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Note

Proof of Grünbaum's Conjecture on the Stretchability of Certain Arrangements of Pseudolines

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We prove Grünbaum's conjecture that every arrangement of eight pseudolines in the projective plane is stretchable, i.e., determines a cell complex isomorphic to one determined by an arrangement of lines. The proof uses our previous results on ordered duality in the projective plane and on periodic sequences of permutations of $[1, n]$ associated to arrangements of n lines in the euclidean plane.

Any finite set of lines in the real projective plane determines a cell mplex; these complexes and their combinatorial properties have been a bject of study at least since 1826 [9]. More recently, Levi [6] considered a pological generalization of this notion, defined as follows: Consider a mple closed curve in RP^2 which does not separate RP^2; this is called a *eudoline*. (It is clear that any two pseudolines must meet, and it is easy to e that if they meet at precisely one point, they must cross there.) If a finite t of pseudolines has the property that any two meet at precisely one point, d that not all pass through a common point, we speak of an *arrangement* pseudolines; the arrangement is *simple* if distinct pairs meet at distinct ints. An arrangement of pseudolines also determines a cell complex, and vi and others showed that a number of properties of linear complexes rry over to pseudolinear ones [6, 4]. Two arrangements are called *morphic* if there is an isomorphism of their associated cell complexes; it

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then follows easily from the Schoenflies theorem [7] that some homeomorphism of RP^2 to itself maps the pseudolines of one arrangement onto those of the other. Finally, an arrangement is called *stretchable* if it is isomorphic to an arrangement of lines.

If all but one of the pseudolines of an arrangement pass through a common point (a "near pencil"), it is clear that the arrangement is stretchable. For more general arrangements, however, this is false: Levi [6] showed, in fact, that there is an arrangement of nine pseudolines in RP^2 which is not stretchable, and in 1956 Ringel [8] gave an example of a *simple* nonstretchable arrangement, also involving nine pseudolines. In 197– Canham [1] and Halsey [5] independently gave a computer enumeration of all arrangements of seven pseudolines and found that they were all stretchable. (Grünbaum, in a verbal communication, has reported that when this enumeration was subsequently extended to a computer enumeration of all simple arrangements of eight pseudolines, these were also found to be stretchable.) On the basis of this evidence, Grünbaum conjectured [4, Conjecture 3.1] that every arrangement of eight pseudolines, simple or not, is stretchable.

Our result, stated in Theorem 1 below, is that Grünbaum's conjecture is correct. It therefore closes the one gap that still remained after Canham and Halsey's enumeration, and in fact provides a proof of their results as well, which does not resort to any computer enumertion.

The methods used in the proof of Theorem 1 are both geometric and combinatorial in nature, and rely on the ideas and results of [2, 3]. Recall that to an arrangement $\mathscr{A}$ of n lines in the euclidean plane, of which none is vertical and no two parallel, we associate a sequence of permutations of $\{1,\dots,n\}$ as follows: An ordered vertical line in general position meets $\mathscr{A}$ in n points, determining a permutation of $[1,n]$. As the line sweeps across the arrangement (from left to right, say), this gives rise to a sequence of permutations in which the move from each to the next consists of reversing one or more nonoverlapping substrings. For example, the arrangement shown in Fig. 1 gives rise to the sequence

$$\cdots 12345 \xrightarrow{23} 13245 \xrightarrow{245} 13542 \xrightarrow{135} 53142 \xrightarrow{14} 53412 \xrightarrow{34,\,12}$$
$$54321 \xrightarrow{32} 54231 \xrightarrow{542} 24531 \xrightarrow{531} 24135 \xrightarrow{41} 21435 \xrightarrow{21,\,43} 12345 \cdots \qquad (1)$$

in which we have indicated the various moves, and also extended the sequence to a periodic sequence by repeating the moves in the same order, with each move reversed, to get back to the original permutation, and then continuing as before. Such a periodic sequence is called the "circular sequence of permutations associated to the arrangement $\mathscr{A}$," and a

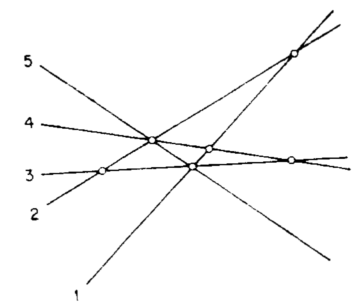

FIGURE 1

periodic sequence of permutations of $[1,n]$ with the properties that (1) each move consists of the reversal of one or more nonoverlapping subrings, and (2) in each half-period, every pair of indices gets reversed exactly once, is called an "allowable sequence of permutations," or simply an "n-sequence" (see [3] for further details). If the n-sequence arises from an arrangement of n lines, we say it is *realizable by lines*; one of the main results coming out of [2, 3]—see [3, Corollary 1.8]—was what we shall call the

REALIZABILITY THEOREM FOR 5-SEQUENCES. *Every 5-sequence is realizable by lines except for the sequence*

$$\cdots 12345 \frac{12}{a} 21345 \frac{34}{a'} 21435 \frac{35}{b} 21453 \frac{14}{b'} 24153 \frac{24}{c} 42153 \frac{15}{c'}$$
$$42513 \frac{13}{d} 42531 \frac{25}{d'} 45231 \frac{45}{e} 54231 \frac{23}{e'} 54321 \cdots$$

and any sequence obtained from it by (1) *renumbering,* (2) *reversing it,* (3) *having any proper subset of the pairs of switches* $\{\{a,a'\}, \{b,b'\}, \{c,c'\}, \{d,d'\}, \{e,e'\}\}$ *occur simultaneously, or some combination of* (1), (2), *and* (3).

Another tool we shall need in the proof of Theorem 1 is the following result of Levi (see [4, Theorem 3.4] for a proof):

LEVI ENLARGEMENT LEMMA. *Given an arrangement* $\mathscr{L}_1,\dots,\mathscr{L}_n$ *of pseudolines and points* P, Q *not both in any* $\mathscr{L}_i$, *there is an arrangement* $\mathscr{L}_0,\dots,\mathscr{L}_n$ *such that* $P, Q \in \mathscr{L}_0$.

It is clear that if $R_1,\dots,R_k$ are any points distinct from P and Q, $\mathscr{L}_0$ can be chosen to avoid $R_1,\dots,R_k$ as well.

THEOREM 1. *Any arrangement of eight pseudolines is stretchable.*

Proof. We shall show first that any nonsimple arrangement of eight pseudolines is stretchable, and then reduce the simple case to the nonsimple case. Thus, let $\mathscr{A} = (\mathscr{L}_1, ..., \mathscr{L}_8)$ be a nonsimple arrangement of eight pseudolines and let $\mathscr{L}_{k+1}, ..., \mathscr{L}_8$ $(1 \leqslant k \leqslant 5)$ pass through P, while $\mathscr{L}_1, ..., \mathscr{L}_k$ do not. Let the pseudolines $\mathscr{L}_1, ..., \mathscr{L}_k$ intersect in points $P_1, ..., P_N$ $(N \leqslant \binom{k}{2})$ and let $P_1, ..., P_n$ be all of these which do not lie on $\mathscr{L}_{k+1}, ..., \mathscr{L}_8$. It follows from the Levi enlargement lemma that we can enlarge the arrangement $\mathscr{A}$ to a new arrangement $\mathscr{A}^*$ by adjoining pseudolines $\hat{\mathscr{L}}_1, ..., \hat{\mathscr{L}}_{n+1}$, in such a way that each $\hat{\mathscr{L}}_k$ passes through P and P_k, and through no other points of intersection of pseudolines of $\mathscr{A}$. (Here, P_{n+1} is a point lying on no pseudoline of $\mathscr{A}$.) Using the Schoenflies theorem we can now find a self-homeomorphism of RP^2 which maps the pseudolines of $\mathscr{A}^*$ passing through P (namely, $\mathscr{L}_{k+1}, ..., \mathscr{L}_8, \hat{\mathscr{L}}_1, ..., \hat{\mathscr{L}}_{n+1}$) onto straight lines $L_{k+1}, ..., L_8, \hat{L}_1, ..., \hat{L}_{n+1}$; let the images of $\mathscr{L}_1, ..., \mathscr{L}_k$ be the pseudolines $L_1, ..., L_k$. Clearly the arrangements $A = \{L_1, ..., L_8\}$ and $A^* = \{L_1, ..., L_8; \hat{L}_1, ..., \hat{L}_{n+1}\}$ are isomorphic to $\mathscr{A}$ and $\mathscr{A}^*$, respectively. If we now take the model of the projective plane in which $\hat{L}_{n+1}$ is the line at infinity and P the "vertical point at infinity," the arrangement A^* can be seen rather simply in the euclidean plane: the lines $L_{k+1}, ..., L_8$, $\hat{L}_1, ..., \hat{L}_n$ are vertical, and all points of intersection of pseudolines $L_1, ..., L_k$ lie on them. If we replace each arc joining two points of intersection of $L_1, ..., L_k$ by a line segment (we can do this, proceeding from left to right (say), by sliding the intersection points up or down as necessary), the cell complex does not change. Therefore we may assume, without loss of generality, that the pseudolines of the subarrangement $B = \{L_1, ..., L_k\}$ are piecewise linear, with corners only at their points of intersection. Hence we may associate to B a sequence S of permutations in exactly the same way that a sequence is associated with an arrangement of lines in [3], namely, by having a directed vertical line L sweep across B from left to right and noting the order in which the pseudolines $L_1, ..., L_k$ cross L as a permutation of $[1, k]$. (The arrangement in Fig. 2, for example, determines the sequence

$$12345 \xrightarrow{123,45} 32154 \xrightarrow{15} 32514 \xrightarrow{14} 32541 \xrightarrow{25} 35241 \xrightarrow{35,24} 53421 \xrightarrow{34} 54321.$$

which we can then extend to a full 5-sequence as above.)

If the sequence S can be realized by lines, i.e., is the sequence determined by an arrangement of lines $\bar{L}_1, ..., \bar{L}_k$, take k lines that realize S and insert 8-k additional vertical lines, $\bar{L}_{k+1}, ..., \bar{L}_8$, in the locations prescribed by the arrangement A. Since the cyclic order in which each L_j meets the other pseudolines L_i is now the same as the cyclic order in which $\bar{L}_j$ meets the

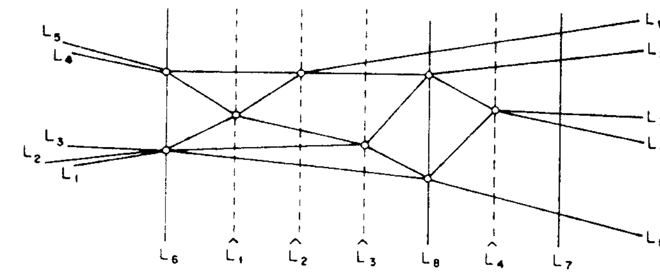

FIGURE 2

corresponding lines $\bar{L}_i$, the resulting arrangement $\bar{A}$ of lines $\bar{L}_1, ..., \bar{L}_8$ is isomorphic to A, and we are done.

We are left with the situation in which S is not realizable by lines. By the realizability theorem, then, it follows—after renumbering and reversing if necessary—that the arrangement B must be as in Fig. 3, or else that the vertical lines L_6, L_7, L_8 force several pairs of switches to take place simultaneously, as L_6 and L_8 did in Fig. 2.

If we could interchange the switches in any one of the pairs $\{a, a'\}, ..., \{e, e'\}$, we would obtain a sequence which is realizable. But clearly one of these five pairs, say $\{c, c'\}$, does not have any of the three vertical lines L_6, L_7, L_8 passing either *between* the locations of the switches c and c', or *through both*. Hence, interchanging the order in which these switches occur by modifying the pseudolines of B (see Fig. 4), we obtain a realizable sequence S' *without affecting the isomorphism class of* A, and the argument concludes as before. Thus we see that every nonsimple 8-arrangement is stretchable.

Finally, if $\mathscr{A}$ is a simple arrangement, then an easy induction argument shows that one of the 2-cells of the arrangement must be a triangle. (There are at least n triangles, in fact, in an arrangement of n pseudolines, by a theorem of Levi [4, Theorem 3.4].) Deform the edges slightly so that the triangle is reduced to a point P, giving a new arrangement $\mathscr{A}'$ which is not simple. By the above, $\mathscr{A}'$ is isomorphic to an arrangement of lines A in

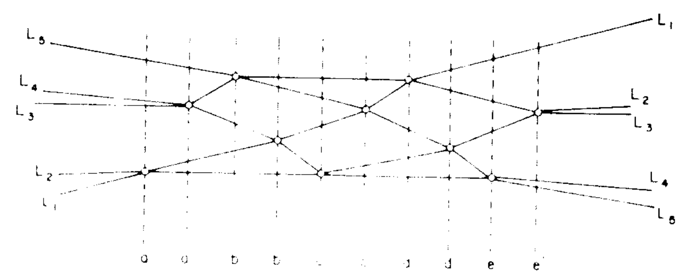

FIGURE 3

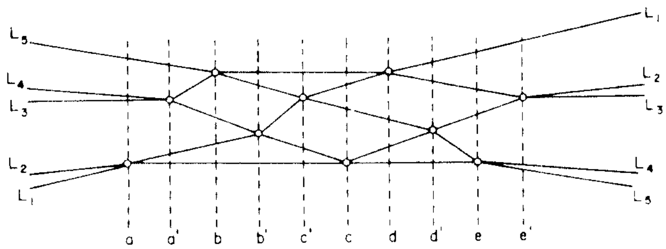

FIGURE 4

which the point P corresponds to the vertical point at infinity; and since A is simple except for the point P, we can tilt one of the three vertical lines slightly to reconstruct the missing triangle. Hence every arrangment of eight pseudolines is stretchable.

Remark. The preceding argument shows that any arrangement of nine pseudolines which has a point common to four of them is also stretchable. In fact, if the arrangement contains a "complete quadrilateral" (a subarrangement of four lines whose vertices can be brought together without crossing any other lines of the arrangement, as was true for the triangle in the last paragraph), the arrangement is stretchable.

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