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Note

Three Points Do Not Determine a (Pseudo-) Plane

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An example is given of an arrangement of eight pseudoplanes, i.e., topological planes, in $\mathbf{P}^3$, and three points which do not lie in any pseudoplane compatible with the arrangement; this provides a counterexample to the "Levi enlargement lemma" in dimension >2.

An arrangement of pseudolines in the projective plane is a finite set of simple closed curves with the property that any two meet at just one point, where they cross. As such, they constitute a natural generalization of arrangements of straight lines, and various authors have investigated the question of which geometric properties of line arrangements carry over to pseudoline arrangements; see, e.g., [4, 7], and above all [6] for an excellent survey of the subject up to 1972.

An indispensable tool in working with arrangements of peudolines is the so-called "Levi enlargement lemma," which says that given any such arrangement $\mathscr{A}$, and any two points P and Q, there is a pseudoline L through P and Q such that $\mathscr{A} \cup \{L\}$ is still a legitimate arrangement (see [6] for a proof). This takes the place of the statement that two points determine a line; of course for pseudolines "determine" means only "determine at least one," not "determine uniquely."

An arrangement of pseudohyperplanes in $\mathbf{P}^n$ may, analogously, be defined

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as a finite set $\mathscr{A}$ of hypersurfaces, each homeomorphic to $\mathbf{P}^{n-1}$, of which any $k \leqslant n$ meet as do k hyperplanes; i.e., if $H_1,..., H_k \in \mathscr{A}$, there are hyperplanes $\bar{H}_1,..., \bar{H}_k$ in P^n such that the cell complex determined by $\{H_1,..., H_k\}$ is isomorphic to that determined by $\{\bar{H}_1,..., \bar{H}_k\}$. (If the *entire* arrangement is isomorphic, in this sense, to an arrangement of hyperplanes, it is called "stretchable"; it is known, for example, that there exist non-stretchable arrangements of k pseudolines in $\mathbf{P}^2$ for $k \geqslant 9$ [8], while every arrangement of $\leqslant 8$ pseudolines is stretchable [3].) It is natural to ask whether the Levi enlargement lemma holds for arrangements in dimension >2, and in fact this question is posed in [5], where the authors point out that a positive answer would be a key step in extending Helly's theorem for pseudoline arrangements from dimension 2 to higher dimensions. Surprisingly enough, the answer turns out to be that it does not hold in dimension >2. The purpose of this note is to exhibit an example of an arrangement of pseudoplanes in $\mathbf{P}^3$ and three points which do not lie on any pseudoplane extending the arrangement.

We first note that the following indirect argument, due to Jim Lawrence (private communication), shows that the Levi enlargement lemma could not hold generally in $\mathbf{P}^3$: If it did, then we could start with an arrangement of pseudoplanes that violates Desargues' theorem and—by successively adjoining (via Levi) new pseudoplanes connecting triples of points of intersection of our arrangement—build up a three-dimensional projective geometry for which Desargues' theorem would automatically hold, giving a contradiction.

Here is an example, also making use of a Desargues configuration, but constructed along somewhat different lines, of an arrangement of eight pseudoplanes, seven of them straight, for which the Levi enlargement lemma does not hold: Let O, A, B, C be four points in general position in $\mathbf{P}^3$ and let A', B', C' be any new points on lines OA, OB, OC, respectively (see Fig. 1). Let $\mathscr{A}$ be the arrangement consisting of the seven planes ABC, OBC, OAC, OAB, $AB'C'$, $A'BC'$, and $A'B'C$. Define points P, Q, R by

$$P = ABC \cap OBC \cap AB'C' \; (= BC \cap B'C'),$$

$$Q = ABC \cap OAC \cap A'BC' \; (= AC \cap A'C'),$$

$$R - ABC \cap OAB \cap A'B'C \; (= AB \cap A'B').$$

We have $P, Q, R \in ABC \cap A'B'C'$; hence there is a plane Π containing O, P, Q, R. Let $\mathscr{B}$ be the arrangement $\mathscr{A} \cup \{\Pi\}$. Since planes Π, ABC, OBC, and $AB'C'$ are all the members of $\mathscr{B}$ which contain P, and since they meet at P in general position (i.e., any three of them meet only at P), we still have a legitimate arrangement if we distort Π slightly, in a neighborhood of P, by pushing it away from P in a direction normal to itself, for example by

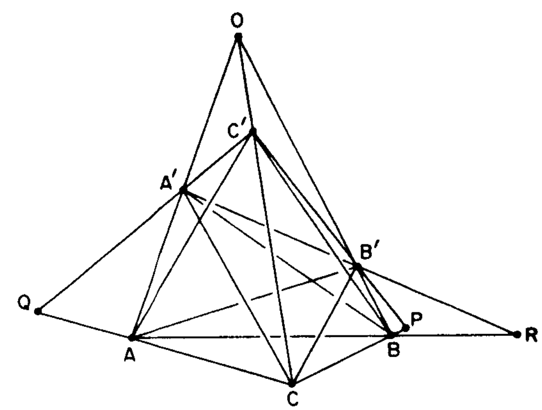

FIGURE 1.

replacing a small circular neighborhood of P in Π by a hemispherical cap of the same radius centered at P. Let Π' be the resulting pseudoplane, and let $\mathscr{C}$ be the arrangement $\mathscr{A} \cup \{\Pi'\}$.

Now consider points A', B', C'. Suppose there were a pseudoplane Π'' containing them, with $\mathscr{C} \cup \{\Pi''\}$ still an arrangement. One property of an arrangement in $\mathbf{P}^3$, which follows immediately from the definition, is that if a pseudoplane contains two points of the intersection of two other pseudoplanes then it contains their entire intersection; we therefore have

$$B', C' \in OBC \cap AB'C' \cap \Pi'', \qquad \text{hence } P \in \Pi'';$$

$$A', C' \in OAC \cap A'BC' \cap \Pi'', \qquad \text{hence } Q \in \Pi'';$$

$$A', B' \in OAB \cap A'B'C \cap \Pi'', \qquad \text{hence } R \in \Pi''.$$

But then $P, Q, R \in ABC \cap \Pi''$; so since $Q, R \in \Pi'$ we must have $P \in \Pi'$, contradiction.

As a corollary, it follows that the arrangement $\mathscr{C}$ is non-stretchable: If $\mathscr{C}$ were isomorphic to an arrangement $\bar{\mathscr{C}} = \{\bar{\Pi}_1,..., \bar{\Pi}_8\}$ of planes, this isomorphism could be extended, by a simple topological argument, to a homeomorphism $f\colon \mathbf{P}^3 \to \mathbf{P}^3$ which would map each member of $\mathscr{C}$ to one of the $\bar{\Pi}_i$. But then if $\bar{\Pi}$ were the plane through $f(A'), f(B'), f(C')$, $f^{-1}(\bar{\Pi})$ would be a pseudoplane through A', B', C' extending $\mathscr{C}$, which is impossible as we have seen. Hence $\mathscr{C}$ is a non-stretchable arrangement.

On the other hand, if there were a non-stretchable arrangement $\mathscr{C}'$ of only seven pseudoplanes, the corresponding oriented matroid [2, Sect. iv] would be nonrealizable (= "noncoordinatizable"), hence so would its dual; but the latter corresponds [2, p. 227] to a (stretchable) arrangement of seven pseudolines in $\mathbf{P}^2$, which gives a contradiction. Hence no non-stretchable

arrangement in $\mathbf{P}^3$ can consist of fewer that eight pseudoplanes. In particular, $\mathscr{C}$ is also extremal as an example of an arrangement for which the Levi enlargement lemma fails, since it certainly holds for stretchable arrangements.

Other non-stretchable arrangements of eight pseudoplanes in $\mathbf{P}^3$ are known; for example one can realize the orientable Vamos matroid [1, p. 110] by an arrangement of pseudoplanes, using the machinery of [2].

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Note

A Family of Pseudo Youden Designs with Row Size Less than the Number of Symbols

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It is shown that if s is a prime or a prime power with $s \equiv 3 \pmod 4$, then there is an $(s(s+1)/2) \times (s(s+1)/2)$ array of s^2 symbols whose rows and columns together form a balanced incomplete block design.

In this note, we construct a family of square designs, each having the property that the rows and columns together form a balanced incomplete block design (BIBD). This consideration arises from the theory of optimum statistical designs. Kiefer [2] generalized the notion of BIBD to *balanced block designs* (BBD), allowing the block size k to be bigger than the number of symbols v. An arrangement of v symbols into b blocks of size k is called a BBD if

(i) each symbol appears in each block $[k/v]$ or $[k/v]+1$ times, where $[x]$ is the largest integer $\leqslant x$;

(ii) each symbol appears bk/v times;

(iii) $\sum_{l=1}^{b} n_{il} n_{jl}$ is a constant, for all $i \neq j$, $1 \leqslant i, j \leqslant v$, where n_{il} is the number of appearances of symbol i in block l.

Note that when $k < v$, a BBD is the same as a BIBD. Kiefer [2] also defined a *generalized Youden design* (GYD) to be a $b_1 \times b_2$ array of v symbols which is a BBD when each of {columns} and {rows} is considered as blocks. This generalizes the notions of Youden squares and Latin squares. This kind of design was proved to have strong optimality properties as a statistical design for the elimination of two-way heterogeneity. Cheng [1] pointed out that when $b_1 = b_2 = b$, the same optimality property preserves as long as the b rows and b columns together form a BBD. Such a design was

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