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HILBERT'S 17TH PROBLEM AND THE CHAMPAGNE PROBLEM.

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Dedicated to Alex Rosenberg on the occasion of his 70^{th} birthday

INTRODUCTION

In 1981, at a conference on Real Algebraic Geometry and Ordered Fields in Rennes, France, E. Becker gave a talk in which he proved that

$$B(t) := \frac{1+t^2}{2+t^2} \in \mathbb{Q}(t)$$

is a sum of $2n$-th powers of elements in $\mathbb{Q}(t)$ for all n. To prove this surprising fact, Becker used his newly developed theory of higher level orderings on fields; the proof was not constructive. He then proposed the following problem: Find an explicit formula giving a representation of $B(t)$ as a sum of $2n$-th powers for all n. Becker promised a bottle of champagne to the first person to solve this, as a result the problem became known as the Champagne Problem. The problem still remains unsolved in the form stated by Becker, however recent work of B. Reznick gives an explicit formula for $B(t)$ as a sum of $2n$-th powers of elements in $\mathbb{R}(t)$.

The theory of higher level orderings on fields, and hence the Champagne Problem, has its genesis in Hilbert's 17th Problem and E. Artin's solution to it. In this paper we trace the history of these roots of the Champagne Problem and briefly describe Reznick's solution.

FROM HILBERT'S 17TH PROBLEM TO ORDERED FIELDS

The Champagne Problem is part of a class of problems concerned with representations of positive semi-definite rational functions as sums of squares of rational functions or, more generally, sums of even powers. A rational function $f \in \mathbb{R}(X) := \mathbb{R}(x_1, \ldots, x_k)$ is *positive semi-definite* (psd) if $f \geq 0$ at every point in $\mathbb{R}^k$ for which it is defined. D. Hilbert, in a paper published in 1888 [H1], showed that there exist psd polynomials that are not sums of squares of polynomials. In 1900, at the International Congress of Mathematics in Paris, Hilbert gave a lecture in which he proposed 23 open problems. Most of these have since been solved, and the solutions have led to fundamental discoveries in mathematics.

Hilbert's work on sums of squares of polynomials was the impetus for the 17th problem: "A rational integral function or form in any number of variables with real coefficients such that it becomes negative for no real values of these variables, is said to be *definite*. But since, as I have shown, not every definite form can be compounded by addition from squares of forms, the question arises – which I have answered affirmatively for ternary

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forms ([H2]) – whether every definite form may not be expressed as a quotient of sums of squares of forms" [H3]. In other words, given a psd polynomial $f \in \mathbb{R}[x_1, \dots, x_k]$, can f be written as a sum of squares of elements in $\mathbb{R}(X)$?

In 1927, Artin solved the 17th Problem in the affirmative [A], using the theory of ordered fields. Artin realized that it is important to study sums of squares in arbitrary fields. For a field F, let $\sum F^2 = \{y_1^2 + \cdots + y_r^2 \mid r \in \mathbb{N}, y_i \in F\}$. If F has characteristic 2, then $\sum F^2 = F^2$, and if char $F \neq 2$ and $-1 \in \sum F^2$, it follows from the formula $x = (\frac{x+1}{2})^2 - (\frac{x-1}{2})^2$ that $\sum F^2 = F$. This leaves the case $-1 \notin \sum F^2$; such fields are called *formally real.* Notice that if F is formally real, then F must have characteristic 0 since char $F = n$ would imply $-1 = (n-1) \cdot 1 \in \sum F^2$. Artin, along with O. Schreier, showed that formally real fields are precisely those that have an order.

The definition of an ordered field goes back to Hilbert, however he did not develop the theory. It was Artin and Schreier who laid out the fundamentals of the theory in two papers published in 1927 [AO1,AO2]. We briefly desribe their major results.

An *order* on a field F is given by a subset $P \subseteq F$, sometimes called the *positive cone* of the order, which satisfies: $P \cdot P \subseteq P$, $P + P \subseteq P$, $P \cap -P = \{0\}$, and $P \cup -P = F$. We will write "P is an order" to mean P is the positive cone of an order. Given P an order on F we can define binary relations $\leq$ and $<$ on F by $x \leq y$ iff $y - x \in P$, and $x < y$ iff $y - x \in P$ and $y \neq x$. One can check that $<$ is then a total order on F and the usual rules for inequalities hold for $<$ and $\leq$. Note that we can recover P from $\leq$ via $P = \{x \in F \mid 0 \leq x\}$.

The easiest examples of orders are the obvious orders on $\mathbb{Q}$ and $\mathbb{R}$, and it is not too hard to show that these are the only orders on these fields. Suppose F is a subfield of K, and K has an order P, then it is easy to see that $F \cap P$ is an order on F. However, not all orders on F arise in this fashion. To see this, consider $F = \mathbb{Q}[\sqrt{2}]$, then in addition to the order arising from the order on $\mathbb{R}$, one can check that $\{a + b\sqrt{2} \mid 0 \leq a - b\sqrt{2} \text{ in } \mathbb{R}\}$ is also an order.

For a subset $S \subseteq F$, we write $\dot{S}$ to denote $S \backslash \{0\}$. It is easy to see that $\sum \dot{F}^2$ is a subgroup of $\dot{F}$ using the fact that if $y = \sum y_i^2 \in \sum F^2$, then $1/y = \sum (y_i/y)^2 \in \sum F^2$. Given any order P on F and $x \in F$, since $x \in P$ or $-x \in P$, it follows that $x^2 = (\pm x)(\pm x) \in F$. Hence, by additive closure, $\sum F^2 \subseteq P$. Then $\dot{P}$ is a subgroup of $\dot{F}$, since $x \in \dot{P}$ implies $1/x = x \cdot 1/x^2 \in \dot{P} \cdot \dot{P} \subseteq \dot{P}$. Note that $\dot{P}$ is a subgroup of index 2 in $\dot{F}$ which is additively closed. It is easy to see that conversely, for any subgroup Q of index 2 in $\dot{F}$ which is additively closed, $Q \cup \{0\}$ is an order on F.

For any order P, we have just seen that $\sum F^2 \subseteq P$. Thus $1 \in P$ and hence, if F has an order, F must be formally real. Artin and Schreier proved the converse of this:

Artin-Schreier Theorem. *A field F admits an order iff F is formally real.*

The key idea needed for the proof of this theorem is that of a real closed field: F is *real closed* if F is formally real and no proper algebraic extension of F is formally real. Note that by Zorn's Lemma, any formally real field admits a maximal algebraic extension which is formally real, hence every formally real field is contained in a real closed field. Artin and Schreier proved the following characterization of real closed fields:

Theorem. *For a field F the following are equivalent:*

(i) *F is real closed,*
(ii) *$-1 \notin F^2$ and $F[\sqrt{-1}]$ is algebraically closed,*
(iii) *F is formally real, $\dot{F}^2$ has index 2 in $\dot{F}$ and is additively closed, and any polynomial of odd degree in $F[x]$ has a root in F.*

Using this theorem, the proof of the Artin-Schreier theorem is easy: Given a formally real field F, then as stated above, F is contained in a real closed field R. From (iii) of the theorem, the set R^2 is an order on R, hence $R^2 \cap F$ is an order on F.

The Fundamental Theorem of Algebra can also be recovered from the theorem: Since $\mathbb{R}$ satisfies (iii), we have that $\mathbb{R}$ is real closed and then, by (ii), $\mathbb{C}$ is algebraically closed.

We note in passing the amazing "meta-theorem" of A. Tarski, now known as Tarski's Principal. Before stating it, we define a *formula of the language of ordered fields* as any formula expressible using field operations, inequalities, and the logical symbols $\vee$ (disjunction), $\wedge$ (conjunction), negation, and quantifiers. Then Tarski's Principal says that any formula in the language of ordered fields which holds over $\mathbb{R}$, holds over every real closed field [T].

Artin proved the following theorem relating orders in F to sums of squares:

Theorem. *If F is formally real, then $\sum F^2 = \bigcap P$, where the intersection is over all orders P in F.*

Let's verify the theorem in the case where $F = \mathbb{Q}[\sqrt{2}]$. First we claim that the only orders on F are the two mentioned above: $P_1 := \{a + b\sqrt{2} \mid a + b\sqrt{2} \geq 0 \text{ in } \mathbb{R}\}$ and $P_2 := \{a + b\sqrt{2} \mid a - b\sqrt{2} \in P_1\}$. Suppose P is an order, then $\mathbb{Q}^+ \subseteq P$ and $\sqrt{2} \in \pm P$. If $\sqrt{2} \in P$, then $P_1 \subseteq P$, from which follows $P = P_1$ since both have index 2 in $\dot{F}$. If $-\sqrt{2} \in P$, then a similar argument shows $\dot{P} = \dot{P}_2$. Now $P_1 \cap P_2 = \{a + b\sqrt{2} \mid a \geq 0$ and $a^2 \geq 2b^2\}$. We can show directly that this is precisely $\sum F^2$. For any $c, d \in \mathbb{Q}$, we have $(c + d\sqrt{2})^2 = (c^2 + 2d^2) + 2cd\sqrt{2} \in P_1 \cap P_2$, hence $\sum F^2 \subseteq P_1 \cap P_1$. Now suppose $a + b\sqrt{2} \in P_1 \cap P_2$. If $b = 0$, then $a \geq 0$ in $\mathbb{Q}$, hence $a \in \sum \mathbb{Q}^2 \subseteq \sum F^2$. If $b < 0$ and we can write $a - b\sqrt{2}$ as a sum of squares, then taking "conjugates" we can do it for $a + b\sqrt{2}$, so we may as well assume $b > 0$. Consider the square $(x + (b/2x)\sqrt{2})^2 = x^2 + b^2/4x^2 + b\sqrt{2}$. As a function of x^2, $x^4 + b^2/4x^2$ attains its minimum at $x^2 = b/\sqrt{2}$ and has value $b\sqrt{2} \leq a$. Hence we can find a rational q so that $q^2 + b^2/4q^2 \leq a$. Then $a + b\sqrt{2} = (q + (b/2q)\sqrt{2})^2 + (a - (q^2 + b^2/4q^2)) \in \sum F^2$. Thus we have verified the theorem for the case of $\mathbb{Q}[\sqrt{2}]$.

It only remained for Artin to show that if $f \in \mathbb{R}(X)$ is psd then f is in every order on $\mathbb{R}(X)$, hence is a sum of squares in $\mathbb{R}(X)$. In fact, Artin proved more than this. He showed that given any subfield $F \subseteq \mathbb{R}$ which has only one order and any $f \in F(X) := F(x_1, \ldots, x_k)$ such that $f \geq 0$ (in the unique ordering on F) at every point at which it is defined, then f is in every order of $F(X)$. To show this, Artin proved a series of "specialization lemmas" by using Sturm's Theorem, which is an algorithm for counting exactly the number of real roots of a polynomial, see [St].

FROM ORDERS TO HIGHER LEVEL ORDERINGS

In the late 1970's, Becker discovered a generalization of the notion of an order on a field, which led to a natural, far-reaching extension of the Artin-Schreier theory. As stated above, if $P \subseteq F$ is an order, then $\dot{P}$ is a subgroup of $\dot{F}$ of index 2 which is additively closed. Furthermore, this characterization is equivalent to the previous definition of order. An *ordering of higher level* on a field F is a subset $P \subseteq F$ such that $\dot{P}$ is an additively closed subgroup of $\dot{F}$ for which $\dot{F}/\dot{P}$ is finite cyclic. Since $\dot{P}$ is additively closed, $-1 \notin \dot{P}$ and hence $\dot{P}$ must have even index in $\dot{F}$. The *level* of P is $\frac{1}{2}[\dot{F} : \dot{P}]$. Note that ordinary orders are simply orderings of level 1; in other words Becker replaced the requirement that $\dot{P}$ have index 2 in $\dot{F}$ by requiring it to have index $2n$.

For an example of a field with orderings of all levels, consider $\mathbb{R}((x))$, the field of formal power series in x over $\mathbb{R}$. Elements are formal series $\sum_{i=m}^{\infty} \alpha_i x^i$, where $m \in \mathbb{Z}$ and $\alpha_i \in \mathbb{R}$. Fix $n \in \mathbb{N}$ and let z be a primitive $2n$-th root of 1. Then $P_z := \{\sum_{i=m}^{\infty} \alpha_i x^i \mid \alpha_m z^m = 1\} \cup \{0\}$ is an ordering of level n, as is easily checked.

Becker [B1] obtained generalizations of the Artin-Schreier and Artin theorems:

Higher Level Artin-Schreier Theorem. *A field F has an ordering of some level n iff $-1 \notin \sum F^{2n}$ iff F is formally real. If F is formally real, then for all n, $\sum F^{2n} = \bigcap P$, where the intersection is over all orderings of level dividing n.*

It should be noted that before Becker proved this theorem, J. Joly [J] proved that F is formally real iff $-1 \notin \sum F^{2n}$ for some n iff $-1 \notin \sum F^{2n}$ for all n, without making use of the notion of a higher level ordering. (In fact, Joly proved this for any commutative ring.)

When working with orders and orderings, it is impossible to avoid valuation theory, since this is one of the main tools for studying formally real fields. The intimate connections between valuation theory and the theory of orders were first seen in the work of R. Baer [Ba1], [Ba2] and W. Krull [K], soon after the theory of ordered fields was developed.

A subring $V \subseteq F$ is a *valuation ring* if V contains x or x^{-1} for every nonzero $x \in F$. In this case, the set $\mathcal{M} = \{x \in V \mid x^{-1} \notin V\}$ forms the unique maximal ideal in V and the field $V/\mathcal{M}$ is called the *residue field* of V. We say V is a *real valuation ring* if the residue field is formally real. One can show that F has a real valuation ring iff F is itself formally real.

Valuation theory arises naturally in ordered fields in the following way: Given an ordering P on F (of some level), let $A(P) = \{x \in F \mid q \pm x \in P$ for some $q \in \mathbb{Q}^+\}$, and let $I(P) = \{x \in F \mid q \pm x \in P$ for all $q \in \mathbb{Q}^+\}$. Then we have

Theorem. *$A(P)$ is a valuation ring in F with maximal ideal $I(P)$, and the set $\bar{P} := \{x + I(P) \mid x \in A(P) \cap P\}$ is an order in the residue field $A(P)/I(P)$.*

The proof of this theorem for orders is a straightforward calculation, however the proof for higher level orderings is much harder. Becker makes use of the Kadison-Dubois representation theorem from Functional Analysis. In the case where the ordering has level a power of 2, there is a direct proof due to A. Wadsworth (unpublished). The theorem is the key result Becker needed for the proof of the Higher Level Artin-Schreier Theorem. Given that result, the proof of the first part now proceeds as follows: If F has an ordering of some level, then F has a real valuation ring, hence, as noted above, F is formally real.

Later work by Becker, J. Harman, and A. Rosenberg [BHR] shows that the orderings of F of all levels can be described using only the (level 1) orders in F and the real valuation rings in F.

Given F formally real, the *real holomorphy ring* of F, $H := H(F)$, is the intersection of all real valuation rings in F. Now let $\mathbb{E} := \mathbb{E}(F)$ denote the units in $H(F)$, and set $\mathbb{E}^+ := \mathbb{E} \cap \sum F^2$. Although these definitions at first seem to have little to do with the higher level theory, in [B3] and [B4] Becker shows that there is an intimate connection between the structure of $\mathbb{E}$ and $\sum F^{2n}$.

Theorem. *[[B4],1.2,1.6,1.9] Let F be formally real and H, $\mathbb{E}$ as above. Then*

(i) $\mathbb{E}^+ = \{r\frac{s+q}{t+q} \mid r,s,t \in \mathbb{Q}^+, q \in \sum F^2\}$.

(ii) $\mathbb{E}^+ \subseteq \bigcap_{n\in\mathbb{N}} \sum F^{2n}$

(iii) *For all* $n \in \mathbb{N}$, $\sum F^{2n} = \mathbb{E}^+ \cdot (\sum F^2)^n$

It follows from (i) and (ii) that

$$B(t) = \frac{1+t^2}{2+t^2} \in \sum \mathbb{Q}(t)^{2n} \text{ for all } n$$

and hence we arrive at the Champagne Problem: Find an explicit formula expressing $B(t)$ as a sum of $2n$-th powers for all n.

The theorem above allows one to construct many examples of sums of $2n$-th powers. Give one!!
Further, for certain fields including $\mathbb{R}(X)$, Becker shows that $\mathbb{E}^+ = \cap_{n\in\mathbb{N}} \sum F^{2n}$.

Notice that it follows from (iii) of the theorem that for any n,

$$(\sum F^2)^n \subseteq \sum F^{2n},$$

a highly non-obvious fact! For example, over $\mathbb{R}$ we have:

$$(*) \qquad (x^2+y^2)^3 = \frac{4}{5}\left(x^6 + \left(\frac{x+y}{\sqrt{2}}\right)^6 + y^6 + \left(\frac{-x+y}{\sqrt{2}}\right)^6\right),$$

which can be checked by hand!

Using the higher level theory, Becker extends this to show that given $n, m \in \mathbb{N}$, there exist identities

$$(**) \qquad (x_1^{2n} + \cdots + x_k^{2n})^m = f_1^{2nm} + \cdots + f_r^{2nm},$$

where $f_i \in \mathbb{Q}(x_1, \ldots, x_k)$. For details, see [B2]. It should be noted that Becker showed only the existence of the identities; they are not given in any explicit way. Hilbert proved the existence of identities $(**)$ in the case $n = 1$, see [H4]; in this case they are usually called Hilbert Identities. We shall see that the Hilbert Identities, or, more precisely, an explicit version of them over $\mathbb{R}$, will play a key role in Reznick's solution to the Champagne Problem.

The Champagne Problem Solved

Artin's solution to Hilbert's 17th problem is not constructive, nor are the proofs of Becker's theorems on sums of even powers in formally real fields. Thus a natural question that arises from the 17th Problem is to what extent can we find an explicit representation of a psd f as a sum of squares of rational functions or, more generally, as a sum of $2n$-th powers for any n. The Champagne Problem is thus a specific example of this type of question. We discuss some computational aspects of the 17th Problem, and conclude with a sketch of Reznick's solution (over $\mathbb{R}$) to the Champagne Problem.

A year after Artin's solution to the 17th problem, G. Pólya [P] (see also [HLP,pp.57–59]) found an explicit solution in a special case. He showed that if $f \in \mathbb{R}(X)$ is positive definite, i.e., $f(a) > 0$ for all $a \in \mathbb{R}^k$, and even (as a function), then for large enough r, $f \cdot (\sum x_i^2)^r$ is a sum of squares of monomials, in particular f is a sum of squares of rational functions with common denominator $(\sum x_i^2)^{r/2}$. Recent work of J. de Loera and F. Santos [dLS] gives algorithms for finding a representation and bounds for r. In 1940, W. Habicht [Ha], using Pólya's result, showed directly that any positive definite polynomial f can be written as a sum of squares of rational functions, and that if f has only rational coefficients then so do the monomials. Recently, Reznick [R2] has extended this to show that any positive definite f can be written as a sum of squares of rational functions with common denominator $(\sum x_i^2)^r$. In other words, Reznick has extended Pólya's result to the more general setting of Habicht's result. The computations used for this result enables Reznick to obtain a representation for $B(t)$ as a sum of $2n$-th powers over $\mathbb{R}$.

As mentioned in the previous section, a key idea needed is the Hilbert Identities: Given n and s, let $N = \binom{n+2s-1}{n-1}$. Then there exist $0 < \lambda_i \in \mathbb{Q}$ and $\alpha_{ij} \in \mathbb{Q}$ for $1 \le i \le N$ such that

$$(x_1^2 + \cdots + x_k^2)^s = \sum_{i=1}^{N} \lambda_i (\alpha_{i1}x_1 + \cdots + \alpha_{ik}x_k)^{2s}.$$

There are no known explicit formulas except in the cases $s = 1, 2, 3$ (see [R1,§8,§9]). However, if we allow formulas over $\mathbb{R}$ instead of $\mathbb{Q}$, we can obtain

$$(\dagger) \qquad (x^2+y^2)^s = \frac{2^{2s}}{v\binom{2s}{s}} \sum_{j=0}^{v-1} \left(\cos\left(\frac{j\pi}{v}\right) x + \sin\left(\frac{j\pi}{v}\right) y \right)^{2s},$$

where s and v are positive integers and $v \ge s+1$. (For a proof of this, see [R2,9.5].) Notice that the identity (**) above is (†) with $s = 3$ and $v = 4$.

Consider the following obvious equality:

$$B(t) = \frac{1+t^2}{2+t^2} = \frac{(1+t^2)(2+t^2)^{2n-1}}{(2+t^2)^{2n}}.$$

Then if we can write $(1+t^2)(2+t^2)^{n-1}$ and $(2+t^2)^n$ as sums of $2n$-th powers of polynomials, their product is a sum of $2n$-th powers, each of which can be divided by $(2+t^2)^{2n}$ to give $B(t)$ as a sum of $2n$-th powers of rational functions. Taking $s = n$, $v = n+2$, $x = \sqrt{2}$, and

$y = t$ in (†), after several pages of calculations Reznick obtains the remarkable formula

$$B(t) = \frac{2^{4n-2}}{n(n+2)^2\binom{2n}{n}^2} \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} \lambda_j \left(\frac{L_i(\sqrt{2},t)L_j(\sqrt{2},t)}{2+t^2} \right)^{2n},$$

where $\lambda_j = 3n - (n+1)\cos\frac{2j\pi}{n+2}$ and $L_i(x,y) = (\cos\frac{2j\pi}{n+2})x + (\sin\frac{2j\pi}{n+2})y$. Thus we have an explicit formula for writing $B(t)$ as a sum of $2n$-th powers in $\mathbb{R}(t)$, in other words, a solution to the Champagne Problem over the reals!

Unfortunately, as noted above, there are no known explicit formulas for the Hilbert Identities in general, hence this method cannot produce a solution over $\mathbb{Q}$. However, Reznick's formula was close enough to a solution to the original problem that when he gave a talk on this work at the AMS/MAA joint winter meeting in Cincinnati in January, 1994 he was presented (by proxy) with a bottle of champagne from Becker.

Acknowledgements and Suggestions for Further Reading

Thanks to Bruce Reznick for many helpful comments and suggestions. His recent paper [R3] contains an exposition of Hilbert's work on sums of squares of polynomials which led to the 17th Problem, detailed information on sums of squares in general, and much more. Our sketch of the derivation of the formula for $B(t)$ is paraphrased from this paper. The present paper has also been influenced by other papers in this area, particularly Lam's excellent expository article on ordered fields [L2].

The text of Hilbert's 1900 lecture at the ICM can be found in [Br], along with descriptions of the mathematical developments arising from the 23 problems he proposed. Much has been written on the 17th Problem and mathematical developments arising from it. Here we mention a few articles that are well worth reading. A. Pfister [Pf] and P. Ribenboim [Ri] wrote surveys of the 17th Problem in the 70's. A more recent survey was written by D. Gondard [G]. See also C. Scheiderer's survey [S], where connections with real algebra and applications to geometry are discussed. A recent article of C. Delzell [D] describes the history of the 17th Problem and its relationship to questions in logic.

For more on the theory of ordered fields, see [L2] mentioned above. The generalization of the notion of an order to commutative rings led to the development of real algebra and real algebraic geometry, see [BCR], [B5] and [L4]. Much of the Artin-Schreier theory has been generalized to semi-local rings by M. Knebusch, Rosenberg, and R. Ware, see [KRW]. The notion of an order and some of the theory was extended to division rings by T. Szele [Sz]. Much of the higher level theory for fields has also been extended, to commutative rings [MW], [Po2], division rings [C], and even to general non-commutative rings [Po1]. The theory of ordered fields and the related valuation theory is intimately connected with the algebraic theory of quadratic forms, see Lam's books [L1] and [L3]. Work of Becker and Rosenberg show that these connections also exist in the higher level theory, see, e.g., [BR].

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