated by two different polynomials on two different subcones of $\mathcal{C} \setminus U$.

The steps of the proof are as follows. (1) locally, $F$ may be expanded by partial fractions when $G$ is in a certain ideal, $\Im_{\mathbf{z}}$, the quotient by which is finite-dimensional (Lemma 4.1). (2) this is true globally in a neighborhood of $D(\mathbf{z})$ (Theorem 4.5). (3) the partial fraction summands are $o_{\exp}(\mathbf{z}^{-\mathbf{r}})$ on $\mathcal{C} \setminus U$ (Lemma 5.2). (4) the coset representatives for analytic functions modulo $\Im_{\mathbf{z}}$ have coefficients in a finite vector space of polynomials. Steps (1) and (2) are carried out in Section 4 and steps (3) and (4) are carried out in Section 5. Two examples serve to illustrate the use of the theorem.

**Example 1** ($d = 2$)

When $d = 2$, any singular point $\mathbf{z} \in \partial\mathcal{D} \cap \mathcal{V}$ is a complete multiple point. This follows from the fact that the leading terms of the expansion of $H$ near $\mathbf{z}$ are all of the same homogeneous degree, which is proved in [PemWil00a, Theorem 6.1]. Unless the branches of $\mathcal{V}$ near $\mathbf{z}$ all intersect tangentially, the cone $\mathbf{dir}(\mathbf{z})$ will have non-empty interior, and the hypotheses of Theorem 3.1 will be satisfied.

If $k$ is the number of factors of $H$ near $\mathbf{z}$, and $\mathbf{z}$ is a complete multiple point, then $k \geq d$. The simplest case, when $k = d$, is worth mentioning as a separate corollary. The proof will be given in Section 5 after the proof of Theorem 3.1.

**Corollary 3.2** *Under the assumptions of Theorem 3.1, suppose $k = d$ and each $n_k = 1$. Then $\mathbf{W}$ is one-dimensional, consisting only of constants.*

**Example 2 (crossing lines)**

Consider the example in figure 1, where $H = (1-(2/3)z-(1/3)w)(1-(1/3)z-(2/3)w)$. The leading term asymptotic is computed in [PemWil00b] to be

$$a_{r,s} = 6 + O(|\mathbf{r}|)^{-1}$$

for $1/2 < r/s < 2$. Further terms are increasingly time-consuming to compute. From Corollary 3.2 we see that in fact there are no more terms of the same exponential order. In this case there is a more elementary method of obtaining a first-order approximation to $a_{rs}$. Because $H$ factors globally, it is possible to represent $a_{r,s}$ as a two-dimensional convolution, resulting in a sum of products of binomial coefficients. A bivariate central limit approximation then recovers the leading term without too much trouble, but gives no indication that $a_{r,s}$ is in fact exponentially well approximated by the constant 6.

**Example 3 (peanut)**

`figure 2 goes here`

Suppose $H = 19 - 20z - 20w + 5z^2 + 14zw + 5w^2 - 2z^2w - 2zw^2 + z^2w^2$. The real part of the zero set is shown in figure 2. The point $(1,1)$ is on the boundary of $\mathcal{D}$, and $\mathbf{dir}(1,1)$ is the set $1/2 \le r/s \le 2$. Thus by Corollary 3.2, we have again

$$a_{r,s} = C + o_{\exp}(1)$$

where the constant $C$ is proportional to $G(1,1)$. This example illustrates that it is the local nature of the singularity that allows us to apply Theorem 3.1 and its corollaries: $H$ factors in the local ring at $(1,1)$, but does not factor globally.

# 4 Partial fraction expansions

If $F = G/H$ has a partial fraction representation as $\sum_{j=1}^{k}(g_j/h_j)$, then clearly $G$ vanishes at $\mathbf{z}$. Amplifying on this, for $S \in \mathcal{S}$ we define $h_S = \prod_{j \in S^c} h_j^{n_j}$, so that

$$\frac{h_S}{H} = \frac{1}{\prod_{j \in S} h_j^{n_j}} .$$

Let $\Im_{\mathbf{z}}$ be the ideal in $\Re_{\mathbf{z}}$ generated by $\{h_S : S \in \mathcal{S}\}$. The following lemma is a local version of the main result of this section, namely the partial fraction representation, Theorem 4.5.

**Lemma 4.1** *The quotient $\Re_{\mathbf{z}}/\Im_{\mathbf{z}}$ is a finite-dimensional complex vector space. Equivalently, there is a finite-dimensional vector space $\mathbf{V} \subseteq \Re_{\mathbf{z}}$ such that for all $g \in \Re_{\mathbf{z}}$,*

$$g = g_0 + \sum_{S \in \mathcal{S}} g_S h_S \tag{4.1}$$

*with $g_0 \in \mathbf{V}$ and each $g_S \in \Re_{\mathbf{z}}$.*

PROOF: First observe that $\mathbf{z}$ is an isolated element of $V(\Im_{\mathbf{z}})$. Indeed, if not, then some variety $A$ of dimension at least 1 containing $\mathbf{z}$ is in $V(\Im_{\mathbf{z}})$. The (possibly empty) set $S_A$ of $j$ for which $A \subseteq \mathcal{V}_j$ is in $\mathcal{S}$, so $h_{S_A} \in \Im_{\mathbf{z}}$ and $h_{S_A}$ does not vanish on $A$, contradicting $A \subseteq V(\Im_{\mathbf{z}})$.

The local ring $\Re_{\mathbf{z}}$ is noetherian [GunRos65, Theorem 9, Ch. IIB] and satisfies the Nullstellensatz (see the discussion after Corollary 16 of Ch. IIE on page 90 of [GunRos65]). From the Nullstellensatz, it follows that the radical of $\Im_{\mathbf{z}}$ is $\mathcal{M}$, the maximal ideal of $\Re_{\mathbf{z}}$. From the noetherian property, it follows that the radical of an ideal is finite-dimensional over the ideal, hence $\Re_{\mathbf{z}}/\Im_{\mathbf{z}}$ is finite-dimensional over $\Re_{\mathbf{z}}/\mathcal{M} \cong \mathbf{C}$. □

To transfer (4.1) to the global setting requires a formulation in terms of sheaves. Let $\omega$ be a neighborhood of $\mathbf{z}$ in which the factors $h_j$ are analytic, and in which $\bigcap_{j=1}^{k} \mathcal{V}_j = \{\mathbf{z}\}$. Since $\mathcal{V}$ does not intersect the interior of $D(\mathbf{z})$, the intersection of $\mathcal{V}_j$ with $\partial\omega$ is disjoint from $D(\mathbf{z})$ and it follows that we may choose a neighborhood $\Omega$ of $D(\mathbf{z})$ containing no such intersection point.

**Lemma 4.2** *Fix any* $\mathbf{x} \in D(\mathbf{z}) \setminus \{\mathbf{z}\}$. *There are functions* $h_j^{\mathbf{x}}$ *analytic on* $\Omega$ *for which the following hold:*

*(1) each* $h_j^{\mathbf{x}}$ *is analytic on* $\Omega$;

*(2)* $h_j^{\mathbf{x}} = u \cdot h_j$ *with* $u$ *a unit in* $\Re_{\mathbf{z}}$;

*(3)* $h_j^{\mathbf{x}}(\mathbf{x}) \neq 0$.

PROOF: Fix $j$ for the entire proof. Let $\mathcal{F}_j$ be the sheaf over $\Omega$ of ideals $< h_j >$. That is, when $\mathbf{w} \in \omega$ and $h_j(\mathbf{w}) = 0$, then $(\mathcal{F}_j)_{\mathbf{w}}$ is the germs of functions divisible by $h_j$ at $\mathbf{w}$, while when $\mathbf{w} \notin \omega$ or $\mathbf{w} \in \omega$ with $h_j(\mathbf{w}) \neq 0$, then $(\mathcal{F}_j)_{\mathbf{w}}$ is all analytic germs at $\mathbf{w}$. The definition of $(\mathcal{F}_j)^{\mathbf{w}}$ is potentially ambiguous when $\mathbf{w} \in \partial\omega$ is in the interior of $\Omega$, but since $h_j$ is nonzero here, there is no problem.

The sheaf $\mathcal{F}_j$ is a subsheaf of the structure sheaf $\mathcal{O}$, hence coherent, so by Cartan's Theorem A (see [GraRem79, page 96-97]) there is a map $\psi_j$ from some $\mathcal{O}^l$ onto $\mathcal{F}_j$, where $\mathcal{O}$ is the sheaf of germs of analytic functions (the structure sheaf) on $\Omega$ and $l \geq 1$. Denote the $l$ generators of $\mathcal{O}^l$ by $\mathbf{1}_i, i \leq l$.

Surjectivity of $\psi_j$ is a local property, but since each $\mathbf{1}_i$ is a global section of $\mathcal{O}^l$, each $f_{ij} := \psi_j(\mathbf{1}_i)$ is an analytic function defined globally on $\Omega$. Surjectivity at $\mathbf{z}$ implies that $h_j$ is in the image of $\psi_j$, which is the ideal generated by the functions $f_{ij}$ at $\mathbf{z}$ as $i$ varies. Thus for some functions $u_i$ in a neighborhood of $\mathbf{z}$,

$$\psi_j(\sum_i u_i \mathbf{1}_i) = \sum_i u_i f_{ij} = h_j \, . \tag{4.2}$$

Surjectivity at any other point implies that $f_{ij}$ do not simultaneously vanish anywhere that $h_j$ does not. By definition of $\psi_j$, each $f_{ij}$ is in the ideal generated by $h_j$ and hence may be written as $u_{ij} h_j$ in $\Re_{\mathbf{z}}$. If each $u_{ij} \in \mathcal{M}$, then each $f_{ij} \in \mathcal{M} \cdot < h_j >$ contradicting the fact that $h_j$ is in the ideal generated by the $f_{ij}$. Thus for some $i$, $f_{ij} \notin \mathcal{M} \cdot < h_j >$.

Given $\mathbf{x}$, if there is an $i$ with $f_{ij} \notin \mathcal{M} \cdot < h_j >$ and $f_{ij}(\mathbf{x}) \neq 0$, then the lemma is proved with $h_j^{\mathbf{x}} := f_{ij}$ and $u = u_{ij}$. If not, then choose $i$ and $i'$ so that $f_{ij} \notin \mathcal{M} \cdot < h_j >$

and $f_{i'j}(\mathbf{x}) \neq 0$. Since it was not possible to choose $i = i'$, we know that $f_{ij}(\mathbf{x}) = 0$ and $u_{i'j} \in \mathcal{M}$. It follows that $u_{ij} + u_{i'j} \notin \mathcal{M}$ and the lemma is proved with $h_j^{\mathbf{x}} := f_{ij} + f_{i'j}$. $\square$

**Corollary 4.3** *There is a finite collection $\{h_\alpha : \alpha \in A\}$ analytic on a neighborhood of $D(\mathbf{z})$ such that for each $S \in \mathcal{S}$ and each $\mathbf{w} \in D(\mathbf{z}) \setminus \{\mathbf{z}\}$ there is an $\alpha \in A$ with*

$$h_\alpha(\mathbf{w}) \neq 0 \text{ and } h_\alpha = h_S u \tag{4.3}$$

*with $u$ a unit of $\Re_{\mathbf{z}}$.*

PROOF: Fix $S \in \mathcal{S}$. The function $h_S^{\mathbf{x}} := \prod_{j \in S^c} (h_j^{\mathbf{x}})^{n_j}$ satisfies (4.3) for all $\mathbf{w}$ in some neighborhood $\mathcal{N}_{\mathbf{x}}$ of $\mathbf{x}$. It also satisfies (4.3) for all $\mathbf{w}$ in some neighborhood $\mathcal{N}$ of $\mathbf{z}$. By compactness of $D(\mathbf{z})$, we may choose finitely many $\mathbf{x}$ for which the collection of sets $\mathcal{N}_{\mathbf{x}}$ covers $D(\mathbf{z}) \setminus \mathcal{N}$. Taking the union of such collections over $S \in \mathcal{S}$ proves the corollary. $\square$

**Lemma 4.4** *Let $\Omega$ be a polydisk containing $z$ and let $\{h_\alpha : \alpha \in A\}$ be a finite collection of functions analytic in $\Omega$. Suppose an analytic function $g$ on $\Omega$ is represented as $\sum_\alpha g_\alpha^{\mathbf{x}} h_\alpha$ in a neighborhood of each $\mathbf{x}$ where each $g_\alpha^{\mathbf{x}}$ is analytic. Then*

$$g = \sum_\alpha g_\alpha^* h_\alpha$$

*with $g_\alpha^*$ analytic in $\Omega$.*

PROOF: This is a straightforward application of Cartan's Theorem B. A sketch of the argument is as follows. Define sheaves over $\Omega$ by $\mathcal{F} = \mathcal{O}^{|A|}$ and $\mathcal{G} = < h_\alpha : \alpha \in A >$. The map $\eta : \mathcal{F} \to \mathcal{G}$ defined by $\eta(f_\alpha : \alpha \in A) = \sum_\alpha f_\alpha h_\alpha$ is a surjection of sheaves. The space of global sections of a sheaf is the zeroth cohomology group, and $H^0(\Omega, \mathcal{F})$ maps onto $H^0(\Omega, \mathcal{G})$ only if the coboundary map from $H^0(\Omega, \mathcal{G})$ to $H^1(\Omega, \mathcal{E})$ is trivial, where $\mathcal{E}$ is the kernel of $\eta$. By Cartan's Theorem B ([GunRos65, Theorem 14, Ch. VIIIA]), since $\mathcal{E}$ is a subsheaf of $\mathcal{O}^{|A|}$ and $\Omega$ is a Stein space, the cohomology groups $H^q(\Omega, \mathcal{E})$ vanish when

$q \geq 1$. Hence the coboundary map is trivial, and there is a global section $(g_\alpha : \alpha \in A)$ of $\mathcal{F}$ mapping by $\eta$ to $g$. $\square$

Let $\Re$ now denote the ring of functions analytic beyond $D(\mathbf{z})$, that is, functions $f$ for which there exists a neighborhood of $D(\mathbf{z})$ on which $f$ is analytic. Putting together the lemmas of this section yields the following result.

**Theorem 4.5** *Let $H$ be analytic on $D(\mathbf{z})$ and have a complete multiple point at $\mathbf{z}$ which is the only zero of $H$ on the closed disk $D(\mathbf{z})$. Let $h_1^{n_1}, \ldots, h_k^{n_k}$, $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be the local factorization of $H$ in a neighborhood of $\mathbf{z}$, and let $\mathcal{S}$ be the family of subsets $S$ of $\{1, \ldots, k\}$ such that $\mathbf{z}$ is not isolated in $\bigcap_{S \in \mathcal{S}} \mathcal{V}_j$. Then there is a finite subset $\{h_\alpha : \alpha \in A\}$ of $\Re$, each localizing to $h_S$ times a unit in $\Re_\mathbf{z}$ for some $S \in \mathcal{S}$, and having the following property.*

*There is a finite-dimensional vector subspace $\mathbf{V}^*$ of $\Re$ such that each $G \in \Re$ may be written as*

$$g_0^* + \sum g_\alpha^* h_\alpha \tag{4.4}$$

*with $g_0^* \in \mathbf{V}^*$ and the dimension of $\mathbf{V}^*$ equal to the dimension of $\Re_\mathbf{z}/\Im_\mathbf{z}$.*

PROOF: The $\{h_\alpha : \alpha \in A\}$ is constructed in Corollary 4.3. Choose coset representatives for a basis of $\Re_\mathbf{z}/\Im_\mathbf{z}$ and let $\mathbf{V}^*$ be their span. We need only to verify the representation property (4.4). By construction, if $G \in \Re$, then $G$ may be written as $g_0^* + g$ with $g_0^* \in \mathbf{V}^*$ and the germ $(g)_\mathbf{z}$ in $\Im_\mathbf{z}$. Evidently, the dimension of $\mathbf{V}^*$ is equal to the dimension of $\Re_\mathbf{z}/\Im_\mathbf{z}$, which is finite by Lemma 4.1.

We now verify the hypotheses of Lemma 4.4. In a neighborhood of $\mathbf{z}$, we know from (4.3) that the functions $\{h_\alpha : \alpha \in A\}$ generate $\Im_\mathbf{z}$. Hence there is a representation $g = \sum g_\alpha^\mathbf{z} h_\alpha$. In a neighborhood of any other $\mathbf{x} \in D(\mathbf{z})$ some $h_\alpha$ is nonzero, so there is trivially a representation $g = \sum g_\alpha^\mathbf{x} h_\alpha$. Applying Lemma 4.4, it follows that $g \in \Im$. $\square$

# 5 Finite-dimensional shift-invariant spaces of arrays must be polynomial

For this section we fix a compact subcone $\mathcal{C}_1 \subseteq \mathcal{C} \setminus U$ with non-empty interior.

Let $\mathbf{W}(d)$ denote the set of complex valued functions on $(\mathbf{Z}^+)^d$. For each $G \in \Re$, let $b_{\mathbf{r};G}$ be the coefficients of the expansion

$$\frac{G(\mathbf{w})}{H(\mathbf{w})} = \sum_{\mathbf{r}} b_{\mathbf{r};G} \mathbf{z}^{\mathbf{r}} .$$

Denote by $q_G$ the element of $\mathbf{W}(d)$ mapping $\mathbf{r}$ to $b_{\mathbf{r};G}$. Thus $q$ is a correspondence between certain meromorphic functions and coefficient arrays. If $S$ is a subset of $\Re$, let $q_S$ denote $\{q_f : f \in S\}$. Let $X \subseteq \mathbf{W}(d)$ denote the vector space $q_\Re$ and let $E \subseteq X$ denote the subspace of functions from $(\mathbf{Z}^+)^d$ to $\mathbf{C}$ that are $o_{\exp}(1)$ uniformly on $\mathcal{C}_1$. For $1 \leq j \leq d$, define a linear map $\sigma_j : \mathbf{W}(d) \to \mathbf{W}(d)$ by $\sigma_j b(\mathbf{r}) = b(\mathbf{r} - e_j)$ where $e_j$ is the vector whose $i^{th}$ component is $\delta_{ij}$ and $f(\mathbf{r})$ is defined to be zero if $\mathbf{r}$ has a negative component.

Recall that $\Im$ denotes the ideal in $\Re$ generated by $\{h_\alpha : \alpha \in A\}$. The next two lemmas state the properties that will be used of the correspondence $q$. The proof of the first one is trivial and is omitted.

**Lemma 5.1** *The map $g \mapsto q_g$ is linear over $\mathbf{C}$ and*

$$\sigma_j(q_g) = q_{w_j g} .$$

□

**Lemma 5.2**

$$q_\Im \subseteq \mathbf{z}^{-\mathbf{r}} E .$$

PROOF: Since $E$ is a vector space, it suffices to show that $q_{gh_\alpha} \in \mathbf{z}^{-\mathbf{r}} E$ for each $\alpha \in A$. This is equivalent to $b_{\mathbf{r};gh_\alpha/H} = o_{\exp}(\mathbf{z}^{-\mathbf{r}})$ on $\mathcal{C}_1$. Each $h_\alpha$ is chosen as $h_S^{\mathbf{x}}$ for some $S \in S$

and $\mathbf{x} \in D(\mathbf{z})$. For such an $S$, the pole set of the meromorphic function $gh_\alpha/H$ is a subset of the set $\mathcal{V}$. Thus $\mathbf{z}$ is on the boundary of the domain of convergence of $gh_\alpha/H$. In a neighborhood of $\mathbf{z}$, the pole set of $gh_\alpha/H$ is simply the union of the sheets $\{\mathcal{V}_j : j \in \mathcal{S}\}$. Thus $\mathbf{dir}_{\frac{gh_\alpha}{H}}(\mathbf{z}) = \mathcal{C}(S)$. We have chosen the cone $\mathcal{C}_1$ to avoid $\mathcal{C}(S)$, so the conclusion of Lemma 2.1 yields the exponential decay of $\mathbf{z}^{\mathbf{r}} q_{gh_\alpha}$ on $\mathcal{C}_1$, and establishes the lemma. □

The ideal $\mathcal{M}$ contains all functions of the form $1 - w_j/z_j$. Write $(1 - \mathbf{w}/\mathbf{z})^{\mathbf{r}}$ to denote $\prod_{j=1}^d (1 - w_j/z_j)^{r_j}$ and similarly write $(1 - \sigma/\mathbf{z})^{\mathbf{r}}$ to denote the product of the operators $(1 - \sigma_j/z_j)^{r_j}$. Since $\mathcal{M}$ is the radical of $\Im_{\mathbf{z}}$, some power of each $(1 - w_j/z_j)$ annihilates $\Re_{\mathbf{z}}/\Im_{\mathbf{z}}$ and hence the set $F := \{\mathbf{r} : (1 - \mathbf{w}/\mathbf{z})^{\mathbf{r}} \notin \Im_{\mathbf{z}}\}$ is finite. From Lemmas 5.1 and 5.2, we see that for any $\mathbf{r} \notin F$ and any $G \in \Re$,

$$(1 - \sigma/\mathbf{z})^{\mathbf{r}} q_G \in E\,. \tag{5.1}$$

The final lemma is as follows.

**Lemma 5.3** *Let $Y \subseteq \mathbf{W}(d)$ be a finite-dimensional subspace such that there is a finite set $F$ for which $\mathbf{r} \notin F$ implies $(1 - \sigma/\mathbf{z})^{\mathbf{r}} Y \subseteq \mathbf{z}^{-\mathbf{r}} E$. Then for each $f \in Y$ there is a polynomial $g$ whose terms have multidegrees in $F$, and for which $f - g\mathbf{z}^{-\mathbf{r}} \in \mathbf{z}^{-\mathbf{r}} E$.*

Assuming this for the moment, the proof of Theorem 3.1 can be finished as follows.

PROOF OF THEOREM 3.1: Let $Y$ be the space $q_{\mathbf{V}^*}$, where $\mathbf{V}^*$ is as in the conclusion of Theorem 4.5. By the conclusion of that theorem, for any $G \in \Re$, we may write $Gu = g_0^* + \sum_{S \in \mathcal{S}} g_S^* h_S^*$. By linearity of $q$, we have written $q_G$ as the coefficients of $g_0^* / \prod_{j=1}^k (h_j^*)^{n_j}$ plus terms of the form $q_{g_S^* h_S^*}$. By Lemma 5.2 these latter terms are in $\mathbf{z}^{-\mathbf{r}} E$. According to (5.1), the hypotheses of Lemma 5.3 are satisfied, and the conclusion of this lemma then proves Theorem 3.1. □

PROOF OF COROLLARY 3.2: The dimension of $\mathbf{V}^*$ is constructed in the proof of Theorem 4.5 to equal the dimension of $\Re_{\mathbf{z}}/\Im_{\mathbf{z}}$. This is at most the cardinality of $F$, though it may be less. In the case where each $n_j = 1$ and the surfaces $\mathcal{V}_j$ intersect transversely at $\mathbf{z}$, the ideal

$\Im_{\mathbf{z}}$ contains $d$ independent linear polynomials, so is equal to $\mathcal{M}$. Hence $|F|$ is the singleton $\{\mathbf{0}\}$ and $\mathbf{W}$ contains only constants. $\square$

It remains to prove Lemma 5.3.

PROOF OF LEMMA 5.3: Replacing each function $f$ in $\mathbf{W}(d)$ by $\mathbf{z}^{-\mathbf{r}}f$, it suffices to prove the lemma for the case $\mathbf{z} = \mathbf{1}$.

Proceed by induction on $|F|$. If $|F| = 1$ then $F = \{\mathbf{0}\}$. In this case, for each $f \in Y$ and $i \leq k$, the function $\mathcal{E}_i := (1 - \sigma_i)f$ is in $E$. The cone $C_1$ has nonempty interior, which implies that $C_1 \cap \mathbf{Z}^d$ has a co-finite subset $C'$ which is a connected subgraph of the integer lattice. For any $\mathbf{r} \leq \mathbf{s} \in C_1$, there is an oriented path $\gamma_0, \gamma_1, \ldots, \gamma_l$ connecting $\mathbf{r}$ to $\mathbf{s}$ in $C'$, where $l = \sum_{i=1}^k (s_i - r_i)$. (An oriented path takes steps only in the increasing coordinate directions.) Then

$$f(\mathbf{s}) - f(\mathbf{r}) = \sum_{j=1}^{l} f(\gamma_j) - f(\gamma_{j-1}) = \sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1})$$

where $m(j) = i$ if $\gamma_{j-1}$ and $\gamma_j$ differ by $e_i$. Sending $\mathbf{s}$ to infinity, we see that $\lim_{\mathbf{s}\to\infty} f(\mathbf{s})$ exists and

$$f(\mathbf{r}) = \lim_{s\to\infty} f(\mathbf{s}) + \sum_{j=1}^{\infty} f(\gamma_j) - f(\gamma_{j-1}) = -\sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1})$$

where $\gamma$ connects $\mathbf{r}$ to infinity. Thus on $C'$, $f$ is a constant plus a tail sum of functions in $E$, and the conclusion is true with $g = \lim_{s\to\infty} f(\mathbf{s})$, the constant polynomial.

The induction step is similar. Let $F_i = \{\mathbf{r} : \mathbf{r} + e_i \in F\}$. Fix $f \in X$. The space $(1 - \sigma_i)X$ satisfies the hypotheses of the lemma with $F_i$ in place of $F$. Since $|F_i| < |F|$, we may apply the induction hypothesis to conclude that $(1 - \sigma_i)f = g_i + \mathcal{E}_i$ where $g_i$ is a polynomial with multi-degrees in $F_i$ and $\mathcal{E}_i \in E$. For any $\mathbf{r} \leq \mathbf{s} \in (\mathbf{Z}^+)^d$, and any oriented path $\gamma$ from $\mathbf{r}$ to $\mathbf{s}$, we have

$$f(\mathbf{s}) - f(\mathbf{r}) \quad = \quad \sum_{j=1}^{l} f(\gamma_j) - f(\gamma_{j-1})$$

$$= -\sum_{j=1}^{l} g_{m(j)}(\gamma_{j-1}) - \sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1}) .$$

If $\mathbf{r}, \mathbf{s} \in C'$ then we have already seen that, as a function of $\mathbf{s}$, the last contribution $\sum_{j=1}^{l} \mathcal{E}_{m(j)}(\gamma_{j-1})$ is equal to a function $C(\mathbf{r})$ plus a term decaying exponentially in $\mathbf{s}$. Fixing $\mathbf{r} \in C'$ so that the set of $\mathbf{s} \in C_1$ not greater than or equal to $\mathbf{r}$ is finite, it remains to show that

$$p(\mathbf{s}) := f(\mathbf{r}) + \sum_{j=1}^{l} g_{m(j)}(\gamma_{j-1})$$

defines a polynomial in $\mathbf{s}$ whose terms have multidegrees in $F$.

We see from the equation

$$(1 - \sigma_i)(1 - \sigma_j) f = (1 - \sigma_j)(1 - \sigma_i) f \tag{5.2}$$

that

$$(1 - \sigma_i) g_j = (1 - \sigma_j) g_i + \mathcal{E}$$

where $\mathcal{E} = (1 - \sigma_j)\mathcal{E}_i) - (1 - \sigma_i)\mathcal{E}_j$. By (5.2), $\mathcal{E}$ is a polynomial, and since it is exponentially small it must vanish entirely. Thus for $\mathbf{x} \in \mathbf{Z}^d$, we see that $g_i(\mathbf{x}) + g_j(\mathbf{x} + e_i) = g_j(\mathbf{x}) + g_i(\mathbf{x} + e_j)$. It follows that the sum defining $p$ is invariant under switching the order of two steps in the path $\gamma$, and hence is independent of the choice of $\gamma$. Choosing $\gamma$ to take first $s_1 - r_1$ steps in direction $e_1$, then $s_2 - r_2$ steps in direction $e_2$ and so on, we may write

$$p(\mathbf{s}) = f(\mathbf{r}) + \sum_{j=1}^{k} \sum_{t=1}^{s_i - r_i} -g_j(s_1, \ldots, s_{j-1}, r_i + t - 1, r_{j+1}, \ldots, r_k) .$$

Each of the inner sums is the sum to $s_i - r_i$ of a polynomial with multi-degrees in $F_i$, which is well known to be a polynomial with multi-degrees in $F$. Hence $p(\mathbf{s})$ is a polynomial with multi-degrees in $F$ and the proof is done. □

**Acknowledgements:** Many thanks to Andrei Gabrielov for patiently explaining to me everything in Section 4. Thanks to Henry Cohn for several conversations and to Bernd Sturmfels for further help with the algebra.

# References


[BenRic83] Bender, E. and Richmond, L. B. (1983). Central and local limit theorems applied to asymptotic enumeration II: multivariate generating functions. *JCT A* **34** 255 - 265.

[CamChaCha91] Campanino, M., Chayes, J.T. and Chayes, L. (1991). Gaussian fluctuations in the subcritical regime of percolation. *Prob. Th. Rel. Fields* **88** 269 - 341.

[ChaCha86] Chayes, J. T. and Chayes, L. (1986). Ornstein-Zernike behavior for self-avoiding walks at all non-critical temperatures. *Comm. Math. Phys.* **105** 221 - 238.

[ChySal96] Chyzak, F. and Salvy, B. (1998). Non-commutative elimination in Ore Algebras proves multivariate identities. *J. Symbol. Comput.* **26** 187 - 227.

[CohElkPro96] Cohn, H., Elkies, N. and Propp, J. (1996). Local statistics for random domino tilings of the Aztec diamond. *Duke Math. J.* **1985** 117 - 166.

[CohPem00] Cohn, H. and Pemantle, R. (2000) A determination of the boundary of the region of fixation for some domino tiling ensembles. *In Preparation.*

[FlaOdl90] Flajolet, P. and Odlyzko, A. (1990). Singularity analysis of generating functions. *SIAM J. Disc. Math.* **3** 216 - 240.

[FlaSed00] Flajolet, P. and Sedgewick, R. (2000). The average case analysis of algorithms. *Book Preprint.*

[Fur67] Furstenberg, H. (1967). Algebraic functions over finite fields. *J. Algebra* **7** 271 - 277.

[GraRem79] Grauert, H. and Remmert, R. (1979). Theory of Stein spaces. Translated from the German by Alan Huckleberry. Grundlehren der Mathematischen Wissenschaften 236. Springer-Verlag: Berlin.

[GunRos65] Gunning, R. and Rossi, H. (1965). Analytic functions of several complex variables. Prentice-Hall: Englewood Cliffs, NJ.

[HauKla71] Hautus, M. and Klarner, D. (1971). The diagonal of a double power series. *Duke Math. J.* **23** 613 - 628.

[LeyTsa00] Leykin, A. and Tsai, H. (2000). *D-modules library for Macaulay 2.* Available at http://www.math.umn.edu/ leykin.

[Lip88] Lipshitz, L. (1988). The diagonal of a D-finite power series is D-finite. *J. Algebra* **113** 373 - 378.

[PemWil00a] Pemantle, R. and Wilson, M. (2000a). Asymptotics for multivariate sequences, part I: smooth points of the singular variety. *In preparation.*

[PemWil00b] Pemantle, R. and Wilson, M. (2000b). Asymptotics for multivariate sequences, part II: multiple points. *In preparation.*

[PemWil00c] Pemantle, R. and Wilson, M. (2000c). Asymptotics for multivariate sequences, part III. *In preparation.*

[Sta99] Stanley, R. (1999). Enumerative combinatorics, vol. 2. Cambridge Studies in Advanced Mathematics no. 62. Cambridge University Press: Cambridge.