DISCRETE AND COMPUTATIONAL GEOMETRY:

TEN YEARS LATER

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Approximating Shortest Paths on a Convex Polytope in Three Dimensions

Pankaj K. Agarwal

Given a convex polytope P with n faces in $\Re^3$, points $s, t \in \partial P$, and a parameter $0 < \varepsilon \leq 1$, we present an algorithm that constructs a path on ∂P from s to t whose length is at most $(1+\varepsilon)d_P(s,t)$, where $d_P(s,t)$ is the length of the shortest path between s and t on ∂P. The algorithm runs in $O(n \cdot \min\{1/\varepsilon^{1.5}, \log n\} + 1/\varepsilon^3)$ time, and is relatively simple to implement. We also present an extension of the algorithm that computes approximate shortest paths from a given source point on ∂P to all vertices of P.

Video Presentation

Four-polytopes and a funeral (for my conjecture)*

Nina Amenta† **Tamara Munzner**‡

A four-dimensional polytope given as the intersection of f halfspaces has $O(f^2)$ vertices. The (incorrect) conjecture of the title is that $o(f^2)$ of these vertices can appear on the outer boundary of a projection to the plane (a "shadow").

The video illustrates the construction of a four-polytope with an $\Omega(f^2)$ shadow [AZ]. It uses an idea, due to Klee and Minty [KlMi] and applied by others, for constructing polytopes with long simplex paths.

The projection of the polytope to R^3 is shown, as it rotates in R^4, so that the four-dimensional object is understood as a continuous family of deformations of a three-dimensional object. Structure is shown by drawing only subsets of faces, related by the edge skeleton which is always shown.

The video appears as part of an annual collection of computational geometry videos [Am]. It was made at The Geometry Center on an SGI IRIS using Geomview [LMP] and the StageManager scripting system [Ce] . The polytope models were constructed using the convex hull program chD [Em].

References

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Visibility with Multiple Reflections

Boris Aronov* Alan R. Davis† Tamal K. Dey[3]
Sudebkumar P. Pal[3] D. Chithra Prasad‡

Abstract

We show that the region lit by a point light source inside a simple n-gon after at most k reflections off the boundary has combinatorial complexity $O(n^{2k})$, for any $k \geq 1$. A lower bound of $\Omega((n/k)^{2k})$ is also established which matches the upper bound for any fixed k. A simple near-optimal algorithm for computing the illuminated region is presented, which runs in $O(n^{2k} \log n)$ time and $O(n^{2k})$ space for any $k > 1$.

Affine perimeter and limit shape

Imre Bárány
Mathematical Institute, Budapest

We prove here that, given a convex compact set $K \subset R^2$, almost all convex $\frac{1}{n}Z^2$–lattice polygons contained in K are very close to a fixed convex set $K_0 \subset K$ as n goes to infinity. The distinguishing property of K_0 is that it has the largest affine perimeter among all convex sets contained in K. The methods can be used to determine the asymptotic behaviour of the probability that k random, independent points drawn uniformly from K form the vertices of a convex k–gon. Actually, there is a limit shape to these convex random k–gons as k increases and this limit shape is the same set K_0.

Computing mixed discriminants, mixed volumes and permanents

Alexander Barvinok

We present a probabilistic polynomial time algorithm that computes the permanent of any given non-negative n by n matrix within a factor $2^{O(n)}$. That is, for any given non-negative matrix A the algorithm computes a number α which satisfies the inequalities c^n per $A \leq \alpha \leq$ per A with the probability at least 0.9, where $c > 0$ is an absolute constant (one can choose $c = 0.28$). The algorithm solves a more general problem of approximating the mixed discriminant of given n positive definite operators on $\mathbf{R}^n$. Similarly, the mixed volume of n given ellipsoids in $\mathbf{R}^n$ can be approximated in polynomial time within a factor c^n for $c = 0.66$. The algorithms are based on a recursive application of the projective version of the kinematic formula which allows us to represent the mixed discriminant (mixed volume) in $\mathbf{R}^n$ as the expectation of the mixed discriminant (mixed volume) in a random hyperplane in $\mathbf{R}^{n-1}$. The paper is available at http://www.math.lsa.umich.edu/~barvinok/papers.html

Bounding the topological complexity of semi-algebraic sets

Saugata Basu

In this talk I will describe a recent result on bounding the sum of the Betti numbers of semi-algebraic sets. This extends a well-known bound due to Oleinik and Petrovsky , Thom and Milnor. In separate papers they proved that the sum of the Betti numbers of a semi-algebraic set $S \subset R^k$, defined by $P_1 \geq 0, \ldots, P_s \geq 0, deg(P_i) \leq d, 1 \leq i \leq s$, is bounded by $(O(sd))^k$. Given a closed semi-algebraic set $S \subset R^k$ defined as the intersection of a real variety, $Q = 0, deg(Q) \leq d$, whose real dimension is k', with a set defined by a quantifier-free Boolean formula with no negations with atoms of the form, $P_i = 0, P_i > 0, P_i < 0, deg(P_i) \leq d, 1 \leq i \leq s$, we prove $P_i = 0, P_i \geq 0, P_i \leq 0, deg(P_i) \leq d, 1 \leq i \leq s$, we prove that the sum of the Betti numbers of S is bounded by $s^{k'}(O(d))^k$. In the special case, when S is defined by $Q = 0, P_1 > 0, \ldots, P_s > 0$, we have a slightly tighter bound of $\binom{s}{k'}(O(d))^k$. This result generalises the Oleinik-Petrovsky-Thom-Milnor bound in two directions. Firstly, our bound applies to arbitrary unions of basic semi-algebraic sets, not just for basic semi-algebraic sets. Secondly, the combinatorial part (the part depending on s) in our bound, depends on the dimension of the variety rather than that of the ambient space.

We also extend this bound to the case where the algebraic complexity is bounded only in terms of the number of monomials appearing in the system of polynomials (the degrees being unbounded). We also note that these bounds are not obtainable by the Thom-Milnor technique and that a new idea of separating the combinatorial part of the complexity from the algebraic part plays a crucial role.

ISOPERIMETRIC INEQUALITIES AND THE DODECAHEDRAL CONJECTURE REVISITED

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The *cd*-index of zonotopes and arrangements

Louis J. Billera, Richard Ehrenborg and Margaret Readdy

We investigate a special class of polytopes, the zonotopes, and show that their flag f-vectors satisfy only the affine relations fulfilled by flag f-vectors of *all* polytopes. In addition, we determine the lattice spanned by flag f-vectors of zonotopes. By duality, these results apply as well to the flag f-vectors of central arrangements of hyperplanes.

FIXING AND HINDERING SYSTEMS FOR CONVEX BODIES

Vladimir Boltyanski

Let $M \subset R^n$ be a compact, convex body and $F \subset \operatorname{bd} M$. A vector $v \neq 0$ *remuves the interior of M from F* if $(\lambda v + \operatorname{int} M) \cap F = \emptyset$ for any $\lambda > 0$. The set $F \subset \operatorname{bd} M$ is a *fixing system* [8] for M if there is no nonzero vector v which removes $\operatorname{int} M$ from F. A fixing system F is *primitive* if no proper subsystem $F' \subset F$ is a fixing system for M. Evidently, a fixing system of the least cardinality is primitive. Denote the minimal cardinality of fixing systems for M by $\varrho(M)$. B. Grünbaum established [7] that for any compact, convex body $M \subset R^n$ the estimate $n + 1 \leq \varrho(M) \leq 2n$ holds.

A vector $v \neq 0$ *remuves the body M from F* if $(\lambda v + M) \cap F = \emptyset$ for any $\lambda > 0$. The set $F \subset \operatorname{bd} M$ is a *hindering system* [9] for the body M if there is no nonzero vector v which removes M from F.

Finally, let $v(t)$, $0 \leq t \leq 1$, be a continuous family of vectors in R^n such that $v(0) = 0$. We say that the family $v(t)$ *remuves the body M from F* if for $0 < t \leq 1$ the equality $(v(t) + M) \cap F = \emptyset$ holds. The set $F \subset \operatorname{bd} M$ is a *strict hindering system* [3] for the body M if there is no continuous family $v(t)$ with $v(0) = 0$ which removes M from F.

In the talk, the following new results by H. Martini, E. Morales and the speaker [3, 4, 6] are formulated (the estimates hold for any compact, convex body $M \subset R^n$; the functional $\operatorname{md} M$ is introduced in [2]):

$$2 \leq \sigma(M) \leq \sigma'(M) \leq \operatorname{md} M + 1 \leq n + 1, \qquad ([3], 1996)$$

$$n + 1 \leq n + \frac{n}{\operatorname{md} M} \leq \varrho(M) \leq 2n + 1 - \operatorname{md} M \leq 2n. \qquad ([4, 6], 1995)$$

For *maximal* cardinalities of primitive fixing and hindering systems of planar, compact, convex figures, the complete analysis conducted by L. Fejes Tóth [8] and P. Many [9].

In the case of the bodies $M \subset R^n$ for $n \geq 3$ the situation is in a sense unexpected. Denote by $\varrho_{\max}(M)$, $\sigma_{\max}(M)$, $\sigma'_{\max}(M)$ the maximal cardinalities of primitive fixing, hindering, and strict hindering systems for the body M, respectively. As B. Bollobás [1] showed, for any integer $k \geq 4$ there exists a body $M_k \subset R^3$ with $\varrho_{\max}(M_k) \geq k$. In our joint paper with H. Martini [5], this result is improved. Namely, consider in R^3 the body $P = \operatorname{conv}(B \cup \{a\})$, where B is a ball and $a \notin B$. Then $\varrho_{\max}(P) = \infty$, i.e., for any integer $k \geq 4$ there is a primitive fixing system $F_k \subset \operatorname{bd} P$ that consists of k points. The same holds for hindering and strict hindering systems [3].

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On the difficulty of vertex and facet enumeration

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A *convex polytope* P can be specified in two ways: as the convex hull of the vertex set $\mathcal{V}$ of P, or as the intersection of the set $\mathcal{H}$ of its facet-inducing halfspaces. The *vertex enumeration problem* is to compute $\mathcal{V}$ from $\mathcal{H}$. The *facet enumeration problem* it to compute $\mathcal{H}$ from $\mathcal{V}$. These two problems are essentially equivalent under point/hyperplane duality. Although there are efficient algorithms for non-degenerate input, it is open whether vertex/facet enumeration for arbitrary input can be solved in time polynomial in $|\mathcal{H}| + |\mathcal{V}|$.

Known methods for facet enumeration (and in the dual setting, vertex enumeration) fall into one or more of the the following categories:

1. lattice producing algorithms compute the entire face lattice,
2. triangulation producing algorithms compute a triangulation of the boundary complex (possibly by perturbation), and
3. incremental algorithms compute the final output by inserting the input points one at a time and maintaining a series of intermediate polytopes.

Each of these methods has a weakness: the face lattice, triangulation, and intermediate polytopes can all have size superpolynomial in $|\mathcal{H}| + |\mathcal{V}|$. In this talk we present a number of families of polytopes hard for one or more of these methods and several families that are hard for all three.

This talk summarizes two recent papers: "How good are convex hull algorithms?" by Avis, Bremner, and Seidel, will appear in *Computational Geometry: Theory and Applications*; "Incremental convex hull algorithms are not output sensitive." by Bremner is currently in preparation.

A polynomial time approximation technique for Norm-Maximization

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Let $K \subseteq \mathbb{R}^d$ be a convex body (i.e. a compact, convex and full-dimensional set) and define for $x = (x_1, .., x_d)^T \in \mathbb{R}^d$ and $p \in \mathbb{N}$, $\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}}$. Then the problem of computing the p^{th} power of $\max_{x \in K} \|x\|_p$ is $\mathbb{NP}$-hard even if K is a parallelotope centered at the origin (*Bodlaender, Gritzmann, Klee* and *van Leeuwen*).

In the realm of the algorithmic theory of convex bodies developed by *Grötschel, Lovász* and *Schrijver*, it can be inferred from work by *Bárány* and *Füredi* that for p=2 it is not possible to compute in (oracle-) polynomial time the maximum within an error of $\gamma_1\sqrt{d/\log d}$, γ_1 a constant independent on d, for convex d-dimensional bodies given by separation oracles.

Inspired by work of *Kochol* we use a 'good' spherical code to show that $\max_{x \in K} \|x\|_2$ can indeed be approximated within an error less than $\gamma_2\sqrt{d/\log d}$ (γ_2 a constant independent on d).

A similar technique can also be applied in general l_p-spaces. The result can further be used to give optimal approximative algorithms for certain inner and outer radii of convex bodies, particularly for inradius.

Computational Geometry: The Next 10 Years

Bernard Chazelle

The last 10 years have been the most exciting in the history of computational geometry. Stunning progress has been made in many areas: the asymptotic complexity of basic geometric computations such as convex hulls, Voronoi diagrams, polygon triangulation, motion planning, low-dimensional optimization, has been resolved. Randomization and a new theory of geometric sampling have produced simplifications and improvements for a large array of geometric algorithms. Computational topology and computational algebraic geometry, as well as the theory of line and surface arrangements have reached a very high level of sophistication.

Less successful has been the application of these theoretical advances to the practice of geometric computing. We will argue that the emphasis on foundations and theoretical investigations was the right course to follow, but that the next decade should see a broadening of the computational-geometric base to encompass problems of practical as well as theoretical relevance.

Some Exotic Species of Aperiodic Tilings in Two and Three Dimensions

Ludwig Danzer

First some basic notions are introduced:
The *protoset* $\mathcal{T} := \{T_1, T_2, \ldots, T_k\}$ (T_i are the *prototiles*);
the *species* $S(\mathcal{T}, \text{cond})$, the family of all global tilings P, such that every tile of P is congruent to some prototile, and P satisfies some condition cond (e.g.: inflation, local matching rule). A species S is said to have property
(I) if S can be defined by an *inflation* infl (the inflation factor is denoted by η),
(D) if (I) and $\text{defl} := \text{infl}^{-1}$ is unique,
(PV) if (I) and η^d is a PV-number,
(LMR) if S can be defined by a local matching rule lmr.

The following example are discussed:
1) AMMAN'S chair. (I), (D), (PV), (LMR), $\eta^2 = \tau$.
2) A species S with (I), (D), (PV) ($\eta^2 = 2$) but a singular inflation matrix (not (LMR)).
3) A species S with three prototiles ($3 = \frac{1}{2}\phi(7)$) each a triangle with all angles of type $k\pi/7$ satisfying (I), (D) and (LMR) (but not (PV)) and η of degree three.
4) BÖRÖCZKY's example in the hyperbolic plane.
5) SCD, a biprism that tiles $\mathbb{E}^3$, but $S(\{\text{SCD}\}, -)$ is aperiodic (cf. problem 1 of the problem session).
6) $S(\{P\}, \text{infl})$, where P is a triangular prism and infl is such that in every tiling belonging to S the copies of P occur in infinitely many orientations, which form a dense subset of $O^+(\mathbb{R}, 3)$ (due to J. H. CONWAY and Ch. RADIN).

Intersection Graphs of Jordan Arcs

H. de Fraysseix and P. Ossona de Mendez *

Abstract

An *intersecting family* of Jordan arcs is a family of Jordan arcs, such that two arcs are neither sharing more than one point nor tangent, and such that one crossing point belongs to exactly two arcs. Such a family defines an *intersection graph*, whose vertices are the arcs, and whose edges are the crossing points. Deciding wether a given graph is the intersection graph of some intersecting family of Jordan arcs is known to belong to the NP-complete class. It has been a challenge for long to prove or disprove that all planar graphs have this property.

Up to now, only those planar graphs representable by contacts (e.g. bipartite and outerplanar graphs) are known to have this property. We extend here this result to 4-connected maximal planar graphs which are either 3-colorable or free of C_4 separating cycle.

Leading Strand

Arc intersection problems arise topological difficulties that vanish when arcs are only in contact.

Recall that a *contact family* of Jordan arcs is a family of Jordan arcs, such that two arcs share at most one point, and such that any point is interior to at most one arc. Besides an intersection graph, called *contact graph* in this context, such a family defines a planar bipartite *point-arc incidence graph*, whose vertices are the arcs and the contact points, and whose edges are defined by inclusion of a contact point into an arc.

We fully characterize, by a pure combinatorial property, those bipartite planar graphs which are point-arc incidence graphs.

To a 4-connected maximal planar graph, either 3-colorable or free of C_4 separating cycles, we associate a bipartite planar graph. Then the result follows from the proof that this graph is a point-arc incidence graph : from the associated contact family, local deformations around the contacts give rise to an intersection family which represents the original graph.

Whenever the graph is not 3-colorable, the proof relies heavily on the four color theorem.

Computing combinatorial models of real smooth hypersurfaces

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Hilbert's 16th problem concerns the classification of topological types of smooth real projective hypersurfaces. An important component on the classification efforts is to have effective methods to produce examples of new topological types. During the 1980's Oleg Viro and his students developed a technique, based on convex triangulations of Newton polytopes, to construct examples. In this talk we present partial results about Viro's construction when it is extended to arbitrary triangulations. The guiding question is whether Viro's construction still produces combinatorial models of real hypersurfaces when the triangulations of the Newton polytopes are not convex (also called regular). We have proved some results hold in dimension two, such as Harnacks inequality and some consequences of Bezout's theorem. We have written software that allowed us to compute several millions of examples that indicate Petrowsky and Arnol'd inequalities could be valid as well for non-convex triangulations. This is joint work with Frederick Wicklin.

Computational Topology

Herbert Edelsbrunner

The speaker believes there is or should be a reasearch area appropriately referred to as 'computational topology'. A more accurate but also more cumbersome title would emphasize the intended relation to discrete and geometric methods. Its agenda includes the identification and formalization of topological questions in computer applications and the study of algorithms for topological problems. It is hoped this talk can contribute to the creation of a computational branch of topology with a unifying influence on computing and computer applications.

The talk illustrates the interaction between application problems, geometric models, and topological methods with four case studies:

1. Molecular conformation; Voronoi complexes, alpha shapes; holes, homology, filtrations.

2. Imperfect holes; Delaunay complexes, pockets; vector fields, stable manifolds.

3. Molecular surfaces; Pedoe vector space, skin; topological duality.

4. Deformation; Minkowski sum, convexification principle; preimage theorem, Morse theory.

Recent Progress on Packing and Covering

Gábor Fejes Tóth

In the talk we give a survey about new developments in the theory of packing and covering. We start with an account about recent plans for proving Kepler's conjecture concerning the densest packing of congruent balls in three-dimensional Euclidean space. We continue with the discussion of bounds for the packing and covering densities of a convex body. The most important new result on this topic is Rush's construction of dense lattice packings via codes for a special metric. For certain bodies his construction yields considerable improvement upon the Minkowski-Hlawka bound. We also address the question about the regularity of optimal arrangements. Roger's conjecture that for sufficiently large d the densest packing of equal balls in E^d cannot be lattice like, is still open. On the other hand, A. Bezdek and W. Kuperberg showed that for $d \geq 3$, there is an ellipsoid E in E^d such that the densest packing of congruent copies of E cannot be realized in a lattice arrangement. We mention some further results indicating that, in general, optimal arrangements are not very regular.

Algorithms for prediction of indoor radio propagation

Steven Fortune
Bell Laboratories

The design of indoor wireless communication networks requires prediction of radio propagation inside a building. Radio propagation inside a building is complex, because each building wall can act both as an obstruction, attenuating a propagation path, and as a mirror, providing an additional reflecting path. We describe two algorithms that simulate radio propagation, a ray tracing algorithm that uses a discrete sample of propagation directions and a beam tracing algorithm that enumerates reflection cones defined by building walls. The two algorithms are compared experimentally and analytically. With a triangulation-based spatial data structure, both algorithms are fast enough to provide propagation simulations in a few minutes of computing time, even for large buildings.

Polygons and polyhedra

Branko Grünbaum

The theory of convex polygons, polyhedra and polytopes is a well-developed discipline. However, even the simplest results of this field have not been extended to more general objects. The main purpose of the talk is to present some of the concepts that may form a suitable starting point for an exploration of not-necessarily- convex polyhedra. The exposition centers on polyhedra, since polygons will be discussed in more detail elsewhere, and the information available on higher-dimensional polytopes is too meager.

Polyhedra are considered not as "solids", but as families of polygons (faces), satisfying certain natural conditions. If the faces of a polyhedron are simple Jordan polygons, and if the faces intersect only along common edges or vertices, the polyhedron is called acoptic. Acoptic polyhedra are closely related to maps on orientable 2-manifolds. One of the most challenging problems is the General Realizability Conjecture, which makes precise the claim that any reasonable cell-complex decomposition of any orientable 2-manifold can be realized by an acoptic polyhedron. Various recent discoveries (such as those by Szilassi, Schwörbel, Wills, Ljubić and others) of acoptic polyhedra realizing maps that cannot be realized by polyhedra with convex faces provide support for the conjecture, although there are also maps of a rather simple character that so far have not been realized.

Other challenging problems are: (i) What is the analogue for acoptic polyhedra of Steinitz's theorem for convex polyhedra? In other words, which graphs are graphs of acoptic polyhedra? (ii) The enumeration of the types of acoptic polyhedra with a given number of faces (or vertices). This depends, naturally, on the definition of "type". It is proposed to distinguish between "combinatorial equivalence" and various levels of "isomorphism", the latter taking into account the geometric features at different levels of detail. (iii) What are the appropriate extensions of Cauchy's "rigidity theorem" to acoptic polyhedra? In particular, how can the movable polyhedra be characterized?

Generalizations to polyhedra with selfintersections, possibly with selfintersecting faces, are also of interest. Again, it is convenient to distinguish several levels of generality. Part of the interest in such polyhedra stems from the most widely known examples of this kind, the Kepler-Poinsot regular polyhedra, as well as from other regular polyhedra that can be constructed using non-simple polygons; but even more relevant is the appearance of polyhedra with selfintersections as polars of many acoptic polyhedra. Some of these aspect are discussed in other venues.

Recent Progress on the Kepler Conjecture

Thomas C. Hales

The Kepler conjecture asserts that the density of a packing of spheres of equal radius in three dimensions cannot exceed that of the face-centered cubic packing.

Recently, there has been significant progress toward a proof of this conjecture, which will be described in this talk.

Two earlier approaches to the Kepler conjecture are based respectively on the Voronoi and Delaunay decompositions of space. This work combines these earlier approaches by defining a hybrid of the Voronoi and Delaunay decompositions. This hybrid appears to retain the best attributes of the earlier approaches, and manages to avoid many of their complexities.

This recent work brings methods of global nonlinear optimization to bear on the Kepler conjecture. This talk will describe these methods and how they have been applied to the Kepler conjecture.

Computational Convexity

Victor Klee

There was a discussion of the relationship of Computational Convexity to the rest of Computational Geometry. Here are some of the main points.

(1) Computational Convexity is the common meeting ground of Convex Geometry, Mathematical Programming, and Computer Science. Rather than being motivated by intrinsicaly low-dimensional problems such as those from computer graphics, its most important problems are intrinsically high- dimensional.

(2) As a consequence of (1), it seems appropriate to focus on the case of variable dimension (i.e., the dimension is part of the input) rather than on an arbitrary fixed dimension.

(3) As a consequence of high dimensions, it seems necessary at present to focus on problems concerning bodies that are *convex*. Extensions to non-convex bodies may come later, but at present the convex case provides plenty of important challenges.

(4) Since (to exaggerate slightly) no computer ever saw an irrational number, it seems appropriate to use the binary (Turing machine) model of computational complexity (supplemented when necessary by a carefully specified oracle), rather than the model of infinite-precision real arithmetic.

(5) Because of the difficulty of its problems, the present status of Computational Convexity may be described as "largely qualitative." There has been considerable success in classifying problems as to their polynomial-time solvability or NP-hardness, but very few optimal algorithms are known. The task remains to find optimal exact algorithms for problems that have been found to be solvable in polynomial time, and to find good heuristics or useful approximation algorithms for problems that are NP- hard.

(6) Significant challenges for the future are provided by several important problems that are, by some, already regarded as "well-solved" because for each *fixed* dimension there is an algorithm of low worst- case complexity. However, when the exponent or the multiplicative constant in the complexity estimate increases exponentially with the dimension, such "good" solutions, still leave room for much improvement from the viewpoint of Computational Convexity and also from the viewpoint of actual use.

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After presenting the above motivation, some results and probems from specific areas of computational convexity were presented. These appear (along with much additional material) in the following two survey articles:

P. Gritzmann & V. Klee, On the complexity of some basic problems in computational convexity: I. Containment problems, Discrete Math. 136 (1994) 129–174;

P. Gritzmann & V. Klee, On the complexity of some basic problems in computational convexity: II. Volume and mixed volumes, in Polytopes: Abstract, Convex and Computational, Kluwer, Boston, 1994, 373–466.

For more detailed studies of inner and outer j-radii, and of largest j-simplices in d-polytopes, see the following papers:

P. Gritzmann & V. Klee, Inner and outer j-radii of convex bodies in finite-dimensional normed spaces, Discrete Comput. Geom. 7 (1992) 255–280;

P. Gritzmann & V. Klee, Computational complexity of inner and outer j- radii of convex polytopes, Math. Programming A 59 (1993) 163–213;

P. Gritzmann, V. Klee & D. Larman, Largest j-simplices in n-polytopes, Discrete Comput. Geom. 13 (1995) 477–515;

M. Hudelson, V. Klee & D. Larman, Largest j-simplices in d-cubes: Some relatives of the Hadamard maximum determinant problem, Linear Algebra Appl. 241–243, 1996, 519–598.

Randomization and derandomization in computational geometry

Jiří Matoušek

In the first part, we discuss mainly randomized incremental algorithms in computational geometry. We formulate several axiomatic frameworks for such algorithms, and point out differences among them. We mention related open problems (analyzing randomized incremental constructions for highly degenerate point configurations, etc.).

In the second part, we discuss techniques for replacing randomized algorithms by deterministic ones with a similar asymptotic running time. All successfully derandomized algorithms are essentially of the divide-and-conquer type. If we allow for an n^δ extra factor ($\delta > 0$ an arbitrarily small constant) in the running time, then derandomization is usually easy using known techniques (computation of ε-nets and ε-approximations). Only partial results and techniques of a limited applicability are known for optimal derandomization, and the successful examples are fairly complex.

Combinatorics of Visibility and Illumination: 36 Open Problems

Joseph O'Rourke*

Abstract

A compendium of thirty-six open problems is presented: on point visibility, floodlight illumination, and visibility graphs.

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Extremal Problems in Combinatorial Geometry

János Pach

We survey several recent developments in combinatorial geometry. The results presented are divided into four categories. (A) Quantitative Helly-Erdős-Szekeres-Tverberg-type theorems, (B) Distribution of distances among n points, (C) Bounds on the number of incidences between points and curves or surfaces, (D) Crossing numbers of graphs.

As an example of a problem belonging to (A), we present a detailed proof of the following result. Let $P_1, \ldots, P_{d+1}$ be pairwise disjoint n-element point sets in general position in d-space. Then there exist a point O and suitable subsets $Q_i \subseteq P_i$ $(i = 1, \ldots, d+1)$ such that $|Q_i| \geq c_d|P_i|$ and every d-dimensional simplex with exactly one vertex from each Q_i contains O in its interior. Here c_d is a positive constant depending only on d. This generalizes a theorem of Vrećica and Živaljevič. The proof is based on the Szemerédi Regularity Lemma.

Universality Theorems for Oriented Matroids and Polytopes

Jürgen Richter-Gebert

The realization space of a rank d oriented matroid $\mathcal{M}$ is the set of all point configurations $X \in \mathbf{R}^{\mathbf{d}}$ that have $\mathcal{M}$ as underlying oriented matroid (modulo linear equivalence). The realization space of a d-dimensional polytope P is the set of all polytopes in $\mathbf{R}^{\mathbf{d}}$ that have the same face lattice as P (modulo affine equivalence). The *Universality Theorem for rank 3 oriented matroids* (Mnëv, 1988) states that for every basic primary semialgebraic set V (over $\mathbf{Z}$) there is a rank 3 oriented matroid whose realization space is stably equivalent to V. The *Universality Theorem for 4-dimensional Polytopes* (Richter-Gebert, 1995) is a similar statement in the category of 4-dimensional polytopes.

The proof in the oriented matroid case can be performed by starting with the defining inequality system of V, performing a (non-trivial) algebraic translation into a *Shor normal form*, and then modelling elementary multiplications and additions by the classical von Staudt constructions (this is, in essence, Mnëv's construction). The proof for the case of 4-polytopes proceedes in a similar way. However the role of von Staudt constructions must be replaced by a relatively complicated polytopal construction.

We demonstrate that it is possible to derive a relatively simple proof for a Universality Theorem for 6-dimensional Polytopes by directly making use of Mnëv's construction. The proof proceeds as follows. Starting from a (basic primary) semialgebraic set V we take the corresponding point configuration $X(V)$ that is constructed by Mnëv's proof. The corresponding *zonotope* $Z = Z(X(V))$ is a 3-dimensional polytope. In the category of zonotopes the realization space of Z is stably equivalent to V. The property of Z beeing a (generalized) zonotope can be fixed by applying *connected sums* and *Lawrence extensions.* By this we end up with a 6-dimensional polytope $P = P(Z(X(V)))$ that contains Z as a 3-face. Every realization of P the face Z is projectively equivalent to a (generalized) zonotope. The realization space of P is stably equivalent to V.

Cinderella, a program for geometric drawing and automated theorem proving (demonstration)

Jürgen Richter-Gebert

Cinderella is a program for doing (projective) incidence geometry interactively on a computer. There are two main tasks for which *Cinderella* is designed:

- *Interactive drawing*

  The system provides facilities to construct geometric configurations containing points, lines and conics by performing mouse actions on a drawing surface. The construction sequence is memorized and can be changed by moving the base elements of the construction. Different ports allow to view the same configuration simultaneously under different aspects (primal, polarized, etc.).

- *Automatic proving*

  The system is equipped with an algebraic theorem prover that allows one to generate short (and readable) proofs for many projective incidence theorems. The prover is based on the method of *binomial final polynomials* that was originally developed to prove non-realizability of oriented matroids. Using invariant theoretic methods à la F. Klein the scope of theorem prover that can be extended also to euclidean, elliptic and hyperbolic geometry.

Interactions between real algebraic geometry and discrete and computational geometry

Marie-Françoise Roy
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The realization space of an order type is a basic semi-algebraic (which we abbreviate as "sa") set of R^{2n} (i.e. a set defined by a finite conjunction of sign conditions on polynomials). It looks like a very special sa set, but according to Mnev's Universality theorem, every basic sa set is (up to multiplication by an affine space) homeomorphic to the space of configurations of an order type. A consequence is that the answer to the isotopy conjecture is no, as a basic sa set may have many connected components.

Sa sets have a lot of remarkable stability and finiteness properties: Projections of sa sets are sa ; a sa set has a finite number of connected components, each of which is sa; sa sets can be finitely stratified; a sa set is sa homeomorphic to a simplicial complex; given a sa family, the set of parameters where the topological type is fixed is sa; sa families have finite VC dimension .

Because of these properties, sa sets are useful in modeling various problems. For example the Piano Movers Problem, (robot motions planning in a sa environment (roadmap construction), or the visual aspects of a sa set in space.

Explicit bounds in sa geometry are frequently based on the following result which follows from results by Petrowsky, Oleinik, Thom and Milnor: The number of connected component of a real algebraic set of degree d in k variables is $O(d)^k$. The number of connected components of sign conditions defined by a list of s polynomials in k variables of degree d on a variety of dimension k' can be proved to be $\binom{O(s)}{k'}O(d)^k$ (Basu-Pollack-Roy 1993). This is useful to get bounds on the number of isotopy types of configurations of n points in R^d and to the number of geometric permutations induced by k transversals to a family of $k+1$–separated compact, convex bodies in R^d (Goodman Pollack Wenger 1993).

In recent work, the complexity of algorithms for sa sets can be separated into an algebraic part (dependence on the degree) and a combinatorial part (dependence on the number). Also, the degrees of the polynomials output is independent of the number of polynomials. These features are useful in discrete and computational geometry, where $d = 1$, or d, k are fixed.

Finding a point in every connected components of sign conditions defined by a list of s polynomials in k variables of degree d on a variety of dimension k' can be done in time $\binom{O(s)}{k'}sd^{O(k)}$ (Basu-Pollack-Roy 1996)

Given $\Phi(Y) = (Q_\omega X^{[\omega]}) \ldots (Q_1 X^{[1]})F(P_1, \ldots, P_s)$, it is possible to compute an equivalent quantifier-free formula in $s^{(\ell+1)\Pi(k_i+1)}d^{(\ell+1)\Pi O(k_i)}$ arithmetic operations in the ring generated by the coefficients of the input polynomials. The degrees in the output are bounded by $d^{\Pi_i O(k_i)}$ which is independent of s. (Basu Pollack Roy 1994).

It is possible to compute a road map on a semi-algebraic set of dimension k' defined by s polynomials in k variables with each polynomial of degree at most d with complexity $\binom{O(s)}{k'}sd^{O(k^2)}$ (Basu-Pollack-Roy 1996).

All these results are based on the critical point method (first introduced by Grigor'ev and Vorobov) and various perturbation tricks.

Software efforts (Canny's toolkit, Rouilliers' Real Solving (POSSO FRISCO project)) are made to implement the critical point method. The challenge is to be able to decide whether or not a small (written on half a page) system of equalities and inequalities is consistent.

The central open algorithmic problems are: Compute the real dimension of a real algebraic set or of a semi algebraic set with algebraic complexity $d^{O(k)}$. Compute stratifications of sa sets in single exponential time.

Various quantities associated to sa sets can be used to give lower bounds for many problems in discrete and computational geometry concerning sorting, element distinctness (Ben-Or, Bjorner-Lovasz- Yao, Recio et al), polytope membership (Grigor'ev- Karpinsky-Vorobjov) linear algebra, sign determination (Lickteig, Lickteig-Roy). The idea is that small sa decision or computation trees cannot define sa sets with many connected components, huge sum of Betti numbers, or high geometric degrees.

Finally, the Signed Newton Diagram is a fascinating object connecting discrete geometry to the topology of real algebraic sets. Two examples have been developed: Viro's method for curves, and a conjecture generalizing Descartes's law of signs (Itenberg-Roy).

Periodic and Aperiodic Tilings

Marjorie Senechal

The rapid and extensive development of tiling theory since 1986 can be traced in large measure to the influence of "Tilings and Patterns", by Branko Grünbaum and Geoffrey Shephard. We show, in particular, how subtle aspects of the relations between local configurations in and the global structure of both periodic and aperiodic tilings are elucidated by considerations developed in that work.

Complexity of Arrangements

Micha Sharir
Tel Aviv University and New York University

We survey recent progress on the problem of bounding the combinatorial complexity of various substructures in arrangements of n algebraic surface or surface patches of constant maximum degree in $d \geq 3$ dimensions. (Informally, such an arrangement is the subdivision of space into cells of various dimensions induced by the given surfaces.) Arrangements of this kind are a central construct in computational geometry, and arise in many applications, such as motion planning in robotics, ray shooting and visibility in three dimensions, Voronoi diagrams, transversals, geometric optimization, and more. In these applications, we are usually interested not in the full arrangement (whose complexity is in general $\Theta(n^d)$), but in various portions, such as the lower or upper envelope of the surfaces, a single cell of the arrangement, a 'zone' of another surface, etc.

To show that such portions have smaller combinatorial complexity, has been a major open problem for nearly a decade, ever since analogous results have been obtained for 2-dimensional arrangements, using the theory of Davenport-Schinzel sequences (which we also briefly review). Considerable progress has been made in the past 3-4 years, where almost tight bounds (close to $O(n^{d-1})$) for the complexity of such arrangement portions have been obtained, including many related results. In the talk we survey these recent developments, demonstrate some of the analysis techniques on some simple examples, and present some combinatorial and algorithmic applications of the new bounds.

New Results on Packings in Grassmannian Space and Connections with Quantum-error-correcting Codes

N. J. A. Sloane

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This work began when R. H. Hardin, J. H. Conway and I set out to investigate the question: How should N n-dimensional subspaces of m-dimensional Euclidean space be arranged so that they are as far apart as possible? The results of extensive computations for modest values of N, n, m are described, as well as a reformulation of the problem that was suggested by these computations. The reformulation gives a way to describe n-dimensional subspaces of m-space as points on a sphere in dimension $(m-1)(m+2)/2$, which provides a (usually) lower-dimensional representation than the Plücker embedding, and leads to a proof that many of the new packings are optimal. The results have applications to the graphical display of multi-dimensional data via Asimov's grand tour method.

In the past few weeks some astonishing connections have been discovered between this problem and the problem of constructing quantum-error-codes, which has led to further discoveries in both fields. The latter is joint work with A. R. Calderbank, E. M. Rains and P. W. Shor.

Geometric Discrepancy Theory

Joel Spencer

Abstract

The discrepancy of a family $\mathcal{A}$ is the least K for which there *exists* a two coloring $\chi : \Omega \to \{-1, +1\}$ so that $|\chi(A)| \leq K$ for each $A \in \mathcal{A}$. Here Ω is the set of underlying vertices and $\chi(A) = \sum_{a \in A} \chi(a)$. We compare and contrast three settings: general set systems, where one gets upper bounds on the discrepancy in terms of $|\Omega|$, the $|A|$, $\deg(x)$, etc., number theory, with $\Omega = \{1, \ldots, n\}$ and $\mathcal{A}$ the family of arithmetc progressions, and our core topic geometry, with $\Omega \subset R^d$ and $\mathcal{A}$ the half spaces, balls, or whatever.

The basic Erdős Existence argument is described and it is noted that the bounds are far from best possible in the number theoretic and geometric realms. The Floating Colors method of Beck and Fiala is given and applied to axis parallel boxes in the geometric setting to give a polylogarithmic bound. A classic 1964 paper of K.F. Roth uses Fourier analysis to give a lower bound $\Omega(n^{1/4})$ in the number theoretic setting and this is contrasted with this author's version of Janos Pach's version of Bernard Chazelle's version of Alexander's 1990 lower bound of $\Omega(n^{1/4})$ for half spaces in the plane. In both cases an appropriate variance of a random set is shown to be high.

An Entropy argument inaugurated by this author and both expanded and simplified by Ravi Boppana and Jiri Matousek is given. It is shown that under certain conditions on the $|A|$ and $|\Omega|$ there exists a partial coloration $(\chi : \Omega \to \{-1, 0, +1\}$ with the $|\chi(A)|$ small and a positive proportion of the $\chi(v) \neq 0$. One can then sometimes iterate this procedure to give a good full coloring. In the initial 1985 application this author showed that any n sets on n vertices have discrepancy $O(n^{1/2})$. With Matousek this author showed in 1994 that the number theory case has discrepancy $O(n^{1/4})$ and Matousek showed that half spaces in the plane have discrepancy $O(n^{1/4})$.

Proof of Reay's conjecture on certain 1-dimensional intersections

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In 1979 John Reay conjectured (see Israel J. Math. 34 (238-244) or conjecture 9.12 in Eckhoff's chapter 2.1 of Handbook of Convex Geometry):

Let $2d(r-1)+2$ points be given in d-space. Then they can be grouped into r parts so that the intersection of the corresponding convex hulls is at least 1-dimensional.

This is proved, and the problem of finding the cases in which one cannot do with one point less, is discussed.

Geometric Transversal Theory

Dr. Rephael Wenger
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The Ohio State University

A k-transversal to a family of convex sets in R^d is an affine subspace of dimension k (such as a point, line, plane or hyperplane) which intersects every member of the family. This talk covers some of the major results in the past ten years on k-transversals and poses open questions related to these major results. In particular, I discuss necessary and sufficient conditions for the existence of transversals, Tverberg's Helly-type theorem for line tranversals of translates in the plane, piercing or Gallai numbers, the combinatorial complexity and the topological structure of the space of transversals, the order in which transversals intersect a family of sets, and a theory of convexity on the affine Grassmannian developed by Goodman and Pollack.

Sphere Packing and Crystal Growth

Let B^d denote the unit ball in euclidean d–space E^d, $d \geq 2$. For a convex body $K \subset E^d$ let $V(K)$ denote its volume. For a finite set $C_n = \{c_1, \ldots, c_n\} \subset E^d$ with $||c_i - c_j|| \geq 2$, $i \neq j$ we call $C_n + B^d$ a finite sphere packing or briefly a packing. If $L \subset E^d$ is a packing lattice for B^d and $C_n \subset L$, then $C_n + B^d$ is a lattice packing. For $\rho > 0$ the parametric density of $C_n + B^d$ is defined by

$$\delta_\rho(B^d, C_n) = nV(B^d)/V(\operatorname{conv} C_n + \rho B^d).$$

Further

$$\delta_\rho(B^d, n) = \max\{\delta_\rho(B^d, C_n) | C_n + B^d \,\text{packing}\}.$$

The optimal C_n with $\delta_\rho(B^d, C_n) = \delta_\rho(B^d, n)$ are denoted by $C_{n,\rho}$.
If $\operatorname{conv} C_n$ is a segment of length $2(n-1)$, we write $C_n = S_n$ and call $S_n + B^d$ a sausage. For each $d \geq 2$ and each $n \geq 3$ there is a critical radius $\rho_{d,n}$ with $C_{n,\rho} = S_n$ for $0 < \rho \leq \rho_{d,n}$ and $C_{n,\rho} \neq S_n$ for $\rho > \rho_{d,n}$.
The parametric density δ, the critical radii $\rho_{d,n}$, the optimal shapes $C_{n,\rho}$ and their analogues δ^L_ρ, $\rho^L_{d,n}$, $C^L_{n,\rho}$ for restriction to lattice packings are the main objects of finite sphere packings. One can show their asymptotic close relation to classical sphere packing, lattice and nonlattice. One can further show how the classical theory fits together with the typical phenomena of parametric density as sausages and sausage catastrophes.

Here we show that for large n and suitable ρ the normalized $C^L_{n,\rho}$, i.e. $n^{-1/d} \operatorname{conv} C^L_{n,\rho}$ tend to Wulff–shapes, i.e. to shapes of real crystals. If one replaces $V(\operatorname{conv} C_n + \rho B^d)$ by $V(\operatorname{conv} C_n + \rho C)$ with a suitable convex body C, one can control unisotropies, which occur in crystals by bonds. This gives a much richer variety of shapes. Even extreme shapes of crystals (e.g. whiskers) can be realized via parametric density.

J.M. Wills (Siegen)

Günter M. Ziegler
(TU Berlin)

Recent Progress on Polytopes

We outline five very active and important areas of research concerned with the (combinatorial) theory of (convex) polytopes, report about recent progress, and present five "challenge" problems that we hope to see solved soon.

Universality Theorems for polytopes of constant dimension: see Richter-Gebert's recent work! *Challenge:* Provide a Universality Theorem for simplicial polytopes of constant dimension.

Triangulations and subdivisions of polytopes. *Challenge:* Decide whether all triangulations on a fixed point set in general position can be connected by bistellar flips.

0/1-polytopes and their combinatorial structure. *Challenge:* Are the numbers of facets of 0/1-polytopes bounded by an exponential function in the dimension?

Neighborly polytopes. Explicit constructions and extremal properties.

Paths. The existence and construction of short monotone paths to the "top vertex" of a polytope is a crucial problem in the analysis of the simplex algorithm for linear programming. We report about joint work with Nina Amenta on "deformed products" of polytopes, a new construction that all the known examples of linear programs with exponential lower bounds. *Challenge:* Decide whether there is a polynomial upper bound for the expected running time of the RANDOM-EDGE simplex algorithm. *Challenge:* The "Monotone Upper Bound Problem": What is the maximal number of vertices of a monotone path on a d-dimensional polytope with n facets?

Discrete and Computational Geometry: Ten Years Later, Mount Holyoke College.
Open Problem Session, July 15, 1996
(Collection of open problems in order of presentation, edited by E. Welzl)

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Does there exist a "strong Einstein"? More precisely: Is there a tile T in $\mathbb{E}^d$ or $\mathbb{H}^d$ with the following properties:

- *int* (T) connected,
- $T = cl\ (int\ (T))$,
- T is compact,
- T does tile the space,
- if P is a T-tiling, then the symmetry group $\mathrm{Sym}(P)$ is finite;

or – somewhat weaker –

- $\mathrm{Sym}(P)$ is finite for every facet-to-facet T-tiling P (∂T dissected into "facets" forming a cell-complex)?

Rephael Wenger `wenger.4@osu.edu`

Let P be a set of n points on a horizontal line l in $\mathbb{R}^3$ labelled $p_1, p_2, \ldots p_n$ although not necessarily in order. We wish to connect the points in the order $p_1, p_2, p_3, \ldots p_n$ by a simple curve (not self-interseetings) which has minimum number of intersections with the line l. In the worst case, $\Theta(n^2)$ intersections are necessary and sufficient.

Give a polynomial time algorithm to find the optimal solution or prove the problem is NP-complete or give an approximation algorithm.

Victor Klee `klee@math.washington.edu`

Determine the smallest number r that has the following property: Whenever a 3-dimensional convex body can be passed through a circle of radius 1 (with "turning" permitted), it can also be passed through a long circular cyclinder of radius r. (It is known that $r > 1$.)

Jack Snoeyink snoeyink@cs.ubc.ca

Let S be a set of n lines in the plane and $T = \bigcup S$ be their union. Let p and q be points in T. Given S, can one determine in $o(n^2)$ time, the shortest (Euclidean) path in T from p to q?

Using a real RAM model of computation that can compute distances, $O(n^2)$ time can be achieved by constructing the arrangement and using standard techniques. The problem is to find and exploit some metric structure in an arrangement of lines to obtain a subquadratic algorithm. Simple arguments lead to an $O(n \log n)$ algorithm for a path that is within factor of 2 of the shortest. (See [Bose et al., Canadian Conf. Comp. Geom. 1996])

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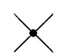
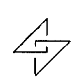
The fact that the graph $K_{3,3}$ is not planar can be abbreviated as follows

$$(K_{3,3} \to \mathbb{R}^2) \implies (2 \mapsto 0\text{-}dim) \qquad or$$

$$(K_{3,3} \to \mathbb{R}^2) \implies$$ 

where $2 \mapsto 0\text{-}dim$ means that the images of some two independent edges of $K_{3,3}$ intersect, i.e. have a common 0-dimensional transversal.

It is also known that

$$(K_6 \hookrightarrow \mathbb{R}^3) \implies$$ 

which means that for any embedding $K_6 \hookrightarrow \mathbb{R}^3$, there exist two linked triangles in the image, [1].

PROBLEM: Find the characterization in terms of forbidden graphs (minors) of all graphs G with the property

$$(G \hookrightarrow \mathbb{R}^3) \implies (4 \mapsto 1\text{-}dim)$$

where $4 \mapsto 1\text{-}dim$ means that some four independent edges of G have a common line transversal.

[1] P. Seymour, Progress on the four-color theorem, *Proc. Int. Cong. Math. Zürich, Switzerland, Birkhäuser,* 1995.

[2] R. Živaljević, The Tverberg-Vrećica problem and the Combinatorial Geometry on vector bundles, *preprint.*

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How many non-simplex facets can an n-vertex 4-polytope have?

Examples with $\Omega(n)$ non-simplex facets are easy to construct, and an upper bound of $O(n^2)$ follows immediately from McMullen's upper bound theorem. As far as I know, these are the tightest known bounds, even for polyhedral 3-spheres.

The best construction I know of is the connected sum of $n/5 - 1$ copies of a bipyramid over a cube, which has n vertices and $11n/5 - 10$ facets, each a square pyramid. Is $3n$ achievable?

There is an asymptotically tight bound of $\Theta(n^2)$ in five dimensions. The lower bound is achieved by the convex hull of a set of integer points on the "weird moment curve" (t, t^2, t^3, t^4, t^6). For example, when n is even, the set $\{-n, 2-n, \ldots, -4, -2, 1, 2, \ldots, n/2\}$ induces a polytope with n vertices and $n^2/16 + \Theta(n)$ bipyramidal facets.

More generally, in d dimensions, we have an upper bound of $O(n^{\lfloor d/2 \rfloor})$, and a generalization of the weird moment curve construction gives us a lower bound of $\Omega(n^{\lceil n/2 \rceil - 1})$. Which bound is correct when d is even?

I am particularly interested in the case of *quasisimplicial* polytopes, each of whose $(d-2)$-faces is a simplex. The weird moment curve examples are quasisimplicial. See [J. Erickson. New lower bounds for convex hull problems in odd dimensions. In *Proc. 12th Ann. ACM Symp. Comptuational Geometry*, pp. 1–9, 1996].

Raimund Seidel seidel@cs.uni-sb.de

Is there an infinite family of combinatorial types of 4-polytopes P with the property that the sum of the number of vertices and facets of P is asymptotically smaller than the sum of the faces of other dimensions, i.e.

$$f_0(P) + f_3(P) = o(f_1(P) + f_2(P))\,?$$

Such classes of "fat-lattice" polytopes are known to exist for dimension $d \geq 6$. For dimension $d = 3$ such classes cannot exist because of Euler's relation.

Peter Mani-Levitska (with Nicolai Mnëv) math@math-stat.unibe.ch

Are there two triangulations of the 7-simplex without a common stellar subdivision?

Jacob E. Goodman jegcc@cunyvm.cuny.edu

Given a family $\mathcal{L}$ of directed lines through the origin in $\mathbb{R}^d$, no two orthogonal, its acuteness graph $\mathcal{A}(\mathcal{L})$ (as defined in joint work with Boris Aronov and Richard Pollack) is the graph whose vertex set is $\mathcal{L}$ in which two vertices are joined by an edge if and only if the corresponding directed lines make an acute angle. It is not difficult to see that every graph G can be realized as the acuteness graph of some family of directed lines, and the lowest dimension d such that the lines can be chosen to lie in $\mathbb{R}^d$ is called the *acuteness number* $\alpha(G)$ of G.

If C_n is the n-cycle (the graph with n vertices joined in cyclic order), it is not hard to see that $\alpha(C_3) = 1$, $\alpha(C_5) = \alpha(C_6) = \alpha(C_7) = 2$, $\alpha(C_4) = \alpha(C_8) = 3$, and $\alpha(C_{10}) = 4$, and that in general $\alpha(C_{2d+2}) = d$. But the value of $\alpha(C_n)$ is not clear for odd $n \geq 9$ (it is either $(n-1)/2$ or $(n-3)/2$, so that $\alpha(C_n)$ is monotone for $n \geq 5$), and the problem we propose is to determine (at least) $\alpha(C_9)$.

Comment: After the problem was presented, Peter Shor and Egon Schulte independently worked out the value of $\alpha(C_n)$ for n even.

Marshall Bern (Problem from Steve Vavasis) bern@parc.xerox.com

Let $P = s_1 s_2 \ldots s_n$ be a simple polygon in the Euclidean plane. Assume you are given the following information about P: (1) for each i, the measure of the angle at s_i, (2) the "names" of all the diagonals in a triangulation of P, that is, $d_1 = s_{i_1} s_{j_1}$, $d_2 = s_{i_2} s_{j_2}$, ..., $d_{n-3} = s_{i_{n-3}} s_{j_{n-3}}$, and (3) for each d_i, the cross-ratio of the quadrilateral formed by the two triangles bounded by d_i. The cross-ratio of a quadrilateral is the product of the lengths of one pair of opposite sides divided by the product of the lengths of the other pair. (For specificity, we may assume that the numerator is always the product of the two sides clockwise from d_i.)

PROBLEM: Does this information uniquely determine P, up to similarity?

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Given finite $S \subset \mathbb{R}^2$, a point $p \in \mathbb{R}^2$ is an S-maximum if $p \in S$ and there is no $q \in S$, $q \neq p$, such that $p_x \leq q_x$ and $p_y \leq q_y$. Given $q \in \mathbb{R}^2$, let $S_q = \{ p \in S \mid p_x \leq q_x \text{ and } p_y \leq q_y \}$. Let $B(S,q)$ denote the number of S_q-maxima. Finally, let $B(S) = \max_q B(S,q)$.

PROVE OR DISPROVE: For any $S \subset \mathbb{R}^2$, $|S| = n$, there exists $T \subset \mathbb{R}^2$, $|T| = O(n)$ such that $B(S \cup T) = O(\log n)$.

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PROVE OR DISPROVE the following generalization of the Sylvester-Gallai Theorem: Given n points in the plane not all of which are colinear, and given any balanced non-Radon partition (P, Q) of the point set, there exists an ordinary line containing exactly one point from each of P and Q?

Note. A partition (P, Q) is called non-Radon if there is a line strongly separating P and Q, and is balanced if their sizes differ at most by one. It is not difficult to see that neither "balanced" nor "non-Radon" can be eliminated for generalization.

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PROBLEM: Are there positive constants c_1, c_2, such that every set P of n points in the plane allows a matching of size $c_1\sqrt{n}$ with no line crossing more than c_2 edges of the matching? (A line *crosses* an edge $\{p, q\}$, if the points p and q lie on opposite sides of the line.)

WHAT IS KNOWN? It is always possible to choose a matching of size $\sqrt{n}$ with no line crossing more than $O(\log n / \log\log n)$ edges. And for any fixed positive $\varepsilon \leq 1/2$, there is a matching of size $n^{(1/2)-\varepsilon}$ with no line crossing more than $O(1)$ edges. (See [E. Welzl, On spanning trees with low crossing numbers, *Lecture Notes in Computer Science* **594** (1991) 233-249].) An old result by Erdős and Szekeres on subsets in convex position [*Compositio Mathematica* **2** (1935) 463-470], gives a matching of size $\Theta(\log n)$ with every line crossing at most two edges. Is $\Theta(\log n)$ optimal for "2"?

DISCRETE AND COMPUTATIONAL GEOMETRY: TEN YEARS LATER
Mt. Holyoke College, July 14–July 18, 1996

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