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On the Combinatorial and Topological Complexity of a Single Cell

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Abstract

The problem of bounding the *combinatorial complexity* of a single connected component (a single cell) of the complement of a set of n geometric objects in R^k of constant description complexity is an important problem in computational geometry which has attracted much attention over the past decade. It has been conjectured that the combinatorial complexity of a single cell is bounded by a function much closer to $O(n^{k-1})$ rather than $O(n^k)$ which is the bound for the combinatorial complexity of the whole arrangement. Currently, this is known to be true only for $k \leq 3$ and only for some special cases in higher dimensions.

A classic result in real algebraic geometry due to Oleinik and Petrovsky, Thom and Milnor, bounds the *topological complexity* (the sum of the Betti numbers) of basic semi-algebraic sets. However, till now no better bounds were known if we restricted attention to a single connected component of a basic semi-algebraic set.

In this paper, we show how these two problems are related. We prove a new bound on the sum of the Betti numbers of one connected component of a basic semi-algebraic set which is an improvement over the Oleinik-Petrovsky-Thom-Milnor bound. This also implies that the topological complexity of a single cell, measured by the sum of the Betti numbers, is bounded by $O(n^{k-1})$. Finally, we show that under a certain natural geometric assumption on the objects (namely that, whenever they intersect the intersection is robustly transversal on average) it is possible to prove a bound of $O(n^{k-1})$ on the combinatorial complexity of a single cell. We also show that this geometric assumption is satisfied by most arrangements and deduce that the *expected* complexity of a single cell in a *randomly* chosen arrangement is $O(n^{k-1})$.

1 Introduction

Arrangements of a finite collection of geometric objects and their combinatorial and algorithmic properties have been fundamental objects of study in computational geometry. The most general setting considered in the computational geometry literature [1] is that of n surface patches $S_1, \ldots, S_n$ in R^k where each surface patch is a semi-algebraic set of dimension $k-1$ described by a quantifier free first-order formula whose atoms are polynomial equalities and inequalities involving a small number of polynomials of small degree. A *cell* is a maximal connected subset of the intersection of a fixed (possibly empty) subset of surface patches that avoids all other surface patches. Thus, a k-dimensional cell is a connected component of the complement of $S_1, \ldots, S_n$. The *combinatorial complexity* of the arrangement is the total number of cells of all dimensions. The combinatorial complexity of an ℓ dimensional cell C is the number of cells of dimension less than ℓ which are contained in the boundary of C.

Usually, some additional *geometric* assumptions are made about the surface patches. One such assumption is that each surface patch is part of a smooth hypersurface and whenever surface patches intersect they do so transversally. Thus, the intersection of any ℓ surface patches is either empty or a $k-\ell$ dimensional manifold. This also implies that no $k+1$ of the surface patches have a non-empty intersection.

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One outstanding problem that has remained opened till now has been to bound the combinatorial complexity of a single k dimensional cell in an arrangement of surface patches. Various special cases have been considered by researchers [16, 2, 17, 11]. In the special case, when each surface patch is a hyperplane, the combinatorial complexity of a single cell (which is a convex polytope with at most n facets) is bounded by $n^{\lfloor k/2 \rfloor}$ by the upper bound theorem [12]. In the more general situation the prevailing conjecture is that the combinatorial complexity of a single cell is bounded by $O(n^{k-1}\beta(n))$ for some extremely slow growing function $\beta(n)$. This is known for $k = 2$ (see [10]). The best known result for $k = 3$ is due to Halperin and Sharir [11] who proved a bound of $O(n^{2+\epsilon})$ for every $\epsilon > 0$. In higher dimensions bounds close to $O(n^{k-1})$ are known only for certain special cases. For instance, a $O(n^{k-1}\log n)$ bound was proved by Aronov and Sharir [2] (see also [17]) for the combinatorial complexity of a single cell in an arrangement of n $(k-1)$-simplices in R^k. These bounds do not extend to general surface patches. The general problem is open in higher dimensions. Note also, that using the lower bound construction given by Wiernik and Sharir [19] one can prove a lower bound of $\Omega(n^{k-1}\beta(n))$, and hence a $O(n^{k-1})$ bound is not possible. For a survey of the numerous algorithmic applications of these bounds in computational geometry see [1].

Another problem that has received considerable attention from researchers interested in real algebraic geometry, is to bound the topological complexity of semi-algebraic sets. The first and classical result in this area is due to Oleinik and Petrovsky [15], Thom [18] and Milnor [14], who independently proved that the sum of the Betti numbers of a basic closed semi-algebraic set in R^k defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with $deg(P_i) \leq d, 1 \leq i \leq s$, is bounded by $O(sd)^k$. This bound was extended to arbitrary closed semi-algebraic sets in [7], where it is shown that the sum of the Betti numbers of a closed semi-algebraic set defined in terms of s polynomials of degree d, which is contained in a variety of dimension k', is bounded by $s^{k'}O(d)^k$.

These bounds are essentially tight as one can easily define a basic semi-algebraic set with s polynomials and degrees bounded by d which has $(sd)^k$ connected components. However, until now no attempt has been made to study the topological complexity of a single connected component of a basic semi-algebraic set. In analogy to the single cell combinatorial complexity results in computational geometry, one might conjecture that the sum of the Betti numbers of a single connected component of a basic semi-algebraic set is bounded by $s^{k-1}O(d)^k$. One cannot hope to do much better as it is easy to construct a basic semi-algebraic set defined in terms of s polynomials of degree d such that it has one connected component the sum of whose Betti numbers is $\Omega(sd)^{k-1}$.

In this paper we prove the above conjecture. Moreover, we show how to approximate any collection of closed semi-algebraic sets in R^k each of them having constant description complexity by semi-algebraic sets bounded by smooth hypersurfaces of small degrees. This enables one to apply Morse theoretic results obtained in [6] and other papers directly to the problem of bounding the combinatorial complexity of a single cell. Together with the bound on the Betti numbers of a single connected component of a basic semi-algebraic, it also implies that the sum of the Betti numbers of a single cell is bounded by $O(n^{k-1})$.

Further, under a certain geometric assumption on the intersections of the surface patches, we prove an $O(n^{k-1})$ bound on the combinatorial complexity of a single cell.

The geometric assumption is that whenever k of the surface patches intersect at a point, the k unit normals to the surfaces at that point (with any orientation) span a (k-dimensional) angle of at least b (see figure 1).

Even though the above condition is stated in *worst case* terms, our results continue to hold with the *weaker* assumption that the *average* angle at the vertices made by the normals is at least b. We also show in section 5 that for randomly chosen surfaces (in a sense to be made precise later) the average angle made by the normals at the vertices is quite large (bounded away from zero by a constant). This together with the previous results imply that for randomly chosen arrangements, the expected combinatorial complexity of a single cell is bounded by $O(n^{k-1})$.

Note that this condition is a natural strengthening of the requirement that the surfaces meet

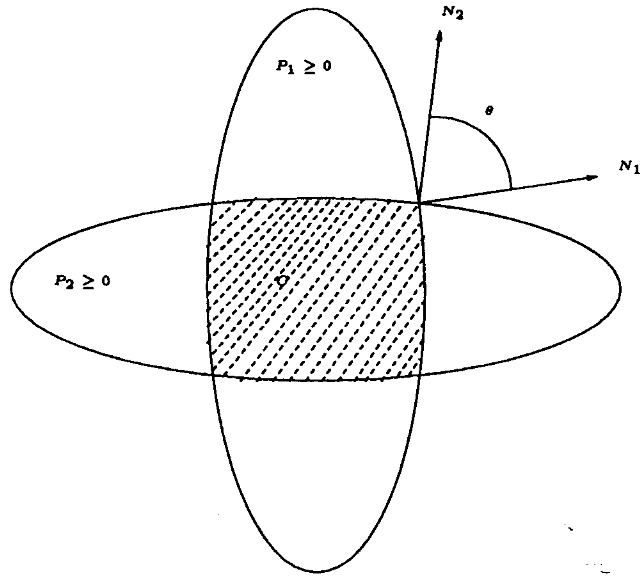


Figure 1: The angle made by the normals is $\theta > b$.

transversally at the vertices. If they are required to meet transversally then the solid angle will be positive. By requiring that the solid angle be at least b we are imposing the extra condition that the transversality be robust.

Finally, we remark that we believe that our method should be useful in extending the Halperin-Sharir proof for the bound on the combinatorial complexity of a single cell in three dimensions to all higher dimensions. We provide the Morse theoretic tools needed to prove an $O(n^{k-1})$ bound on the number of locally X_1-extreme vertices of a cell which was one of the obstructions to extending their proof to higher dimensions (see [1]).

2 New Results

First, we bound the topological complexity of a single connected component of a basic semi-algebraic set.

Theorem 1 *Let $C \subset R^k$ be a connected component of a basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with the degrees of the polynomials P_i bounded by d. Then the sum of the Betti numbers of C is bounded by $\binom{s}{k-1}O(d)^k$.*

Actually, our technique proves the following stronger version of the above theorem.

Theorem 2 *Let $C_1, \ldots, C_m \subset R^k$ be m different connected components of a basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with the degrees of the polynomials P_i bounded by d. Then $\sum_{i,j} \beta_i(C_j)$ is bounded by $m + \binom{s}{k-1}O(d)^k$.*

For single cells we have the following results.

Recall that a *surface patch* S_i is a *closed* semi-algebraic set contained in a hypersurface $Z(Q_i)$ and defined by a first-order quantifier-free formula involving a family of polynomials, $\{P_{i,1}, \ldots, P_{i,r}\}$. We will assume that the degrees of all the polynomials, $Q_i, P_{i,j}$ are bounded by d. Also, the parameters r, d, k are treated as constants as is usual in the computational geometry literature.

Let C be a single connected component of the complement of n surface patches, $S_1, \ldots, S_n$.

A corollary of theorem 1 is the following:

Corollary 1 *The sum of the Betti numbers of C is bounded by $O(n^{k-1})$.*

We are also interested in bounding the combinatorial complexity of C. The complement of the surface patches $S_1, \ldots, S_n$ is open, and we will bound the number of vertices (the 0-dimensional cells) in the boundary of C, $\bar{C} - C$.

Theorem 3 *Let $S_1, \ldots, S_n \subset R^k$ be surface patches as defined above, and let C be a connected component in the complement of the surface patches $S_1, \ldots, S_n$.*

Also let the surface patches $S_1, \ldots, S_n$ satisfy the following geometric conditions.

T1 No $k+1$ patches intersect.

T2 The points of C that are common to k of these hypersurfaces are isolated. They are called the vertices of C. If x is a vertex of C common to k patches $S_{i_1}, \ldots, S_{i_k}$, then the solid angle made by any of the unit normals $N_1, \ldots, N_k$ to these hypersurfaces at p is bounded from below by some constant b.

Then, the number of vertices in $\bar{C} - C$ is bounded by $O(b^{-1}n^{k-1})$.

As before, we can prove a bound of $O(b^{-1}n^{k-1} + m)$ on the sum of the complexities of m different cells. This immediately gives us a bound of $O(b^{-1}n^{k-1})$ on the combinatorial complexity of the *zone* of an algebraic variety of degree d, under the same assumptions on the normal cones. Also, the theorem is still true if the condition T2 is satisfied only on the average, that is if the average of the normal cone angles is bounded from below by some constant b.

Finally we prove that for a randomly chosen arrangement of n surface patches, the expected complexity of a single cell is bounded by $O(n^{k-1})$. In order to show this we need to consider a proper probability measure on the space of all allowable surface patches. The details will appear in the full version of this paper but we give an outline of our arguments in section 5.

3 Mathematical Preliminaries

In the following we will perturb polynomials by various *infinitesimals* so that our geometric objects live over the field of Puiseux series in these infinitesimals. We write $R\langle\zeta\rangle$ for the real closed field of Puiseux series in ζ with coefficients in R [3]. The sign of such a Puiseux series agrees with the sign of the coefficient of the lowest degree term in ζ. This order makes ζ positive and smaller than any positive element of R. The map eval_ζ maps an element of $R\langle\zeta\rangle$ bounded over R (one that has no negative powers of ζ) to its constant term. An element of $R\langle\zeta\rangle$ is infinitesimal over R if it is mapped by eval_ζ to 0. In particular ζ is infinitesimal over R. If S is a semi-algebraic subset of R^k we denote by $S_{R\langle\zeta\rangle}$ the subset of $R\langle\zeta\rangle^k$ defined by the same equalities and inequalities that define S. We refer the reader to [4] for more details on the use of infinitesimals.

The main topological result that we will require is a variant of Morse's lemma which is applicable to basic semi-algebraic sets [6]. For the rest of the paper we will denote by π the projection map on the first co-ordinate. For any set $S \subset R^k$ we let S_x denote $S \cap \pi^{-1}(x)$.

Let $S \subset R^k$ be a basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. Let C be a connected component of S and let $C_{\leq x}$ denote the set $C \cap (X_1 \leq x)$. Then as x varies from $-\infty$ to ∞ the number of connected components of $C_{\leq x}$ changes at only finitely many values. These "critical" values are characterized in terms of the various algebraic sets of the form $P_{i_1} = \cdots = P_{i_l} = 0$ in the following lemmas from [6].

Given a polynomial $Q \in R[X_1, \ldots, X_k]$ we denote by $Z(Q)$ the set of real zeros of Q in R^k. We denote the by $B_p(r)$ the ball of radius r centered at p. Given a bounded algebraic set $Z(Q) \subset B_O(M) \subset$

R^k defined by a polynomial Q of degree $\leq d$, we adapt an idea of Gournay-Risler [9] and consider $Z(Q_1) \subset R\langle\zeta\rangle^k$ where

$$Q_1 = (1-\zeta)Q^2 + \zeta(X_1^{2d+2} + \cdots + X_k^{2d+2} - kM^{2d+2}).$$

The algebraic set $Z(Q_1)$ is a smooth algebraic hypersurface of $R\langle\zeta\rangle^k$ on which π (the projection of $(x_1, \ldots, x_k) \in R\langle\zeta\rangle^k$ to $x_1 \in R\langle\zeta\rangle$) has a finite number of critical points [6].

Lemma 1 *Let $Z(Q) \subset B_O(M)$ be a bounded algebraic set. Let Z_1 be the union of the semi-algebraically connected components of $Z(Q_1)$ which are bounded over R. Then* $\mathrm{eval}_\zeta(Z_1) = Z(Q)$. *Moreover, $Z(Q_1) \setminus Z_1$ contains no point bounded over R.*

Definition 1 *A* special value *of $Z(Q)$ is a $c \in R$ for which there exists $y \in Z(Q_1)$ with* $\mathrm{eval}_\zeta(\pi(y)) = c$, *$g(y)$ infinitesimal over R and y a local minimum of g on $Z(Q_1)$, where*

$$g(X) = \frac{\sum_{i=2}^{k} \frac{\partial Q_1}{\partial X_i}^2}{\sum_{i=1}^{k} \frac{\partial Q_1}{\partial X_i}^2}.$$

The number of special values of $Z(Q)$ is bounded by $O(d)^k$.

Let S be a basic closed bounded semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, where the set of polynomials $\mathcal{P} = \{P_1, \ldots, P_s\}$ is such that $Z(P_i)$ is bounded for $1 \leq i \leq s$. A *special value* of S is a special value of $Z(\mathcal{P}')$ for some $\mathcal{P}' \subset \mathcal{P}$, where we denote by $Z(\mathcal{P}')$ the set of common zeros of the polynomials in $\mathcal{P}'$.

Proposition 1 *Let S be a basic closed bounded semi-algebraic set defined as*

$$S = \{x \in R^k \mid \forall P \in \mathcal{P},\ P(x) \geq 0\},$$

where the polynomials $P \in \mathcal{P}$ are such that $Z(P)$ is bounded. If C is a semi-algebraic connected component of $S_{[a,b]}$ and $[a,b] \setminus \{c\}$ contains no special value of S where $c \in (a,b)$ then C_c is semi-algebraically connected.

4 Proofs of the main Theorems

We first prove a technical result which will play an important role in the proof of theorem 3. Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset R[X_1, \ldots, X_k]$ be a family of polynomials whose degrees are bounded by d.

Let C be a connected component of the basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$.

We assume that the zero sets $Z(P_i)$ are smooth hypersurfaces such that:

T1 No $k+1$ of the hypersurfaces $Z(P_i)$ intersect.

T2 The points of C that are common to k of these hypersurfaces are isolated. They are called the vertices of C. If x is a vertex of C common to the hypersurfaces $Z(P_{i_1}), \ldots, Z(P_{i_k})$, then the solid angle made by the unit normals $N_1, \ldots, N_k$ to these hypersurfaces at p directed away from C, is bounded from below by some constant b.

We prove the following proposition.

Proposition 2 *If C is defined as above and satisfies T1 and T2 then the number of vertices of C is bounded by $b^{-1}\binom{s}{k-1}O(d)^k$.*

Proof: For any real x let $C_{\leq x}$ denote the set $C \cap \pi^{-1}(-\infty, x]$. We let x vary from $-\infty$ to $+\infty$ and study what happens to $C_{\leq x}$.

First observe that the number of connected components of $C_{\leq x}$ changes only at a finitely many points as x varies, and each such x is a special value of S derived from an algebraic set of the form $P_{i_1} = \cdots = P_{i_l} = 0$. This follows from proposition 1. Note that the first co-ordinate of every vertex is a special value.

Let p be a vertex of C and let $N_1, \ldots, N_k$ be the outer normals at p. We call p a *good* vertex if and only if the the vector $(-1, 0, \ldots, 0)$ which is normal to the hyperplane $X_1 = 0$ lies in the positive cone generated by the vectors $N_1, \ldots, N_k$ translated to the origin (see figure 2).

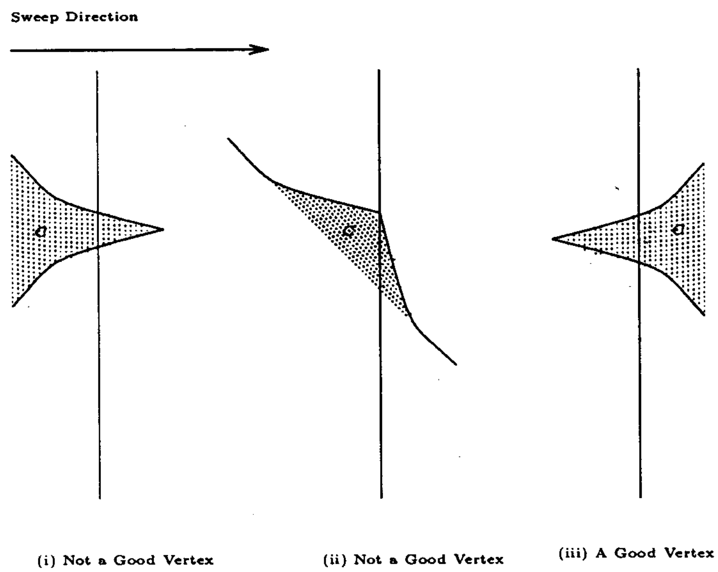


Figure 2: Different kinds of vertices.

We also observe that if p is a vertex of C and $c = \pi(p)$, then the number of connected components of $C_{\leq x}$ changes (increases by one) as we cross c from left to right if and only if p is a good vertex. This follows from the fact that in a small neighborhood of p, the set C can be very closely approximated by a convex cone bounded by the tangent hyperplanes at p to the k hypersurfaces incident on p. The total number of special values coming from algebraic sets defined by less than k polynomials is bounded by $\binom{s}{k-1}(O(d))^k$. Also, for all large enough x , $C_{\leq x}$ is connected.

Thus, at every special value corresponding to a good vertex a new connected component is born, no topological change occurs at special values corresponding to vertices which are not good, and there are only $\binom{s}{k-1}(O(d))^k$ other special values where the number of connected components can change. At these special values only at most two connected components can join together. This allows us to bound the number of good vertices by $\binom{s}{k-1}(O(d))^k$.

Finally, since the outer normal cone at every vertex has (solid) angle bounded from below by b we can choose $O(b^{-1})$ directions for the normal of our sweep hyperplane such that every vertex is good for at least one direction. Thus, the total number of vertices is bounded by $b^{-1}\binom{s}{k-1}O(d)^k$.

□

Remark 1: Note that if we consider the number of vertices in any m connected components, $C_1, \ldots, C_m$ instead of just C, the same argument bounds the number of good vertices by $m + \binom{s}{k-1}(O(d))^k$. This is so because at the end of the sweep we have m components left rather than only one.

Remark 2: The bound on the number of good vertices in a particular direction is independent of the bound on the angle of the normal cones.

Remark 3: The proof is valid not only for a single connected component of a basic semi-algebraic set, but for a connected component of any semi-algebraic set defined as an intersection of s semi-algebraic sets each of which is bounded by a semi-algebraic hypersurface of bounded degree.

Remark 4: The result holds with the weaker restriction that only the average of the normal cone angles be bounded from below by a constant b. If the average is bounded from below then at least a constant proportion of the angles has to be larger than (say) $b/2$. By the same argument as the proof outlined above we can bound the number of such vertices by $O(b^{-1}n^{k-1})$, and hence the total number of vertices is bounded by $O(b^{-1}n^{k-1})$.

We next establish a connection between the single cell problem and the problem of bounding the number of vertices in a connected component of a set defined as the intersection of n sets each of which is bounded by a smooth hypersurface of bounded degree. We will then be in a position to use proposition 2.

To establish this connection we show how to replace each surface patch S_i by a union of sets bounded by smooth hypersurfaces which contains S_i and is infinitesimally close to it. For this we use a classical perturbation technique originally due to Milnor [14].

Let $S \subset R^k$ be a surface patch. For simplicity, we first assume that S is a basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_r \geq 0$, and let D be a connected component of S. We also assume that D is contained in a ball of radius R centered at the origin.

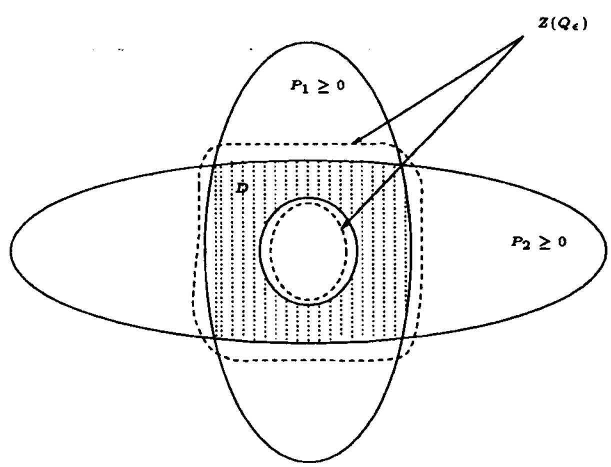
If the surface patch is contained in a variety $Z(Q)$ then we let $P_1 = Q$ and $P_2 = -Q$. Let ϵ be a positive infinitesimal.

Consider the set S_ϵ defined by $Q_\epsilon \geq 0$, where

$$Q_\epsilon = \Pi_{1\leq i\leq r}((1-\epsilon)P_i + \epsilon) - \epsilon^{r+1}(X_1^{2d} + \cdots + X_k^{2d} + 1),$$

where $2d > \sum_{1\leq i\leq r} \deg(P_i)$.

The set S_ϵ contains S, and hence it has a connected component D_ϵ which contains D (see figure 3). Moreover, $D = \mathrm{eval}_\epsilon \mathrm{D}_\epsilon$, and D_ϵ is bounded by connected components of the smooth hypersurface $Z(Q_\epsilon)$. For a proof of these facts see [7].



D is a connected component of $P_1 \geq 0$, $P_2 \geq 0$.

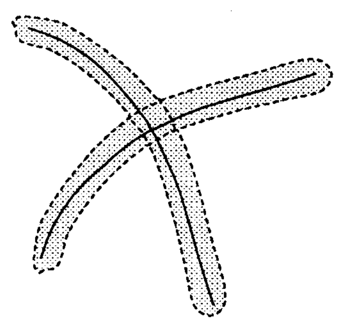
Figure 3: Approximating D (the shaded region) by a connected component of $Q_\epsilon \geq 0$.

Thus, if all the surface patches $S_1, \ldots, S_n$ are basic semi-algebraic sets, then we can replace each connected component $D_{i,j}$ of S_i by the corresponding set $D_{i,j,\epsilon}$ defined as above. If C is a connected component of the complement of $\cup S_i$ then there exists a connected component, $C_\epsilon \subset R\langle\epsilon\rangle^k$, of the complement of $\cup_{i,j} D_{i,j,\epsilon}$ such that $C = C_\epsilon \cap R^k$. Moreover, C_ϵ is a connected component of a set defined as the intersection of sets bounded by smooth hypersurfaces.

The next lemma shows that near smooth points of S_i the unit normals to the approximating hypersurfaces $\partial D_{i,j,\epsilon}$ closely approximates the normals to the original surface.

Lemma 2 *For a smooth point p of a connected component $D_{i,j}$ of a surface patch S_i, let N_p be a unit normal at p. Then, inside any ball $B_p(\rho)$ around p of infinitesimal radius ρ, and any point $q \in B_p(\rho) \cap \partial D_{i,j,\epsilon}$, the unit normal N_q is such that, $1 - |N_p \cdot N_q|$ is infinitesimal.*

Proof: The proof uses Leibniz's rule to compute the partial derivatives of Q_ϵ and is omitted. □



Surface patches shown in solid. The shaded region show the perturbed sets.

Figure 4: Surface patches replaced by sets bounded by smooth hypersurfaces.

It is clear that the number of vertices of C_ϵ is at least the number of vertices of $\bar{C} - C$ which are intersections of k different surface patches. In order to see this, consider a vertex p in $\bar{C} - C$ which is the intersection of k different surface patches, $S_{i_1}, \ldots, S_{i_k}$. Let the corresponding connected components be, $D_{i_1,j_1}, \ldots, D_{i_k,j_k}$. Then there is at least one vertex in C_ϵ which is infinitesimally close to p and which is in the intersection of the k hypersurfaces, $\partial D_{i_1,j_1,\epsilon}, \ldots, \partial D_{i_k,j_k,\epsilon}$.

This shows that, we can replace the single cell C by another semi-algebraic set C_ϵ which is an intersection of sets bounded by smooth hypersurfaces, satisfying T1 and T2 and the number of vertices of C_ϵ bounds the number of vertices of C.

We finally consider the case when the surfaces patches are not necessarily basic, but the surface patch S_i is contained in $Z(Q_i)$ and is defined by some arbitrary Boolean formula with atoms of the form $P_{i,j}\{\leq, \geq\}0$. Since S_i is closed it can be expressed as the union of $O(r^k)$ basic semi-algebraic sets. Thus, $S_i = \cup S_{i,j}$ where each $S_{i,j}$ is basic.

We replace each connected component of each of the basic semi-algebraic sets $S_{i,j}$ by sets bounded by smooth hypersurfaces as earlier. The union of these sets will still be connected.

We have proved the following proposition.

Proposition 3 *Let C be a connected component of the complement of the surface patches $S_1, \ldots, S_n$. Then C is homotopic to another semi-algebraic set, C_ϵ, which is an intersection of $O(n)$ semi-algebraic sets each of which is bounded by a smooth hypersurface. Moreover, the hypersurfaces bounding these sets satisfy the conditions T1 and T2. Further, the number of vertices in the boundary of C (that is $\bar{C} - C$) is $O(n^{k-1} + M)$ where M is the number of vertices of C_ϵ.*

The above proposition together with proposition 2 (see Remark 3) proves theorem 3.

4.1 Bound on the Sum of the Betti Numbers of a Connected Component of a Basic Semi-algebraic Set

In this section R is the field of real numbers. We will be using certain facts from stratified Morse theory due to Goresky and MacPherson which is available only over the reals and not over general real closed fields.

We will continue to use the infinitesimal notation. However, a positive infinitesimal will denote a sufficiently small positive real number.

Let C be a connected component of a basic semi-algebraic set S defined by $P_1 \geq 0, \ldots, P_s \geq 0$. By adding the additional polynomial inequality, $X_1^2 + \cdots + X_k^2 - \Omega \leq 0$, with sufficiently large $\Omega > 0$, we can assume that S is compact.

We now replace the set S by a new set, $S^+(\bar{\epsilon})$, defined by, $P_1 \geq -\epsilon\epsilon_1, \ldots, P_s \geq -\epsilon\epsilon_s$, where $\epsilon_1 \gg \epsilon_2 \gg \cdots \gg \epsilon_s \gg \epsilon > 0$, are positive infinitesimals.

The following lemma appears in [7].

Lemma 3 *$S^+(\bar{\epsilon})$ has the same homology groups as S.*

We also need the following lemma.

Lemma 4 *The sets $Z(P_i + \epsilon\epsilon_i)$ are smooth hypersurfaces intersecting transversally. Moreover, the sign invariant sets of the family of polynomials, $\{P_1 + \epsilon\epsilon_1, \ldots, P_s + \epsilon\epsilon_s\}$ give rise to a Whitney stratification of R^k.*

Proof: It follows from Sard's lemma that for almost all choices of ϵ and ϵ_i (that is outside a set of measure zero) $Z(P_i + \epsilon\epsilon_i)$ are smooth hypersurfaces intersecting transversally. Since the intersection of Whitney stratified sets intersecting transversally is also Whitney stratified the lemma follows. □

Consider the connected component, $C^+(\bar{\epsilon})$, which contains C. By Lemma 3, we have that $\beta_i(C) = \beta_i(C^+(\bar{\epsilon}))$, where $\beta_i(C)$ is the rank of the i^{th} singular homology group of C.

Before giving the proof we recall certain facts from Morse theory of stratified sets [8].

A Whitney stratification of a space X is a decomposition of X into sub-manifolds called strata, which satisfy certain frontier conditions, (see [8] page 37). In particular, given a compact set bounded by a smooth hypersurface, the boundary and the interior form a Whitney stratification.

Now, let X be a compact Whitney stratified subset of R^k, and f a restriction to X of a smooth function. A critical point of f is defined to be a critical point of the restriction of f to any stratum, and a critical value of f is the value of f at a critical point.

The first fundamental result of stratified Morse theory is the following.

Theorem 4 *(Goresky-MacPherson, [8]) As c varies in the open interval between two adjacent critical values, the topological type of $X \cap f^{-1}((-\infty, c])$ remains constant.*

Stratified Morse theory actually gives a recipe for describing the topological change in $X \cap f^{-1}((-\infty, c])$ as c crosses a critical value. This is given in terms of Morse data, which consists of a pair of topological spaces $(A, B), A \supset B$, with the property that as c crosses the critical value $v = f(p)$, the change in $X \cap f^{-1}((-\infty, c])$ can be described by gluing in A along B.

A function is called a Morse function if it has only non-degenerate critical points when restricted to each strata, and all its critical values are distinct.(There is an additional non-degeneracy condition which states that the differential of f at a critical point p of a strata S should not annihilate any limit of tangent spaces to a stratum other than S. However, in our simple situation this will always be true.) By a suitable change of coordinates we can assume that the projection map π onto the first coordinate is a Morse function when restricted to each of the strata of $C^+(\bar{\epsilon})$.

In stratified Morse theory the Morse data is presented as a product of two pairs, called the tangential Morse data and the normal Morse data. The notion of product of pairs is the standard one in topology, namely

$$(A, B) \times (A', B') = (A \times A', A \times B' \cup B \times A').$$

The tangential Morse data at a critical point p is then given by $(B^\lambda \times B^{k-\lambda}, (\partial B^\lambda) \times B^{k-\lambda})$ where B^k is the closed k dimensional disk, ∂ is the boundary map, and λ is the index of the Hessian matrix of f (restricted to the stratum containing p) at p.

Let $p = (p_1, \ldots, p_k)$, be a critical point in some ℓ dimensional stratum S of a stratified subset Z of R^k.

Let N' be any $k - \ell$ dimensional hyperplane passing through the point p which is transverse to S which intersects the stratum S locally at the single point p.

Let $B_p(\delta)$ be a closed ball centered at p of radius δ, for some sufficiently small δ. Then, the ***normal slice***, $N(p)$ at the point p is defined to be,

$$N(p) = N' \cap Z \cap B_p(\delta).$$

Choose $\delta \gg \epsilon' > 0$. The lower *halflink* of Z at the point p is tha pair of spaces,

$$(\ell^-, \partial\ell^-) = (N(p) \cap B_p(\delta) \cap \pi^{-1}(p_1 - \epsilon'), N(p) \cap \partial B_p(\delta) \cap \pi^{-1}(p_1 - \epsilon').)$$

The normal Morse data has the homotopy type of the pair $(\mathrm{cone}(\ell^-), \ell^-)$.

The second fundamental theorem of stratified Morse theory states the following:

Theorem 5 *(Goresky-MacPherson, [8]) Let f be a Morse function on a compact Whitney stratified space X. Then, Morse data measuring the change in the topological type of $X \cap f^{-1}((-\infty, c])$ as c crosses the critical value v of the critical point p is the product of the normal Morse data at p and the tangential Morse data at p.*

We are finally in a position to use the above machinery to bound the Betti numbers of the set $C^+(\bar{\epsilon})$.

We first bound the number of good vertices. Using the same argument as in section 4.1 it is easy to see that the number of good vertices is bounded by $\binom{s}{k-1}O(d)^k$. It is also clear that no topological change occurs as the sweep hyperplanes cross vertices which are not good.

Finally, we need to bound the change in the Betti numbers as the sweep hyperplane crosses a critical point belonging to a stratum of dimension > 0. In order to do this we need to examine the normal component of the Morse data. Suppose the critical point belonged to a stratum of dimension ℓ.

Now the lower halflink is either empty or the pair $(B^{k-\ell-1}, \partial B^{k-\ell-1})$. To see this consider the $k - \ell$ dimensional plane N' through p which intersects the critical stratum transversally. The critical stratum is locally the intersection of $k - l$ hypersurfaces. Thus, locally around p, the set $N' \cap C^+(\bar{\epsilon})$ is like a convex $k - \ell$ dimensional cone. Thus, the lower half link, is either empty or homeomorphic to $(B^{k-\ell+1}, \partial B^{k-\ell+1})$ depending on whether $N(p) \cap B_p(\delta) \cap \pi^{-1}(p_1 - \epsilon')$ is empty or not respectively.

The following theorem measures the change in topology as we cross a critical value.

Theorem 6 *(Goreski-MacPherson, [8], page 69) Let Z be a Whitney stratified space, $f : Z \to R$ a proper Morse function, and $[a, b] \subset R$ an interval which contains no critical values except for an isolated critical value $v \in (a, b)$ which corresponds to a critical point p which lies in some stratum S of Z. Let λ be the Morse index of the critical point p. Then, the space $Z_{\leq b}$ has the homotopy type of a space which is obtained from $Z_{\leq a}$ by attaching the pair $(B^\lambda, \partial B^\lambda) \times (\mathrm{cone}(\ell^-), \ell^-)$.*

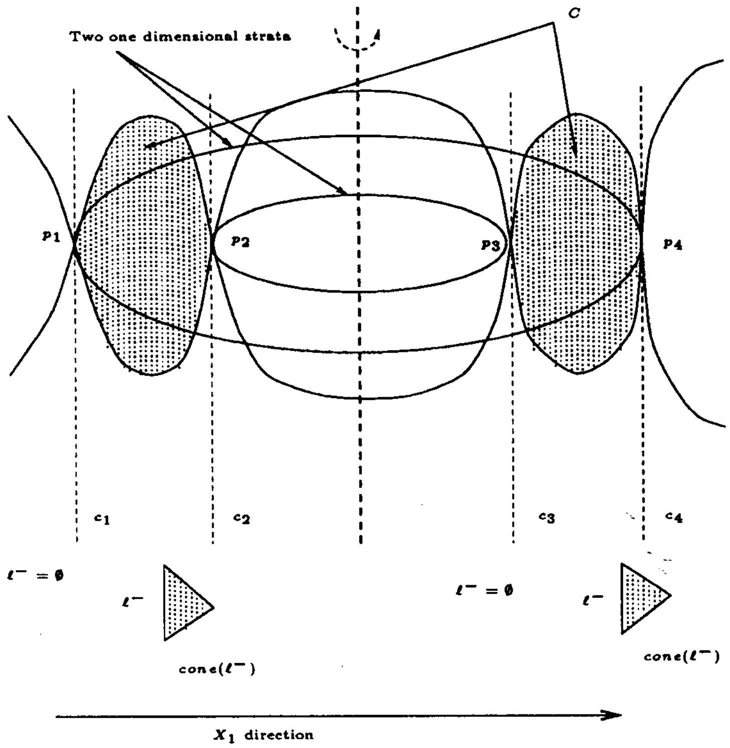


Figure 5: Example of a non-smooth torus in R^3.

In our situation the normal Morse data is either empty or is homotopic to $(\text{cone}(B^{k-\ell-1}), B^{k-\ell-1})$.

In the first case, the Morse data is just the tangential Morse data. In the second case, there is no topological change as the pair that is being added is contractible.

The following example should be helpful.

Example: Consider the solid torus in Figure 5 formed by rotating the shaded region enclosed by two curves, around the dotted line at the center. The resulting set has one 3-dimensional stratum, two 2-dimensional strata, and two 1-dimensional strata. There are four critical points p_1, p_2, p_3, p_4, with respect to the projection map onto X_1 co-ordinate, all on the 1-dimensional strata. The corresponding critical values are c_1, c_2, c_3, c_4. All the critical points are of index one. The corresponding lower halflinks ℓ^- are shown in Figure 5 for each critical point. Notice, that the normal data $(\text{cone}(\ell^-), \ell^-)$ is contractible for the critical points p_2 and p_4. Thus no topological change occurs as the sweep plane crosses c_2 and c_4. At c_1 and c_3 a one-dimensional ball gets added. This shows that the solid torus as the same homotopy type as S^1 as one would expect.

This shows that the sum of the Betti numbers of $C^+(\bar{\epsilon})_{\leq x}$ increases by at most 1 as x crosses a critical value corresponding to a critical point of a stratum of dimension > 0. Since, there are at most $\binom{s}{k-1}O(d)^k$ of such critical points and we already know that there are only $\binom{s}{k-1}O(d)^k$ good vertices, this proves that the sum of the Betti numbers of $C^+(\bar{\epsilon})$ and hence that of C is bounded by $\binom{s}{k-1}O(d)^k$.

5 On the Expected Complexity of a Single Cell

We first formalize what is meant by randomly choosing n surface patches in R^k.

We assume that each surface patch is defined by a first order formula of a certain *shape*. For example each surface patch can be defined by a formula,

$$Q = 0, P_1, \ldots, P_r \geq 0,$$

with the polynomials Q and P_i having degree at most d. By choosing different coefficients for the polynomials $Q, P_1, \ldots, P_r$ we obtain different surface patches.

By a randomly chosen surface patch we will mean a surface patch defined by a first order formula of a fixed shape, whose coefficients are chosen uniformly at random. Since multiplying the polynomials by some positive constant leaves the inequalities intact the space of such formulas is compact and we can consider a uniform probability measure.

Let A denote n surface patches chosen randomly as defined above. Let dA denote the uniform probability measure and let $V(A)$ denote the number of vertices of A.

We have the following theorem.

Theorem 7 *The expectation* $E[V(A)] = O(n^{k-1})$.

Here we only give an outline of the proof. The full details will appear in a later paper.

Let $b(A)$ denote the average normal cone angles of A. Then previously we have shown that, $V(A) = b(A)^{-1}O(n^{k-1})$.

Now,

$$E[V(a)] = \int V(A)dA \leq O(n^{k-1}) \int (b(A)^{-1})dA = O(n^{k-1})E[b(A)^{-1}].$$

We now prove that, $E[b(A)^{-1}] = O(1)$.

Let $c(A)$ denote the harmonic mean of the normal angle cones of A. Then, $E[b(A)^{-1}] \leq E[c(A)^{-1}]$. Also, if $\theta_1, \ldots, \theta_N$, are the normal cone angles, then, $E[c(A)^{-1}] = E[\theta_i^{-1}]$, by the linearity of expectation.

But, $E[\theta_i^{-1}] = O(1)$ because if k randomly chosen surface patches intersect at a point, they will with high probability intersect at a large angle. (The detailed proof of this fact involves using Crofton's formula to bound the volume (in the coefficient space) of the set of points corresponding to k surface patches intersecting at a vertex with the normal cone angle less than b, and will appear in the full paper).

This completes the sketch of the proof.

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