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Constructive Approaches to Representation Theorems in Finitely Generated Real Algebras

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ABSTRACT. A constructive approach to the Kadison-Dubois Representation Theorem is given, with degree bounds and an algorithm for finding a representation. This is applied to give a constructive approach to M. Marshall's representation theorem for certain non-compact semialgebraic sets. Our approach is based on work of M. Schweighofer.

1. Introduction

Given $\{f_1, \ldots, f_k\}$, a finite set of polynomials in $\mathbb{R}[X] := \mathbb{R}[x_1, \ldots, x_n]$, let S be the basic closed semialgebraic set $\{\alpha \in \mathbb{R}^n \mid f_1(\alpha) \geq 0, \ldots, f_k(\alpha) \geq 0\}$. Suppose that a polynomial p can be written as a finite sum of products of the f_i's and sums of squares in $\mathbb{R}[X]$, then clearly p is non-negative on S. A "representation theorem" is a converse to this fact, namely, a theorem that says that if a polynomial p is positive on S, then there is a representation of p in terms of the f_i's and (possibly) sums of squares. Such a representation can be thought of as a "certificate of positivity" from which the positivity condition is immediately apparent.

In [**13**], K. Schmüdgen proved a remarkable and far-reaching representation theorem, namely, he showed that if S as above is compact and $p > 0$ on S, then p is in the **preorder** generated by the f_i's, i.e., the set of finite sums of elements of the form $s_\epsilon f_1^{\epsilon_1} \cdot \cdots \cdot f_k^{\epsilon_k}$, where s_ϵ is in $\sum \mathbb{R}[X]^2$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{0,1\}^k$. The proof, which uses functional analysis techniques, is not constructive. Thus no information is obtained on the degree of the sums of squares involved, nor on how to find a representation for a specific p. T. Wörmann [**15**] gave an algebraic proof of this result but this proof is also non-constructive. The main tool used in Wörmann's proof is the Kadison-Dubois representation theorem for archimedean partially ordered rings. Recently, M. Schweighofer [**14**] found a constructive approach to Schmüdgen's Theorem, using instead of Kadison-Dubois a representation theorem of Pólya for polynomials positive on the standard simplex. Unlike the Kadison-Dubois Theorem, degree bounds for the "output data" are known for Pólya's Theorem [**12**].

In this paper we use the ideas of Schweighofer to give a constructive approach to the Kadison-Dubois Theorem, with degree bounds and an algorithm for finding a representation. We then apply this to give a constructive approach to a result of M. Marshall [**9**] which gives a representation theorem for certain non-compact semialgebraic sets. As in Schweighofer's work, our algorithm depends on having representations of polynomials of the form $N \pm x_i$ for all i, and thus is not completely constructive. We discuss one possible approach to finding such representations in specific cases.

Schmüdgen's theorem was proven for a real polynomial ring, however Wörmann's proof and Schweighofer's algorithm both generalize easily to the finitely generated real algebra case. Our results will also hold in this more general case. The full generality will be needed for the proof of Marshall's theorem.

Some of the results in this paper are part of the first author's Ph.D thesis at Emory University, written under the direction of the second author.

2. Preliminaries on Archimedean Preprimes and Preorders

We work in the coordinate ring $R = \mathbb{R}[V]$ of an algebraic set V in $\mathbb{R}^n$. Then R is an $\mathbb{R}$-algebra generated by the coordinate functions $x_i : V \to \mathbb{R}$. Given $f_1, \dots, f_k \in R$, let $S(f_1, \dots, f_k)$ be the **basic closed semialgebraic set** generated by the f_i's, i.e., $S(f_1, \dots, f_k) := \{\alpha \in V \mid f_i(\alpha) \geq 0 \text{ for } 1 \leq i \leq k\}$.

A subset P of R is a **preprime** if

$$P + P \subseteq P, \quad P \cdot P \subseteq P, \quad \mathbb{R}^+ \subseteq P, \quad -1 \notin P.$$

Let $\sum$ denote the set of sums of squares in R. A preprime P is a **preorder** if $\sum \subseteq P$. Given $f_1, \dots, f_k \in R$, $PP(f_1, \dots, f_k)$ denotes the preprime generated by the f_i's, i.e., the set of finite sums of elements in R of the form $a_\epsilon f_1^{\epsilon_1} \dots f_k^{\epsilon_k}$, where $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbb{N}^k$ and $a_\epsilon \in \mathbb{R}^+$.

DEFINITION 2.1. Given a finitely-generated preprime $P := PP(f_1, \dots, f_k)$ and suppose $g \in P$. A P**-representation** of g is a an equation of the form

$$g = \sum_{\epsilon \in \mathbb{N}^k} a_\epsilon f_1^{\epsilon_1} \cdot \dots \cdot f_k^{\epsilon_k},$$

where $a_\epsilon \in \mathbb{R}^+$ for all ϵ. The **degree** of the representation is the maximum of $\{|\epsilon|\}$.

REMARK 2.2. Another way to view elements of $PP(f_1, \dots, f_k)$ is as the set of all elements of R which can be written $F(f_1, \dots, f_k)$, where F is a polynomial in k variables with non-negative coefficients. If we have a representation $g = F(f_1, \dots, f_k)$ of $g \in PP(f_1, \dots, f_k)$, then the degree of the representation is the degree of F as a polynomial.

The preorder generated by the f_i's, denoted $PO(f_1, \dots, f_k)$, is the set of all elements of the form

$$\sum_{\epsilon \in \{0,1\}^k} s_\epsilon f_1^{\epsilon_1} \cdot \dots \cdot f_k^{\epsilon_k},$$

where $s_\epsilon \in \sum$ for all ϵ.

REMARK 2.3. Preorders and preprimes are the fundamental algebraic objects associated to semialgebraic sets. In some sense, preorders play a role in semialgebraic geometry that is comparable to the role played by ideals in algebraic geometry. Computationally, a finitely generated preprime is much simpler to work with than a finitely generated preorder, since sums of squares are not involved. However, for the most general cases, the preprime is not sufficient, as we shall see.

We say that a subset P of R is **archimedean** if for each $g \in R$ there exists a non-negative integer m such that $m - g \in P$. Suppose $P := PP(f_1, \dots, f_k)$ and $S := S(f_1, \dots, f_k)$ and P is archimedean. Then it is easy to see that S is compact: There must be some $N \in \mathbb{N}$ such that $N \pm x_i \in P$ for all i, from which it follows that S is contained in the compact set $\{\alpha \in V \mid N \pm x_i(\alpha) \geq 0 \text{ for all } i\}$ and hence is also compact. For preorders, the converse also holds:

THEOREM 2.4. *Let S be as above and $P := PO(f_1, \dots, f_k)$. Then S is compact if and only if P is archimedean.*

PROOF. The proof that P archimedean implies S compact was given above. The converse follows from [**15**]. □

REMARK 2.5. In general, the theorem is not true if we replace the preorder P by the preprime generated by the f_i's. This can be shown by the following example, which is due to D. Handelman [**5**]: In $\mathbb{R}[x, y]$, consider $PP(x, y, 1 - x^2 - y^2)$ then the corresponding semialgebraic set is the semi-circle $\{x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$, which is obviously compact. On the other hand, an elementary argument shows that for any positive integer N, it is not possible to write $N - x$ in the form $F(x, y, 1 - x^2 - y^2)$, where F is a polynomial in three variables with non-negative coefficients. Thus the preprime is not archimedean.

It turns out that in the linear case, the preprime is enough:

PROPOSITION 2.6. *Given linear functions $l_1, \dots, l_k \in R$, suppose $S = S(l_1, \dots, l_k)$ is compact. Then $PP(l_1, \dots, l_k)$ is archimedean.*

PROOF. This follows from [**5**, 1.3], see also [**12**]. □

Now suppose that P is an archimedean preprime. Then there is some non-negative integer N such that for $N \pm x_i \in P$ for $i = 1, \dots, n$. We say that a minimal such N is the **archimedean bound** of P. We fix representations of $N \pm x_i$ in P.

REMARK 2.7. For a preorder, we can construct from representations of $N \pm x_i$, a representation of $nN^2 - \sum x_i^2$, and, conversely, we can construct representations of $\frac{N+1}{2} \pm x_i$ in P, using a representation of $N - \sum x_i^2$. This follows from the identity

$$\frac{N+1}{2} \pm x_i = \frac{1}{2}\left((x_i \pm 1)^2 + (N - \sum_i x_i^2) + \sum_{j \neq i} x_j^2\right) \tag{2.1}$$

Thus a preorder P is archimedean iff $M - \sum x_i^2 \in P$ for some $M \in \mathbb{N}$. However, if P is a preprime it is not true that $M - \sum x_i^2$ in P for some M implies that P is archimedean. This follows from the example $PP(x, y, 1 - x^2 - y^2)$ above.

THEOREM 2.8. *Suppose P is an archimedean preprime, with archimedean bound N and we have fixed representations of $\{N \pm x_i\}$ in P. Let D be the maximum degree of the fixed representations.*

(1) *For any $g \in R$, $M - g \in P$ for some $M \in \mathbb{N}$ and a P-representation of $M - g$ in P can be constructed explicitly, in terms of the representations of $\{N \pm x_i\}$.*
(2) *Suppose we have $g = F(x_1, \dots, x_n)$, where F is a real polynomial with $\deg F = d$ and L is the sum of the absolute values of the coefficients of F. Then we can take $M = LN^d$ and the degree of the P-representation will be at most dD.*

PROOF. We prove this for a real polynomial ring and then, using an epimorphism $\mathbb{R}[X_1,\dots,X_n] \to R$, obtain the general result. So suppose g is a real polynomial with degree d. We will proceed by induction on d. The proposition is obviously true for $d=0$. Since preprimes preserve addition, it is enough to prove the theorem for g a monomial. Suppose $g = aX^\epsilon$, where $\epsilon = (\epsilon_1,\dots,\epsilon_n) \in \mathbb{N}^n$ and assume w.l.o.g. that $\epsilon_1 \neq 0$. Then $g = X_1 \cdot aX^\beta$, where $|\beta| = |\epsilon| - 1$. Then

$$(2.2)\quad |a|N^{|\epsilon|} - g = \frac{1}{2}\Big((N+X_1)(|a|N^{|\beta|} - aX^\beta) + (N-X_1)(|a|N^{|\beta|} + aX^\beta)\Big).$$

Thus we are done by our induction hypothesis. Further, this gives us a recursive algorithm for constructing a P-representation for a specific g. □

COROLLARY 2.9. *Suppose P is a preprime in R, then P is archimedean iff there exists $N \in \mathbb{N}$ such that $N \pm x_i \in P$ for all i.*

Generally speaking, we will give algorithms for constructing a representation of an element in an archimedean preprime P in terms of the polynomials $\{N \pm x_i\}$. For ease of exposition, we want to enlarge the generating set of the preprime by adding these elements. Note that adding to the generators of P a polynomial p which is in P does not change P or S. Also, by Theorem 2.8, if $\{f_1,\dots,f_k\}$ is the set of generators of P, then we have $M - \sum f_i \in P$ for some M. Replacing each f_i by f_i/M and adding $1 - \sum f_i/M$, we have a set of generators for which $\sum f_i = 1$. For our constructions, it will be necessary to add to our set of generators elements of the form $N/M - x_i/M$, where $N, M \in \mathbb{N}$ and to have $\sum f_i = 1$. It will be convenient to replace the generators of P with this (possibly) enlarged set. As a final step in any construction, we can get a representation in terms of the original generators by using our fixed representations of the set $\{N \pm x_i\}$ and the representation of $1 - \sum f_i/M$ from Theorem 2.8.

DEFINITION 2.10. Suppose $P = PP(f_1,\dots,f_k)$ is an archimedean preprime. We say that the set of generators $\{f_1,\dots,f_k\}$ is **full** if

(1) The first n generators $\{f_1,\dots,f_n\}$ are $\{N/M - x_1/M,\dots,N/M - x_n/M\}$ for some $N, M \in \mathbb{N}$.
(2) $\sum f_i = 1$

3. Pólya's Theorem and the Kadison-Dubois Theorem

Pólya's Theorem is a representation theorem for homogeneous polynomials positive on the standard n-simplex. In this section, we show how other representation theorems can be reduced to Pólya's Theorem, in particular, the Kadison-Dubois Theorem. Since explicit bounds for the degree of a representation have been given for Pólya's Theorem, this yields degree bounds for the theorems reduced to Pólya. This technique was first used by M. Schweighofer [**14**] to study Schmüdgen's Theorem. In [**12**], it was used to study representations of polynomials positive on compact polyhedra.

We write Δ_n for the n-simplex $\{(x_1,\dots,x_n) \mid x_i \geq 0, \sum_i x_i = 1\}$. Pólya's Theorem ([**11**], [**6**, pp.57-59]) says that if $f \in \mathbb{R}[X]$ is homogeneous and positive on Δ_n, then for sufficiently large N all the coefficients of $(x_1+\cdots+x_n)^N f(x_1,\dots,x_n)$ are positive. In [**12**], an explicit bound for N is given:

THEOREM 3.1. *Given homogeneous $f \in \mathbb{R}[X]$ of degree d, say*

$$f(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

where $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Let $L = L(f) := \max_{|\alpha|=d} |b_\alpha|$ and $\lambda = \lambda(f) := \min_{X \in \Delta_n} f(X)$. If

$$N > \frac{d(d-1)}{2} \frac{L}{\lambda} - d,$$

then $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ has positive coefficients.

We now fix the following set-up: Given an archimedean preprime P in R with $P := PP(f_1, \ldots, f_k)$ and assume that the set of generators is full. Let $S := S(f_1, \ldots, f_k)$. Let $\mathbb{R}[Y]$ denote the polynomial ring $\mathbb{R}[y_1, \ldots, y_k]$ in k variables and define $\phi : \mathbb{R}[Y] \to R$ by $\phi(y_i) = f_i$. Since the generators contain $\{N/M - x_i/M\}$, the map ϕ is onto. Let $\{r_1, \ldots, r_t\}$ be a basis for $\ker \phi$. To construct such a basis, we can compute a Gröbner Basis B for the ideal generated by $\{y_1 - f_1, \ldots, y_k - f_k\}$ in the polynomial ring $\mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_k]$ using lex order with $x_1 > \cdots > x_n > y_1 > \cdots > y_k$. Then $B \cap \mathbb{R}[y_1, \ldots, y_k]$ is the desired basis.

Given $g \in R$, we can construct explicitly $\tilde{g} \in \mathbb{R}[Y]$ such that $\phi(\tilde{g}) = g$ and $\deg \tilde{g} = \deg g$. For example, if the first n generators of P are $\{N/M - x_1/M, \ldots, N/M - x_n/M\}$, then we can set $\tilde{g} = g(N - My_1, \ldots, N - My_n)$.

The following lemma is the key to reducing to Pólya's Theorem. It was first proven in [**14**] for preorders in polynomial rings and in [**12**] it was proven for preprimes associated to polyhedra. The argument generalizes immediately to our setting.

LEMMA 3.2. *Suppose S, P, and ϕ are above. Given $g \in R$ such that $g > 0$ on S. Then there exists a homogeneous $G \in \mathbb{R}[Y]$ such that $G > 0$ on Δ_k and $\phi(G) = g$. Furthermore, there is an algorithm for constructing G and there is a constant D which depends only on the generators of P such that $\deg G \leq \max\{\deg g, D\}$.*

PROOF. Let $\tilde{g}$ be as above, then $\phi(\tilde{g}) = g$. Also, let $\{r_1, \ldots, r_t\}$ be a basis for $\ker \phi$, which can be constructed as above. Set $r := \sum r_i^2$ and let $D = \deg r$. The argument of [**14**, 3.1] shows that $r > 0$ on the compact set $U := \Delta_k \cap \{\tilde{g} \leq 0\}$. Now choose a real number $c > \max\{0, \frac{-m_1}{m_2}\}$, where m_1 is the minimum of $\tilde{g}$ on Δ_k and m_2 is the minimum of r on U. Then $\tilde{g} + cr > 0$ on Δ_k. Note that $\deg(\tilde{g} + cr) = \max\{\deg g, D\}$. Finally, we let G be the homogeneous polynomial obtained from $\tilde{g} + cr$ by multiplying monomials by appropriate powers of $(y_1 + \cdots + y_k)$. □

The above theorem shows that instead of working in the archimedean preprime in R, we can use the map ϕ to transform to a polynomial ring and then apply Pólya's Theorem. In particular, we want to apply this to the Kadison-Dubois Theorem.

The Kadison-Dubois Theorem [**7**], [**4**] is a representation theorem for archimedean preprimes in commutative rings; it can be viewed as a far-reaching generalization of the fact that a field with an archimedean order has an order-preserving embedding into $\mathbb{R}$. For the most general statement of the theorem, see [**1**].

In our setting, the theorem is as follows:

THEOREM 3.3 (Kadison-Dubois). *Given $f_1, \ldots, f_k \in R$ and suppose that $P := PP(f_1, \ldots, f_k)$ is archimedean. Then for any $h \in R$, the following are equivalent:*

(1) $h(\alpha) \geq 0$ *for each* $\alpha \in S := S(f_1, \dots, f_k)$
(2) $h + \gamma \in P$ *for all positive* $\gamma \in \mathbb{R}$.

PROOF. It is clear that (ii) implies (i); we prove (i) implies (ii). Given $h \geq 0$ on S and real $\gamma > 0$, let $g = h + \gamma$. We can assume that the given set of generators for P is full, by the discussion in the previous section. Let $\mathbb{R}[Y]$ and ϕ be as above. Given $g > 0$ on S, by Lemma 3.2, there is homogeneous G in $\mathbb{R}[Y]$ such that $G > 0$ on Δ_k and $\phi(G) = g$. By Pólya, there is some non-negative integer M such that $(\sum y_i)^M G$ has only positive coefficients in $\mathbb{R}[Y]$ and hence, with $m = M + \deg G$, we have

$$(\sum y_i)^M G = \sum_{\epsilon \in \mathbb{N}^k, |\epsilon| = m} a_\epsilon y_1^{\epsilon_1} \cdot \dots \cdot y_k^{\epsilon_k},$$

where $a_\epsilon \in \mathbb{R}^+$. Applying ϕ to both sides, we obtain a representation of g in P, of degree m. □

EXAMPLE 3.4. Let $S = \{1 - x^2 \geq 0\} = [-1, 1]$, and

$$P = PP(\frac{1-x}{3}, \frac{1-x^2}{3}, \frac{1+x+x^2}{3}).$$

Note $2 + x = (1 - x^2) + (1 + x + x^2) \in P$ and so P is archimedean. The given set of generators $\{f_1, f_2, f_3\} = \{\frac{1-x}{3}, \frac{1-x^2}{3}, \frac{1+x+x^2}{3}\}$ is full.

Let $f = 7x^4 + 7x^3 + 7x^2 + 6$, then by Kadison-Dubois, $f \in P$. We want to find a representation of f in P. Set $g(y_1, y_2, y_3) = 27(y_1^2 + y_2^2 + y_3^2) - 9(y_1y_2 + y_1y_3 + y_2y_3)$, then $g(f_1, f_2, f_3) = f$ and $g > 0$ on Δ_3.

It turns out that g has Pólya exponent 2:

$$(y_1 + y_2 + y_3)^2 g =$$
$$27y_1^4 + 45y_1^3y_2 + 36y_1^2y_2^2 + 45y_1y_2^3 + 27y_2^4 + 45y_1^3y_3 + 9y_1^2y_2y_3 + 9y_1y_2^2y_3 +$$
$$45y_2^3y_3 + 36y_1^2y_3^2 + 9y_1y_2y_3^2 + 36y_2^2y_3^2 + 45y_1y_3^3 + 45y_2y_3^3 + 27y_3^4.$$

This yields the following representation of f in P:

$$7x^4 + 7x^3 + 7x^2 + 6 =$$
$$27(\frac{1-x}{3})^4 + 45(\frac{1-x}{3})^3(\frac{1-x^2}{3}) + 36(\frac{1-x}{3})^2(\frac{1-x^2}{3})^2 + 45(\frac{1-x}{3})(\frac{1-x^2}{3})^3$$
$$+ 27(\frac{1-x^2}{3})^4 + 45(\frac{1-x}{3})^3(\frac{1+x+x^2}{3}) + 9(\frac{1-x}{3})^2(\frac{1-x^2}{3})(\frac{1+x+x^2}{3})$$
$$+ 9(\frac{1-x}{3})(\frac{1-x^2}{3})^2(\frac{1+x+x^2}{3}) + 45(\frac{1-x^2}{3})^3(\frac{1+x+x^2}{3}) + 36(\frac{1-x}{3})^2(\frac{1+x+x^2}{3})^2$$
$$+ 9(\frac{1-x}{3})(\frac{1-x^2}{3})(\frac{1+x+x^2}{3})^2 + 36(\frac{1-x^2}{3})^2(\frac{1+x+x^2}{3})^2 + 45(\frac{1-x}{3})(\frac{1+x+x^2}{3})^3$$
$$+ 45(\frac{1-x^2}{3})(\frac{1+x+x^2}{3})^3 + 27(\frac{1+x+x^2}{3})^4,$$

as is easily checked!

REMARKS 3.5. (1) Note that the preprime $PP(1-x^2)$ is not archimedean. We could start with the preorder $PO(1 - x^2)$, so that Schmüdgen's Theorem applies. This must be archimedean, in fact $1-x = \frac{1}{2}[(1-x^2)+(1-x)^2]$ and $1 + x = \frac{1}{2}[(1 - x^2) + (1 + x)^2]$. Let $P = PP(1 - x^2, (1 - x)^2, (1 + x)^2)$. Then P contains $1 \pm x$ and thus is archimedean by Corollary 2.9. We now proceed as above. This shows that we do not need the "full" preorder to obtain the conclusion of Schmüdgen's Theorem.

In general, if we start with compact $S = S(f_1, \ldots, f_k)$ and the preorder $P = PO(f_1, \ldots, f_k)$, we know that P is archimedean. Suppose we have representations of $N \pm x_i$ in P, and say that $\{s_1, \ldots, s_m\}$ are all the sums of squares that occur in the representations. Then the *preprime* $PP(f_1, \ldots, f_k, s_1, \ldots s_m)$ is archimedean and hence Kadison-Dubois applies.

(2) The proof of the Kadison-Dubois Theorem, along with Lemma 3.2, yields an algorithm for constructing a representation of $h + \gamma$ in P, assuming the set of generators for P is full. Thus we have an algorithm for the Kadison-Dubois Theorem modulo representations of $\{N \pm x_i\}$ in P. However, it should be noted that to calculate the constant c needed in Lemma 3.2 is in general computationally difficult, probably as difficult as finding a P-representation. In [**14**], an alternate method is suggested: Instead of trying to calculate c, make it a parameter. For some c and some N, $(\sum y_i)^N(\tilde{g} + cr)$ has only positive coefficients, and the coefficients of this are linear in c. We can then write this as a linear programming problem for a fixed N and increase N until a solution exists. We hope to explore this idea in further work.

(3) We do not know an algorithm for finding the archimedean bound N and representations of $N \pm x_i$ in archimedean P. There is a procedure that could be used in specific cases, which comes from the fact that the problem of finding a P-representation of fixed degree for an element of P can be rewritten as a semidefinite programming (SDP) problem, which can then be solved using SDP software. Such problems can involve parameters, thus the archimedean bound N could be taken as a parameter. Then we could look for the existence of P-representations of $N \pm x_i$ of fixed degree for non-negative N. If no such P-representations exist, then one could raise the degree bound and try again. This would yield a sequence of SDP problems which must eventually have a solution. For details on the relationship between representations in preorders and preprimes and semidefinite programming, the reader should consult the work of J. B. Lasserre [**8**] and P. Parillo [**10**].

4. A Representation Theorem in the Non-compact Case

Recently, M. Marshall proved a generalization of Schmüdgen's Theorem to the non-compact case. The proof uses a generalization of Wörmann's proof, i.e., an algebraic argument is used to reduce to the archimedean case so that the Kadison-Dubois Theorem applies. In this section, we apply our proof of Kadison-Dubois to give a constructive version of this result.

Given $f_1, \ldots, f_k \in R$, set $S = S(f_1, \ldots, f_k)$ and $P := PO(f_1, \ldots, f_k)$. We do not assume that S is compact, hence P need not be archimedean. We need an element p in $1 + P$ such that there exists non-negative integers M and l with $Mp^l \pm x_1, \ldots, Mp^l \pm x_k \in P$. Note that such a p always exists, e.g., we can always let $p = 1 + \sum x_i^2$ and $M = l = 1$. Also note that if S is compact, then we can take $p = 1$.

The main theorem of this section is:

THEOREM 4.1 (Marshall). *Given P, S, p, M, and l as above. Then, for any $f \in R$ of degree D, the following are equivalent.*

(1) $f(\alpha) \geq 0$ *for each* $\alpha \in S(p_1, \ldots, p_k)$

(2) *For each* $\epsilon > 0$ *there is some sufficiently large integer* $t \geq 0$ *such that* $p^t(f + \epsilon p^{lD}) \in P$

PROOF. The proof of [**9**] shows that we can reduce to the archimedean case by localizing. We include the proof here for completeness.

Consider the following $\mathbb{R}$-algebra

$$C := \mathbb{R}[\frac{x_1}{p^l}, \ldots, \frac{x_n}{p^l}, \frac{1}{p}].$$

Let $d_i = \deg p_i$. Then we can decompose each p_i as $\sum_{j=0}^{d_i} p_{i_j}(x_1, \ldots, x_n)$ where each p_{i_j} is a homogeneous polynomial of degree j. It follows that

$$\frac{p_i}{p^{ld_i}} = \sum_{j=0}^{d_i} \frac{1}{p^{l(d_i - j)}} p_{i_j}(\frac{x_1}{p^l}, \ldots, \frac{x_n}{p^l}) \in C \tag{4.1}$$

Define the following preprime in C:

$$\tilde{P} := PP(\frac{p_1}{p^{ld_1}}, \ldots, \frac{p_k}{p^{ld_k}}, \frac{1}{p}, \frac{Mp^l \pm x_1}{p^l}, \ldots, \frac{Mp^l \pm x_n}{p^l}, \frac{p \pm 1}{p}) \tag{4.2}$$

By Corollary 2.9, $\tilde{P}$ is archimedean in C since we have $M \pm \frac{x_i}{p^l} = \frac{Mp^l \pm x_i}{p^l} \in \tilde{P}$ $(1 \leq i \leq n)$ and $1 \pm \frac{1}{p} = \frac{p \pm 1}{p} \in \tilde{P}$.

Decompose f given in our hypothesis as $f = \sum_{j=0}^{D} f_j(x_1, \ldots, x_n)$ where each f_j is homogeneous of degree j. Then

$$\tilde{f} := \frac{f}{p^{lD}} = \sum_{j=0}^{D} \frac{1}{p^{l(D-j)}} f_j(\frac{x_1}{p^l}, \ldots, \frac{x_n}{p^l}) \in C. \tag{4.3}$$

Define the semialgebraic set associated to $\tilde{P}$ as follows:

$$\tilde{S} := S(\frac{p_1}{p^{ld_1}}, \ldots, \frac{p_k}{p^{ld_k}}, \frac{1}{p}, \frac{Mp^l \pm x_1}{p^l}, \ldots, \frac{Mp^l \pm x_n}{p^l}, \frac{p \pm 1}{p}) \subseteq \mathbb{R}^n$$

then we will show that $\tilde{f} \geq 0$ on $\tilde{S}$. Let $\alpha \in \tilde{S}$ then $\frac{p_i(\alpha)}{p(\alpha)^{ld_i}} \geq 0$ for each i, and $\frac{1}{p(\alpha)} > 0$ and $p(\alpha) \neq 0$ since p is positive definite. Hence, $p_i(\alpha) \geq 0$ for every i which yields that $\alpha \in S$. Thus by our hypothesis $f(\alpha) \geq 0$. From equation (4.3), we obtain $\tilde{f}(\alpha) \geq 0$. We then complete the proof by applying Kadison-Dubois to $\tilde{f} + \epsilon$ with $\tilde{P}$ and $\tilde{S}$. □

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