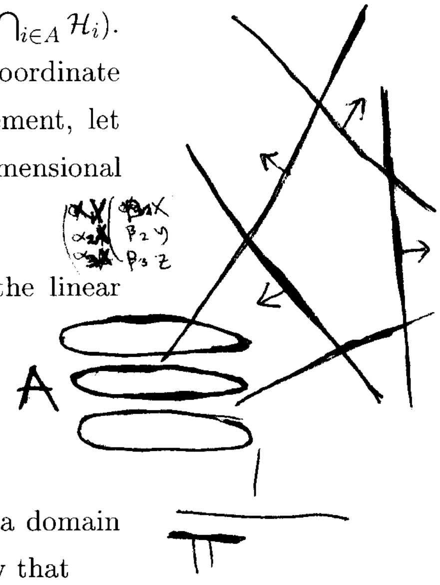
Begin with k linear polynomials $l_1, \ldots, l_k$ in n variables. These define a (noncentral) complex hyperplane arrangement, where $\mathcal{H}_i$ is the hyperplane where l_i vanishes. The lattice of flats, $\mathcal{A}$, is a sublattice of the Boolean lattice on $\{1, \ldots, n\}$ (we identify the element $A \subseteq \mathcal{A}$ with the flat $\bigcap_{i \in A} \mathcal{H}_i$). Assume that $l_i \neq 0$ for all i. Let axes be the union of the coordinate planes, let poles be the union of the hyperplanes in the arrangement, let $X := \mathbb{C}^n \setminus (\text{poles} \cup \text{axes})$, and let $H_n(X, \infty)$ denote the middle-dimensional homology of X relative to its intersection with a large sphere.



Let $F(\mathbf{z}) : \mathbb{C}^d \to \mathbb{C}$ denote the inverse of the product of the linear polynomials. Thus

$$F(\mathbf{z}) := \prod_{i=1}^{k} l_i(\mathbf{z})^{-1} = \sum_{\mathbf{r} \in (\mathbb{Z}^+)^n} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} ,$$

the last equality holding as a formal power series, converging in a domain $\mathcal{D}$. We are interested in asymptotics for $\{a_{\mathbf{r}}\}$ as $\mathbf{r} \to \infty$. We know that

$$(2\pi i)^n a_{\mathbf{r}} = \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) \, d\mathbf{z}$$

why ?

Meaningful conversation
Deep Understanding

where $d\mathbf{z}$ is the holomorphic volume form and T is a torus inside $\mathcal{D}$.

By the use of Stratified Morse Theory, the torus T may be replaced by a sum of quasi-local cycles. These all have the following form. Let A be a flat of $\mathcal{A}$. Fix $\mathbf{r}$ and let $\mathbf{z}(A, \mathbf{r})$ be the unique stationary point on A for the function $\mathbf{z}^{\mathbf{r}}$ (the unique point $\mathbf{x} \in A$ such that $\nabla(\mathbf{z}^{\mathbf{r}})(\mathbf{x}) \perp A$). We assume throughout that $\mathbf{r}$ satisfies the genericity condition that $\mathbf{z}(B, \mathbf{r})$ is not on a proper subflat of B for any $B \in \mathcal{A}$. Let j be the codimension of A (thus $j = 1$ for $|A| = 1$ and so forth). Then the local j-dimensional homology of X at $\mathbf{z}(A, \mathbf{r})$ is independent of $\mathbf{r}$ and is generated by small tori near $\mathbf{z}(A, \mathbf{r})$. Let $\mathbf{z}' = (1 - \epsilon)\mathbf{z}(A, \mathbf{r})$, for some sufficiently small ϵ, and let $C(Z, \mathbf{r})$ be the relative $(n - j)$-cycle

$$C := \{\mathbf{z}' + \alpha : \alpha \in (i\mathbb{R})^d \cap A^{\perp}\} ,$$

where $A^\perp$ is the space spanned by $\{\mathbf{v}_i : i \in A\}$, and $\mathbf{v}_i$ is the vector of coefficients of l_i, that is, $l_i(\mathbf{z}) = b_i + \mathbf{z} \cdot \mathbf{v}$. The quasi-local cycles at A are the cartesian products $\beta \times C(A, \mathbf{r})$ where β is one of the local j-cycles at $\mathbf{z}(A, \mathbf{r})$.

Let $\mathcal{A}_A$ be the sub-arrangement of hyperplanes containing the flat A, that is the collection $\{\mathcal{H}_i : i \in A\}$. Its complement (still removing the axes as well) is denoted X_A. The inclusion of X in X_A induces a map on homology which is an isomorphism on the local homology at $\mathbf{z}(Z, \mathbf{r})$. Our job is therefore to compute

$$\int_{\beta \times C} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z}$$

on each quasi-local cycle. We compute it by first integrating over β then over C. The second integral is a standard oscillatory integral, and it is the integral over β which is interesting. So far this is what I told you when we last talked.

To compute this, we need to determine a better representative for the local cohomology class defined by $\eta := \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F\, d\mathbf{z}$. By the above argument, we assume without loss of generality that all the hyperplanes pass through a common point, say for specificity through $\mathbf{1}$, and that their common intersection is the single point $\mathbf{1}$. We rely on the following fact:

Where did this come from?

Fact 1: In the case of n hyperplanes whose common intersection is $\mathbf{1}$, the asymptotics of $a_\mathbf{r}$ are given by $g(\mathbf{r}) + R(\mathbf{r})$, where

$$g(\mathbf{r}) = [\det(\mathbf{v}_i : i = 1, \ldots, n)]^{-1}$$

in the convex cone spanned by the $\{\mathbf{v}_i : 1 \le i \le n\}$ and is zero otherwise, and where $R(\mathbf{r})$ is exponentially decaying on compact sets away from poles $\cup$ axes.

Let $\mathcal{M}$ be the matroid of linearly independent subsets of $\{\mathbf{v}_i : 1 \le i \le n\}$. It is known that one basis for the local cohomology is the No Broken Circuit basis $\{F_A := d\mathbf{z} \prod_{i \in A} l_i^{-1}, A \in nbc\}$, where A ranges over $\mathcal{B}$, the set of bases of $\mathcal{M}$ not containing any circuit with its greatest element deleted. Given a local cohomology element

$$\omega := G \, d\mathbf{z} \prod_{i \in A_\omega} l_i^{-d_i} ,$$

it is not apparent to see how to write it in the above basis. That is, it should be cohomologically equal to the sum of constant multiples of the F_A, $A \in \mathcal{B}$, but we need to figure out how. There are two steps.

Step 1 is to use the dependence relations among the l_i to get rid of denominators containing forbidden products. Suppose $B \cup \{j\}$ is a circuit with greatest element j. Then there is a linear relation expressing $l_j = \sum c_{ij} l_i$, and therefore

How to do it fast?

$$\begin{aligned} F_B &= \frac{l_j}{\prod_{i \in B \cup \{j\}} l_i} \\ &= \sum_{i \in B} \frac{c_{ij} l_j}{\prod_{i \in B \cup \{j\}} l_i} \\ &= \sum_{i \in B} c_{ij} F_{B(i)} \end{aligned}$$

where $B(i)$ is the exchange basis $B \cup \{j\} \setminus \{i\}$. If A_ω contains B, then isolating a factor of F_B and applying this exchange replaces a factor of a forbidden broken circuit in the denominator by a linear combination of terms in which this factor has been exchanged for a non-forbidden factor. Thus for example, if $l_1 + l_2 = l_3$ and one starts with $(l_1 l_2)^{-1}$, this process gives $l_3/(l_1 l_2 l_3) = (l_1 + l_2)/(l_1 l_2 l_3) = 1/(l_2 l_3) + 1/(l_1 l_3)$. Iterating this process yields a representation of ω as the sum of terms where the set of linear factors in the denominator (ignoring powers) contains no broken circuit. At each step, the span of the factors in the denominator is preserved in each new denominator, so for all of the terms once the iteration stops, the set of linear factors in the denominator is in $\mathcal{B}$.

Step 2 is to get rid of the powers greater than 1. Given a term $G/\prod_{i\in A} l_i^{d_i}$ where $A \in \mathcal{B}$, we proceed as follows. Fix $j \in A$. Let $\mathbf{w}$ be a vector orthogonal to each $\mathbf{v}_i$ with $i \in A \setminus \{j\}$. Let η be the wedge of forms $\mathbf{v}_i \cdot d\mathbf{z}$ as i varies over $A \setminus \{j\}$, so that $dl_i \wedge \eta$ is nonzero only when $i = j$. Then the differential of $G\eta$ is computed as:

$$\begin{aligned} d(G\eta) &= \frac{dG \wedge \eta}{\prod_{i\in A} l_i^{d_i}} + \sum_{i\in A} \left[\frac{G}{\prod_{s\neq i} l_s^{d_s}} \frac{-d_i}{l_i^{d_i+1}} dl_i \wedge \eta \right] \\ &= d\mathbf{z} \left[\frac{\nabla_{\mathbf{w}} G}{\prod_{i\in A} l_i^{d_i}} + \frac{-d_j G}{\prod_{i\in A} l_i^{d_i+\delta_{ij}}} \right] . \end{aligned}$$

Now use the fact that $d(G\eta) = 0$ in the local cohomology ring to set

$$\frac{G}{\prod_{i\in A} l_i^{d_i+\delta_{ij}}} \equiv \frac{-(1/d_j)\nabla_{\mathbf{w}} G}{\prod_{i\in A} l_i^{d_i}}$$

in the local cohomology ring. Thus whenever a denominator is not square-free, one may lower the exponent by one, provided one multiplies by a constant and differentiates the numerator in the appropriate direction. Iterating this, one arrives at a linear combination of terms of the form

$$G_{\mathbf{w}_1,\ldots,\mathbf{w}_{d-n}} F_A$$

where the subscripts on G refer to repeated directional differentiation. Finally, we use the fact that when G is analytic in a neighborhood of the point $\mathbf{1}$ where all the hyperplanes meet, then the asymptotics for the integral of $\int G F_A d\mathbf{z}$ are known, and are the constant $G(\mathbf{1})$ times a determinant, as quoted above.

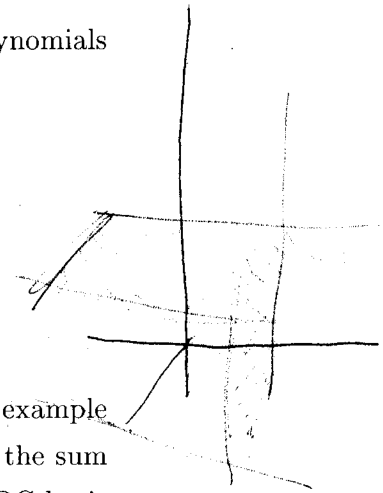
Recalling that we are interested in the case where $G(\mathbf{z}) = \mathbf{z}^{-\mathbf{r}-\mathbf{1}}$, we see that the derivative $G_{\mathbf{w}_1,\ldots,\mathbf{w}_{d-n}}$ evaluates to a sum of products of rising

1-X=0

factorials in the quantities $r_i + 1$, of total degree $d - n$. We have therefore computed the integral we seek, modulo the exponentially decreasing terms, and arrived at a polynomial of degree $d - n$ in the coordinates of $\mathbf{r}$.

As an example, I tried letting $n = 3$ and $k = 5$, with linear polynomials

$$\begin{aligned} l_1 &= 1 - x\,; \\ l_2 &= 1 - y\,; \\ l_3 &= 1 - z\,; \\ l_4 &= 2 - x - y\,; \\ l_5 &= 2 - x - z\,. \end{aligned}$$



These are the tangent planes to the sheets of the toric variety in your example in the paper. Some easy compuation (no computer needed) yields the sum of polynomials over four different cones, the cones being a dual NBC basis for the indicator functions of the five chambers mentioned in your paper. While my generating function is different from yours, being a linearization, I was hoping they might agree up to exponentially decaying terms, and therefore that my asymptotic polynomials would be the same as yours. If I did my computations right, they differ in some of the non-leading terms. Thus my algorithm for producing asymptotics in the product linear case does not extend to the toric case, at least not in this easy way. It does work to produce the leading term in each cone, which must be a much easier problem.

I should add one more thing: the result of all this computation is pretty easy to describe. Define an operation Φ on polynomials by taking each monomial $\mathbf{z}^{\mathbf{r}}$ to $\prod (z_i)^{r_i}$, where this denotes the ascending product $(z_i + 1) \cdots (z_i + r_i)$. Extend this linearly to all polynomials. Step 1 of the algorithm results in a linear combination of terms all of which have denominators which are monomials of degree k in the linear polynomials l_i. Step 2 converts this to a linear combination of terms with numerators that are mixed partials of G, all having order $k-n$, and denominators that are monomials of order n in $\mathcal{B}$. Applying Fact 1, we obtain $\Phi(Q)$ for the asymptotics, where Q is homogeneous of degree $k-n$. I don't have a good proof at the moment, but Q may be describedc as follows. Map $\mathbb{R}^k$ to $\mathbb{R}^n$ by mapping $\mathbf{x}$ to $\sum_{i=1}^k x_i\mathbf{v}_i$ (recall that $l_i(\mathbf{x}) = b_i + \mathbf{v}_i \cdot \mathbf{x}$). Then Q is the homogeneous polynomial of degree $k-n$ which gives the density (with respect to n-dimensional Lebesgue measure) of the image under this mapping of k-dimensional Lebesgue measure.

Birkhoff p.