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Best regards,
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# Foundations of a Theory of Convexity on Affine Grassmann Manifolds

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## 0. Introduction.

One historical trend in geometry has been to proceed from the study of objects of lower dimension and codimension to those of higher. Thus, in algebraic geometry, from roots of polynomials to curves to surfaces to higher-dimensional varieties, and from hypersurfaces to varieties of intermediate dimension; or in combinatorial geometry from configurations of points to arrangements of lines in the plane to arrangements of hyperplanes in $\mathbb{R}^d$ and, only recently, to arrangements of intermediate-dimensional flats.

The starting point for the present paper is the following question, which—in the same spirit—asks whether points can be replaced by flats (translates of linear subspaces of arbitrary dimension) as the basic objects in a convexity structure on $\mathbb{R}^d$:

*Is there a simple convexity structure for lines in $\mathbb{R}^3$, or, more generally, for k-flats in $\mathbb{R}^d$, that extends the standard one for points in $\mathbb{R}^d$ in a natural way, and that is invariant under the action of the affine group?*

Since the parametrizing space for $k$-flats in $\mathbb{R}^d$ is the "affine Grassmannian", $\mathfrak{G}'_{k,d}$, whose points represent $k$-flats and whose topology is inherited from that of $\mathbb{R}^d$ in the natural way (a neighborhood of the $k$-flat spanned by points $x_0, \ldots, x_k$ in general position consisting of all $k$-flats spanned by points $y_0, \ldots, y_k$ with $y_i$ in a neighborhood of $x_i$ for each $i$), what we are asking is whether there exists a convexity structure on $\mathfrak{G}'_{k,d}$ that extends the convexity structure on $\mathbb{R}^d$ $(= \mathfrak{G}'_{0,d})$ given by the usual convex hull operator conv $S$, and that satisfies the same basic properties that conv does on $\mathbb{R}^d$.

Surprisingly, the answer turns out to be "yes"; the purpose of this paper is to describe this convexity structure and to develop its most important properties.

This is not mere "generalization for its own sake": there are many questions involving configurations that have been answered in the case of points, or in the case of hyperplanes, but that become difficult or intractable when the points are replaced by intermediate-dimensional flats, such as lines in $\mathbb{R}^3$. Among those that come to mind are problems in geometric transversal theory such as finding a generalization of the higher-dimensional Hadwiger theorem on hyperplane transversals [7] to transversals of intermediate dimension, or of recent topological results on the space of common tangent hyperplanes to a separated family of convex sets [4] to lower-dimensional tangents, or of the recent proof of the Hadwiger-Debrunner conjecture [2] and its hyperplane generalization [1] to the intermediate-dimensional case. Our aim, in generalizing the basic concepts of convexity so as to encompass these higher-dimensional flats, is to make it possible for the tools of convexity theory to be applied to questions such as these.

Why $\mathfrak{G}'_{k,d}$? Convexity is an *affine* concept, not a projective one: To talk about the convex hull of a pair of points as being the linear segment *between* the points, one must have a distinguished "hyperplane at infinity". For this reason $\mathfrak{G}'_{k,d}$, rather than the (full) Grassmann manifold $\mathfrak{G}_{k,d}$, is the natural space to which to extend the standard convexity structure on $\mathbf{R}^d$.

What properties do we want in a convex hull operator "conv" on subsets of $\mathfrak{G}'_{k,d}$? Obviously conv should be monotone (increasing) and idempotent, i.e.,

($A_0$) $\mathcal{F} \subset \operatorname{conv} \mathcal{F}$ for $\mathcal{F} \subset \mathfrak{G}'_{k,d}$

($A_1$) $\mathcal{F}_1 \subset \mathcal{F}_2 \implies \operatorname{conv} \mathcal{F}_1 \subset \operatorname{conv} \mathcal{F}_2$

($A_2$) $\operatorname{conv}(\operatorname{conv} \mathcal{F}) = \operatorname{conv} \mathcal{F}$

In addition, the following "anti-exchange" property is usually required of a convex hull operator, which says essentially that conv induces a (partial) ordering on the complement of a convex set:

($A_3$) If $x, y \in \mathfrak{G}'_{k,d}$ and $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ such that $x, y \notin \operatorname{conv} \mathcal{F}$, $y \in \operatorname{conv}(\mathcal{F} \cup \{x\})$, and $x \in \operatorname{conv}(\mathcal{F} \cup \{y\})$, then $y = x$.

Since the usual operator conv, on subsets of $\mathbf{R}^d$, is defined with reference to line segments:

*The convex hull of a set $\mathcal{F}$ is the smallest set $\mathcal{G}$ containing $\mathcal{F}$ with the property that the line segment joining any two points of $\mathcal{G}$ lies in $\mathcal{G}$*

it commutes with affine transformations; i.e.,

($A_4'$) If $\sigma : \mathbf{R}^d \to \mathbf{R}^d$ is affine then $\mathrm{conv}(\sigma\mathcal{F}) = \sigma(\mathrm{conv}\,\mathcal{F})$

When we extend conv to sets of $k$-flats in $\mathbf{R}^d$, however, it no longer makes sense to talk about $\mathrm{conv}(\sigma\mathcal{F})$ since $\sigma\mathcal{F}$ may include flats of various dimensions. On the other hand, the affine group

$$\mathbf{A}(d,\mathbf{R}) = \left\{ \begin{bmatrix} L & \alpha \\ 0 & 1 \end{bmatrix} \;\middle|\; L \in \mathrm{GL}(d,\mathbf{R}),\, \alpha \in \mathbf{R}^d \right\}$$

induces an action on $\mathfrak{G}'_{k,d}$ since the image of a $k$-flat under any *nonsingular* affine transformation is again a $k$-flat. We shall therefore restrict property ($A_4'$) to nonsingular affine transformations $\sigma$, and require, for the sake of compatibility with the underlying linear structure,

($A_4$) If $\sigma_k : \mathfrak{G}'_{k,d} \to \mathfrak{G}'_{k,d}$ is induced by $\sigma \in \mathbf{A}(d,\mathbf{R})$, then $\mathrm{conv}(\sigma_k\mathcal{F}) = \sigma_k(\mathrm{conv}\,\mathcal{F})$ for $\mathcal{F} \subset \mathfrak{G}'_{k,d}$.

The standard operator conv on subsets of $\mathbf{R}^d$, which satisfies these conditions, can be characterized in various ways, not all of which are available for extension to $\mathfrak{G}'_{k,d}$ for $k > 0$. For example, to mimic the definition above, we would need nonsingular-affine-invariant geodesics (to replace the line segments in the standard definition). Now while rigid-motion-invariant geodesics do exist on $\mathfrak{G}'_{k,d}$ (see, for example, [**6**]), it is easy to see that these are not preserved under nonsingular affine transformations, nor is there a *unique* geodesic joining every pair of points.

Instead, we will define a convex hull operator on subsets of $\mathfrak{G}'_{k,d}$ by means of a duality operator $^*$ that assigns to each set $\mathcal{F}$ of $k$-flats the set of all convex point sets meeting all the members of $\mathcal{F}$, and that, likewise, assigns to each set $\mathcal{S}$ of convex point sets the set of all $k$-flats meeting all the members of $\mathcal{S}$ (the so-called "common transversals" of $\mathcal{S}$). The convex hull $\mathrm{conv}\,\mathcal{F}$ of a set $\mathcal{F}$ of $k$-flats will then be defined to be the double dual $\mathcal{F}^{**}$ of $\mathcal{F}$. With this definition, which agrees trivially with the usual definition in the case where $k = 0$, it turns out that conv satisfies all the conditions ($A_0$) through ($A_4$), and in fact—as we will see—extends many of the other properties of the standard convexity operator on $\mathbf{R}^d$ to $\mathfrak{G}'_{k,d}$.

The convex hull operator obtained in this way also turns out to have a quite different-looking geometric description, which captures the idea that the convex hull of a set should consist of all the objects that the set "surrounds" in a suitable sense.

Let us say that a point $x \in \mathrm{R}^d$ is *surrounded* by a set $\mathcal{S}$ of points if there is some flat $G$ of dimension $k$ $(0 \leq k < d)$ containing $x$ within which the following holds: If $H \subset G$ is any flat of dimension $k-1$ containing $x$, then $H$ lies strictly between two parallel $(k-1)$-flats, $H_1$ and $H_2$, also contained in $G$, each of which contains members of $\mathcal{S}$. (For $k = 0$, we interpret this to mean that $x$ itself belongs to $\mathcal{S}$.) Thus in the plane, for example, $\mathcal{S}$ surrounds $x$ if either

(i) no line $H$ containing $x$ can be translated continuously to infinity without passing through some point of $\mathcal{S}$, or

(ii) there is a line $G$ containing $x$ within which $x$ cannot escape to infinity without passing through some point of $\mathcal{S}$, or

(iii) $x$ itself belongs to $\mathcal{S}$.

One sees easily that a point $x$ lies in the convex hull of a set $\mathcal{S}$ if and only if $\mathcal{S}$ surrounds $x$ in this sense.

This concept of a point being "surrounded" by a set of points generalizes in a straightforward way to the idea of a $k$-flat being "surrounded" by a set of $k$-flats, and turns out to be equivalent to the "double-dual" definition of the convex hull operator on $\mathfrak{G}'_{k,d}$. This equivalence is proved in Theorem 1.1, where we also characterize the convex hull operator in terms of $k$-semispaces (maximal convex point sets disjoint from some $k$-flat) and in terms of the values of linear functionals on $k$-flats (appropriately defined). In particular, given a $k$-flat $F'$ and a set $\mathcal{F}$ of $k$-flats, we define the "face $\mathcal{F}^{F'}$ of $\mathcal{F}$ on the side of $F'$" and show that $F' \in \operatorname{conv}\mathcal{F}$ if and only if $F' \in \operatorname{conv}\mathcal{F}^{F'}$. Theorem 1.1 leads immediately to the definition of a *convex set* in $\mathfrak{G}'_{k,d}$ as a set $\mathcal{F}$ for which $\operatorname{conv}\mathcal{F} = \mathcal{F}$.

The double-dual characterization of the convex hull operator is interesting also from the point of view of its connection to geometric transversal theory. Recall [8] that what we are calling the $k$-dual of a family of convex point sets is also known as its set of ***k-transversals***. A good deal is known about these transversals for $k = 0$ and $d - 1$, but relatively little for intermediate values of $k$; we have in mind here combinatorial results, results on the geometric complexity of the space of transversals, and results of a topological nature on this same space [4]. The convexity structure defined in the present paper provides an answer to the question "What conditions does a family of $k$-flats have to satisfy in order to be the set of all common transversals to some family of convex point sets", and hopefully will make it possible to help solve some of the other open problems in this area.

In §2 we present ten examples of convex sets, ranging from rulings on a hyperboloid to affine Schubert varieties in $\mathfrak{G}'_{k,d}$, each exhibiting some features of interest. (Example 2.1, for instance, shows explicitly the connection between convex sets of $k$-flats and geometric transversal theory.)

Perhaps the most surprising of these is Example 2.8, a finite set of mutually non-parallel lines in $\mathbb{R}^3$, which turns out always to be convex(!). How can a finite, disconnected set turn out to be convex? The answer lies in the fact that the lines in question are mutually non-parallel; for points in $\mathbb{R}^d$, this phenomenon cannot arise, since any two points are automatically *parallel*, in the sense that there is a translation of $\mathbb{R}^d$ taking one into the other!

Another reason why this anomaly is inevitable is given by

PROPOSITION 0.1. *There is no nonsingular-affine-invariant convexity structure on $k$-flats in $\mathbb{R}^d$ that satisfies the anti-exchange axiom ($A_3$) and in which every convex set is connected.*

PROOF: It is enough to prove this for $k = 1$ and $d = 2$. Consider two intersecting lines $l_1, l_2$ in the plane. If the convex hull of the set $\{l_1, l_2\}$ is connected, it must include either a third line through $l_1 \cap l_2$ or else a third line cutting $l_1$ and $l_2$ at distinct points. In the former case, it would follow from the non-singular-affine-invariance that given any three concurrent lines the convex hull of of any two of them would contain the third, thereby violating the anti-exchange axiom; in the latter case, the same would follow for any three lines in general position. (Notice that just as any simple 3-line arrangement in the plane can be mapped to any other by a nonsingular affine transformation, the same holds for any pair of *non*-simple 3-line arrangements.) □

Although Example 2.8 shows that a convex set of $k$-flats with $k > 0$ may not be connected, if the $k$-flats are parallel their convex hull *is* always connected, and is in fact exactly the set we would expect it to be—see Example 2.4.

§3 continues the development of the basic properties of general convex sets in $\mathfrak{G}'_{k,d}$: their closure under intersection, nonsingular affine transformations, restriction to subspaces, and restriction to direction, and develops the relations between convex sets in $\mathfrak{G}'_{k,d}$ and $\mathfrak{G}'_{l,d}$ for $l \neq k$. We also extend to convex sets of $k$-flats the property of convex sets of points that the complement of such a set is connected if and only if the set contains no hyperplane (Theorem 3.2).

In §4 we restrict our attention to compact convex subsets of $\mathfrak{G}'_{k,d}$, and prove that any such set has the Krein-Milman property of being the convex hull of its set of extreme points (Theorem 4.1). Along the way we define the concept of a *supporting hyperplane* to a compact convex set $\mathcal{F}$ of $k$-flats, which plays a role in the proof.

While the portion of Theorem 1.1 that characterizes sets of $k$-transversals in terms of the intrinsic "surrounding" criterion can be viewed as an existence theorem for solutions to the "inverse problem of geometric transversal theory" (*When is a set of $k$-flats the complete set of $k$-transversals to some family of convex point sets?*), §5 contains what can be thought of as a "uniqueness" result about the same inverse problem: Can we find a "canonical" family of convex point sets whose dual is a given set of $k$-flats? Here, "canonical" is taken to mean both "maximal" and "irredundant": *maximal* in the sense that no convex point set in the family can be shrunk without costing us transversals, and *irredundant* in the sense that no convex point set can be discarded without our gaining transversals. It is easy to see (Proposition 5.1) that for $\mathcal{F}$ the dual of a finite set of compact convex point sets, we can find such a minimal, irredundant presentation of $\mathcal{F}$, and in particular, in the special case where $\mathcal{F}$ is a finite set of hyperplanes in general position, such a finite, minimal, irredundant presentation can be explicitly constructed (Theorem 5.1).

Sets of $k$-flats closed under parallels are considered in §6, where we describe a simple geometric condition that is necessary and sufficient (Proposition 6.1) for such a set to be convex. This, in turn, yields a "non-decomposition theorem" (Corollary 6.1) about the impossibility of partitioning $\mathfrak{G}'_{k,d}$ into fewer than $d-k+1$ parallel-closed convex sets. In Proposition 6.3 we extend this to arbitrary convex sets in $\mathfrak{G}'_{1,3}$, leaving open the question of its extendibility to arbitrary $k$ and $d$.

Finally, in §7, we discuss several problems that still remain to be resolved.

One important property of convex sets in $\mathbf{R}^d$ that does not extend to convex sets in $\mathfrak{G}'_{k,d}$ for $k > 0$ is Helly's theorem. For $\mathfrak{G}'_{1,2}$, for example, the existence of a Helly number would imply, when restricted to "principal" convex sets of lines (the duals of single convex point sets) the existence of a number $h$ such that if every $h$ members of a family of compact convex point sets have a common transversal line then there is a line meeting the entire family. But this is known to be false, in general [**10**, **16**], and even Hadwiger's extension of Helly's theorem to lines in the plane, which requires the additional hypothesis that the lines meet the convex sets in a consistent order, is false beyond dimension 2 [**8**].

The reason for the failure of Helly's theorem on $\mathfrak{G}'_{k,d}$ for $k > 0$, and—for that matter—of Carathéodory's theorem and Radon's theorem, can be traced to the fact that our convex hull operator is not "domain finite" in the sense of Hammer [**12**], in general, i.e., $\operatorname{conv}\mathcal{F}$ is not simply the union of the convex hulls of the finite subsets of $\mathcal{F}$. If a point $x \in \mathbf{R}^d$ is surrounded by points of a set $S$ in a flat $G$ of dimension $l$, then $x$ is already surrounded by points of $S$ in a "simplex set of directions" around $x$; i.e., a point of $S$ that prevents an $(l-1)$-flat through $x$ from escaping to infinity will prevent nearby flats through $x$ as well. But the corresponding fact does not hold for $k$-flats with $k > 0$: we may need infinitely many to surround a single one. Of course if we retain the hypothesis—automatic for points—that our $k$-flats are parallel, then all of these combinatorial theorems are trivially true, since they hold for the projections of our $k$-flats to $\mathsf{R}^{d-k}$ and there we have only to deal with *points*.

Let us mention, finally, that there have been several previous attempts to define a convexity structure on higher-dimensional Grassmann manifolds, the most notable being that of Busemann, Ewald, and Shephard. In a substantial series of papers written in the 1960's [**3**], they explored several notions of convexity for real-valued functions defined on Grassmann cones of $a$-vectors in a real projective space. But the goals they were after were different from ours, and their results—while quite extensive—do not include the kind of affinely invariant convexity structure developed here.

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## 1. Basic definitions and properties.

We work in $\mathbf{R}^d$ for some $d > 0$, and fix a value of $k$ with $0 \leq k < d$. Let $\mathfrak{G}'_{k,d}$ denote the affine Grassmannian consisting of all $k$-flats in $\mathbf{R}^d$, with its natural topology: a neighborhood of a $k$-flat $F$ is obtained by choosing $k+1$ points of $F$ in general position and taking all the $k$-flats passing through neighborhoods of these points. We will define several conditions relating a $k$-flat $F'$ and a set $\mathcal{F}$ of $k$-flats, and prove them equivalent in Theorem 1.1; this equivalence will then be used to define our convex hull operator on $\mathfrak{G}'_{k,d}$.

We begin by defining a duality operator, $^{*_k}$, or simply $^*$ if $k$ is understood, between subsets of $\mathfrak{G}'_{k,d}$ and families of convex point sets in $\mathbf{R}^d$, as follows:

DEFINITION 1.1. Let $\mathcal{F}$ be a set of $k$-flats and $\mathcal{S}$ a family of convex point sets in $\mathbf{R}^d$. Then
(i) $\mathcal{F}^* = \{S \mid S \text{ is convex and } S \cap F \neq \emptyset \forall F \in \mathcal{F}\}$, and
(ii) $\mathcal{S}^* = \mathcal{S}^{*_k} = \{F \in \mathfrak{G}'_{k,d} \mid F \cap S \neq \emptyset \forall S \in \mathcal{S}\}$.

This duality between sets of $k$-flats and sets of points satisfies the following conditions:

PROPOSITION 1.1. *(i)* $\mathcal{F}_1 \subset \mathcal{F}_2 \implies \mathcal{F}_1^* \supset \mathcal{F}_2^*$; $\mathcal{S}_1 \subset \mathcal{S}_2 \implies \mathcal{S}_1^* \supset \mathcal{S}_2^*$

*(ii)* $\mathcal{F} \subset \mathcal{F}^{**}$; $\mathcal{S} \subset \mathcal{S}^{**}$

*(iii)* $\mathcal{F}^{***} = \mathcal{F}^*$; $\mathcal{S}^{***} = \mathcal{S}^*$

*(iv)* $\mathcal{F} = \mathcal{S}^*$ *for some* $\mathcal{S} \iff \mathcal{F}^{**} = \mathcal{F}$; $\mathcal{S} = \mathcal{F}^*$ *for some* $\mathcal{F} \iff \mathcal{S}^{**} = \mathcal{S}$.

PROOF: (i) is immediate from the definitions; (ii) follows from (i); (iii) follows by applying (ii) to $\mathcal{F}^*$ (resp. $\mathcal{S}^*$) and (i) to the inclusion in (ii); and the direct implications in (iv) follow by taking the double dual of both sides and applying (iii). □

The concept of "surrounding", which is central, is perhaps most easily visualized in the context of $\mathfrak{G}'_{1,3}$, lines in $\mathbf{R}^3$. For a line $L'$ to be "surrounded" by a family $\mathcal{L}$ of lines means that either (i) every plane through $L'$ strictly separates two members of $\mathcal{L}$ (this is the "generic" way in which $\mathcal{L}$ can surround $L'$), or (ii) $L'$ itself strictly separates two members of $\mathcal{L}$ parallel to it in a plane containing all three, or (iii) $L'$ actually belongs to $\mathcal{L}$.

In general, we have:

DEFINITION 1.2. Given $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ and a particular $k$-flat $F' \in \mathfrak{G}'_{k,d}$ contained in some $l$-flat $G$, we say that $F'$ *is surrounded by* $\mathcal{F}$ *in* $G$ if every $(l-1)$-flat containing $F'$ and lying in $G$ strictly separates two members of $\mathcal{F}$ also lying in $G$. (In the degenerate case in which $G = F'$ we interpret this to mean that $F'$ itself is a member of $\mathcal{F}$.)

Next, we generalize the notion of a "semispace", as defined in [**11**] or [**14**], where this term is used to mean a convex point set maximal among those not containing a given point.

DEFINITION 1.3. A convex point set in $\mathbb{R}^d$ that is maximal among those not meeting a given $k$-flat $F$ is called a *$k$-semispace* at $F$. The set of all $k$-flats meeting a $k$-semispace at $F$ is called a *co-flat* of $F$.

PROPOSITION 1.3. *If $F \in \mathfrak{G}'_{k,d}$ and $S$ is any convex point set not meeting $F$, there exists a maximal convex set $\overline{S}$ containing $S$ and not meeting $F$. Moreover, any such $\overline{S}$ must have the following form: Choose a (partial) flag*

$$F = F_k \subset F_{k+1} \subset \cdots \subset F_d = \mathbb{R}^d$$

*of flats with $\dim F_i = i$ and, for each $i$, $k \leq i < d$, choose one of the two open half-spaces in $F_{i+1}$ determined by $F_i$; call it $F_i^+$. Then*

$$S = \bigcup_{i=k}^{d-1} F_i^+$$

*is a $k$-semispace maximally disjoint from $F$.*

PROOF: The first part is an easy consequence of Zorn's lemma. The second follows by projecting along $F$ and applying the characterization of (0-)semispaces in [**14**]. □

As in Definition 1.1, we can dualize sets of $k$-flats with respect to $k$-semispaces, rather than arbitrary convex point sets:

DEFINITION 1.4. Let $\mathcal{F} \subset \mathfrak{S}'_{k,d}$. The *restricted dual* $\mathcal{F}^\dagger$ of $\mathcal{F}$ is the set of all $k$-semispaces meeting all the members of $\mathcal{F}$.

As in Proposition 1.1, we have $\mathcal{F}_1 \subset \mathcal{F}_2 \implies \mathcal{F}_1^\dagger \subset \mathcal{F}_2^\dagger$. It is also clear from the definition that $\mathcal{F}^\dagger \subset \mathcal{F}^*$.

We can relate these ideas to $\mathcal{L}^d$, the space of linear functionals of $\mathbb{R}^d$, as follows.

DEFINITION 1.5. If $g \in \mathcal{L}^d$ and $F \in \mathfrak{S}'_{k,d}$, we define $g(F)$ by

$$g(F) = \begin{cases} g(x) \text{ for } x \in F, \text{ if } g \text{ is constant in } F \\ -\infty \text{ otherwise.} \end{cases}$$

Given a $k$-flat $F'$ and a set $\mathcal{F}$ of $k$-flats, the set

$$\mathcal{L}_{F',\mathcal{F}} = \{g \in \mathcal{L}^d \mid g \text{ is constant on } F' \text{ and } g(F') \geq g(F) \text{ for all } F \in \mathcal{F}\}$$

is called the set of $F'$-*maximizing* functionals with respect to $\mathcal{F}$. The set

$$\mathcal{F}^{F'} = \{F \in \mathcal{F} \mid g(F) = g(F') \text{ for all } g \in \mathcal{L}_{F',\mathcal{F}}\}$$

is called the *face of $\mathcal{F}$ on the side of $F'$*. (Intuitively, $\mathcal{F}^{F'}$ consists of the flats of $\mathcal{F}$ that "keep up with" $F'$ in every direction in which $F'$ "leads" all of $\mathcal{F}$.)

We then have

THEOREM 1.1. *If $F'$ is a $k$-flat, and $\mathcal{F}$ a set of $k$-flats, in $\mathbb{R}^d$, the following are equivalent:*

*(i) $F' \in \mathcal{F}^{**}$*

*(ii) $F' \in \mathcal{F}^{\dagger *}$*

*(iii) there exists an $l$-flat $G$, for some $k \leq l \leq d$, such that $F'$ is surrounded by $\mathcal{F}$ in $G$*

*(iv) there exists an $l$-flat $G$, for some $k \leq l \leq d$, such that $F'$ is surrounded by $\mathcal{F}^{F'}$ in $G$.*

*In (iii) and (iv), moreover, $G$ can be taken to be the same $l$-flat.*

PROOF: We prove (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i) and (iii) $\iff$ (iv).

(i) $\implies$ (ii) by Proposition 1.1 (i), since $\mathcal{F}^\dagger \subset \mathcal{F}^*$.

(ii) $\implies$ (iii): Suppose $F'$ is not surrounded by $\mathcal{F}$ in any $l$-flat $G$. We proceed, as follows, to construct a $k$-semispace $S$ meeting all the $k$-flats in $\mathcal{F}$ but missing $F'$. Since $F'$ is not surrounded by $\mathcal{F}$ in $\mathbf{R}^d$, we can find a $(d-1)$-flat $G_{d-1}$ containing $F'$ that has no member of $\mathcal{F}$ entirely on one side; i.e., the other side, call it $G_{d-1}^+$, together with $G_{d-1}$ itself, meets (or contains fully) every member of $\mathcal{F}$. Since $F'$ is not surrounded by $\mathcal{F}$ in $G_{d-1}$, we can find a $(d-2)$-flat $G_{d-2} \subset G_{d-1}$ containing $F'$ that has no member of $\mathcal{F}$ entirely on one side, i.e., the other side, call it $G_{d-2}^+$, together with $G_{d-2}$ itself, meets (or contains fully) every member of $\mathcal{F}$. Continuing in this way, we construct a flag

$$\mathbf{R}^d = G_d \supset G_{d-1} \supset \cdots \supset G_k = F'$$

such that the associated $k$-semispace

$$S = \bigcup_{i=k}^{d-1} G_i^+$$

meets every member of $\mathcal{F}$. But since, by construction, $S$ does not meet $F'$ itself, we have a contradiction.

(iii) $\implies$ (i): Suppose first that $G = \mathbf{R}^d$. If $S$ is a convex point set meeting all the members of $\mathcal{F}$, we must show that $S$ meets $F'$ as well. If not, then extending $S$ to a maximal $k$-semispace $\overline{S}$ disjoint from $F'$, as in Proposition 1.2, we get a flag

$$F' = G_k \subset \cdots \subset G_d = \mathbf{R}^d,$$

as in that proposition, with $\overline{S} = \bigcup G_i^+$ meeting all the members of $\mathcal{F}$. Then, in particular, $F' \subset G_{d-1}$, yet no member of $\mathcal{F}$ lies entirely in $G_{d-1}^-$, which contradicts the assumption that $F'$ is surrounded by $\mathcal{F}$ in $G$.

If $G$ is an $l$-flat with $l < d$, replace $\mathbf{R}^d$ by $G$, $S$ by $S \cap G$, and $\mathcal{F}$ by the subset consisting of all $k$-flats of $\mathcal{F}$ lying in $G$. The conclusion then follows as above.

(iii) $\implies$ (iv): Since $F'$ is surrounded by $\mathcal{F}$ in $G$, for any functional $g \in \mathcal{L}_{F',\mathcal{F}}$ we must have $g(x) = g(F')$ for all $x \in G$: otherwise $G$ would be transversal to the hyperplane

$g(x) = 0$, and then $F'$ would not "lead $\mathcal{F}$ in the direction $g$", because of the surrounding. Thus $G$ is contained in the intersection of the hyperplanes through $F'$ corresponding to the functionals in $\mathcal{L}_{F',\mathcal{F}}$, so that any flat of $\mathcal{F}$ lying in $G$ also lies in $\mathcal{F}^{F'}$. Hence, since $F'$ is surrounded by $\mathcal{F}$ in $G$, it is also surrounded by $\mathcal{F}^{F'}$ in $G$.

(iv) $\implies$ (iii): Since $\mathcal{F}^{F'} \subset \mathcal{F}$, this is trivial. □

We can now define our convex hull operator on $\mathfrak{G}'_{k,d}$.

DEFINITION 1.6. If $\mathcal{F}$ is a set of $k$-flats in $\mathbb{R}^d$, the *convex hull* $\operatorname{conv}\mathcal{F}$ of $\mathcal{F}$ is the set of all $k$-flats $F'$ satisfying the equivalent conditions of Theorem 1.1.

COROLLARY 1.1. *If $\mathcal{F} \subset \mathfrak{G}'_{k,d}$, the following are equivalent:*

*(i)* $\operatorname{conv}\mathcal{F} = \mathcal{F}$

*(ii) $\mathcal{F} = S^*$ for some family $S$ of convex point sets*

*(iii) $\mathcal{F}$ is an intersection of co-flats of dimension $k$.*

PROOF: This follows immediately from Propositions 1.1 and 1.2 and Theorem 1.1. □

DEFINITION 1.7. A set $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ is *convex* if it satisfies the equivalent conditions of Corollary 1.1.

This concept of convexity clearly extends the usual one for point sets, since criterion (i) of Theorem 1.1, applied to a point $x$ and a point set $S$, says simply that every convex point set containing $S$ also contains $x$, i.e., that $x$ lies in the (usual) convex hull of $S$. It is easy to see, as well, that the monotonicity, idempotence, and anti-exchange properties that hold for the convex hull operator on point sets continue to hold in this more general setting:

PROPOSITION 1.3. *(i) If $\mathcal{F}_1, \mathcal{F}_2 \subset \mathfrak{G}'_{k,d}$, then $\mathcal{F}_1 \subset \mathcal{F}_2 \implies \operatorname{conv}\mathcal{F}_1 \subset \operatorname{conv}\mathcal{F}_2$*

*(ii) if $\mathcal{F} \subset \mathfrak{G}'_{k,d}$, then $\mathcal{F} \subset \operatorname{conv}\mathcal{F}$ and $\operatorname{conv}(\operatorname{conv}\mathcal{F}) = \operatorname{conv}\mathcal{F}$*

*(iii) if $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ and $F_1, F_2$ are $k$-flats such that $F_1, F_2 \notin \operatorname{conv}\mathcal{F}$, $F_2 \in \operatorname{conv}(\mathcal{F} \cup \{F_1\})$, and $F_1 \in \operatorname{conv}(\mathcal{F} \cup \{F_2\})$, then $F_2 = F_1$.*

PROOF: (i) and (ii) follow immediately from Proposition 1.1, (ii) and (iii) respectively. (iii) is seen as follows:

Suppose first that $F_2$ is generically surrounded by $\mathcal{F} \cup \{F_1\}$. Since $F_2$ is *not* surrounded by $\mathcal{F}$ alone, some hyperplane $H \supset F_2$ "escapes" $\mathcal{F}$ in some direction but is "trapped" by $\mathcal{F} \cup \{F_1\}$, hence by $F_1$, in that direction. It follows that the translate of $H$ passing through $F_1$ escapes not only $\mathcal{F}$, but $F_2$ as well, giving a contradiction.

Now suppose $F_2$ is surrounded by $\mathcal{F} \cup \{F_1\}$ within some flat $G$; in particular, $F_2 \subset G$. Letting $\mathcal{F}'$ be the set of all flats of $\mathcal{F}$ lying in $G$, and observing that $F_2$ is still not surrounded by $\mathcal{F}'$ in $G$, hence that $F_1 \subset G$ as well, we see that we can apply the previous argument to $\mathcal{F}'$, $F_1$, and $F_2$ to obtain the desired conclusion. □

If a convex set $\mathcal{F}$ of $k$-flats is given as the dual $\mathcal{S}^*$ of a family $\mathcal{S}$ of convex point sets, we will see in §2 (Example 2.4) that there need not be a finite family $\mathcal{S}$ of this kind. If there is, we call $\mathcal{F}$ "finitely presented":

DEFINITION 1.8. If $\mathcal{F} = \mathcal{S}^*$ with $\mathcal{S}$ finite, $\mathcal{F}$ is said to be a *finitely presented* convex set. In particular, if $\mathcal{F} = \mathcal{S}^*$ with $|\mathcal{S}| = 1$, $\mathcal{F}$ is said to be a *principal* convex set.

## 2. Examples.

We present next a number of examples that we have found useful in guiding our intuition when thinking about convex sets of *points* proved inadequate. Many of the examples involve lines in $\mathbb{R}^3$, since that setting is already rich enough to illustrate a number of the phenomena arising in our generalization. We omit the verification of the properties of the example, which is usually straightforward.

EXAMPLE 2.1. Let $\mathcal{S} = \{S\}$, with $S$ a convex body in $\mathbb{R}^3$, and let $\mathcal{F} = \mathcal{S}^{*_1}$. Then $\mathcal{F}$, consisting of all the lines meeting $S$, is a principal convex set. If $\mathcal{S} = \{S_1, \ldots, S_n\}$, each $S_i$ convex, and $\mathcal{F} = \mathcal{S}^{*_1} = \bigcap_{i=1}^{n} \{S_i\}^{*_1}$, so that $\mathcal{F}$ is the set of convex (line) transversals of the $S_i$, then $\mathcal{F}$ is finitely presented. Similarly, $\mathcal{S}^{*_2} = \bigcap_{i=1}^{n} \{S_i\}^{*_2}$ is the set of hyperplane transversals of the $S_i$, and is a finitely presented convex family of planes in $\mathbb{R}^3$.

EXAMPLE 2.2. Let $Q$ be the one-sheeted hyperboloid obtained by rotating the line $F$ : $x = 1, z = y$ in $\mathbb{R}^3$ about the $z$-axis. $Q$ has equation $x^2 + y^2 = z^2 + 1$, and contains, in addition to the family $\mathcal{F}$ of disjoint rulings consisting of the rotates of $F$, also the family $\mathcal{G}$ consisting of the line $x = 1, z = -y$ and all its rotates; notice that each line in $\mathcal{F}$ meets each line in $\mathcal{G}$ except the unique line parallel to it.

Then, by the surrounding criterion, it is easily seen that a line $F' \in \operatorname{conv} \mathcal{F}$ if and only if either $F' \in \mathcal{F}$ or $F'$ is disjoint from $Q$. Thus $\operatorname{conv} \mathcal{F}$ consists of $\mathcal{F}$ together with all the lines strictly "inside" the hyperboloid $Q$, i.e., those cutting the open disk $x^2+y^2 < 1$, $z = 0$ and having slope $> 1$.

EXAMPLE 2.3. If we remove one or more rulings from the family $\mathcal{F}$ in Example 2.2, the new family $\mathcal{F}_0$ obtained is convex: We can find a plane through any line previously in the convex hull (but not in $\mathcal{F}$) that can now "escape" through a missing line. (Notice that if we remove *three* rulings from $\mathcal{F}$, the resulting family $\mathcal{F}_1$, which is again convex, now has a particularly simple (finite) presentation: it is the dual of the three rulings in $\mathcal{G}$ parallel to those removed!)

In particular, if we let $\mathcal{F}_t (0 \leq t < 2\pi)$ be the family of rulings cutting the circle $x = \cos\theta, y = \sin\theta, z = 0$ at the points parametrized by all values of $\theta$ with $0 \leq \theta \leq t$, we see that $\mathcal{F}_t$ is an increasing chain of convex sets of $k$-flats whose union is nevertheless not convex; this is an important difference from the situation for points.

EXAMPLE 2.4. In $\mathbb{R}^d$, consider any family $\mathcal{F}$ of mutually parallel $k$-flats, and let $\sigma : \mathbb{R}^d \to \mathbb{R}^{d-k}$ be projection in the corresponding direction. Then it follows immediately from the "convex dual" criterion that $\operatorname{conv}\mathcal{F} = \{F' \in \mathfrak{G}'_{k,d} \mid \sigma(F') \in \operatorname{conv}\sigma(\mathcal{F})\}$, so that in particular $\mathcal{F}$ is a convex set of $k$-flats if and only if $\sigma(\mathcal{F})$ is a convex point set. Hence, for example, the set of all lines in $\mathbb{R}^3$ parallel to the $z$-axis and cutting the disk $x^2 + y^2 < 1$, $z = 0$ is convex, as is the set consisting of two parallel lines and all the parallels lying between them. It is easy to see that this last example, for instance, is *not* finitely presented: Any finite set of convex point sets meeting all the lines in the family would have to meet additional lines that "spread out" far away.

EXAMPLE 2.5. Let $\mathcal{F}$ be a planar pencil of lines, considered as a subset of $\mathfrak{G}'_{1,3}$. Then $\mathcal{F}$ is convex, since the lines of $\mathcal{F}$ surround no line *not* in $\mathcal{F}$. (It is also easy to see that $\mathcal{F}$ is finitely presented, in fact by *two* convex point sets.) The same holds, more generally, for any so-called "spread" of lines [**9**], or continuous planar family, one in each direction.

EXAMPLE 2.6. If $C$ is a convex body in a plane $\Pi$, however, and $\mathcal{F}$ is the set of all lines in $\Pi$ that support $C$, again considered as a subset of $\mathfrak{G}'_{1,3}$, then $\operatorname{conv}\mathcal{F}$ includes not only the members of $\mathcal{F}$ but also all the lines passing through interior points of $C$, whether these lines lie in $\Pi$ or not. (If such a line lies in $\Pi$ it is surrounded by $\mathcal{F}$ in $\Pi$, and if it is transversal to $\Pi$ it is surrounded by $\mathcal{F}$ in the full space $\mathbb{R}^3$.)

EXAMPLE 2.7. As a consequence, if $\mathcal{F}$ is the set of all lines lying in a plane $\Pi$, considered as a subset of $\mathfrak{G}'_{1,3}$, then $\operatorname{conv}\mathcal{F}$ consists of all the lines that cut (or are contained in) $\Pi$. More generally, if $\mathcal{F}$ consists of all the lines lying in a "slab" between two parallel planes, $\operatorname{conv}\mathcal{F}$ consists of $\mathcal{F}$ plus all the lines transversal to that slab. This extends immediately to all the $k$-flats in $\mathbf{R}^d$ lying on or between two parallel hyperplanes, or in a single hyperplane, plus all the $k$-flats crossing those hyperplanes. Such a convex set will, by extension, also be referred to as a *closed slab*. An *open slab*, similarly, consists of all the $k$-flats lying strictly between two parallel hyperplanes, plus all the $k$-flats crossing them.

EXAMPLE 2.8. If a family $\mathcal{F}$ of mutually non-parallel lines in $\mathbf{R}^3$ surround a line $F' \notin \mathcal{F}$, there must be *at least* a semi-circle of directions around $F'$ in each of which a plane through $F'$ can be translated to contain a line of $\mathcal{F}$; if no member of $\mathcal{F}$ is parallel to $F'$, there is in fact an entire circle of such directions. In particular, it follows that no finite (or even countable) set $\mathcal{F}$ of mutually non-parallel lines in $\mathbf{R}^3$ can surround another line, hence that every such set $\mathcal{F}$ is convex. (Notice that this phenomenon does not occur in the usual convexity structure for *points*, since any two points are parallel!)

EXAMPLE 2.9. Let $\mathcal{F}$ be a (necessarily discontinuous) section of the tangent bundle to a 2-sphere in $\mathbf{R}^3$, i.e., choose a line tangent to $\mathbf{S}^2$ at each point. Then $\operatorname{conv}\mathcal{F}$ consists of the original family $\mathcal{F}$ together with all the lines meeting the interior of the sphere. (Here it does not matter whether $\mathcal{F}$ includes any parallels.)

EXAMPLE 2.10. Let $\Phi : F_0 \subset F_1 \subset \cdots \subset F_k$, where the inclusions are strict, be a (partial) flag of flats in $\mathbf{R}^d$. The *affine Schubert variety* $\Omega(F_0, \ldots, F_k)$ determined by $\Phi$ (see [**13**, **15**] for the corresponding definition in the projective case), which consists of all the $k$-flats $F' \in \mathfrak{G}'_{k,d}$ with $\dim(F' \cap F_i) \geq i$ for all $i = 0, \ldots, k$, and the algebraic set $\Omega^0(F_0, \ldots, F_k)$ determined by the *strict* Schubert conditions coming from $\Phi$, which consists of all the $k$-flats $F' \in \mathfrak{G}'_{k,d}$ with $\dim(F' \cap F_i) = i$ for all $i = 0, \ldots, k$, are both convex.

Notice that these are affine, not projective, Schubert conditions: If $F'$ and $F_i$ are parallel for some $i$, then $F'$ cannot belong to $\Omega(F_0, \ldots, F_k)$ or to $\Omega^0(F_0, \ldots, F_k)$. In $\mathbf{R}^3$, for example, if $F_0$ is a point and $F_1$ a plane, $\Omega(F_0, F_1)$ and $\Omega^0(F_0, F_1)$ each consist of the planar pencil in $F_1$ through $F_0$; if $F_0$ is a line and $F_1$ a plane, $\Omega(F_0, F_1)$ consists of all the lines in $F_1$ meeting $F_0$ and $\Omega^0(F_0, F_1)$ of the same lines with the exception of $F_0$ itself.

**3. Further properties.**

We collect here some additional properties of convex sets of $k$-flats, dealing principally with topological questions, with the relationships among convex sets of various dimensional flats, and with the preservation of convexity by various operations.

We begin with a transitivity property that holds for the "surrounding" relation.

PROPOSITION 3.1. *Suppose $F' \subset G \subset H$ with $F', G, H$ flats of dimensions $k, l, m$ respectively, and suppose $\mathcal{F} \subset \mathfrak{G}'_{k,d}$. Let $\sigma_G : \mathsf{R}^d \to \mathsf{R}^{d-l}$ be projection along $G$. If (i) $F'$ is surrounded by $\mathcal{F}$ in $G$, and (ii) $\sigma_G(G)$ is surrounded by $\sigma_G(\mathcal{F}||_G)$ in $\sigma_G(H)$, where $\mathcal{F}||_G$ consists of all the flats of $\mathcal{F}$ parallel to $G$, then $F'$ is surrounded by $\mathcal{F}$ in $H$.*

PROOF: Let $H_0 \subset H$ be any $(m-1)$-flat containing $F'$. If $H_0 \supset G$ then apply $\sigma_G$; the result then follows from (ii). If $H_0 \not\supset G$ then $H_0 \cap G$ must have dimension $l-1$, and the result follows from (i). □

Just as for point sets, convex subsets of $\mathfrak{G}'_{k,d}$ are closed under intersection. (Example 2.3 shows, however, that—unlike the situation for point sets—the union of an increasing chain of convex sets of $k$-flats may not be convex.)

PROPOSITION 3.2. *If $\mathcal{F}_i \subset \mathfrak{G}'_{k,d}$ is convex for each $i \in I$ then $\bigcap_{i\in I} \mathcal{F}_i$ is convex.*

PROOF: Since $\mathcal{F}_i = \mathcal{F}_i^{**}$, we have $\bigcap \mathcal{F}_i = (\bigcup \mathcal{F}_i^*)^*$; the result then follows from Theorem 1.1. □

Moreover, convex sets are preserved under nonsingular affine transformation, under restriction to subspaces, and under restriction to direction:

PROPOSITION 3.3. *Let $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ be convex.*

*(i) If $\sigma_k : \mathfrak{G}'_{k,d} \to \mathfrak{G}'_{k,d}$ is induced by $\sigma \in \mathbf{A}(d, \mathsf{R})$, then $\sigma_k(\mathcal{F})$ is convex.*

*(ii) If $G$ is an $l$-flat in $\mathsf{R}^d$, the restriction $\mathcal{F}|_G$ of $\mathcal{F}$ to $G$, consisting of all the members of $\mathcal{F}$ that lie in $G$, is a convex subset of $\mathfrak{G}'_{k,l}$.*

*(iii) If $G$ is an $l$-flat in $\mathbb{R}^d$ with $l \leq k$, then the set $\mathcal{F}\|_G$ of all flats in $\mathcal{F}$ parallel to $G$ is convex.*

PROOF: (i) follows, for example, from Corollary 1.1 ((i) $\iff$ (ii)) and the fact that the affine image of a convex point set is convex.

(ii) is immediate, by the surrounding criterion. □

(iii) follows from the fact that any $k$-flat surounded by $\mathcal{F}\|_G$ must also be parallel to $G$.

Notice that in Proposition 3.3 (ii) it is not true, in general, that $\mathcal{F}|_G$ is a convex subset of $\mathfrak{G}'_{k,d}$; see Example 2.7 above. The same example also shows that (iii) does not hold, in general, for $l > k$.

DEFINITION 3.1. If $\mathcal{S}_1, \mathcal{S}_2$ are sets of convex point sets, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *$k$-equivalent*, $\mathcal{S}_1 \sim_k \mathcal{S}_2$, if $\mathcal{S}_1^{*k} = \mathcal{S}_2^{*k}$. The corresponding equivalence class of a set $\mathcal{S}$ is denoted by $[\mathcal{S}]_k$.

Thus a convex set $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ is principal (resp. finitely presented) if $\mathcal{F}^* \sim_k \{S\}$ for some convex point set $S$ (resp. $\mathcal{F}^* \sim_k \{S_1, \dots, S_n\}$ for some $S_1, \dots, S_n$).

DEFINITION 3.2. If $\mathcal{S}_1, \mathcal{S}_2$ are sets of convex point sets, we say that $\mathcal{S}_1 \prec_k \mathcal{S}_2$ if $\mathcal{S}_2^{*k} \subset \mathcal{S}_1^{*k}$.

Thus this weak partial order becomes a strong partial order, which we denote as well by $\prec_k$, on $k$-equivalence classes of sets of convex sets, by letting $[\mathcal{S}_1]_k \prec [\mathcal{S}_2]_k$ if $\mathcal{S}_1 \prec \mathcal{S}_2$.

It is easy to see that, in general, there is no implication between $k$-equivalence and $l$-equivalence for sets of convex point sets; for example a convex point set may miss a point, yet meet every line through that point! But for *closed* convex point sets we have

PROPOSITION 3.4. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets of closed point sets, then $\mathcal{S}_1 \sim_k \mathcal{S}_2 \implies \mathcal{S}_1 \sim_l \mathcal{S}_2$ for $l < k$.*

PROOF: If $\mathcal{S}_1$ has some $l$-flat $F$ ($l < k$) as a transversal, then any $k$-flat $G$ containing $F$ is also a transversal to $\mathcal{S}_1$. But if $\mathcal{S}_2$ did not have $F$ as a transversal, then there would be a $k$-flat (in fact even a hyperplane!) $G$ through $F$ that missed $\mathcal{S}_2$ (since $\mathcal{S}_2$ is closed), contradiction. □

Thus for sets of closed sets, at least, the equivalence relation $\sim_k$ gets finer as $k$ goes up. Related to this is the notion of the core of a convex set.

DEFINITION 3.3. If $\mathcal{F}$ is a convex set in $\mathfrak{G}'_{k,d}$ and $l < k$, the *l-core* $\mathcal{F}_l$ of $\mathcal{F}$ is the set of all flats of dimension $l$ whose entire pencils of $k$-flats are contained in $\mathcal{F}$.

We then have

THEOREM 3.1. *(i) If $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ is convex and $l < k$ then $\mathcal{F}_l \subset \mathfrak{G}'_{l,d}$ is convex;*

*(ii) (Transitivity:) If $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ is convex and $m < l < k$ then $(\mathcal{F}_l)_m = \mathcal{F}_m$;*

*(iii) If $\mathcal{S}$ consists of closed convex point sets and $l < k$ then $(\mathcal{S}^{*k})_l = \mathcal{S}^{*l}$.*

PROOF: (i) Suppose $L$ is an $l$-flat surrounded by $\mathcal{F}_l$ within some flat $G \supset L$; we must show that $L \in \mathcal{F}_l$, i.e., that every $k$-flat $F$ containing $L$ lies in $\mathcal{F}$.

If $F$ lies in $G$, first, we claim that $F$ is surrounded ***within*** $G$ by members of $\mathcal{F}$. Take any codimension 1 subspace $H$ of $G$ containing $F$. Since $L$ itself is surrounded by $\mathcal{F}_l$ within $G$, there are codimension 1 subspaces $H'$ and $H''$ of $G$ parallel to and on both sides of $H$ that contain $l$-flats $L', L''$ (resp.) in $\mathcal{F}_l$. But then, by the definition of $\mathcal{F}_l$, they also contain $k$-flats $F', F'' \in \mathcal{F}$, which proves the assertion in this case.

On the other hand, if $F$ does not lie in $G$, we claim that $F$ is *generically* surrounded by members of $\mathcal{F}$. Take any hyperplane $H$ in $\mathbb{R}^d$ containing $F$. Since $H$ does not contain $G$, $H_0 = H \cap G$ is a codimension 1 subspace of $G$. Hence, as above, it lies between parallel subspaces $H_0'$ and $H_0''$ in $G$, each containing an $l$-flat $L', L''$ (resp.) belonging to $\mathcal{F}_l$. But then, by parallel extension, there are hyperplanes $H'$ and $H''$ on both sides of $H$ as well, containing $L'$ and $L''$ (resp.), so that (again, as above) the assertion follows by the definition of $\mathcal{F}_l$.

(ii) follows from the definitions, and (iii) is immediate from Proposition 3.4. □

Notice that (iii) $\implies$ (i) if $\mathcal{F}$ is the dual of a set of *closed* convex point sets, but—as we have shown—(i) holds in general. (It is easy to see that (iii) does not.)

Finally, we have the following result, which extends a property of convex point sets; for the definition of a "slab" see Example 2.7 above.

THEOREM 3.2. *If an open convex set* $\mathcal{F} \subset \mathfrak{G}'_{k,d}$, *with* $k \leq d-2$, *is not a slab, then the complement of* $\mathcal{F}$ *is connected, and conversely.*

PROOF: The converse is immediate, since a slab splits the set of $k$-flats parallel to it into two disconnected sets. For the direct implication, suppose $F_1, F_2 \notin \mathcal{F}$. Then there is a hyperplane $H_i$ containing $F_i$ that "escapes to infinity", i.e., such that one open halfspace, say $H_i^+$, contains no flat of $\mathcal{F}$. If $H_1 \nparallel H_2$ then letting $H_i'$ be the hyperplane parallel to $H_i$ and one unit away that is contained in $H_i^+$, we see that each of $F_1, F_2$ can be moved continuously to $H_1' \cap H_2'$ by a path of $k$-flats in $H_1^+$ (resp. $H_2^+$), so that $F_1$ and $F_2$ can be connected to each other by a path that stays outside $\mathcal{F}$.

Suppose, then, that $H_1 \parallel H_2$. If there is any $k$-flat $F \subset H_1^- \cap H_2^-$ with $F \notin \mathcal{F}$ such that some hyperplane $H$ containing $F$ "escaping to infinity" is not parallel to $H_1$ and $H_2$, we can connect each of $F_1, F_2$ to $F$ as above. Hence we may assume that every $k$-flat $F \subset H_1^- \cap H_2^-$ with $F \notin \mathcal{F}$ has an escaping hyperplane $H \supset F$ with $H$ parallel to $H_1$ and $H_2$ (and lying between them). Replacing one of $H_1, H_2$ by $H$, and continuing (and passing to the linit) if necessary, we arrive at the following situation: $F_i \subset H_i$, $H_1^+ \cap H_2^+ = \emptyset$, and every $k$-flat $F \subset H_1^- \cap H_2^-$ belongs to $\mathcal{F}$. But then $\mathcal{F}$ contains the open slab determined by $H_1$ and $H_2$, and it follows that $\mathcal{F}$ must, in fact, *be* this slab, since no member of $H_i^+$ belongs to $\mathcal{F}$. □

## 4. Compact convex subsets of $\mathfrak{G}'_{k,d}$.

The fact that convexity in $\mathfrak{G}'_{k,d}$ is not "domain finite" [**12**] for $k > 0$, i.e., that the convex hull of a set $\mathcal{F}$ of $k$-flats is not the union of the convex hulls of its finite subsets, means that several standard facts valid for convex point sets do not hold for convex sets of higher-dimensional flats. For example, the closure of a convex set need not be convex (consider the set $\mathcal{F}$ in Example 2.3), and the convex hull of a compact set of $k$-flats need not be compact (Example 2.6).

The dualizing operator on point sets, moreover, does not preserve closed sets in general. If $S = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 1 \text{ and } x > 0\}$ and $\mathcal{S} = \{S\}$, then $\mathcal{S}^{*_1}$ contains the lines $y = c$ for $c > 0$, but not the line $y = 0$.

It does, however, preserve *compactness*: If $\mathcal{S}$ is a set of compact convex point sets in $\mathbb{R}^d$, then $\mathcal{S}^{*_k}$ is a compact convex subset of $\mathfrak{G}'_{k,d}$, since for any $S \in \mathcal{S}$ the set of all $k$-flats meeting $S$ is compact.

In addition, the Krein-Milman property for convex sets of points, that a compact convex set is the convex hull of its extreme points, holds for convex sets in $\mathfrak{G}'_{k,d}$ as well.

DEFINITION 4.1. A $k$-flat $F'$ is an *extreme flat* of a convex set $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ if (i) $F' \in \mathcal{F}$ and (ii) $F' \notin \operatorname{conv}(\mathcal{F} \setminus \{F'\})$. The set of all extreme flats of $\mathcal{F}$ is denoted by $\operatorname{ext} \mathcal{F}$.

For convex sets of points, this definition is equivalent to the one more commonly seen, that a point $x$ is an extreme point of a convex set $S$ if $x$ is not contained properly in the convex hull of any pair of points of $S$. For $k$-flats, however, the corresponding condition would not be equivalent to the one in the definition, again because of the absence of domain-finiteness. This can be seen by looking at a slight modification of Example 2.2. In that example, replace the single ruling $F$ by the line $x = 2, z = y$ (keeping the rest of $\mathcal{F}$), and call the new family $\mathcal{F}'$. It is then easily seen that the resulting convex set $\operatorname{conv} \mathcal{F}'$ contains the line $x = 0, y = 0$, but that this line does not lie between two parallels belonging to $\operatorname{conv} \mathcal{F}'$.

DEFINITION 4.2. For a compact convex set $\mathcal{F}$ of $k$-flats we define a *supporting hyperplane* $H$ to be one that contains at least one member of $\mathcal{F}$ and which is such that one of the halfspaces determined by $H$ meets all the members of $\mathcal{F}$.

LEMMA 4.1. *If $\mathcal{F}$ is a compact convex subset of $\mathfrak{G}'_{k,d}$ and $H$ is a supporting hyperplane to $\mathcal{F}$, then* $\operatorname{ext} \mathcal{F}|_H \subset \operatorname{ext} \mathcal{F}$.

PROOF: Recall (Proposition 3.3 (ii)) that $\mathcal{F}|_H$ is a convex set in the ambient space $H$. Suppose $F' \in \operatorname{ext} \mathcal{F}|_H$. Then $F' \in \mathcal{F}|_H$, and $F' \notin \operatorname{conv}_H(\mathcal{F}|_H \setminus \{F'\})$, where $\operatorname{conv}_H$ means the relative convex hull in $H$. In particular, $F' \in \mathcal{F}$, and we must show that $F' \notin \operatorname{conv}(\mathcal{F} \setminus \{F'\})$.

To begin with, $F'$ cannot be *generically* surrounded by $\mathcal{F} \setminus \{F'\}$, since $F' \in H$ and $H$ supports $\mathcal{F}$. For the same reason, $F'$ cannot be surrounded by $\mathcal{F} \setminus \{F'\}$ within any flat $G$ transversal to $H$. But if $F'$ were surrounded by $\mathcal{F} \setminus \{F'\}$ within a flat $G \subset H$, then the surrounding flats would lie in $H$ as well, and we would have $F' \in \operatorname{conv}_H(\mathcal{F} \setminus \{F'\})$. Hence $F' \notin \operatorname{conv}(\mathcal{F} \setminus \{F'\})$. □

LEMMA 4.2. *Every non-empty compact convex set $\mathcal{F}$ of $k$-flats contains at least one extreme flat.*

PROOF: This follows from Lemma 4.1 and Proposition 3.3 (ii) by induction on the dimension $d$ of the ambient space. If $d = k + 1$, any $k$-flat $F \in \mathcal{F}$ can be translated as far as possible (in either direction) to an extreme $k$-flat. For $d > k+1$, choose a $k$-flat $F \in \mathcal{F}$ and a hyperplane $H$ containing $F$, and translate $H$ as far as possible to a supporting hyperplane $H'$. Proposition 3.3 (ii) (applied to the set $\mathcal{F}|_{H'}$), the induction hypothesis (applied to $H'$, which is of dimension $d-1$), and Lemma 4.1 (applied to the resulting extreme flat) then give the result. □

THEOREM 4.1. *If $\mathcal{F}$ is a compact convex subset of $\mathfrak{G}'_{k,d}$ then* $\mathcal{F} = \operatorname{conv}(\operatorname{ext} \mathcal{F})$.

PROOF: Since $\operatorname{ext} \mathcal{F} \subset \mathcal{F}$ and $\mathcal{F}$ is convex, it is immediate that $\mathcal{F} \supset \operatorname{conv}(\operatorname{ext} \mathcal{F})$. For the opposite inclusion, we must show that each $F' \in \mathcal{F}$ is surrounded by extreme flats of $\mathcal{F}$ in some flat $G$.

If $F'$ is extreme, we are done. If not, then by definition $F' \in \text{conv}(\mathcal{F} \setminus \{F'\})$. Suppose first that $F'$ is generically surrounded by $\mathcal{F} \setminus \{F'\}$, i.e., in $\mathbf{R}^d$. Then any hyperplane $H$ through $F'$ can be moved a positive distance in either direction so as to contain some member of $\mathcal{F} \setminus \{F'\}$. Choose either direction; then—by the compactness of $\mathcal{F}$ and the fact that $H$ is moving *away* from $F'$—we can move $H$ to a *farthest* hyperplane $H'$ in the same direction that contains members of $\mathcal{F} \setminus \{F'\}$. $H'$ will thus be a supporting hyperplane of $\mathcal{F}$, hence by Lemmas 4.1 and 4.2 and Proposition 3.3 (ii) will contain at least one extreme flat of $\mathcal{F}$. Since this holds for every hyperplane $H$ through $F'$, we have shown in this case that $F'$ is surrounded by ext $\mathcal{F}$.

Now suppose $F'$ is surrounded by $\mathcal{F} \setminus \{F'\}$ in some flat $G$ of dimension $< d$, *but not in* $\mathbf{R}^d$. Then, in particular, some hyperplane $H$ through $F'$ must support $\mathcal{F}$. Now $G$ must be contained in $H$, since otherwise $G$ would extend *on both sides* of $F$, and the fact that $F'$ is surrounded by $\mathcal{F} \setminus \{F'\}$ in $G$ would contradict the fact that $H$ is a supporting hyperplane. Let $\mathcal{F}|_H$ be the set of all members of $\mathcal{F}$ lying in $H$; in particular, $F' \in \mathcal{F}|_H$. By Proposition 3.2 (ii), $\mathcal{F}|_H$ is a convex set of $k$-flats in the ambient space $H$. Notice that $F'$ is still surrounded by $\mathcal{F}|_H \setminus \{F'\}$ in $G$. By induction on the dimension of the ambient space, it therefore follows that $F' \in \text{conv}_H(\text{ext}\,\mathcal{F}|_H)$. But by Lemma 4.1 we therefore have $F' \in \text{conv}(\text{ext}\,\mathcal{F})$, and the theorem is proved.

## 5. Minimal and irredundant presentations.

DEFINITION 5.1. If $\mathcal{F}$ is a convex set of $k$-flats in $\mathbb{R}^d$ and $\mathcal{S}$ a family of convex point sets, we say that the presentation $\mathcal{F} = \mathcal{S}^*$ is *irredundant* if $\mathcal{F} \subsetneqq \mathcal{S}_0^*$ for every proper subset $\mathcal{S}_0$ of $\mathcal{S}$. On the other hand, if $\mathcal{S}^*$ becomes strictly smaller whenever any $S \in \mathcal{S}$ is replaced by a proper subset, we say that the presentation $\mathcal{F} = \mathcal{S}^*$ is *minimal.*

It is easy to see that not every convex set $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ has a presentation that is both minimal and irredundant: In $\mathbb{R}^2$, for example, let $\mathcal{F}$ consist of two parallel lines, $l_1$ and $l_2$, together with all the lines lying between them; then one sees easily that any set of segments joining points of $l_1$ and $l_2$ and going to infinity in either direction gives a minimal presentation of $\mathcal{F}$, but that such a presentation can never be irredundant since we can always remove a finite subset of it without enlarging the dual beyond $\mathcal{F}$.

If $\mathcal{F}$ is finitely presented by compact convex point sets, however, we can prove

PROPOSITION 5.1. *If $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ is the dual of a finite set $\mathcal{S}$ of compact convex point sets, then $\mathcal{S}$ can be refined to a family $\mathcal{S}_1$ of compact convex point sets giving a minimal, irredundant presentation of $\mathcal{F}$.*

PROOF: The fact that each $S \in \mathcal{S}$ can be reduced to a minimal set follows easily from Zorn's lemma if we observe that the intersection of any descending chain of compact convex point sets each transversal to $\mathcal{F}$ is also transversal to $\mathcal{F}$, by the compactness of the intersection of each set with each member of $\mathcal{F}$.

Notice that if we replace each member of $\mathcal{S}$ by such a minimal compact convex set, and call the resulting collection $\mathcal{S}_0$, then we still have $\mathcal{S}_0^* = \mathcal{F}$; this is clear since $\mathcal{S}_0^* \subset \mathcal{F}$ on general principles, yet each member of $\mathcal{S}_0$ is transversal to all of $\mathcal{F}$.

Finally, removing redundant members of $\mathcal{S}_0$ one at a time, we reach an irredundant set $\mathcal{S}_1 \subset \mathcal{S}_0$, each of whose members is still a minimal compact convex point set meeting all the flats in $\mathcal{F}$. □

In the special case where $\mathcal{F}$ is a finite subset of $\mathfrak{G}'_{d-1,d}$ consisting of hyperplanes in general position (such a set is convex by the surrounding criterion), we have

THEOREM 5.1. *If $\mathcal{F}$ is a finite subset of $\mathfrak{G}'_{d-1,d}$ consisting of hyperplanes in general position, then $\mathcal{F}$ has a finite, minimal, irredundant presentation by compact convex point sets.*

PROOF: By Proposition 5.1, it is enough to show that $\mathcal{F}$ is finitely presented by compact convex point sets. We will actually describe such a presentation that is minimal.

Let $F_1, \dots, F_n$ be the members of $\mathcal{F}$, and let us assume to begin with that each $F_i \in \mathcal{F}$ passes through the origin $O$. In that case let $l_i$, $i = 1, \dots, n$ be the line through $O$ orthogonal to $F_i$, and let $\{x_i, x_{n+i}\}$ be the intersection of $l_i$ with the unit sphere $\mathbf{S}^{d-1}$ centered at $O$. By straightforward techniques, we can find a centrally-symmetric triangulation of $\mathbf{S}^{d-1}$ having the points $x_1, \dots, x_{2n}$ as its complete vertex set. For each facet $\Delta_j$ of this triangulation ($j = 1, \dots, 2N$, with antipodal facets numbered $\Delta_j, \Delta_{N+j}$), choose a point $y_j \in \text{int}\,\Delta_j$ (taking care to choose the antipodal point $\overline{y}_{N+j}$ for the facet antipodal to $\Delta_j$) and let $G_j$ be the hyperplane through $O$ orthogonal to the line $y_j y_{N+j}$. The resulting hyperplanes $G_1, \dots, G_N$ have the following properties: (i) $G_j \notin \mathcal{F}$ for $j = 1, \dots, N$; (ii) every hyperplane $G$ in general position with respect to $\mathcal{F}$ is *equivalent* to a unique $G_j$ in the sense that the line $l$ through $O$ orthogonal to $G$ cuts $\mathbf{S}^{d-1}$ in a pair of points belonging to the same antipodal pair of facets $(\Delta_j, \Delta_{N+j})$ as the intersection with $\mathbf{S}^{d-1}$ of the line through $O$ orthogonal to $G_j$.

For each $j = 1, \dots, 2N$, let $H_j$ be the hyperplane tangent to $\mathbf{S}^{d-1}$ at $y_j$ (so that $H_j$ and $H_{N+j}$ are each parallel to $G_j$ and on opposite sides of it), and let $\Sigma_j, \Sigma_{N+j}$ be the simplices formed by intersecting these two hyperplanes with the hyperplanes $F_i$ whose normals $l_i$ pass through the vertices of $\Delta_j$; each $\Sigma_j$ ($j = 1, \dots, 2N$) is the projection from $O$ into $H_j$ of the simplex on $\mathbf{S}^{d-1}$ polar to $\Delta_j$. For each $j = 1, \dots, 2N$, choose a point on every (relatively open) facet of $\Sigma_j$, and choose these points so that *each lies sufficiently close to a distinct vertex of* $\Sigma_j$. Then the convex hull $S_j$ of these points is itself a simplex inscribed in $\Sigma_j$, and approximates $\Sigma_j$. Let $\mathcal{S} = \{S_1, \ldots, S_{2N}\}$. We claim that $\mathcal{F} = \mathcal{S}^*$ and that the sets in $\mathcal{S}$ are minimal with this property.

It is clear from the construction that each $S_j$, $j = 1, \ldots, 2N$, meets every member of $\mathcal{F}$: $S_j$ has for its vertices points chosen from the flats $F_i$ that provide the boundary of $\Sigma_j$, and meets the remaining flats $F_i$, all of which cut $\Sigma_j$, since $S_j$ approximates $\Sigma_j$.

It is also clear that if we shrank any $S_j$ to a smaller set $S_j'$, $S_j'$ would miss some $F_i \in \mathcal{F}$ that provides one of the bounding facets of $\Sigma_j$.

To finish the proof (in the case where each $F_i \in \mathcal{F}$ passes through $O$) we must therefore show that any hyperplane $G \notin \mathcal{F}$ misses some $S_j$.

Suppose first that $G$ passes through $O$ and is in general position with respect to $\mathcal{F}$. Then, as observed above, $G$ is equivalent to one of the hyperplanes $G_j$, $j = 1, \ldots, N$. Since the line $l$ through $O$ orthogonal to $G$ can be moved continuously to the line $l_i$ without leaving the simplex pair $\Delta_j, \Delta_{N+j}$, it follows that $G$ can be moved continuously to $G_j$ without passing through the simplex-pair $\Sigma_j, \Sigma_{N+j}$. Since $S_j, S_{N+j}$ are inscribed in $\Sigma_j, \Sigma_{N+j}$ respectively and $G_j$ separates them, it follows that $G$ misses both $S_j$ and $S_{N+j}$ and separates them as well.

If $G$ passes through $O$ but is not in general position with respect to $\mathcal{F}$, the same result follows from the observation that while $G$ may now "touch" $\Sigma_j$ and $\Sigma_{N+j}$, it still misses $S_j$ and $S_{N+j}$, and in fact still separates them, since $S_j, S_{N+j}$ are "strictly inscribed" in $\Sigma_j, \Sigma_{N+j}$ respectively: Since each vertex of $S_j$ is a point in the relative interior of $\Sigma_j$, the only way a hyperplane through $O$ missing int $\Sigma_j$ can pass through that vertex is to contain a full facet of $\Sigma_j$, i.e., to belong to $\mathcal{F}$.

Finally, if $G$ does not pass through $O$, it is parallel to a hyperplane $G'$ that does, hence—assuming that $G'$ lies between $S_j$ and $S_{N+j}$ and misses both—$G$ will have to miss either $S_j$ *or* $S_{N+j}$. (If $G$ is parallel to $F \in \mathcal{F}$ then we need only observe that if any $S_j$ touches $F$ on one side then $S_{N+j}$ touches it on the other, hence $G$ misses either $S_j$ or $S_{N+j}$.)

We have now proved the theorem in the special case where all the $F_i \in \mathcal{F}$ pass through

$O$. If they do not, by rescaling we may assume that all the $F_i \in \mathcal{F}$ pass within some $\epsilon \ll 1$ of $O$ and, rescaling further, that the intersections of any subset also pass within $\epsilon$ of $O$. We now proceed as before, choosing our triangulation of $\mathbf{S}^{d-1}$ based on the *directions* of the planes in $\mathcal{F}$, but choosing each representative hyperplane $G_j$ ($j = 1, \ldots, N$) so as to pass through the intersection $v_j$ of the hyperplanes whose normals pass through the vertices of $\Delta_j$. Any plane in general position is then *parallel* to a plane equivalent to one of the $G_j$'s. We again choose $H_j$ and $H_{N+j}$ at unit distance from $G_j$, and $\Sigma_j$ and $S_j$ ($j = 1, \ldots, 2N$) as before. The argument is then the same as before, except that $O$ is replaced by $v_j$. □

REMARK 5.1. One sees easily that if $\mathcal{F}$ consists of $n$ hyperplanes then the construction in the proof of Theorem 5.1 yields a presentation of $\mathcal{F}$ by precisely $2(d-1)(n-d)+2^d$ convex point sets.

In general, as we have seen (Example 2.7), a set $\mathcal{F}$ of $k$-flats lying in an $l$-flat $G$ in $\mathbf{R}^d$ may "surround" flats not contained in $G$. But if $\mathcal{F}$ is finite we have

COROLLARY 5.1. *If $\mathcal{F}$ is a finite set of $k$-flats in (relatively) general position in a $(k+1)$-flat $G$, then $\mathcal{F}$ has a finite, minimal, irredundant presentation by compact convex sets, and these can be chosen to lie in $G$ as well.*

PROOF: This follows immediately from Theorem 5.1, with $d = k+1$, as soon as we observe that if a $k$-flat not contained in $G$ met all of the convex point sets constructed in the proof of that theorem then its intersection $H$ with $G$, which has dimension $\leq k-1$, would also meet them, and the same would then hold for the entire infinite pencil of $k$-flats in $G$ through $H$, contradicting the finiteness of $\mathcal{F}$. □

In the special case of lines in the plane we have the following particularly "pretty" presentation:

COROLLARY 5.2. *If $\mathcal{F}$ is a set of $n$ lines in the plane, no two parallel, then $\mathcal{F}$ has a minimal, irredundant presentation by $2n$ line segments forming a $2n$-pointed star if $n$ is even and two $n$-pointed stars if $n$ is odd.*

PROOF: We carry out the construction of Theorem 5.1 in a special way. Number the lines from 1 to $n$ in order of their slopes, and let the two half-lines on $l_i$ based at (say) the closest point of $l_i$ to $O$ be $r_i, r_{n+i}$, with the numbering from 1 to $2n$ being in order of direction. Let $\Gamma$ be a large circle centered at $O$, and for each $i$, $1 \leq i \leq 2n$, let $p_i = r_i \cap \Gamma$. For each $j$, $1 \leq j \leq 2n$, let $S_j$ be the line segment joining $p_j$ and $p_{n+j}$. As in Theorem 5.1, the set $\mathcal{S}$ consisting of these segments $S_j$ provides a minimal presentation of $\mathcal{F}$ by compact convex point sets, which is in fact already irredundant in this case, since removing $S_j$ from $\mathcal{S}$ would allow the line extending $S_{n+j}$ to meet all the remaining sets in $\mathcal{S}$. If one draws the figure, one sees immediately that if $n$ is even the segments $S_1, \ldots, S_{2n}$ form a $2n$-pointed star inscribed in $\Gamma$, while if $n$ is odd they form two "opposite" $n$-pointed stars inscribed in $\Gamma$. $\square$

We do not know if the result corresponding to Theorem 5.1 holds more generally in $\mathfrak{G}'_{k,d}$, for $k < d - 1$.

## 6. Parallel-closed sets and convex partitions.

One phenomenon that distinguishes convexity for higher-dimensional sets of flats from convexity for point sets is the existence of non-trivial *parallel-closed* convex sets. In the case of point sets, since all points are parallel, there are no parallel-closed sets beyond the empty set and the entire ambient space; in $\mathfrak{G}'_{k,d}$ there are many between these two extremes. Proposition 6.1 provides a characterization of those that are convex.

DEFINITION 6.1. A set $\mathcal{F}$ of $k$-flats in $\mathbf{R}^d$ is called *parallel-closed* if $F, G \in \mathfrak{G}'_{k,d}$, $F \in \mathcal{F}$, $G \parallel F \implies G \in \mathcal{F}$. The *direction set* $\delta(\mathcal{F})$ of $\mathcal{F}$ is the set of all $(k-1)$-flats in $\mathbf{P}^{d-1}$ obtained by applying to the flats in $\mathcal{F}$ passing through $O$ the canonical mapping $\delta : \mathbf{R}^d \setminus \{O\} \to \mathbf{P}^{d-1}$.

PROPOSITION 6.1. *A parallel-closed set $\mathcal{F} \subset \mathfrak{G}'_{k,d}$ is convex if and only if every $(k-1)$-flat not in $\delta(\mathcal{F})$ is contained in a hyperplane in $\mathbf{P}^{d-1}$ that contains no members of $\delta(\mathcal{F})$. (In particular, a parallel-closed set $\mathcal{F} \subset \mathfrak{G}'_{1,3}$ is convex if and only if $\delta(\mathcal{F})$ is the complement of a union of lines.)*

PROOF: By Corollary 1.1, if $\mathcal{F}$ is convex then for any $F \notin \mathcal{F}$ there is a co-flat of $F$ that contains $\mathcal{F}$, hence a hyperplane $H$ containing $F$ that has no members of $\mathcal{F}$ strictly on one side of it. Since $\mathcal{F}$ is parallel-closed, it follows that no member of $\mathcal{F}$ can be parallel to $H$, so that the hyperplane in $\mathbf{P}^{d-1}$ corresponding to $H$ contains no members of $\delta(\mathcal{F})$.

Conversely, given any $F \notin \mathcal{F}$ it follows from the given condition that there is a hyperplane $H$ in $\mathbf{R}^d$ parallel to $F$ and containing no members of $\mathcal{F}$, and—since $\mathcal{F}$ is parallel-closed—$H$ can be taken to pass through $F$ itself. Again, since $\mathcal{F}$ is parallel-closed, any co-flat of $F$ with hyperplane $H$ will also contain no member of $\mathcal{F}$, and the convexity of $\mathcal{F}$ again follows from Corollary 1.1. □

We get, as an immediate corollary, that it is impossible to partition $\mathfrak{G}'_{k,d}$ into fewer than $d-k+1$ parallel-closed convex sets.

COROLLARY 6.1. *If $\mathfrak{G}'_{k,d} = \bigcup_{i=1}^{n} \mathcal{F}_i$ is a partition of $\mathfrak{G}'_{k,d}$ into parallel-closed convex sets, then either $n = 1$ or $n \geq d - k + 1$.*

PROOF: If $n > 1$ then since each $\mathcal{F}_i$ provides at least one hyperplane in $\mathbf{P}^{d-1}$ containing no member of $\delta(\mathcal{F}_i)$, $n \leq d - k$ would imply that the intersection of these hyperplanes, which has dimension at least $(d-1) - n \geq k - 1$, would be large enough to contain at least one $(k-1)$-flat, so that the sets $\mathcal{F}_i$ could not cover $\mathfrak{G}'_{k,d}$. □

On the other hand, it is not difficult to see that $\mathfrak{G}'_{k,d}$ *can* be partitioned into $\binom{d-1}{k} + 1$ parallel-closed convex sets. The following decomposition of $\mathbf{P}^{d-1}$ into Schubert cells provides such a partition.

PROPOSITION 6.2. *Let $G_0 \subset G_1 \subset \cdots \subset G_{d-1}$ be a flag of subspaces of $\mathbf{P}^{d-1}$, with $\dim G_i = i$, and set $G_{-1} = \emptyset$. Let $S$ be the set of all subsets $\{i_0, \ldots, i_{k-1}\}$ with $i_0 < \cdots < i_{k-1}$ of the integers $0, \ldots, d-1$, and let $S_{d-1}$ consist of all $k$-sets $i_0 < \cdots < i_{k-1}$ in $S$ with $i_{k-1} = d-1$. For each element $\sigma = \{i_0, \ldots, i_{k-1}\}$ of $S$, put $i_{-1} = 1$ and $i_k = d$ and let $\Phi_\sigma$ be the Schubert cell defined by*

$$\Phi_\sigma = \{\phi \in \mathfrak{G}_{k-1,d-1} \mid \dim \phi \cap G_i = j \text{ for } i_j \leq i < i_{j+1},\ -1 \leq j \leq k-1\}.$$

*Let $\mathcal{F}_\sigma = \delta^{-1}(\Phi_\sigma)$. Then each $\mathcal{F}_\sigma$ is a parallel-closed convex set of $k$-flats, as is the union*

$$\mathcal{F}_{d-1} = \bigcup \mathcal{F}_{i_0,\ldots,i_{k-2},d-1},$$

*and*

$$\mathfrak{G}'_{k,d} = \Big(\bigcup_{\sigma \in S \setminus S_{d-1}} \mathcal{F}_\sigma\Big) \cup \mathcal{F}_{d-1}$$

*is a partition of $\mathfrak{G}'_{k,d}$ into $\binom{d-1}{k} + 1$ parallel-closed convex sets.*

PROOF: This follows from Proposition 6.1; we omit the details, which are straightforward. □

REMARK 6.1. The partition of $\mathfrak{G}'_{k,d}$ provided by Proposition 6.2 is not *full-dimensional*, i.e., the sets $\mathcal{F}_\sigma$, for $\sigma \in S \setminus S_{d-1}$, all have lower dimension than $\mathcal{F}_{d-1}$; in particular this holds even for $\mathfrak{G}'_{1,3}$, where Proposition 6.2 yields a partition into three parallel-closed convex sets (the minimum attainable, according to Corollary 6.1). It is possible also to describe a partition of $\mathfrak{G}'_{1,3}$ into three full-dimensional parallel-closed convex sets. This can be achieved by appropriately 3-coloring the faces, edges, and vertices of an octahedron centered at the origin in $\mathbb{R}^3$, and taking for $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ the sets of lines joining the origin to the points in each color class, so as to satisfy the condition at the end of Proposition 6.1. The conditions needed on the coloring turn out to be that (i) the color of each vertex agrees with that of some incident edge; (ii) the color of each edge agrees with that of some incident face; and (iii) whenever a vertex and an incident face have the same color then every edge incident to both has the same color as well. We do not know whether this construction can be generalized.

In the case of $\mathfrak{G}'_{1,3}$, Corollary 6.1 can be strengthened to show that there is no partition into two convex subsets, parallel-closed or not.

PROPOSITION 6.3. *The lines in $\mathbb{R}^3$ cannot be partitioned into two non-empty convex sets.*

PROOF: Suppose $\mathfrak{G}'_{1,3} = \mathcal{F}_1 \cup \mathcal{F}_2$ is such a partition, and suppose $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ are such that $F_1 \parallel F_2$. By Proposition 3.3 (iii), the members of each set $\mathcal{F}_i$ parallel to $F_1$ themselves form a convex set of lines; call these sets $\mathcal{F}'_1, \mathcal{F}'_2$ (resp.). Since the two convex point sets obtained by projecting $\mathcal{F}'_1, \mathcal{F}'_2$ into a plane orthogonal to $F_1$ are disjoint, they can be (weakly) separated by a line in that plane. Hence $\mathcal{F}'_1, \mathcal{F}'_2$ themselves can be (weakly) separated by a plane $H$ in $\mathbb{R}^3$; i.e., the lines of $\mathcal{F}_1$ lie in one closed halfspace of $H$, say $H^+$, and those of $\mathcal{F}_2$ in the other, $H^-$. It follows that if $F'_1$ (resp. $F'_2$) is *any* member of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) parallel to $H$ then $F'_1 \subset H^+$ (resp. $F'_2 \subset H^-$), for otherwise, if (say) $F'_1 \subset \operatorname{int} H^-$, any member of $\mathcal{F}_2$ sufficiently close to $H$ would be surrounded by the lines in $\mathcal{F}_1 \cup \{F'_1\}$, hence would have to belong to $\mathcal{F}_1$. Thus $H$ (weakly) separates *all* the lines parallel to it: those of $\mathcal{F}_1$ lie in $H^+$, and those of $\mathcal{F}_2$ in $H^-$. But then any line that crosses $H$ is surrounded by both $\mathcal{F}_1$ and $\mathcal{F}_2$, which is impossible.

Hence each direction contains only lines from one of the sets $\mathcal{F}_1, \mathcal{F}_2$, so we are done by Corollary 6.1. $\square$

In addition to narrowing the gap between the results of Corollary 6.1 and Proposition 6.2, it would also be interesting to know whether Proposition 6.3 can be extended to the corresponding statement for $\mathfrak{G}'_{k,d}$. We venture

CONJECTURE 6.1. *$\mathfrak{G}'_{k,d}$ cannot be partitioned into fewer than $d-k+1$ non-empty convex sets.*

## 7. Open questions.

In addition to several questions and conjectures posed earlier, we list here six problems whose solution will be important for the further development of the ideas presented here.

PROBLEM 7.1. Probably the most important problem to resolve concerns the separation theorem for convex sets, which—for point sets—asserts (in one version) that any two disjoint convex sets can be separated by a hyperplane; if one adds the assumption that the sets are compact, the conclusion can be strengthened to "strictly separated".

Is there a corresponding "separation theorem" for convex sets in $\mathfrak{G}'_{k,d}$?

The point-set version of the theorem can be stated: If $S_1, S_2$ are disjoint convex sets, each $S_i$ can be enlarged to a convex set $\overline{S}_i$ so that $\overline{S}_1, \overline{S}_2$ partition $\mathsf{R}^d$. But as Proposition 6.3 shows, already in the case of $\mathfrak{G}'_{1,3}$, no such partition of $\mathfrak{G}'_{k,d}$ into two convex sets may be possible for $k > 0$. Perhaps the fact that a partition into $\binom{d-1}{k} + 1$ convex sets can always be found (see Proposition 6.2 and Remark 6.1) can be used to find the "correct" generalization of the separation theorem to $k$-flats.

PROBLEM 7.2. A second open problem relates to connectedness. We have observed above (Example 2.8) that convex sets on $\mathfrak{G}'_{k,d}$ need not, in general, be connected. But what if we take the convex hull of a *connected* set of $k$-flats—must such a set be connected? It is not hard to see that this question is equivalent to asking whether a connected component of a convex set must necessarily be convex.

PROBLEM 7.3. As pointed out in §4, the closure of a convex set in $\mathfrak{G}'_{k,d}$ need not be convex, and the convex hull of a closed (even a compact) set need not be closed. To define the "closed convex hull" of a set $\mathcal{F}$ of $k$-flats, we must therefore take the limit of the sequence

$$\ldots(\mathrm{cl}\,(\mathrm{conv}\,(\mathrm{cl}\,(\mathrm{conv}\,\mathcal{F}))))\ldots.$$

Does this sequence of iterations stop after a finite number of steps, the number depending only on $k$ and $d$?

PROBLEM 7.4. A question about the relation between compact point sets and their $k$-duals that we have been unable to answer is: If a finitely presented convex set of $k$-flats is compact, must it have a finite presentation by *compact* sets of points? (It is easy to see that the converse is true.)

PROBLEM 7.5. Corollary 5.2 shows that a finite set of lines in the plane without parallels is finitely presented. Does the converse hold, in the sense that a finitely presented set of lines in the plane without parallels must be finite? More generally, if a set of hyperplanes in $\mathsf{R}^d$ in general position is finitely presented, must it be finite? (This would be a converse to Theorem 5.1.)

Finally, here is a problem of a combinatorial nature about lines in "convex position":

PROBLEM 7.6. A result of Erdős and Szekeres [5] says that from any set of $\binom{2n-4}{n-2}$ points in general position in the plane one can always select $n$ points in convex position, i.e., extreme points of some convex body (e.g., their convex hull). Does a similar result hold for lines in $\mathsf{R}^2$ (or, more generally, for $k$-flats in $\mathsf{R}^d$?): is there a number $f(k,d,n)$ such that from any set of $f(k,d,n)$ $k$-flats in general position in $\mathsf{R}^d$ we can always select $n$ which are in "convex position", i.e., which support some convex body?

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