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Loïc Pottier

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Abstract: We study here Gröbner bases of ideals which define toric varieties. We connect these ideals with the sub-lattices of Z^d, then deduce properties on their Gröbner bases, and give applications of these results. The main contributions of the report are a bound on the degree of the Gröbner bases, the fact that they contain Minkowski successive minima of a lattice (in particular shortest vector), and the algorithm (derived from Buchberger algorithm), which starts with ideal of polynomials with less variables than usual .

Key-words: Standard bases, Gröbner bases, toric varieties, sucessive minima

(Résumé : tsvp)

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Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Bases de Gröbner d'idéaux toriques

Résumé : Nous étudions ici les bases de Gröbner d'idéaux définissant des variétés toriques. Nous mettons en relation ces idéaux avec les sous-réseaux de Z^d, déduisons des propriétés de leurs bases de Gröbner , et donnons des applications de ces résultats. Les principales contributions de ce rapport sont une borne sur le degré des bases de Gröbner , le fait qu'elles contiennent les minimaux successifs de Minkowski d'un réseau (en particulier le plus court vecteur), et l'algorithme (dérivé de celui de Buchberger), qui prend en entrée un idéal de polynômes ayant moins de variables qu'usuellement.

Mots-clé : Bases standard, bases de Gröbner , variétés toriques, minimaux sucessifs

1 Toric varieties and toric ideals

We will consider here only toric varieties as algebraic varieties parametrized by monomials (see [14] or [6] for general studies of toric varieties):

$$X = \{(X_1, \dots, X_n) \in \mathbb{C}^n | X_1 = T^{a_1}, \dots, X_n = T^{a_n},\ T_1, \dots, T_d \in \mathbb{C}\} \subset \mathbb{C}^n$$

where a_i are points in $\mathbb{Z}^d$.
The variety X is the zero-set of the ideal I, kernel of the map

$$\begin{array}{rccc} \phi: & k[X] & \longrightarrow & k[T^{\pm 1}] \\ & X_i & \longmapsto & T^{a_i} \end{array}$$

Such a prime ideal is called now "toric ideal".
To the set $\mathcal{A} = \{a_1, \dots, a_n\}$ we associate the integer lattice $H = \{v \in \mathbb{Z}^n | \sum_i v_i a_i = 0\}$, which is related to I by the

Proposition 1

$$X^\alpha - X^\beta \in I \iff \alpha - \beta \in H$$

As a consequence, we have:

Proposition 2

$$X^\alpha - X^\beta \in I \implies X^{(\alpha-\beta)^+} - X^{(\alpha-\beta)^-} \in I$$

where v^+ denotes the vector v in which the negative coordinates are set to zero, and $v^- = (-v)^+$.

Basic properties of toric ideals can be found in [11]. We recall some of them:

Proposition 3 *[11] I is generated by the binomials $X^{v^+} - X^{v^-}$ where $v \in H$. Reduced Gröbner bases of I contains only binomials $X^{v^+} - X^{v^-}$ where $v \in H$, and have degree lesser than $n(n-d)a^d$ where $a = sup\{\|a_i\|\}$.*

2 Properties of Gröbner bases of toric ideals

First we recall the definition of a Gröbner basis of an ideal.
Let $<$ be a total order on monomials, compatible with multiplication (e.g. lexicographic order, reverse lexicographic order,etc). We denote $in(f)$ the greatest monomial of a polynomial f. In the following, when a polynomial is written $X^a + \dots$, we suppose that X^a is the greatest monomial.

Definition 1 *A familly F of polynomials of an ideal J is a Gröbner basis of J iff the set $in(F) = \{in(f) | f \in F\}$ generates the ideal $in(J) = \{in(f) | f \in J\}$.*

The basis is "reduced" if no monomial of its polynomials is a multiple of the greatest monomial of one of its polynomials.

2.1 Degree of toric Gröbner bases

In general the degree of Gröbner basis is doubly exponential in the number of variables [7]. But in the toric case this complexity falls to a simply exponential degree, as mentionned above ([11]). We prove now another simply exponential bound:

Theorem 1 *Let A be the $d \times n$ matrix whose columns are the a_is, with generic term a_{ij}. Then any reduced Gröbner basis of I has degree lesser than*

$$\prod_i (1 + \sum_j |a_{ij}|)$$

Proof : Let H^+ the set of vectors of H with positive coordinates. It is a monoid of finite type (by Gordan's lemma), generated by its minimal elements for the order $v \leq w \iff \forall i\ v_i \leq w_i$.
In [8] is proved that these minimal elements verify:

$$\sum_i v_i \leq \prod_i (1 + \sum_j |a_{ij}|)$$

More generally consider now the vectors of H with any given choice for the signs of their coordinates. They form a monoid to which the previous results can be applied (just change the sign of the corresponding columns of A): they are finitely generated by minimal vectors verifying $\sum_i |v_i| \leq \prod_i (1 + \sum_j |a_{ij}|)$
Suppose now that $X^{\alpha^+} - X^{\alpha^-}$ is in a reduced Gröbner basis of I, with $X^{\alpha^+} > X^{\alpha^-}$. The vector α belongs to one of the previous monoids. If it is not minimal, there we can write $\alpha^+ = \beta^+ + \gamma^+$ and $\alpha^- = \beta^- + \gamma^-$, with $\beta, \gamma \in H - \{0\}$. Because $X^{\alpha^+} > X^{\alpha^-}$, we can suppose $X^{\beta^+} > X^{\beta^-}$. But $X^{\beta^+} - X^{\beta^-} \in I$, so $\alpha^+ = \beta^+, \alpha^- = \beta^-$ because the Gröbner basis is reduced, and then $\gamma = 0$, which contradicts the hypotheses.
So α is minimal and then verify $\sum_i \alpha_i \leq \prod_i (1 + \sum_j |a_{ij}|)$ $\square$.

2.2 Geometric caracterization

We show here how any reduced Gröbner basis of the ideal I can be read on the lattice H, giving an explicit caracterization which does not use Buchberger algorithm.
Let H_+ the part of H constitued with vectors v of H such that $X^{v^+} >_w X^{v^-}$ (for example, with lexicographic ordering, those vectors are vectors with first non zero coordinate which is positive).

Theorem 2 *Let B be a part of H_+ such that $\forall v \in H_+, \exists w \in B, w^+ \preceq v^+$.*
Then the familly $\{X^{v^+} - X^{v^-} \mid v \in B\}$ is a Gröbner basis of I.

To prove this assertion, we will need some intermediate propositions.
First, a direct consequence of the definition of a Gröbner basis:

Proposition 4 *F is a Gröbner basis of an ideal J iff every monomial of $in(J)$ is divided by a monomial $in(f)$ where $f \in F$.*

Let $\mathcal{H}$ the familly of polynomials $\{X^{v^+} - X^{v^-} \mid v \in H_+\}$.

Proposition 5 *Every monomial of $in(I)$ is divided by a monomial of $in(\mathcal{H})$.*

Proof : Let P a non zero polynomial of I. By definition of I and H_+, $\mathcal{H}$ generates I, P can be written $P = \sum_{i=1\ldots p} a_i X^{\gamma_i}(X^{v_i^+} - X^{v_i^-})$ where the v_is are in H_+.
The following lemma precises this:

Lemma 1 *One can choose such an expression of P such that $in(P) = X^{\gamma_j} X^{v_j^+}$ for a certain j.*

Proof : By induction on p, the number of terms of the expression of P.
For $p = 1$ this is clear.
For $p > 1$, let $Q = \sum_{i=1\ldots p-1} a_i X^{\gamma_i}(X^{v_i^+} - X^{v_i^-})$.
By induction hypothesis, $in(Q) = X^{\gamma_j} X^{v_j^+}$ for a certain $j < p$.
If $in(Q) < X^{\gamma_p} X^{v_p^+}$, then $in(P) = X^{\gamma_p} X^{v_p^+}$ $\square$
If $in(Q) > X^{\gamma_p} X^{v_p^+}$, then $in(P) = in(Q) = X^{\gamma_j} X^{v_j^+}$ $\square$
If $in(Q) = X^{\gamma_p} X^{v_p^+}$ and $a_p + a_j \neq 0$, then $in(P) = X^{\gamma_p} X^{v_p^+}$ $\square$
Finally if $in(Q) = X^{\gamma_p} X^{v_p^+} = X^{\gamma_j} X^{v_j^+}$ and $a_p + a_j = 0$, then $P = a_p(X^{\gamma_j} X^{v_j^-} - X^{\gamma_p} X^{v_p^-}) + \sum_{i=1\ldots p-1, i \neq j} a_i X^{\gamma_i}(X^{v_i^+} - X^{v_i^-})$.

But $a_p(X^{\gamma_j}X^{v_j^-} - X^{\gamma_p}X^{v_p^-})$ can be written $\pm a_p X^\delta(X^{v^+} - X^{v^-})$ with $v = \pm(v_p - v_j) \in H_+$; the expression of P has then $p-1$ terms, and we conclude by the induction hypothesis $\square$

Now, we come back to the proposition: by the lemma, $in(P) = X^{\gamma_j}X^{v_j^+}$, we have $X^{v_j^+} \in in(\mathcal{H})$, and then $in(P)$ is divided by a monomial of $in(\mathcal{H})$ $\square$

We now prove the theorem: the definition of B shows that every monomial of $in(\mathcal{H})$ is divided by a monomial X^{v^+} where v is in B. Then property 4 gives the conclusion $\square$

Definition 2 *We call "Gröbner basis of H" (for the ordering $<_w$ on monomials), the part $Bs(H,<_w)$ of H_+ whose elements are vectors v of H verifying:*

- *(i)* $\forall u \in H_+, u^+ \preceq v^+ \implies u^+ = v^+$
- *(ii)* $\forall u \in H_+, u^+ \not\preceq v^-$

This definition is justified by the

Theorem 3 *The familly $\{X^{v^+} - X^{v^-} \mid v \in Bs(H,<_w)\}$ is the reduced Gröbner basis of I for the order $<_w$ on monomials.*

which is an easy consequence of the previous theorem.

3 Algorithms for toric Gröbner bases

3.1 A good ideal to begin with

The usual techniques, used in [4] or [10], is to compute a Gröbner basis of the ideals $J = (X_1 - T^{a_1}, \ldots, X_n - T^{a_n}, T_1{T_1}^{-1} - 1, T_m{T_m}^{-1} - 1)$, or $J = (X_1 - T^{a_1}, \ldots, X_n - T^{a_n}, UT_1 \ldots T_m X_1 \ldots X_n - 1)$, with Buchberger algorithm and an order eliminating U,T_j and ${T_j}^{-1}$, and then keep the binomials where U,T_j and ${T_j}^{-1}$ do not appear, which form a Gröbner basis for I.

These methods use Gröbner bases computations with $n+2d$ or $n+d+1$ variables. We present here a method which uses only $n+1$ variables, and then is much more efficient in practice.

Theorem 4 *Let $\{v_1, \ldots, v_p\}$ be a basis of the lattice H . Let J be the ideal*

$$(X^{v_1^+} - X^{v_1^-}, \ldots, X^{v_p^+} - X^{v_p^-}, UX_1 \ldots X_n - 1)$$

Let B be a Gröbner basis of J for an order eliminating the variable U. Then the binomials of B where U does not appear form a Gröbner basis of I.

Proof :
Consider the morphisms

$$\begin{array}{ccccccc} & \phi_1 & & \phi_2 & & \phi_3 & \\ k[X] & \longrightarrow & k[X,U]/(UX_1\ldots X_n-1) & \longrightarrow & k[X^{\pm 1}] & \longrightarrow & k[T^{\pm 1}] \\ X_i & \longmapsto & X_i & \longmapsto & X_i & \longmapsto & T_i^{a_i} \\ & & U & \longmapsto & X_1{}^{-1}\ldots X_n{}^{-1} & & \end{array}$$

We have $I = Ker(\phi_3 \circ \phi_2 \circ \phi_1)$ (ϕ_2 is an isomorphism keeping the X_is invariant). Let $J = Ker(\phi_3)$. Then $I = \phi_2{}^{-1}(J) \cap k[X]$, and $J = (X^v - 1)_{v \in H}$.
Now, we prove that $J = (X^{v_1} - 1, \ldots, X^{v_p} - 1)$, which gives the result, because that it implies $J = (X^{v_1^+} - X^{v_1^-}, \ldots, X^{v_p^+} - X^{v_p^-})$
Let $v = \sum_j \alpha_j v_j$ a vector of H.
We have $X^v - 1 = X^{\sum_j \alpha_j v_j} - 1$
$= (X^{\sum_j \alpha_j v_j} - X^{\alpha_1 v_1}) + (X^{\alpha_1 v_1} - 1)$
$= X^{\alpha_1 v_1}(X^{\sum_{j \geq 2} \alpha_j v_j} - 1) + (X^{v_1 a_1} - 1)$
if a_1 is nonnegative, $(X^{v_1} - 1)$ divides $X^{v_1 a_1} - 1$, otherwise $(X^{-v_1} - 1)$ divides $X^{v_1 a_1} - 1$.
Then we are led to the same problem for the vector $\sum_{j \geq 2} \alpha_j v_j$, and by induction we conclude $X^v - 1 \in (X^{v_1} - 1, \ldots, X^{v_p} - 1)$ □
The computation of a basis of H can be performed in polynomial time (by computing the Hermite normal form of the matrix A) but coefficients of the v_i can be very large in regard with those of the matrix A. However, in practice the reduction of the number of variables in the binomials suffices to drastically reduce the time of the computations.

3.2 Improving Buchberger algorithm for the toric case

Buchberger algorithm [2] works with general polynomials. But the toric case is very specific:

Remark 1 *Polynomials are only binomials, and reductions of binomials are again binomials.*

Remark 2 *Binomials of the ideal I have no variable in common: if $X^{a+c} - X^{b+c}$ belongs to I then $X^a - X^b$ belongs to I. So we can perform these simplifications at any step of the algorithm. More, we can then represent (and implement) a binomial $X^a - X^b$ of I by the vector $a - b$ of H.*

Remark 3 *Divisions of binomials correspond to additions or subtractions in H:*

- *if X^{b^+} divides X^{a^+}, then $X^{a^+} - X^{a^-}$ reduces to $X^{(a-b)^+} - X^{(a-b)^-}$. We denote this property on vectors by $a \xrightarrow{+}_b a - b$.*
- *if X^{b^+} divides X^{a^-}, then $X^{a^+} - X^{a^-}$ reduces to $X^{(a+b)^+} - X^{(a+b)^-}$. We denote this property on vectors by $a \xrightarrow{-}_b a + b$.*
- *The S-polynomial (called also critical pair) of two binomials $X^{a^+} - X^{a^-}$ and $X^{b^+} - X^{b^-}$ corresponds to the vector $a - b$.*

Then we have the

Lemma 2 *Let B be a set of binomials.*
If $a_1 \xrightarrow{+} \ldots \xrightarrow{+} a_r \xrightarrow{-} \ldots \xrightarrow{-} a_l$ by binomials of B,
if a_r is irreducible for any reduction $\xrightarrow{+}$ by binomials of B, and a_l is irreducible for any reduction $\xrightarrow{-}$ by binomials of B,
then a_l is irrecucible by the binomials of B.

This lemma discards many unnecessary tests for division.
Each couple of binomial P_i and P_j gives a critical pair, ties to a monomial denoted lcm_{ij}, which is the lcm of the two leading monomials of $in(P_i)$ and $in(P_j)$.
It is well-known [2] that a critical pair between P_i and P_j can be discarded in essentially two cases:

- $in(P_i)$ and $in(P_j)$ are relatively prime (no variable in common).
- lcm_{kl} strictly divides lcm_{ij}, i.e. lcm_{ij} is not minimal for the component-wise order.

4 Applications

4.1 Successive minima of a lattice

We show here how the Minkowski successive minima of the lattice H are given by a Gröbner basis of I.
Given a norm on $\mathbb{R}^n$, the successive minima $\lambda_1, \ldots, \lambda_p$ of a lattice H of dimension p are defined in the following way: λ_k is the radius of the smallest ball (for the choosen norm) contaning k independant vectors of H.
In[3], general results on successive minima and lattices are given.

One can show that there exists a free familly v_i of H such that $\|v_i\| = \lambda_i$.
The second fondamental theorem of Minkowski's geometry of numbers gives upper and lower bounds on $\prod_i \lambda_i$ for the euclidian norm.
Here, we will use two norms:
the 1norm: $\|x\|_1 = \sum_i \mid x_i \mid$
and the -1norm: $\|x\|_{-1} = sup(\|x^+\|_1, \|x^-\|_1)$
Some easy remarks will be useful:

Remark 4 *Suppose that $X^a - X^b$ reduces to 0 by binomials $\{X^{a_1} - X^{b_1}, \ldots, X^{a_q} - X^{b_q}\}$. Then the vectors $a - b, a_1 - b_1, \ldots, a_q - b_q$ are not independant.*

We note $lad(I)$ "the ladder of I ", i.e. the minimal monomials (for the partial order of division of monomials) of $in(I)$.

Remark 5 *Let $X^a - X^b$ be a binomial of I such that $X^a \in lad(\mathcal{I})$.Then $a = (a-b)^+$ and $b = (a-b)^-$.*

Theorem 5 (Homogeneous case) *Suppose that H is "homogeneous",i.e. contained in the hyperplane $x_1 + \ldots + x_n = 0$.*
Let $\lambda_1, \ldots, \lambda_p$ be the successive minima of H for the 1norm.
Let B be a Gröbner basis of I for $<_{(1,\ldots,1)}$ (i.e. total degree order).
Then there exists independant vectors $a_1, \ldots, a_p$ of H such that $\forall k$, $\|a_k\|_1 = \lambda_k$ and $X^{a_k^+} - X^{a_k^-}$ is in B.
In other terms, a Gröbner basis of I for $<_{(1,\ldots,1)}$ contains the successive minima of H for the 1norm.

Proof : First remark that H is homogeneous, then the ideal I is also homogeneous, so for all binomial $X^{v^+} - X^{v^-}$ of I, we have $deg(X^{v^+}) = deg(X^{v^-}) = \|v\|_1/2$.
We will build the vectors $a_1, \ldots, a_p$ by induction.

- Case $k = 1$:

  Let a in H such that $\|a\|_1 = \lambda_1$. The binomial $X^{a^+} - X^{a^-}$ is in I, then there exists a binomial $X^{a_1^+} - X^{a_1^-}$ of B such that $X^{a_1^+} \in lad(I)$. Then $X^{a_1^+}$ divides X^{a^+}, and $deg(X^{a_1^+}) \leq deg(X^{a^+})$. By definition of λ_1, we have then $\|a_1\|_1 = \lambda_1$, so $a = a_1 \in B$ $\Box$

- Case $k > 1$:

Let $a_1, \ldots, a_{k-1}$ giving the successive minima $\lambda_1, \ldots, \lambda_{k-1}$, then there exists a in H such that $\|a\|_1 = \lambda_k$, and a independent of $a_1, \ldots, a_{k-1}$ (such an a exists).

Let B_k be the part of B formed by the binomials $X^{b^+} - X^{b^-}$ where $a_1, \ldots, a_{k-1}, b$ are not independent.

If the binomial $X^{a^+} - X^{a^-}$ reduces to 0 by division by binomials of B_k, by Remark 4, $a_1, \ldots, a_{k-1}, a$ are not independent. Then $X^{a^+} - X^{a^-}$ reduced by B_k is a non zero binomial $X^{a'^+} - X^{a'^-}$, irreducible by B_k, with $\|a'\|_1 = \|a\|_1 = \lambda_k$ because the binomials of B are homogeneous.

As $X^{a'^+} - X^{a'^-} \in \mathcal{I}$, there exists a binomial $X^{a_k^+} - X^{a_k^-}$ of $B - B_k$ such that $X^{a_k^+} \in lad(I)$, $X^{a_k^+}$ divides $X^{a'^+}$, and $deg(X^{a_k^+}) \leq deg(X^{a'^+})$, $\|a_k\|_1 \leq \lambda_k$. But $a_1, \ldots, a_{k-1}, a_k$ are independant vectors, and then by definition of λ_k, $\|a_k\|_1 = \lambda_k$, $a_k \in B$ $\square$

We can easily generalize the theorem to the non homogeneous case:

Theorem 6 (General case) *Suppose H is not necessarily homogeneous. A Gröbner basis of I for $<_{(1,\ldots,1)}$ contains the successive minima of H for the -1norm.*

4.2 Delaunay triangulation

A triangulation of the set of points $a_1, ..., a_n$ of $\mathbb{Z}^d$ is said "regular" iff there exists a vector w of $\mathbb{R}^n$ such that this triangulation is obtained as the projection of the lower convex envelop of the points $(a_1, w_1), ..., (a_n, w_n)$ of $\mathbb{R}^{d+1}$.
A theorem of Sturmfels [10] gives relations between regular triangulations of $a_1, ..., a_n$ and the Gröbner bases of the toric ideal I.
Suppose that the points $a_1, ..., a_n$ are in an affine hyperplane and generate $\mathbb{Z}^d$. Let Δ_w be the triangulation defined by $w \in \mathbb{R}^n$. Let I_Δ be the ideal generated by the monomials $X_{\sigma_1} ... X_{\sigma_p}$ where $\{\sigma_1, ..., \sigma_p\}$ is not a face of Δ_w (the Stanley-Reisner ideal of Δ_w) .

Theorem 7 ([10]) *I_Δ is the radical of $in_w(I)$*

The Delaunay triangulation is regular, as obtained with the vector $(||a_1||^2, ..., ||a_n||^2)$. Then, a consequence of the theorem is:

Corollary 1 *Let $w = (||a_1||^2, ..., ||a_n||^2)$, then a Gröbner basis of I for $<_w$ gives the Delaunay triangulation of $a_1, ..., a_n$.*

The complexity of this computation depends directly from the degree of the Gröbner basis, bounded by $(1+na)^d$ where a is the maximum coordinate of the a_is. We cannot *a priori* compete with specialized algorithms for Delaunay triangulation (which have however a theoretical complexity of the same order), but we can expect to interpret them in the context of polynomials.

Example:

Let $a_1 = (0,0,1), a_2 = (6,0,1), a_3 = (5,2,1), a_4 = (1,4,1), a_5 = (2,2,1), a_6 = (4,3,1)$ in the plane identified to $\{z = 1\}$ in $\mathbb{R}^3$.

Computations with softwares Macaulay [1] or bastat [9], give a Gröbner basis for I with initial monomials

$\{X_1^2X_6^6, X_1X_2X_6^6, X_1X_3^2, X_1X_3X_4, X_2^2X_6^6, X_3^6X_4^2, X_2X_4, X_3^5X_4^3, X_3^4X_4^2X_5\}$

and then $I_\Delta = (X_1X_6, X_1X_3, X_2X_6, X_3X_4, X_2X_4)$,

the Delaunay triangulation is $\Delta = \{\{1,2,5\},\{1,4,5\},\{2,3,5\},\{3,5,6\},\{4,5,6\}\}$.

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Unité de recherche INRIA Lorraine, Technôpole de Nancy-Brabois, Campus scientifique,
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