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Dear Jesus,

Here are the slides from my talk which I promised to send you. Sorry it took so long! I hope all is well with you. ~~Are~~ Are you snowed in yet? We have been having cold weather here, but I suppose it's much worse in Minnesotta.

Best,
Vicki

SUMS OF SQUARES OF REAL POLYNOMIALS

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NOTATION:

$R = \mathbb{R}[x_1, \dots, x_n]$

$\sum R^2 = \{ \sum_1^k f_i^2 \mid f_i \in R, k \in \mathbb{N} \}$

$\Lambda_m = \{ (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}^+)^n \mid \sum \alpha_i \leq m \}$

$f \in R$, $\deg f = m$, then $f = \sum_{\alpha \in \Lambda_m} a_\alpha X^\alpha$

$X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

If $f \in \sum R^2$, then $\deg f = 2m$

and $f = \sum h_i^2$, $\deg h_i \leq m$

<u>length</u> $(f) = \min \{ k \mid f = \sum_1^k h_i^2 \}$

QUESTIONS/PROBLEMS:

1. Given $f \in R$, decide if $f \in \Sigma R^2$ or not. If so, find an explicit representation $f = \sum h_i^2$
2. Given $f \in \Sigma R^2$, find length(f)
3. Given $S \subseteq R$, find ~~the~~ $P(S)$, the Pythagoras number of S,

$$P(S) := \sup \{ \text{length}(f) \mid f \in S \cap \Sigma R^2 \}$$

Remark: Let $K_n = \mathbb{R}(x_1, \dots, x_n)$, then

$P(K_1) = 2$, $P(K_2) = 4$,

$n+2 \leq P(K_n) \leq 2^n$ (Pfister)

$P(\mathbb{R}[x_1]) = 2$

BUT $P(\mathbb{R}[x_1, \dots, x_n]) = \infty$ if $n \geq 2$

(Choi, Dai, Lam, Reznick)

Suppose $f \in \Sigma\mathbb{R}^2$, $\deg f = 2m$

$$f = \sum_{\alpha \in \Lambda_{2m}} a_\alpha x^\alpha = \sum_{i=1}^{k} h_i^2$$

$$h_i = \sum_{\beta \in \Lambda_m} u_\beta^{(i)} x^\beta$$

Order Λ_m, say $\Lambda_m = \{\beta_1, \beta_2, \dots, \beta_N\}$

Set $u_j := (u_{\beta_j}^{(1)}, u_{\beta_j}^{(2)}, \dots, u_{\beta_j}^{(k)}) \in \mathbb{R}^k$

Then $f = \sum h_i^2$ can be written

$$f = (x^{\beta_1}, \dots, x^{\beta_N}) \cdot \underbrace{\begin{bmatrix} - u_1 - \\ - u_2 - \\ \vdots \\ - u_N - \end{bmatrix} \cdot \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_N \\ | & | & & | \end{bmatrix}} \cdot \begin{pmatrix} x^{\beta_1} \\ \vdots \\ x^{\beta_N} \end{pmatrix}$$

GRAM MATRIX of f associated to $\{h_1, \dots, h_k\}$

The Gram matrix of f (associated to the h_i's) is

$$V = (U_j \cdot U_k)$$

where $U_j = (u_{\beta_j}^{(1)}, \dots, u_{\beta_j}^{(k)})$

V is symmetric and psd (positive semi-definite) and the entries $v_{i,j}$ satisfy:

$$(*) \quad \sum_{\beta_i + \beta_j = \alpha} v_{i,j} = a_\alpha \quad \text{for all } \alpha$$

Given $f \in R$, a symmetric matrix $V = (v_{i,j})$ is a Gram matrix for f if V is psd and the entries satisfy $(*)$

Theorem [Choi, Lam, Reznick]: $f \in \Sigma R^2$ if and only if there exists a Gram matrix for f. If f has a Gram matrix of rank k, then we we can construct h_i with

$$f = \sum_{i=1}^{k} h_i^2$$

(T. Wörmann, V.P.) : This yields an

Algorithm

Input : $f = \sum a_\alpha x^\alpha$

Output : Yes, $f \in \Sigma R^2$ or No, $f \notin \Sigma R^2$

If Yes, an explicit representation

$$f = \sum h_i^{\,2}$$

find the minimum!!!

Procedure:

1. Solve the linear system

$$\sum_{\beta_i + \beta_j = \alpha} v_{i,j} = a_\alpha$$

where $\{v_{i,j}\}$ are variables and $v_{j,i} = v_{i,j}$. Solution given by l independent parameters $\lambda_1, \dots, \lambda_l$ where $l = \frac{d(d+1)}{2} - e$, $d = \#$ of β's, $e = \#$ of α's

$\binom{d}{2} - e$??

2. Find values of $\lambda_1, \dots, \lambda_l$ which make $V = (v_{i,j})$ psd, or not.
If no such values $\longrightarrow$ NO
If values exist $\longrightarrow$ get representation

Given $V = (v_{ij})$, where each v_{ij} is linear in $\lambda_1, \dots, \lambda_e$.

To find values for λ_i's which make V psd:

$F(y)$ = characteristic poly. of V

$F(y) = y^s + b_{s-1} y^{s-1} + \cdots + b_0$, where $b_i \in \mathbb{R}[\lambda_1, \dots, \lambda_e]$

By Descartes rule of Signs, $F(y)$ has only non-negative roots iff $(-1)^i b_i \geq 0 \ \forall i$. So we look for a point in $S = \{(\lambda_1, \dots, \lambda_e) \mid (-1)^i b_i(\lambda_1, \dots, \lambda_e) \geq 0 \ \forall i\}$

Unfortunately, "solving" a semi-algebraic system $\{f_1 \geq 0, \dots, f_e \geq 0\}$ is highly non-trivial.

<u>Recall</u>: rank V = number of h_i's in $f = \sum h_i^2$

$f(x,y,z) = x^4 + 2x^2y^2 + x^3z + z^4$

A Gram matrix for f would be of the form

$$\begin{bmatrix} 1 & 0 & 2 & \lambda \\ 0 & 2 & 0 & 0 \\ 2 & 0 & -2\lambda & 0 \\ \lambda & 0 & 0 & 1 \end{bmatrix}.$$

In this case, S contains $-8-4\lambda+4\lambda^3$ and $-8-4\lambda$, which cannot both be ≥ 0. Hence S is empty and f is not a sos.

$f(x,y,z) = x^6 + 4x^3y^2z + y^6 + 2y^4z^2 + y^2z^4 + 4z^6$

Note $|\Lambda_3| = 10$, but in this case there are only 5 exponents that can occur in the h_i's. We get

$$V = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & r & s \\ 2 & 0 & 2-2r & -s & t \\ 0 & r & -s & 1-2t & 0 \\ 0 & s & t & 0 & 4 \end{bmatrix}$$

as the general form of a Gram matrix.

Then $S =$

$$-2r - 2t + 9 \geq 0,$$

$$-r^2 + 4rt - 14r - 2s^2 - t^2 - 16t + 25 \geq 0,$$

$$2r^3 - 7r^2 + 2rs^2 + 24rt - 30r + 2s^2t - 10s^2 + 2t^3 - 3t^2 - 34t + 19 \geq 0,$$

$$10r^3 + r^2t^2 - 10r^2 - 2rs^2t + 4rs^2 + 36rt - 26r + s^4 + 6s^2t - 10s^2 + 4t^3 - 3t^2 - 4t - 6 \geq 0,$$

$$8r^3 + r^2t^2 + 8r^2 + -2rs^2t + 2rs^2 + 16rt - 8r + s^4 - 4s^2t - 2s^2 + 2t^3 - t^2 + 16t - 8 \geq 0$$

Using "ad hoc" methods, we see that S is nonempty, in fact $(-1, 0, 0), (-2, 0, -3/2) \in S$.

Using $(-1, 0, 0)$,

$$V = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Note that rank $V = 3$, so this gives f as a sum of 3 squares. We have $Q(y) = y_1^2 + 4y_1y_3 + y_2^2 - 2y_2y_4 + 4y_3^2 + y_4^2 + 4y_5^2 = (y_1 + 2y_3^2)^2 + (y_2 - y_4)^2 + (2y_5)^2$. This yields

$$f = (x^3 + 2y^2z)^2 + (y^3 - yz^2)^2 + (2z^3)^2.$$

Using $(-2, 0, -3/2)$,

$$V = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 2 & 0 & 6 & 0 & -3/2 \\ 0 & -2 & 0 & 4 & 0 \\ 0 & 0 & -3/2 & 0 & 4 \end{bmatrix}.$$

Note rank $V = 4$. Proceeding as before we get

$$f = (x^3 + 2y^2z)^2 + (y^3 - 2yz^2)^2 +$$
$$(\sqrt{2}y^2z - 3\sqrt{2}/4z^3)^2 + (\sqrt{23/8}z^3)^2.$$

Example:

$$f = x^2y^2 + y^2z^2 + x^2z^2 - 4xyz + 1$$

Note: f is psd (arithmetic-geo. inequality)

Suppose $f = \sum h_i^2$, $\deg h_i = 2 \ \forall i$

so h_i's involve the following monomials:

$$\cancel{x^2} \quad xy \quad xz \quad \cancel{y^2} \quad yz \quad \cancel{z^2}$$

$$\cancel{x} \quad \cancel{y} \quad \cancel{z}$$

$$1$$

If x^2 appears $\Rightarrow$ f has non-trivial x^4 term ↯

If x appears $\Rightarrow$ non-trivial x^2 term ↯

Remaining monomials cannot yield an xyz term!

"Term Inspection"

Newton Polytopes and Sums of Squares

$$f = \sum_{\alpha \in \Lambda_{2m}} a_\alpha x^\alpha$$

The Newton polytope of f is the convex hull of $\{\alpha \mid a_\alpha \neq 0\}$, $C = C(f)$

$L(C) := C \cap (\mathbb{Z})^n$

$E(C) := C \cap (2\mathbb{Z})^n$

$\frac{1}{2}C := \{\frac{1}{2}c \mid c \in L(C)\}$

Get your act together
Fight for what you want.

Theorem (Reznick): If f is psd, then $C(f) =$ convex hull of $E(C(f))$

If $f = \sum h_i^2$, then $C(h_i) \subseteq \frac{1}{2} C(f)$

$C(h_i) \subseteq \frac{1}{2} C(f)$

Does this define a subdivision?? EXAMPLES!!!

$A(C) = \{\alpha \in L(C) \mid \alpha = \beta + \beta'$ for some $\beta, \beta' \in \frac{1}{2}C\}$

Note: $L(\frac{1}{2}C) = \frac{1}{2}E(C)$, so $A(C) =$ "averages" of pairs of even lattice points in C.

Proposition: If $f \in \Sigma R^2$, then $C = C(f)$

i) If $\alpha \in L(C) \setminus A(C)$, then $a_\alpha = 0$

ii) If $\alpha \in E(C)$ and α can be written in only <u>one</u> way as $\alpha = \beta + \beta'$, $\beta, \beta' \in \frac{1}{2}C$, then $a_\alpha \geq 0$

$(\alpha = \alpha/2 + \alpha/2)$

Note: The easiest example of (ii) is $\alpha = (2m, 0, \ldots, 0)$

$\alpha = (m, 0, \ldots, 0) + (m, 0, \ldots, 0)$

$\Rightarrow$ exponent of x_1^{2m} in f must be ≥ 0.

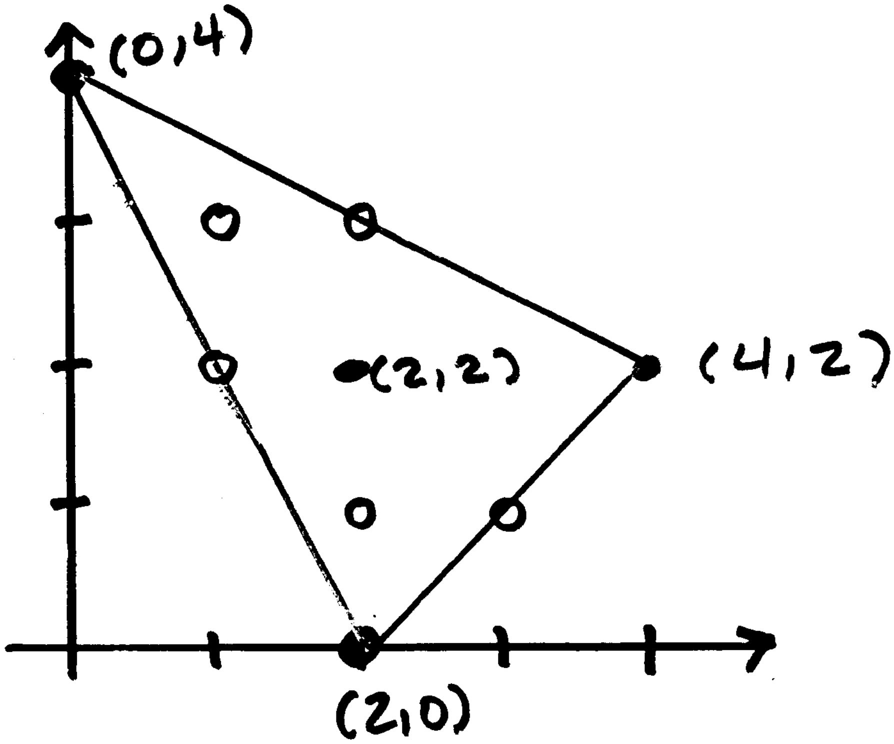
Example: $f = x^2y^2 + y^2z^2 + x^2z^2 - 4xyz + 1$

$E(C) = \{(2,2,0), (0,2,2), (2,0,2), (0,0,0)\}$

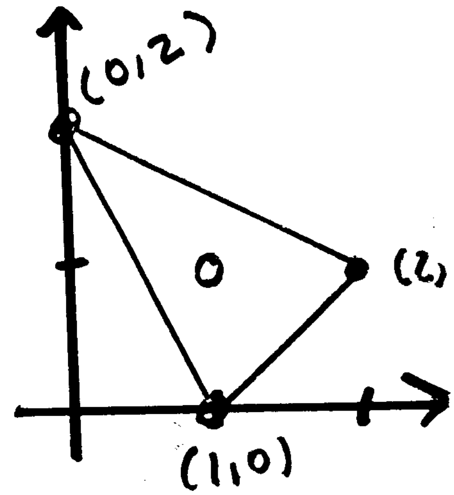
$A(C) =$ averages of these points

$(1,1,1) \notin A(C) \Rightarrow f \notin \Sigma R^2$

$$f(x,y) = x^4y^2 + y^4 + x^2 - 3x^2y^2$$



$C = C(f)$



$\frac{1}{2}C$

$L(C) = \{(4,2), (2,0), (0,4), (3,1), (2,3), (2,2), (2,1), (1,3), (1,2)\} = A(C)$

$E(C) = \{(4,2), (0,4), (2,0), (2,2)\}$

$L(\frac{1}{2}C) = \frac{1}{2}E(C) = \{(2,1), (1,1), (1,0), (0,2)\}$

$(2,2) = (1,1) + (1,1)$ ← only way

But coefficient of x^2y^2 is < 0

$\Longrightarrow f \notin \Sigma R^2$

PYTHAGORAS NUMBERS

$f = \sum h_i^2$, fix $C := C(f)$

$P(C) := P(\{g \in R \mid C(g) = C\})$

$P(n, 2m) := P(\{g \in R \mid \deg g = 2m\})$

$= P(\{\text{convex hull of } \Lambda_{2m}\})$

Set $a := |A(C)| = \#$ of monomials in f

$e := |E(C)| = |\tfrac{1}{2} E(C)|$

<u>Note</u>: $C(h_i) \subseteq \tfrac{1}{2} E(C) \Rightarrow$

$\{\#$ of monomials that occur in the h_i's$\} \leq e$.

<u>Proposition</u>: $P(C) \leq e$

<u>Proof</u>: $f = \sum h_i^2$, V = Gram matrix

$\Rightarrow$ V is $e \times e$ matrix $\Rightarrow$

rank $V \leq e$ $\Rightarrow$ f is a sum of $\leq e$ squares.

Theorem (Choi, Lam, Reznick)

$p = p(c)$, $\quad a = |A(c)|$, $\quad e = |E(c)|$

Let $\Lambda = \frac{\sqrt{1+8a}-1}{2}$, the positive

root of $x^2 + x - 2a$. Then

$$p \le \Lambda \le e.$$

"**Proof**": We know $p \le e$. Since

$A(c) = \{\beta + \beta' \mid \beta, \beta' \in \tfrac{1}{2} E(c)\}$,

$$a \le \frac{e(e+1)}{2} \Rightarrow e(e+1) - 2a \le 0$$

$\Rightarrow \Lambda \le e$. Now show

$$\frac{p(p+1)}{2} \le a \quad \ldots$$

(Look at Gram matrix and use quadratic forms)

Look at $P(n, 2m)$

Note: $|\Lambda_m| = \binom{n+m}{n}$

Well-known:

$$P(1,m) = 2$$

$$P(n,2) = n-1$$

Hilbert showed

$$P(2,4) = 3$$

Let $\lambda(n,m) :=$ positive root of $x^2 + x - 2a$, $a = |\Lambda_m|$

so that $P(n,m) \leq \lambda(n,m)$

$n=2$: $|\Lambda_m| = \binom{2+m}{2} = \frac{(m+1)(m+2)}{2}$

$\Rightarrow 1 + 8a = (2m+3)^2$

Hence $\boxed{P(2,m) \leq m+1}$

LOWER BOUNDS

Theorem (C., L., R.): $P = P(C)$, a, e as before. Let λ be the smaller root of $x^2 - (2e+1)x + 2a$, then $P \geq \lambda \geq a/e$.

Note: $g(x) := x^2 - (2e+1)x + 2a$, then $g(a/e) = \frac{a(a-e)}{e^2} \geq 0$

$\Rightarrow a/e \geq \lambda$

Theorem: There exist constants $\gamma_1(m)$, $\gamma_2(m)$ such that

$$\gamma_1(m)(n+1)^{m/2} \leq P(n,m) \leq \gamma_2(m)(n+1)^{m/2}$$

n	m	$a(n,m)$	$\lambda(n,m)$	$\Lambda(n,m)$
1	2	3	2	2
1	4	5	2	2.702
1	6	7	2	3.275
1	8	9	2	3.772
2	2	6	3	3
2	4	15	3	5
2	6	28	3.135	7
2	8	45	3.242	9
3	2	10	4	4
3	4	35	4.156	7.882
3	6	84	4.618	12.471
3	8	165	5	17.673

E.g. $p(2,6) \in \{4, 5, 6, 7\}$