LAGRANGE MULTIPLIERS AND SOME PROBLEMS IN COMPUTATIONAL REAL ALGEBRAIC GEOMETRY

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Introduction

Problems in computational real algebraic geometry tend to be very difficult to solve unless they can be reduced to (complex) zero-dimensional problems. In the special case of deciding if a polynomial f over a real closed field has a real zero, one way to do this is to use the method of Lagrange multipliers, i.e., look for critical points of a "distance function" ϕ subject to the constraint $f = 0$. This set will be zero-dimensional almost always. This idea is not new, see e.g. [GV]. In this paper, we look at two new applications of this method, namely deciding if a polynomial is positive semi-definite and deciding if a basic closed semi-algebraic set is empty.

The Ideal of Lagrange Multipliers

Let R be a real closed field, set $R[X] := R[x_1, \dots, x_n]$, and let C be the algebraic closure of R. For an ideal $I \subset R[X]$, $V(I) := \{x \in C^n | g(x) = 0 \text{ for all } g \in I\}$. Then I is **zero-dimensional** if $V(I)$ is a finite set. For $f_1, \dots, f_k \in R[X]$, we write $V(f_1, \dots, f_k)$ to denote $V(I)$, where I is the ideal generated by the f_i's. The set of **real points** of $V(I)$, denoted $V_R(I)$, is $V(I) \cap R$.

Definition. Given $f, \phi \in R[X]$ and λ a new indeterminate. The **ideal of Lagrange multipliers of f with respect to ϕ**, denoted $L(f, \phi)$, is

$$(f, \lambda f_{x_1} - \phi_{x_1}, \dots, \lambda f_{x_n} - \phi_{x_n}),$$

the ideal in $R[X, \lambda]$ generated by f and the partial derivatives of $\lambda f - \phi$.

Proposition 1. *Given $f, \phi \in R[X]$.* f must be smooth

(i) *$V_R(f) \neq \emptyset$ iff $V_R(L(f, \phi)) \neq \emptyset$.*
(ii) *$L(f, \phi)$ is not zero-dimensional iff an irreducible component of $V(f)$ which is not zero-dimensional is contained in $V(\phi - c)$ for some $c \in C$.*

Proof. Set $L := L(f, \phi)$.
(i): From elementary analysis we know that the real points of L consist of the extremal points of ϕ under the constraint $f = 0$. Suppose $V_R(f)$ is not empty, then since $V_R(f)$ is a closed set, ϕ must attain a minimum on $V_R(f)$. Hence $V_R(L)$ is not empty. Since $V_R(L) \subseteq V_R(f)$, the opposite implication is trivial.

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(ii): For any $g \in R[X]$, let $\Sigma(g) := \{x \in C^n \mid \frac{\partial f}{\partial x_i}(x) = 0 \text{ for all } x_i\}$. Let $g = \lambda f - \phi$, then $\Sigma(g) = L(f, \phi)$. It follows from Sard's theorem that $g(U)$ is constant for any irreducible component U of $\Sigma(g)$. (For a proof of this, see e.g. [W,6.2].) Thus any irreducible component of $L(f, \phi)$ must be contained in $V(\phi - c)$ for some $c \in C$ and the result follows. □

Since $V(f)$ has only finitely many irreducible components, it follows from the proposition that for a randomly chosen ϕ, the probability is 1 that $L(f, \phi)$ will be zero-dimensional. In the case where $V(I)$ is zero-dimensional, there are practical and efficient algorithms for deciding if $V_R(I)$ is empty or not, c.f. [BW],[PRS]. For our examples we used the RealSolving software, developed by M.-F. Roy and F. Rouillier. For details on RealSolving, see [R].

To determine if a smooth polynomial f has a real root or not, we proceed as follows: Choose a "distance function" ϕ. If $L(f, \phi)$ is zero-dimensional, then we can test whether $V_R(L(\phi, f)$ is empty. If $L(f, \phi)$ is not zero-dimensional we can change the distance function ϕ with the aim of making $L(f, \phi)$ zero-dimensional. In practice, we use $\phi = x_1^2 + \ldots x_n^2$, the square of the Euclidean distance function.

Testing positivity of a polynomial

A polynomial $f \in R[X]$ is positive semi-definite (psd) iff $f(x) \geq 0$ for all $x \in R^n$. In this section, using the results above, we give an algorithm for determining whether f is psd.

Given $f \in R[X]$, let t be a new indeterminant and define $F := ft^2 + 1 \in R[x_1, \ldots, x_n, t]$. Then clearly f is psd iff $V_R(F) = \emptyset$. Thus we choose $\phi \in R[X, t]$ and by Prop. 1, f is psd iff $V(L(F, \phi))$ has a real point. If we take ϕ to be the square of the Euclidean distance function, we can simplify $L(F, \phi)$ somewhat:

Proposition 2. *Given $f \in R[X]$, let $F = t^2 f + 1 \in R[x_1, \ldots, x_n, t]$, $f_i = \frac{\partial f}{\partial x_i}$, and $\phi = x_1^2 + \cdots + x_n^2 + t^2$. Then f is psd iff $V_R(F, t^4 f_1 + 2x_1, \ldots, t^4 f_n + 2x_n) = \emptyset$.*

Proof. Set $V := V_R(F, t^4 f_1 + 2x_1, \ldots, t^4 f_n + 2x_n)$ and $L := L(F, \phi) = (F, \lambda t^2 f_1 - 2x_1, \ldots \lambda t^2 f_n - 2x_n, 2tf - 2\lambda t)$. By Prop. 1 and the above, we have f is psd iff $V_R(L) = \emptyset$. Suppose $x \in R^n$, $\lambda_0, t_0 \in R$ such that $\alpha = (x, \lambda_0, t_0) \in V_R(L)$, then since $F(\alpha) = 0$ we must have $f(x) \neq 0$ and $t_0 \neq 0$. Thus $F(\alpha) = 0$ implies $t_0^2 = -1/f(x)$ and $2t_0 f(x) - 2\lambda_0 t_0 = 0$ implies $1/f = \lambda_0$. Hence $\lambda_0 = -t_0^2$ and so $V \neq \emptyset$. Conversely, given $(x, t_0) \in V$, setting $\lambda_0 = -t_0^2$ yields $(x, \lambda_0, t_0) \in V(L(F, \phi))$. □

Example. Let $f = 2x^6 + y^6 - 3x^4y^2 + x^2y^2 - 6y + 5$ and $F = t^2 f + 1$. As above, we have f is psd iff $V := V(F, t^4(12x^5 - 12x^3y^2 + 2xy^2) + 2x, t^4(6y^5 - 6x^4y + x^2y - 6) + 2y)$ has no real point. V is zero-dimensional, as is easily checked. The RealSolving software calculates that V has 4 real points, hence f is not psd.

The polynomial f is a special case of the following: For each $a \in \mathbb{R}^+$, set

$$f_a := 2x^6 + y^6 - 3x^4y^2 - 6y + 5 + ax^2y^2.$$

We have just shown that f_1 is not psd, and in a similar manner we can show that f_2 is psd. Thus there exists b, $1 < b \leq 2$, such that f_a not psd for $a \leq b$ and psd for $a > b$. We would like to find b. We cannot follow the exact procedure used for f_1, f_2 since the RealSolving software cannot handle parameters. However we can get rid of the parameter using the following "trick", due to Reznick: First note that $f_a(x,y) \geq 0$ trivially when $xy = 0$, and also $f_a(x,y) \geq f_a(x,|y|)$, hence it suffices to assume $x > 0$ and $y > 0$. Then we have $f_a(x,y) \geq 0$ iff $(2x^6 + y^6 - 3x^4y^2 - 6y + 5)/x^2y^2 + a \geq 0$. Hence b is the minimum of the rational function $G(x,y) := 2x^4y^{-2} - 3x^2 - 6x^{-2}y^{-2} + 5x^{-2}y^{-2}$. Set $t := x^2$, take derivatives and clear denominators to get that the critical points of G are $V(g_1, g_2)$, where $g_1 = y^6 - 3t^2y^2 + +6y + 4t^3 - 5$ and $g_2 = 4y^6 + 6y - 4t^3 - 10$. A Groebner Basis of (g_1, g_2) in lex order contains a polynomial in t only of degree 13 which has a root at 0 and 3 other real roots. Solving numerically, we find that G has one real critical value between 1 and 2, and that b is approximately 1.2099.

Basic Closed Semi-algebraic Sets

A basic closed semi-algebraic set in R^n is a set of the form $K(f_1, \ldots, f_k) := \{\alpha \in R^n \mid f_i(\alpha) \geq 0 \text{ for all } i, 1 \leq i \leq k\}$, where $f_1, \ldots, f_k \in R[X]$. In this section we use the idea of Lagrange multipliers to give an algorithm for a test of emptyness for a basic closed semi-algebraic set, i.e., given a basic closed semi-algebraic set S, the algorithm decides whether S is empty or not.

Proposition 3. *Given $S = K(f_1, \ldots, f_k)$, defined as above and $\phi \in R[X]$. Assume $V_R(f_i) \neq \emptyset$ for all i. For each subset $A \subseteq \{f_1, \ldots, f_k)$ set $f_A = \sum_{f_i \in A} f_i^2$ and let $L_A = \{\alpha \in V_R(L(f_A, \phi)) \mid f_j(\alpha) \geq 0$ for all $f_j \notin A\}$. Then $S = \emptyset$ iff $L_A = \emptyset$ for all i.*

Proof. Suppose S is not empty, then ϕ attains a minimum on S since S is closed. Clearly, ϕ attains a minimum on some $V_R(f_i)$. Suppose a minimum occurs at α, then let $A = \cup\{f_i \mid f_i(\alpha) = 0\}$. Then α is a minimum of ϕ on $V_R(A) = V_R(f_A)$, and since $\alpha \in S$, we have $\alpha \in L_A$. Since $L_A \subseteq S$ for all A, the the opposite implication is immediate. □

If $V(f_A)$ is zero-dimensional, then there are practical algorithms for deciding if L_A is empty or not. In fact, the RealSolving software mentioned in the previous section can do this. Thus we proceed as follows: Choose a suitable ϕ, then for each non-empty $A \subseteq \{f_1, \ldots, f_k\}$, check whether L_A is empty or not. (In practice, one should start with the smallest A's.)

Example. In [PW], an algorithm is given for determining whether or not a polynomial is a sum of squares (of polynomials). The algorithm works as follows: Given a polynomial f, associated to f is a matrix, the Gram matrix of f, which is of the form $B = B_0 + t_1B_1 + \cdots + t_mB_m$, where each B_i is a square matrix (all of the same size) and the t_i's are new variables. Then f is a sum of squares iff there exist values for the t_i's for which the matrix B is a positive semi-definite (psd) matrix. Now B is psd iff all eigenvalues of B are non-negative. Hence, using Descarte's rule of signs, we have that f is a sum of squares iff the basic closed semi-algebraic set $\{(-1)^{i+k}b_i \geq 0\}$ is non-empty, where $b_0, \ldots, b_k \in R[t_1, \ldots, t_m]$ are the coefficients of the characteristic polynomial of B.

Let $f = 2x^6 + 2x^3y^2 + y^6 + 1$. We would like to determine whether f is a sum of squares in $\mathbb{R}[x, y]$ or not. Using the algorithm in [PW], we find that the Gram matrix of f is

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & r & s \\ 1 & 0 & -2r & -s & t \\ 0 & r & -s & -2t & 0 \\ 0 & s & t & 0 & 1 \end{bmatrix}$$

The corresponding semi-algebraic set is
$S = \{4r^3 + r^2 + 2s^4 - 2s^2 + 4t^3 + 2t + 8rt - 2s^2t - 4rs^2t + 2r^2t^2 \geq 0,$
$6r^3 + r^2t^2 - r^2 - 2rs^2t + 4rs^2 + 20rt - 4r + s^4 + 4s^2 - 4s^2 + 6t^3 - 2t^2 - 1 \geq 0,$
$2r^3 - 3r^2 - 10r + 2rs^2 + 16rt + 2s^2t - 6s^2 + 2t^3 - 3t^2 \geq 0,$
$-r^2 + 4rt - 8r - 2s^2 - t^2 - 8t + 4 \geq 0, -2r - 2t + 4 \geq 0\}.$

Using RealSolving, we find that in this case $L_A \neq \emptyset$, where $A = \{f_3\}$ and thus f is a sum of squares.

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