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Barvinok's algorithm and the Todd class of a toric variety[1]

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Abstract

In this paper we prove that the Todd class of a simplicial toric variety has a canonical expression as a power series in the torus-invariant divisors. Given a resolution of singularities corresponding to a nonsingular subdivision of the fan, we give an explicit formula for this power series which yields the Todd class. The computational feasibility of this procedure is implied by the additional fact that the above formula is compatible with Barvinok decompositions (virtual subdivisions) of the cones in the fan. In particular, this gives an algorithm for determining the coefficients of the Todd class in polynomial time for fixed dimension. We use this to give a polynomial-time algorithm for computing the number of lattice points in a simple lattice polytope of fixed dimension, a result first achieved by Barvinok. © 1997 Elsevier Science B.V.

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1. Introduction

1.1. Overview

The Todd class of a toric variety is important both in the theory of lattice polytopes and in number theory. The early researchers in the field of toric varieties realized that a formula for the Todd class of a toric variety yields directly a formula for the number of lattice points in a convex lattice polytope (cf. [4]). On the other hand, it has been shown more recently that Dedekind sums and their generalizations appear naturally in formulas for the Todd class [2, 3, 8, 12, 13]. Our purpose here is to give a canonical power series expression for the Todd class of a complete simplicial toric variety. We use this expression together with Barvinok's polynomial time algorithm for finding nonsingular subdivisions of cones [1] to give an effective algorithm for computing the

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coefficients of this power series which is polynomial time in fixed dimension. We show how this algorithm may be used to give a polynomial time algorithm for computing the number of lattice points in a polytope of fixed dimension, a result first achieved in [1].

1.2. Toric varieties and lattice points

A toric variety is an algebraic variety X in which the algebraic torus $T = (\mathbb{C}^*)^n$ sits as a dense open subset, such that the action of T on itself extends to all of X. Such varieties are classified combinatorially by *fans* which are collections of cones in a lattice. In addition, any lattice polytope (with integral vertices) determines a toric variety via the inner normal fan. This establishes a strong link between lattice polytopes and algebraic geometry, a link which has been quite fruitful in both directions ever since the introduction of toric varieties more than 20 years ago. Discussion of these connections as well as an introduction to the subject of toric varieties can be found in the books [6, 11] or the survey article [4].

An example of this link is the relation between counting lattice points in a polytope and finding the Todd class of the corresponding toric variety. Given a lattice polytope it is natural to ask how many lattice points it contains. On the other hand, as every algebraic variety has a naturally defined Todd class [5], one may ask for an expression for this characteristic class. A well-known application of the Hirzebruch–Riemann–Roch theorem shows that such an expression can be easily translated into an answer to the first question of enumerating the lattice points in the given polytope. Much progress has been made in the search for expressions for the Todd class of a toric variety, including the important work of Morelli [9] as well as the papers cited in Section 1.1, which relate the Todd class to classical number-theoretic invariants, Dedekind sums, and their generalizations.

An important subclass of toric varieties is the class of *simplicial* toric varieties. A simplicial toric variety is one for which every cone in the fan is simplicial, that is, generated by linearly independent rays. Geometrically, this implies that the variety is locally a quotient of affine space by a finite abelian group. It follows that with rational coefficients the homology, cohomology and Chow ring of a simplicial toric variety all coincide. This ring has an explicit description as a quotient of the Stanley–Reisner ring of the fan (the polynomial ring in the rays modulo products of distinct rays which do not form a cone of the fan).

In this paper, we show that the Todd class of a simplicial toric variety has a canonical expression as a power series in the rays of the fan. This power series lives naturally in the completion of the Stanley–Reisner ring of the fan, which is isomorphic to the equivariant cohomology ring of the toric variety. We also give a reciprocity relation which expresses the behavior of this power series under subdivisions. By extending these relations to the virtual subdivisions considered by Barvinok, and applying Barvinok's polynomial-time subdivision algorithm, we show how to compute the coefficients of the above power series in polynomial time if the dimension is fixed.

1.3. A Todd class formula

To begin, let Σ be a complete, simplicial fan with corresponding toric variety X_Σ. Let $\Sigma_{(1)} = \{\rho_1, \dots, \rho_l\}$ be the set of rays of Σ, and let A_Σ denote the Stanley–Reisner ring

$$A_\Sigma = \frac{\mathbb{Q}[x_1, \dots, x_l]}{I},$$

with I equal to the ideal generated by $\{x_{i_1} \cdots x_{i_r} \mid \langle \rho_{i_1}, \dots, \rho_{i_r} \rangle \notin \Sigma\}$.

If r is a power series in $x_1, \dots, x_l$, then modulo I, r is determined by its *restrictions* r_σ to the n-dimensional cones $\sigma \in \Sigma_{(n)}$ defined as follows: If $\sigma = \langle \rho_{i_1}, \dots, \rho_{i_n} \rangle$, then set

$$r_\sigma(x_{i_1}, \dots, x_{i_n}) = r(y_1, \dots, y_l),$$

where $y_j = x_j$ if ρ_j is an extreme ray of σ, and $y_j = 0$ otherwise.

Theorem 1. *Given an n-dimensional lattice N, there is a canonical assignment to each n-dimensional simplicial cone $\sigma = \langle \rho_1, \dots, \rho_n \rangle$ a function $t_\sigma(x_1, \dots, x_n)$ which is a power series in $x_1, \dots, x_n$ defined by a rational function of x_i, e^{x_i}, $i = 1, \dots, n$ with the following property: For any complete simplicial fan Σ in N, let $t_\Sigma(x_1, \dots, x_l)$ be power series (in variables x_i corresponding to the rays ρ_i of Σ) whose restriction to each cone $\sigma \in \Sigma_{(n)}$ is t_σ. Then for any such fan, the Todd class $Td\, X_\Sigma$ is given by evaluating $t_\Sigma(x_1, \dots, x_l)$ in the divisor classes $\{x_i = [V(\rho_i)] \in A^1 X_\Sigma \mid \rho_i \in \Sigma_{(1)}\}$.*

The next theorem shows how to compute the t_σ given a nonsingular subdivision of σ. First of all, we renormalize the power series t_σ by defining

$$s_\sigma(x_1, \dots, x_n) = \frac{1}{(mult\, \sigma) x_1 \cdots x_n} t_\sigma(x_1, \dots, x_n).$$

Here $mult\, \sigma$ denotes the multiplicity of the cone σ. This is defined as the index of the group generated by the primitive elements of the rays of σ in the linear sublattice $\sigma + (-\sigma)$. (Here this linear sublattice is all of N since σ is n-dimensional.)

Our next theorem states that the s_σ are *additive* with appropriate natural changes of coordinates. Before stating the theorem, we introduce the matrices which give these changes of coordinates: Let $\sigma = \langle \rho_1, \dots, \rho_n \rangle$ and $\gamma = \langle \tau_1, \dots, \tau_n \rangle$ be any two simplicial cones in the lattice N. We define $A_{\sigma,\gamma}$ to be the $n \times n$ matrix whose (i,j) entry is

$$(A_{\sigma,\gamma})_{i,j} = \langle w_i, \rho_j \rangle,$$

where $w_1, \dots, w_n$ is the basis of $Hom(N, \mathbb{Q})$ dual to $\tau_1, \dots, \tau_n$.

Theorem 2. *Let $\sigma = \langle \rho_1, \dots, \rho_n \rangle$ be an n-dimensional simplicial cone in N, and let Γ be any simplicial subdivision of σ. Then letting X denote the (column) vector $(x_1, \dots, x_n)$, we have*

$$s_\sigma(X) = \sum_{\gamma \in \Gamma_{(n)}} s_\gamma(A_{\sigma,\gamma} X).$$

If furthermore Γ is a nonsingular subdivision of σ, then the s_γ above are given by

$$s_\gamma(y_1,\dots,y_n) = \prod_{i=1}^{n} \frac{1}{1-e^{-y_i}},$$

so the above equation gives a formula for the s_σ as a rational function of e^{x_i} and hence t_σ is expressed as a rational function of x_i, e^{x_i}, $i = 1,\dots,n$.

1.4. Behavior under Barvinok subdivisions

While in general it may require the addition of many new rays to desingularize a given cone, Barvinok [1] has shown that the situation is different if we allow *virtual* subdivisions. Specifically, he gives a polynomial-time algorithm for constructing a nonsingular virtual subdivision of polynomial size in the bit complexity of the coordinates of the given cone. The next theorem shows that the additivity formula of Theorem 2 may be adapted to virtual subdivisions as well. This, together with Barvinok's results, yields an algorithm for computing the Todd class which is polynomial time in fixed dimension.

We first establish some notation. If $\sigma = \langle \rho_1,\dots,\rho_n\rangle$ is an n-dimensional cone, then given any ray ρ_0 such that $\rho_0,\dots,\rho_n$ are contained in a half-space of N, we may "virtually subdivide" σ with respect to ρ_0 by replacing σ with the collection

$$\{\sigma_i = \langle \rho_0,\dots,\hat{\rho}_i,\dots,\rho_n\rangle \mid i = 1,\dots,n, \text{ and } \sigma_i \text{ is an } n\text{-dimensional cone}\}.$$

For each such i, we define

$$\delta_i = \begin{cases} 1 & \text{if } (\rho_1,\dots,\rho_n), (\rho_1,\dots,\rho_{i-1},\rho_0,\rho_{i+1},\dots,\rho_n) \text{ have the same orientation,} \\ -1 & \text{otherwise.} \end{cases}$$

Modulo smaller-dimensional cones, we have

$$\sigma = \sum \delta_i \sigma_i.$$

We may then continue by picking one of the σ_i and virtually subdividing it with respect to a new ray ρ_0', and so on. By a *virtual subdivision* of σ we mean an expression

$$\sigma = \sum_\gamma \delta(\gamma)\gamma$$

obtained by a finite sequence of such operations.

We have the following formula expressing t_σ in terms of the t_γ:

Theorem 3. *Let the notation be as in Theorem 2, except let Γ be a virtual subdivision as above. Then*

$$s_\sigma(X) = \sum_{\gamma\in\Gamma_{(n)}} \delta(\gamma) s_\gamma(A_{\sigma,\gamma}X).$$

1.5. Finding the Todd class in polynomial time

The above theorems may be used to give an algorithm for computing the Todd class which is polynomial time in fixed dimension.

Let X_Σ be a complete, simplicial toric variety of dimension n. In order to compute the Todd class of X_Σ, it suffices to compute the coefficients of t_Σ in degrees not exceeding n because higher-order terms represent zero in the Chow groups (or homology) of X_Σ. We compute these coefficients as follows.

Algorithm 1. (i) For each n-dimensional cone $\sigma \in \Sigma$ apply Barvinok's algorithm to find in polynomial time a nonsingular virtual subdivision Γ which is of polynomial size in the bit complexity of the rays defining σ.

(ii) For each n-dimensional cone γ of Γ, use the second equation of Theorem 2 to express the power series t_γ up to order n.

(iii) The coefficients of t_σ may then be computed inductively as follows: The virtual subdivision Γ determines a sequence of virtual subdivisions $\Gamma_0 = \{\sigma\}, \Gamma_1, \ldots, \Gamma_q = \Gamma$ where each Γ_{i+1} is obtained from Γ_i by the addition of a single ray. Let γ be an n-dimensional cone of Γ_i, and let $\gamma_1, \ldots, \gamma_k$ be the n-dimensional cones of Γ_{i+1} into which γ is subdivided. We now compute the coefficients of t_γ up to order n from the coefficients of the t_{γ_i} up to order n using Theorem 3. (More explicitly, we may use Lemma 7 which is Theorem 3 in the special case of a virtual subdivision obtained by adding a single ray.)

(iv) After we have computed t_σ up to order n for each $\sigma \in \Sigma_{(n)}$, we are done, as all the desired coefficients of t_Σ are now determined.

1.6. Example

To illustrate the above results, we consider the following very simple example. Let $N = \mathbb{Z}^2$, and $\sigma = \langle \rho_1, \rho_2 \rangle$, where $\rho_1 = (1,0)$, and $\rho_2 = (1,n)$ for some integer $n > 0$. Consider the virtual subdivision induced by the ray $\rho_0 = (0,1)$. Letting $\sigma_1 = \langle \rho_0, \rho_2 \rangle$ and $\sigma_2 = \langle \rho_0, \rho_1 \rangle$, this virtual subdivision may be expressed as $\sigma = \sigma_2 - \sigma_1$.

To compute t_σ up to order 2, we will apply the equation of Theorem 3 (or the special case, Lemma 7) to obtain

$$t_\sigma(x_1,x_2) = \frac{x_1}{x_1+x_2} t_{\sigma_2}(nx_2, x_1+x_2) + \frac{x_2}{x_1+x_2} t_{\sigma_1}(-nx_1, x_1+x_2).$$

Note that this equation comes with the guarantee that the right-hand side is actually a power series in x_1, x_2. Further it is clear that the coefficients of t_σ in any degree may be computed easily given the coefficients of t_{σ_1} and t_{σ_2} in the same degree.

Since σ_1 and σ_2 are both nonsingular, we may use the second equation of Theorem 2 to get

$$t_{\sigma_1}(y_0, y_2) = 1 + \tfrac{1}{2}(y_0 + y_2) + \tfrac{1}{12}(y_0{}^2 + y_2{}^2) + \tfrac{1}{4} y_1 y_2 + \cdots,$$

with a similar expression for $t_{\sigma_2}(y_0, y_1)$.

From these equations, we obtain

$$t_\sigma(x_1,x_2) = 1 + \tfrac{1}{2}(x_1 + x_2) + \tfrac{1}{12}({x_1}^2 + {x_2}^2) + (\tfrac{1}{12}n^2 + \tfrac{1}{6})x_1x_2 + \cdots.$$

The interesting coefficient above, that of x_1x_2, actually comes from the classical Dedekind sum $s(1,n)$ (see Section. 1.8 below).

1.7. Relation to counting lattice points

As mentioned above, a formula for the Todd class of a toric variety translates directly into a formula for the number of lattice points in a lattice polytope. This connection has been well understood conceptually for quite some time, and is rather simple computationally. For completeness, we will briefly describe the algebraic geometry involved, and give a recipe for converting the above Todd class formulas into lattice point formulas.

Given an n-dimensional lattice N, and dual lattice $M = Hom(N,\mathbb{Z})$, we begin with any integral convex polytope Δ in M. Via the inner normal fan, Δ determines a toric variety X_Δ. This variety comes equipped with a canonical line bundle a basis of whose sections is given by the lattice points in Δ. Higher cohomology of this line bundle is seen to vanish. Thus, the number of lattice points equals the Euler characteristic of the line bundle, which may be computed via Hirzebruch–Riemann–Roch. The necessary ingredients for this computation are: (1) the Chern character of the line bundle, which turns out to have a nice expression in terms of the volumes of faces of Δ, and (2) the Todd class of the variety X_Δ.

We now give a recipe for translating our formula for the Todd class of a simplicial toric variety into a lattice point formula. Let us assume that Δ is a *simple* polytope, that is, no more than n facets meet at a vertex. This means that the induced fan Σ is simplicial. We may compute the number of lattice points in Δ as follows:

(i) Use the above algorithm to compute the Todd class $t_\Sigma(x_1,\ldots,x_l)$ up to degree n. This is a polynomial T in the variables x_i which correspond to rays ρ_i of Σ, and hence to facets of the polytope Δ.

(ii) Let J be the ideal generated by the set

$$\left\{ \sum_{i=1}^{l} \langle m, \rho_i \rangle x_i \;\middle|\; m \in M \right\}.$$

Consider T as an element of A_Σ/J. This ring is a well-known presentation for the Chow ring of X_Σ. Now find a square-free representative for T. This can be accomplished as follows: For any monomial of degree at most n which is not square-free, we may use an element of J to express this monomial as a sum of terms that are closer to being square-free. That is, each new term will involve more of the variables x_i with nonzero exponent. Proceeding inductively in this way, one arrives at a square-free representative for T.

(iii) We will next replace each monomial appearing in this square-free expression by a rational number. Given a monomial $x_1 \cdots x_k$, we may assume that the corresponding cone $\sigma = \langle \rho_1, \ldots, \rho_k \rangle$ is in Σ, for otherwise this monomial vanishes in A_Σ. Thus, there is a corresponding $(n-k)$-dimensional face F of Δ, namely the intersection of the facets $F_1, \ldots, F_k$ corresponding to the rays $\rho_1, \ldots, \rho_k$. Replace $x_1 \cdots x_k$ by

$$\frac{1}{mult\,\sigma} Vol(F).$$

In this formula, the volumes are computed with respect to the sublattice consisting of those points of M lying within the $(n-k)$-dimensional affine space determined by F.

That the above recipe yields the number of lattice points is well-known (cf. [6, p. 112]). Since the Todd class computation is polynomial time, it is clear that the above yields a polynomial time algorithm for computing the number of lattice points in a simple lattice polytope when the dimension is fixed. Such an algorithm was first given by Barvinok in [1].

One can rephrase the above recipe to avoid computation of the volumes of the faces of Δ. To do this, we assume that Δ is given as the solution to the linear inequalities:

$$\Delta = \{m \in M \mid \langle m, \rho_i \rangle \geq h_i\}.$$

With this setup, we give a second formulation of our polynomial-time algorithm for computing the number of lattice points, as follows:

(i) Compute T above, and let C denote $\exp(\sum -h_i x_i)$ truncated to a polynomial of degree n. This represents the Chern character of the line bundle associated to Δ.

(ii) Let N be the degree n part of the product TC.

(iii) Choose any vertex of Δ and let $\sigma = \langle \rho_1, \ldots, \rho_n \rangle$ be the corresponding cone of Σ. Choosing any Gröbner basis for the ideal J above, we compute the normal forms for N and for $x_1 \cdots x_n$. The degree n part of A_Σ / J is known to be a one-dimensional vector space. Therefore, these two normal forms are rational multiples of the same monomial, and hence their ratio is a rational number. The number of lattice points is then given by this ratio divided by $mult\,\sigma$:

$$\#(\Delta \cap M) = \frac{nf(N)}{nf(x_1 \cdots x_n)\, mult\,\sigma}.$$

Again, the correctness of the above algorithm is well-known and follows from the discussion in [6, Section 5.3].

1.8. Relation with previous work

Theorems 1 and 2 are quite natural extensions of the results of [13]. In [13], it was shown that the Todd class of a simplicial toric variety has a canonical lattice-invariant expression as a polynomial in the torus-invariant divisors. This polynomial is naturally an element of the Stanley–Reisner ring of the fan, which is isomorphic to the equivariant cohomology ring of the toric variety. A reciprocity relation corresponding to the addition of a single ray to the fan was given.

Theorems 1 and 2 of this paper show that the above polynomial in the rays which represents the Todd class may be expressed nicely as a power series in the rays. The first equation of Theorem 2 reformulates the reciprocity formula [13, Theorem 3] in terms of these power series, and extends this reciprocity formula to subdivisions in which an arbitrary number of rays are added. In particular, the coefficients of the power series t_σ, which were denoted $f(\rho_1^{(a_1)}, \ldots, \rho_n^{(a_n)})$ in [13] (where ρ_i are the generators of σ), are generalizations of the classical Dedekind sum $s(p,q)$. Indeed, if a two-dimensional cone σ is isomorphic to the cone $\langle(1,0),(p,q)\rangle$ in $\mathbb{Z}^2$, then we can express the coefficient of x_1x_2 in the power series t_σ in terms of the classical Dedekind sum by the formula $q(s(p,q)+\frac{1}{4})$. Moreover, the higher coefficients in the power series t_σ for a two-dimensional cone σ are also important in number theory. The recent work of Solomon [14] examines power series quite similar to these two-dimensional t_σ in relation to zeta functions of real quadratic number fields. This connection is developed further in [7].

There is also a close link between the formulas of this paper and previous formulas of Morelli [9]. Morelli introduced a certain function μ_k from the set of k-dimensional cones in the n-dimensional lattice N to rational functions on the Grassmannian $Gr_{n-k+1}(N \otimes \mathbb{R})$. He showed that this function is additive and satisfies the relation

$$Td_k\ X_\Sigma = \sum_{\sigma \in \Sigma_{(k)}} \mu_k(\sigma)[V(\sigma)]$$

for any complete fan Σ in N. (In the formula, $[V(\sigma)]$ denotes the closed orbit corresponding to the cone σ.) This settled an old question of Danilov about the existence of such a formula for the Todd class.

If σ is a simplicial cone of dimension n, then we can relate s_σ and $\mu_k(\sigma)$ as follows: Suppose that $u_1, \ldots, u_n \in M$ are the primitive linear functionals defining σ (i.e, the primitive generators of the dual cone $\check{\sigma}$.) Define

$$r_\sigma = s_\sigma(u_1, \ldots, u_n).$$

We then have

Proposition 4. *For any n-dimensional simplicial cone σ, $\mu_n(\sigma)$ is exactly the degree 0 part of r_σ.*

Proof. For nonsingular σ, this is a consequence of Morelli's construction of μ_n, for he gives μ_n in terms of the Todd polynomials as

$$\mu_n(\sigma) = \frac{td_n(u_1, \ldots, u_n)}{u_1 \cdots u_n},$$

which agrees with the degree 0 part of the second equation of Theorem 2. For general σ, the proposition follows from additivity of both expressions: Theorem 2 implies that the function sending σ to r_σ is additive. □

Indeed, the same reasoning can be used to show that Morelli's $\mu_n^{td_1}(\sigma)$ coincides with our $s_\sigma(u_1 t, \ldots, u_n t)$. Thus, the above theorems establish a link between Morelli's formulas and the Todd class formulas of [12, 13], as well as linking Morelli's formulas with generalized Dedekind sums.

2. A Todd class formula

In this section, we give the proofs of Theorems 1 and 2, which provide a recipe for computing the Todd class of a simplicial toric variety given a nonsingular subdivision of the fan.

Let us consider Theorem 1 first. Given a lattice N, the existence of a canonical assignment of power series t_σ to the n-dimensional simplicial cones σ in N follows from Theorem 1 of [13]. This theorem asserts that the Todd class of a simplicial toric variety has a canonical expression as a polynomial in the torus-invariant divisors with rational coefficients:

$$\sum f(\rho_1^{a_1}, \ldots, \rho_k^{a_k})[V(\rho_1)]^{a_1} \cdots [V(\rho_k)]^{a_k},$$

with the sum taken over all tuples of rays of Σ and all multiplicities $a_i > 0$.

We may then define t_σ for an n-dimensional cone $\sigma = \langle \rho_1, \ldots, \rho_n \rangle$ to be the power series

$$t_\sigma(x_1, \ldots, x_n) = \sum f(\rho_1^{a_1}, \ldots, \rho_n^{a_n}) x_1^{a_1} \cdots x_n^{a_n}$$

taken over all nonnegative integers $a_1, \ldots, a_n$.

With this definition, the assertions of Theorem 1 follow immediately, with the exception of the claim that the above power series is always a rational function of x_i, e^{x_i}, $i = 1, \ldots, n$. This, however, will follow from Theorem 2.

Theorem 2 is proved by induction using the following lemma which states the behavior of the t_σ under a subdivision obtained by adding a single ray. This formula is essentially a restatement of Theorem 2 of [13] in terms of power series.

Lemma 5. *Let $\sigma = \langle \rho_1, \ldots, \rho_n \rangle$ be an n-dimensional simplicial cone, and let $\rho_0 \in \text{int}\langle \rho_1, \ldots, \rho_d \rangle$. For $i = 1, \ldots, d$, let $\sigma_i = \langle \rho_0, \ldots, \hat{\rho}_i, \ldots, \rho_n \rangle$, $m_i = \text{mult}\, \sigma_i$, and let $m_0 = -\text{mult}\, \sigma$, so that $m_0\rho_0 + m_1\rho_1 + \cdots + m_d\rho_d = 0$. Then*

$$t_\sigma(x_1, \ldots, x_n) = \sum_{i=1}^{d} t\left(-\frac{m_0}{m_i}x_i, x_1 - \frac{m_1}{m_i}x_i, \ldots, x_d - \frac{m_d}{m_i}x_i, x_{d+1}, \ldots, x_n\right)$$
$$\times \frac{x_1 \cdots \hat{x}_i \cdots x_d}{\left(x_1 - \frac{m_1}{m_i}x_i\right) \cdots \left(x_{i-1} - \frac{m_{i-1}}{m_i}x_i\right)\left(x_{i+1} - \frac{m_{i+1}}{m_i}x_i\right) \cdots \left(x_d - \frac{m_d}{m_i}x_i\right)},$$

where t is the power series whose restriction to σ_i is t_{σ_i}.

Proof. The proof relies on the following push-forward formula, which is a power series version of [13, Theorem 2].

Push-forward formula: *Let Σ be a complete simplicial fan in a lattice N. Let $\rho_1,\dots,\rho_r$ denote the rays of Σ. Suppose that $\sigma=\langle\rho_1,\dots,\rho_d\rangle\in\Sigma$, and let $\rho_0\in int\langle\rho_1,\dots,\rho_d\rangle$. Let Σ' be the fan obtained from Σ by adding the ray ρ_0, and let $\pi: X_{\Sigma'}\to X_\Sigma$ be the induced map of toric varieties. For $i=1,\dots,d$, let $\sigma_i=\langle\rho_0,\dots,\hat{\rho}_i,\dots,\rho_n\rangle$ and $m_i=mult\,\sigma_i$. Let $m_0=-mult\,\sigma$ so that $m_0\rho_0+m_1\rho_1+\cdots+m_d\rho_d=0$. For any $i=1,\dots,d$, let X_i denote $[V(\rho_i)]/m_i\in A^1X_\Sigma\otimes\mathbb{Q}$, and for $i>d$ let $X_i=[V(\rho_i)]$. Similarly, define classes $Y_i\in A^1X_{\Sigma'}\otimes\mathbb{Q}$ by $Y_i=[V(\rho_i)]/m_i$ for $i=0,\dots,d$, and $Y_i=[V(\rho_i)]$ for $i>d$. Then for any polynomial P, we have*

$$\pi_*P(Y_0,\dots,Y_r)=\sum_{i=1}^{d}P(-X_i,X_1-X_i,\dots,X_d-X_i,X_{d+1},\dots,X_r)$$
$$\times\frac{X_1\cdots\hat{X}_i\cdots X_d}{(X_1-X_i)\cdots(\widehat{X_i-X_i})\cdots(X_d-X_i)}.$$

(*In particular the right-hand side is always a polynomial in the X's.*)

Proof of the push-forward formula: Let $X_0=0$. We have the following identities:

(i) For $i>d$, and any polynomial Q,

$$\pi_*(Y_iQ(Y_0,\dots,Y_r))=X_i\pi_*(Q(Y_0,\dots,Y_r)).$$

(ii) For any $k,j\in\{0,\dots,d\}$, and any polynomial Q, we have

$$\pi_*((Y_k-Y_j)Q(Y_0,\dots,Y_r))=(X_k-X_j)\pi_*(Q(Y_0,\dots,Y_r)).$$

These identities follow, respectively, from [13, Theorem 2] (Part B) and the lemma used in the proof of this theorem.

By (i), we may assume that P is a polynomial in $Y_0,\dots,Y_d$, and by linearity, we assume P is a monomial. We will prove the equation of the theorem by induction on C, the number of coincidences in the monomial P: $C=\deg P-\#$ {nonzero exponents in P}. By (ii),

$$\pi_*(Y_kQ(Y_0,\dots,Y_r))=\pi_*(Y_jQ(Y_0,\dots,Y_r))+(X_k-X_j)\pi_*(Q(Y_0,\dots,Y_r)),$$

and it is easy to check that if the equation of Theorem 6 holds for both summands on the right-hand side above, it must also hold for the left-hand side. The above equation allows us to reduce inductively to the case $C=0$, which may be checked using the relations

$$\pi_*(Y_0\cdots Y_{d-1})=X_1\cdots X_d \quad\text{and}\quad \pi_*(Y_1\cdots Y_l)=X_1\cdots X_l$$

for $l < d$, together with the algebraic identities:

$$\sum_{i=1}^{r} \frac{z_1 \cdots \hat{z}_i \cdots z_r}{(z_1 - z_i) \cdots (\widehat{z_i - z_i}) \cdots (z_i - z_r)} = 1,$$

$$\sum_{i=1}^{r} \frac{1}{(z_1 - z_i) \cdots (\widehat{z_i - z_i}) \cdots (z_i - z_r)} = 0.$$

This completes the proof of the push-forward formula.

It is now quite easy to prove Lemma 5. First, we note that with the change of variables:

$$y_i = m_i Y_i \quad \text{and} \quad x_i = m_i X_i,$$

we can rewrite the equation of the push-forward formula as

$$\pi_* Q(y_0, \ldots, y_r) = \sum_{i=1}^{d} Q\left(-\frac{m_0}{m_i}x_i, x_1 - \frac{m_1}{m_i}x_i, \ldots, x_d - \frac{m_d}{m_i}x_i, x_{d+1}, \ldots, x_r\right)$$
$$\times \frac{x_1 \cdots \hat{x}_i \cdots x_d}{\left(x_1 - \frac{m_1}{m_i}x_i\right) \cdots \left(\widehat{x_i - \frac{m_i}{m_i}x_i}\right) \cdots \left(x_d - \frac{m_d}{m_i}x_i\right)}$$

for any polynomial Q. In this equation, x_i and y_i represent the torus invariant divisor classes (now not scaled by multiplicities) on X_Σ and $X_{\Sigma'}$, respectively.

Now by construction of the t_σ, these functions are compatible with the above push-forward formula. (Indeed the function f of [13] was defined in this way.) Thus, the above equation immediately implies Lemma 5. □

We now finish the proof of Theorem 2. We first remark that the changes of coordinates appearing in the statement of the theorem are natural in the following sense:

Claim 6. *Given three n-dimensional simplicial cones σ, γ, and δ,*

$$A_{\sigma,\delta} = A_{\gamma,\delta} A_{\sigma,\gamma}.$$

Proof. Let δ be generated by $\beta_1, \ldots, \beta_n$, and let $u_1, \ldots, u_n$ be the basis dual to $\beta_1, \ldots, \beta_n$. Comparing i, j-entries, we wish to show that

$$\langle u_i, \rho_j \rangle = \sum_{k=1}^{n} \langle u_i, \tau_k \rangle \langle w_k, \rho_j \rangle.$$

So it is enough to show

$$u_i = \sum_{k=1}^{n} \langle u_i, \tau_k \rangle w_k.$$

But this follows from the fact that either side paired with τ_l yields $\langle u_i, \tau_l \rangle$. □

To prove Theorem 2, we observe that any two simplicial subdivisions of σ are equivalent by a sequence of stellar subdivisions (such as those of Lemma 5). Indeed, any fan may be desingularized by a sequence of such stellar operations (cf. [4]), and a theorem of Morelli [10] (proved also by Wlodarczyk, though not published) implies that any two nonsingular fans with the same support are connected by such a sequence. Thus, to prove Theorem 2, it suffices to show that if Γ is a simplicial subdivision of σ and we add a ray $\rho_0 \subset \sigma$ to form a new subdivision Γ', then if the equation of the theorem holds for Γ, it holds also for Γ', and conversely.

Fix a cone $\gamma \in \Gamma$. The addition of the new ray ρ_0 subdivides γ into a number of maximal cones $\gamma_1, \ldots, \gamma_r$ of Γ' (possibly $r = 1$). If we apply Lemma 5 to this situation, we obtain

$$s_\gamma(Y) = \sum_{i=1}^{r} s_{\gamma_i}(A_{\gamma,\gamma_i} Y).$$

This follows by checking that the changes of coordinates in Lemma 5 really coincide with those given by Theorem 2 in the case that a single ray is added.

But now if we change variables by letting $Y = A_{\sigma,\gamma}X$, and sum over all cones $\gamma \in \Gamma$, we obtain

$$\sum_{\gamma \in \Gamma_{(n)}} s_\gamma(A_{\sigma,\gamma}X) = \sum_{\gamma' \in \Gamma'_{(n)}} s_{\gamma'}(A_{\gamma,\gamma'}A_{\sigma,\gamma}X).$$

Hence, by Claim 6, we conclude that

$$\sum_{\gamma \in \Gamma_{(n)}} s_\gamma(A_{\sigma,\gamma}X) = \sum_{\gamma' \in \Gamma'_{(n)}} s_{\gamma'}(A_{\sigma,\gamma'}X).$$

This implies immediately that the equation of the theorem holds for Γ if and only if it holds for Γ'. The proof of Theorem 2 is complete.

3. The Todd class and Barvinok subdivisions

The purpose of this section is to prove Theorem 3 which states the behavior of the functions t_σ under virtual subdivisions.

Theorem 3 will follow from Lemma 7 below which describes the behavior of the s_σ under a virtual subdivision induced by the addition of a single ray. The proof that Lemma 7 implies Theorem 3 is similar to the proof in the preceding section that Theorem 2 follows from Lemma 5.

Lemma 7. *Let $\sigma = \langle \rho_1, \ldots, \rho_n \rangle$ be an n-dimensional simplicial cone, and let ρ_0 be any ray such that $\rho_0, \ldots, \rho_n$ are contained in a half-space. Let $m_0 = -\mathrm{mult}\,\sigma$ and suppose ρ_0 is in the d-plane defined by $\rho_1, \ldots, \rho_d$, with $\sum_{i=0}^{d} m_i \rho_i = 0$, as above, and all $m_i \neq 0$*

for $i = 1, \ldots, d$. *Let*

$$\sigma = \sum_{i=1}^{d} \delta_i \sigma_i$$

be the virtual subdivision Γ *so obtained. (So that now we have* $m_i = \delta_i \, mult\, \sigma_i$.) *Then with these sign changes, the formula of Lemma* 5 *holds for the virtual subdivision* Γ:

$$t_\sigma(x_1, \ldots, x_n) = \sum_{i=1}^{d} t_{\sigma_i}\left(-\frac{m_0}{m_i}x_i, x_1 - \frac{m_1}{m_i}x_i, \ldots, x_d - \frac{m_d}{m_i}x_i, x_{d+1}, \ldots, x_n\right)$$
$$\times \frac{x_1 \cdots \hat{x}_i \cdots x_d}{\left(x_1 - \frac{m_1}{m_i}x_i\right)\cdots\left(x_{i-1} - \frac{m_{i-1}}{m_i}x_i\right)\left(x_{i+1} - \frac{m_{i+1}}{m_i}x_i\right)\cdots\left(x_d - \frac{m_d}{m_i}x_i\right)}.$$

Proof. Let $\rho_0, \ldots, \rho_n$ be as in the statement of the lemma. By reordering, let us assume that

$$\delta_1, \ldots, \delta_r = -1 \quad \text{and} \quad \delta_{r+1}, \ldots \delta_d = 1.$$

Set $\delta_0 = -1$. We then have

$$m_i = \delta_i \, mult\, \sigma_i$$

for all $i = 0, \ldots, d$.

Let β be the ray defined by the primitive element of N lying in the one-dimensional cone

$$\langle \rho_0, \ldots, \rho_r \rangle \cap \langle \rho_{r+1}, \ldots, \rho_d \rangle.$$

Let K denote the convex hull

$$K = \text{conv}\{\rho_0, \rho_1, \ldots, \rho_n\}.$$

We introduce two fans Σ^- and Σ^+ with support K. Σ^- and Σ^+ have the following sets of maximal cones:

$$\Sigma^-_{(n)} = \{\sigma, \sigma_1, \ldots, \sigma_r\}, \qquad \Sigma^+_{(n)} = \{\sigma_{r+1}, \ldots, \sigma_d\}.$$

For $i \in \{0, \ldots, r\}$, and $j \in \{r+1, \ldots, d\}$, define a cone

$$\sigma_{ij} = \langle \beta, \rho_0, \ldots, \hat{\rho}_i, \ldots, \hat{\rho}_j, \ldots, \rho_n \rangle,$$

and let Σ be the fan whose set of maximal cones is

$$\Sigma_{(n)} = \{\sigma_{ij} \mid i \in \{0, \ldots, r\} \text{ and } j \in \{r+1, \ldots, d\}\}.$$

Then Σ is a common refinement of Σ^- and Σ^+, so that we have natural maps

$$\pi^- : X_\Sigma \to X_{\Sigma^-}, \qquad \pi^+ : X_\Sigma \to X_{\Sigma^+}.$$

In fact, Σ may be obtained from either Σ^- or Σ^+ by adding the single ray β.

Let us fix $i \in \{0,\dots,r\}$. Then the cone σ_i is subdivided into the cones

$$\{\sigma_{ij} \mid j \in \{r+1,\dots,d\}\}$$

of Σ. Applying Lemma 5 to this situation we get, with the notation of Section 2 and with $Y=(y_0,\dots,\hat{y}_i,\dots,y_n)$,

$$s_{\sigma_i}(Y) = \sum_{j=r+1}^{d} s_{\sigma_{ij}}(A_{\sigma_i,\sigma_{ij}}Y),$$

and therefore with the change of variables $Y = A_{\sigma,\sigma_i}X$, we obtain

$$s_{\sigma_i}(A_{\sigma,\sigma_i}X) = \sum_{j=r+1}^{d} s_{\sigma_{ij}}(A_{\sigma_i,\sigma_{ij}}A_{\sigma,\sigma_i}X),$$

which by Claim 6 reduces to

$$s_{\sigma_i}(A_{\sigma,\sigma_i}X) = \sum_{j=r+1}^{d} s_{\sigma_{ij}}(A_{\sigma,\sigma_{ij}}X).$$

Similarly, for each $j \in \{r+1,\dots,d\}$, we get

$$s_{\sigma_j}(A_{\sigma,\sigma_j}X) = \sum_{i=0}^{r} s_{\sigma_{ij}}(A_{\sigma,\sigma_{ij}}X).$$

Now we sum this equation over all $j \in \{r+1,\dots,d\}$ and the previous equation over all $i \in \{0,\dots,r\}$. Since the right-hand sides are identical after interchanging the order of summation, we may set the left-hand sides equal and obtain

$$\sum_{i=0}^{r} s_{\sigma_i}(A_{\sigma,\sigma_i}X) = \sum_{j=r+1}^{d} s_{\sigma_j}(A_{\sigma,\sigma_j}X).$$

And since $A_{\sigma,\sigma}$ is the identity, we may rewrite this as

$$s_{\sigma}(X) = \sum_{i=1}^{d} \delta_i s_{\sigma_i}(A_{\sigma,\sigma_i}X),$$

which is the content of the lemma. □

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