A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra

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1 Introduction

Fix a positive integer n and let $\mathbf{R}[X] := \mathbf{R}[x_1, \ldots, x_n]$. We write Δ_n for the simplex $\{(x_1, \ldots, x_n) \mid x_i \geq 0, \sum_i x_i = 1\}$.

Pólya's Theorem ([6], [4, pp.57-59]) says that if $f \in \mathbf{R}[X]$ is homogeneous and positive on Δ_n, then for sufficiently large N all the coefficients of

$$(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$$

are positive. In this note, we give an explicit bound for N and give an application to some special representations of polynomials positive on polyhedra. In particular, we give a bound for the degree of a representation of a polynomial positive on a convex polyhedron as a positive linear combination of products of the linear polynomials defining the polyhedron.

We use the following multinomial notation: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, let X^α denote $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Let us fix homogeneous $f \in \mathbf{R}[X]$ of degree d,

$$f(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L = L(f) := \max_{|\alpha|=d} |b_\alpha|$ and $\lambda = \lambda(f) := \min_{X \in \Delta_n} f(X)$.

Our main theorem is:

Theorem 1. *Suppose that $f \in \mathbf{R}[X]$ is a form as above. If*

$$N > \frac{d(d-1)}{2}\frac{L}{\lambda} - d,$$

then $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ has positive coefficients.

Note in particular that the bound does not depend on n, the number of variables. This bound improves (by a factor of roughly $4n$) a the bound due to J. de Loera and F. Santos, see [1] and [2]. In [7, Ex. 3.5], we considered a special case equivalent to $f(x,y) = x^2 - (2-\delta)xy + y^2$, for which $L = \max\{1, 1 - \frac{2-\delta}{2}\} = 1$ and

$$\lambda = \min_{0 \le t \le 1} (t - (1-t))^2 + \delta t(1-t) = \min_{0 \le t \le 1} 1 - (4-\delta)t(1-t) = \frac{\delta}{4},$$

thus Theorem 1 gives $N > \frac{4}{\delta} - 2$. In fact, $(x+y)^N f(x,y)$ has positive coefficients precisely when $N \ge 2\lceil \frac{2}{\delta} \rceil - 3$, so Theorem 1 is sharp.

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2 The bound

In this section, we prove Theorem 1. We begin with some notation. For a positive number t, a non-negative integer m, and a single real variable x, define

$$(x)_t^m := x(x-t)\cdots(x-(m-1)t) = \prod_{i=0}^{m-1} (x - it)\,.$$

Note for later reference that

$$(ty)_t^d = \prod_{i=0}^{d-1} \bigl(ty - (i-1)t\bigr) = t^d (y)_1^d, \tag{1}$$

and if $m > n$ are both integers, then $(n)_1^m = 0$, since one of the factors in the definition is zero. It follows immediately that in the special case that $x = k/M$ and $t = 1/M$, where M is a positive integer, we have

$$\left(\frac{k}{M}\right)_{1/M}^m = \frac{1}{M^m}\prod_{i=0}^{m-1}(k-i) = \begin{cases} \frac{1}{M^m}\frac{k!}{(k-m)!} = \frac{m!}{M^m}\binom{k}{m}, & \text{if } m \le k; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We fix $f = \sum a_\alpha X^\alpha$ and suppose that $f > 0$ on Δ_n. We assume throughout that $d = \deg f > 1$; the $d = 1$ case is trivial. Following Pólya, we make the explicit computation:

$$
(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n) = \sum_{|\beta|=N} \frac{N!}{\beta_1! \cdots \beta_n!} x_1^{\beta_1} \cdots x_n^{\beta_n} \times \sum_{|\alpha|=d} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}.
$$

Write $\alpha \preceq \beta$ if $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$. For $|\beta| = N + d$, denote the coefficient of $x_1^{\beta_1} \ldots x_n^{\beta_n}$ in $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ by A_β. Then

$$
\begin{aligned}
A_\beta &= \sum_{|\alpha|=d,\ \alpha \preceq \beta} \frac{N!}{(\beta_1 - \alpha_1)! \cdots (\beta_n - \alpha_n)!} \cdot a_\alpha \\
&= \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} \sum_{|\alpha|=d,\ \alpha \preceq \beta} a_\alpha \prod_{\ell=1}^{n} \frac{\beta_\ell !}{(\beta_\ell - \alpha_\ell)!(N+d)^{\alpha_\ell}}.
\end{aligned}
$$

We now express A_β using the $(x)_t^m$ notation and (2):

$$
A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} \sum_{|\alpha|=d} a_\alpha (\tfrac{\beta_1}{N+d})_{(N+d)^{-1}}^{\alpha_1} \cdots (\tfrac{\beta_n}{N+d})_{(N+d)^{-1}}^{\alpha_n} \tag{3}
$$

(If $\alpha \not\preceq \beta$, then the extra terms added in (3) are just 0.) Still following Pólya, define

$$
f_t(x_1, \ldots, x_n) := \sum_{|\alpha|=d} a_\alpha (x_1)_t^{\alpha_1} \cdots (x_n)_t^{\alpha_n}.
$$

Clearly, $f_t \to f$ uniformly on Δ_n as $t \to 0$, hence for t sufficiently small, f_t is also positive on Δ_n. In view of the foregoing, this means that for N sufficiently large, and all β with $\sum_i \beta_i = N + d$ (so that $f_{(N+d)^{-1}}$ is evaluated on Δ_n),

$$
A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} f_{(N+d)^{-1}}(\tfrac{\beta_1}{N+d}, \ldots, \tfrac{\beta_n}{N+d}) > 0. \tag{4}
$$

It follows that all the coefficients of $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ are positive.

We now extend Pólya's work. Let us drop the constant factor in (4) and set $t = \frac{1}{N+d}$, $y_k = \frac{\beta_k}{N+d}$, and keep in mind that $\sum_k y_k = 1$. We have

$$
f_t(y_1, \ldots, y_n) = f(y_1, \ldots, y_n) - \sum_{|\alpha|=d} a_\alpha \left(y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)_t^{\alpha_1} \cdots (y_n)_t^{\alpha_n} \right).
$$

Using the information about f, we see that

$$
f_t(y_1, \ldots, y_n) \geq \lambda - L \sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} \left| y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)_t^{\alpha_1} \cdots (y_n)_t^{\alpha_n} \right| \tag{5}
$$

If $\alpha_k > \beta_k$, then $(y_k)_t^{\alpha_k} = 0$, so $y_k^{\alpha_k} \geq (y_k)_t^{\alpha_k} \geq 0$ for all k; hence we may drop the absolute value in (5)

By the Multinomial Theorem,

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} y_1^{\alpha_1} \cdots y_n^{\alpha_n} = (y_1 + \cdots + y_n)^d = 1.$$

By the iterated Vandermonde-Chu identity (see below for a proof),

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)_t^{\alpha_1} \cdots (y_n)_t^{\alpha_n} = (y_1 + \cdots + y_n)_t^d = \prod_{k=0}^{d-1} (1 - kt). \tag{6}$$

By (5), we are done if we can show that

$$\lambda - L\big(1 - (1-t) \cdots (1-(d-1)t)\big) \geq 0. \tag{7}$$

Suppose now that

$$t = \frac{1}{N+d} < \frac{2}{d(d-1)} \frac{\lambda}{L}.$$

It is easy to prove by induction that if $0 \leq w_j \leq 1$, then $\prod(1 - w_j) \geq 1 - \sum w_j$. Since $\lambda \leq f(1, 0, \ldots, 0) \leq L$ and $d \geq 2$, we have $t < \frac{1}{d-1}$, hence

$$(1 - (1-t) \cdots (1-(d-1)t)) < t(1 + 2 + \cdots + (d-1)) = t\frac{(d-1)d}{2} < \frac{\lambda}{L},$$

and we are done.

What remains is to prove the iterated Vandermonde-Chu identity (reference thanks to Doron Zeilberger):

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (X_1)_t^{\alpha_1} \cdots (X_n)_t^{\alpha_n} = (X_1 + \cdots + X_n)_t^d. \tag{8}$$

We first prove (8) combinatorially in the special case that $t = 1$ and $X_k = y_k$ is a non-negative integer:

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)_1^{\alpha_1} \cdots (y_n)_1^{\alpha_n} = (y_1 + \cdots + y_n)_1^d. \tag{9}$$

Consider n sets $S_1, \ldots, S_n$ of distinct elements, where $|S_k| = y_k$, and let $S = \cup S_k$. Then $|S| = \sum y_k := y$, and the number of d-tuples of distinct elements from S is plainly $y(y-1)\cdots(y-(d-1))$, which is the right-hand side of (9). We now count the number of d-tuples in a different way. For each n-tuple α with $|\alpha| = d$, consider the number of such d-tuples in which there are α_k distinct elements from

S_k, $1 \le k \le n$. There are $\binom{y_k}{\alpha_k} = (y_k)_1^{\alpha_k}/\alpha_k!$ ways to choose these elements, and $d!$ ways to arrange them, and so, altogether,

$$d! \prod_{k=1}^{n} \frac{(y_k)_1^{\alpha_k}}{\alpha_k!} = \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)_1^{\alpha_1} \cdots (y_n)_1^{\alpha_n}$$

d-tuples. We sum over all possible choices of α to get the left-hand side of (9), completing the proof in the special case that the y_k's are non-negative integers. But both sides of (9) are polynomials in the y_k's, and their difference is a polynomial which vanishes on $\mathbf{N}^n$. It's easy to see that such a polynomial must vanish identically, and so (9) is in fact an identity for all real y_k. Finally, let $y_k = X_k/t$ in (9) and multiply through by t^d, keeping (1) in mind. Then we have proved (8).

3 Polynomials positive on polyhedra

Suppose $P \subseteq \mathbf{R}^n$ is a convex, compact polyhedron with non-empty interior, bounded by linear polynomials $\lambda_1, \ldots, \lambda_k \in \mathbf{R}[X]$. We always choose the sign of the λ_i's so that $P = \{X \mid \lambda_i(X) \ge 0 \text{ for all } i\}$. Then it is an remarkable fact that any polynomial which is strictly positive on P can be written as a positive linear combination of powers of the λ_i's:

Theorem 2. *Given P as above and suppose $f \in \mathbf{R}[X]$ is strictly positive on P. Then for some $m \in \mathbf{N}$, f has a representation*

$$f = \sum_{|\alpha| \le m} b_\alpha {\lambda_1}^{\alpha_1} \ldots {\lambda_k}^{\alpha_k}, \tag{10}$$

where all $b_\alpha \in \mathbf{R}^+$.

This was first proven by Handelman [3]. His proof is non-constructive; it uses a representation theorem similar to the Kadison-Dubois Theorem. In this section, using our bound for Pólya's Theorem, we give an upper bound for the degree m of a representation (10).

We begin with the case where P is a simplex. In this case, a degree bound for the representation follows almost immediately from the bound in Pólya's Theorem. Let S be an n-simplex in $\mathbf{R}^n$, with vertices $\{v_0, \ldots, v_n\}$ and let $\{\lambda_0, \ldots, \lambda_n\}$ be the set of barycentric coordinates of S, i.e., each $\lambda_i \in \mathbf{R}[X]$ is linear and

$$X = \sum_{i=0}^{n} v_i \lambda_i(X),$$

$$1 = \sum_{i=0}^{n} \lambda_i(X),$$

$$\lambda_i(v_j) = \delta_{ij}.$$

Given $f \in \mathbf{R}[X]$ of degree d, then for any $m \geq d$, there exists a homogeneous polynomial $\tilde{f}_m$ in $n+1$ variables of degree m such that $\tilde{f}_m(\lambda_0, \ldots, \lambda_n) = f(X)$. We can construct $\tilde{f}_m$ as follows: Suppose $f(X) = \sum_{|\alpha| \leq d} a_\alpha X^\alpha$, then

$$f(X) = \sum_{|\alpha| \leq d} a_\alpha \big(\sum_{i=0}^{n} v_i \lambda_i(X) \big)^\alpha \big(\sum_{i=0}^{n} \lambda_i(X) \big)^{m-|\alpha|},$$

thus we set

$$\tilde{f}_m(y_0, \ldots, y_n) = \sum_{|\alpha| \leq d} a_\alpha \big(\sum_{i=0}^{n} v_i y_i \big)^\alpha \big(\sum_{i=0}^{n} y_i \big)^{m-|\alpha|}. \tag{11}$$

Note that for $d = \deg f$, $\tilde{f}_d$ is the Bernstein-Bézier form of f with respect to S.

The following theorem, without the bound and with a different proof, is [5, 2.6]. It is also a generalization of [7, Thm. 6]:

Theorem 3. *Suppose f of degree d is strictly positive on S and $\tilde{f}_d(y_0, \ldots, y_n)$ is as defined as in* (11)*. Let λ be the minimum of f on S and $L = L(\tilde{f}_d)$. Then for*

$$N \geq \frac{d(d-1)}{2} \frac{L}{\lambda} - d,$$

f has a representation of the form (10) *of degree N.*

Proof. Since $f > 0$ on S, it follows easily that $\tilde{f}_d > 0$ on Δ_{n+1}. Thus we can apply Pólya's Theorem to $\tilde{f}_d$ to find N such that $(\sum y_i)^N \tilde{f}_d(Y)$ has positive coefficients. Then we have

$$\big(\sum y_i\big)^N \tilde{f}_d(Y) = \sum_{|\beta| = N} b_\beta Y^\beta,$$

where $b_\beta \geq 0$ for all β. Substituting λ_i for y_i yields $f(X)$ on the left, and a representation of degree N on the right. The bound on N comes from Theorem 1 if we first note that the minimum of $\tilde{f}_d$ on Δ_{n+1} is the same as the minimum of f on S. $\square$

Now we turn to the more general case of the polyhedron P described in the beginning of the section. Given f strictly positive on P, we want to use the same technique as for simplices, i.e., find a homogeneous polynomial g which is positive on Δ_k such that when we "plug in" the λ_i's we obtain f. In this case, however, finding g is not quite so straightforward.

We fix P as above and $\{\lambda_1, \ldots, \lambda_k\}$ such that $P = \{\lambda_i \geq 0\}$. We first note that by [3], there must exist positive constants $c_i \in \mathbf{R}$ such that $\sum_i c_i \lambda_i = 1$. Constructing the c_i's comes down to finding a positive solution to an under-determined linear system. We replace each λ_i by $c_i\lambda_i$ so that we have

$$\sum_i \lambda_i = 1 \tag{12}$$

Furthermore, there exist constants $b_{i,j} \in \mathbf{R}$ so that, for $j = 1, \ldots, n$,

$$x_j = \sum_{j=1}^{k} b_{i,j}\lambda_i.$$

Explicitly finding the $b_{i,j}$'s is an easy linear algebra problem. Thus we are almost in the situation for simplices, although the $b_{i,j}$'s need not be positive. Let B be the real $n \times k$ matrix $(b_{i,j})$, then

$$B \cdot (\lambda_1, \ldots, \lambda_k)^T = (x_1, \ldots, x_n) \tag{13}$$

As in [8], we formalize the notion of "plugging in" the λ_i's. Let $\mathbf{R}[Y] := \mathbf{R}[y_1, \ldots, y_k]$ and define $\phi : \mathbf{R}[Y] \to \mathbf{R}[X]$ by $y_i \mapsto \lambda_i$. By (12) and (13), ϕ is onto. More explicitly, given a polynomial $f = \sum_{|\alpha| \leq d} a_\alpha X^\alpha$, define homogeneous $\tilde{f} \in \mathbf{R}[Y]$ by

$$\tilde{f} := \sum_{|\alpha| \leq d} a_\alpha (B \cdot Y^T)^\alpha (\sum_{j=1}^{k} y_j)^{d-|\alpha|}. \tag{14}$$

Then $\phi(\tilde{f}) = f$.

Suppose now that $f > 0$ on P and we have a point $\gamma \in \Delta_k$. Then $\tilde{f}(\gamma) = f(B \cdot \gamma)$. Since the point $B \cdot \gamma$ need not be in P, we do not necessarily have that $\tilde{f}(\gamma)$ is positive. Thus we cannot apply Polya's Theorem directly to $\tilde{f}$. However, by a theorem of Schweighofer [8], it turns out that there is a polynomial positive on Δ_k of the form $\tilde{f} + c(\sum_j r_j^2)$, where $\{r_1, \ldots, r_t\}$ is any basis for the kernel of ϕ. Note that any g of this form has the property $\phi(g) = f$. The following result is (essentially) [8, Lemma 3.1]:

Lemma 4. *Suppose P and ϕ are as above and $f > 0$ on P. Let $\{r_1, \ldots, r_t\}$ be a basis for the kernel of ϕ, set $r := \sum_{j=1}^{t} r_j^2$, and define $\tilde{f}$ be as in (14). Then for sufficiently large c, $\tilde{f} + cr$ is strictly positive on Δ_k. More explicitly, this holds for $c > \frac{-m_1}{m_2}$, where m_1 is the minimum of $\tilde{f}$ on Δ_k and m_2 is the minimum of r on $\Delta_k \cap \{\beta \in \mathbf{R}^k \mid \tilde{f}(\beta) \leq 0\}$.*

Proof. Let U be the compact set $\Delta_k \cap \{\beta \in \mathbf{R}^k \mid \tilde{f}(\beta) \leq 0\}$. By [8, §3], $r > 0$ on U and hence, since U is compact, the minimum m_2 of r on U exists and is positive. Then on U, $\tilde{f} + cr \geq m_1 + cm_2 > 0$. On $\Delta_k \setminus U$, $\tilde{f} + cr \geq \tilde{f} > 0$. □

Once we replace $\tilde{f}$ by $F := \tilde{f} + cr$, our new polynomial need not be homogeneous. However, we can multiply monomials of F by powers of $y_1 + \cdots + y_k$ to obtain a homogeneous $\tilde{F}$ of the same degree as F which retains the two essential properties of F: $\phi(\tilde{F}) = f$ and $\tilde{F} > 0$ on P.

Theorem 5. *Given P, ϕ, r, f, and $\tilde{f}$ as above. Fix c such that $F := \tilde{f} + cr > 0$ on Δ_k and let $\tilde{F}$ be the homogeneous polynomial obtained from F as above. Let d be the degree of f and let λ be the minimum of $\tilde{F}$ on Δ_k. For*

$$N \geq \frac{d(d-1)}{2} \frac{L(\tilde{F})}{\lambda} - d,$$

f has a representation

$$f = \sum_{|\alpha|=N} b_\alpha \lambda_1^{\alpha_1} \ldots \lambda_k^{\alpha_k},$$

where $b_\alpha \geq 0$ for all α.

Proof. Since $\phi(\tilde{F}) = f$, this follows from Theorem 1 applied to $\tilde{F}$, exactly as in the proof of Theorem 3. □

Remark. Note that for a specific P and f, we can calculate all elements needed for the bound in the theorem, and then can find a representation for f. Thus the theorem yields an algorithm for finding a representation for f of the form (10).

Algorithm. Given a compact, convex polyhedron $P \subseteq \mathbf{R}^n$ defined by $\{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0\}$, where $\sum_i \lambda_i = 1$, and $f > 0$ on P. We will describe a procedure for constructing a representation of f of the form (10). We proceed as follows:

1. Using (14), construct homogeneous $\tilde{f} \in \mathbf{R}[Y]$ with the same degree as f such that $\phi(\tilde{f}) = f$.
2. Construct a basis $\{r_1, \ldots, r_t\}$ for the kernel of ϕ. We can use the following well-known procedure for this: Construct a Groebner Basis G for the ideal generated by $\{y_1 - \lambda_1, \ldots, y_k - \lambda_k\}$, using lex order with $x_1 > \cdots > x_n > y_1 > \cdots > y_k$. Then $G \cap \mathbf{R}[y_1, \ldots, y_k]$ is the desired basis.
3. Calculate the minima m_1 and m_2 needed for the c of Lemma 4, e.g., by using Lagrange multipliers. Set $F := \tilde{f} + cr$ and find the homogeneous $\tilde{F}$.
4. Calculate $L(\tilde{F})$ and the minimum of $\tilde{F}$ on Δ_k and then find N as in Theorem 5. Use the coefficients of $Y^N \tilde{F}$ to obtain the desired representation.

Example. Let P be the square unit square centered at the origin in $\mathbf{R}^2$, and let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{1/4 + 1/4\,x, 1/4 - 1/4\,x, 1/4 + 1/4\,y, 1/4 - 1/4\,y\}$. With ϕ the map defined above, we have that $\{r_1, r_2\} := \{y_1 + y_2 - 1/2, y_3 + y_4 - 1/2\}$ is a Groebner Basis for the kernel of ϕ. Consider $f := 3/2 - x^2 + y^2 > 0$ on P, then

$$\tilde{f} = -\frac{5}{2}y_1^2 + 11y_1y_2 - \frac{5}{2}y_2^2 + 3y_1y_3 + 3y_2y_3 + \frac{11}{2}y_3^2 + 3y_1y_4 + 3y_2y_4 - 5y_3y_4 + \frac{11}{2}y_4^2$$

The minimum of $r := r_1^2 + r_2^2$ on $\{\tilde{f} \leq 0\} \cap \Delta_4$ is 1 and the minimum of $\tilde{f}$ on Δ_4 is $-5/2$. Thus we need $c > 5/2$. We choose $c = 3$, and set $F := \tilde{f} + 3r$. Then

$$\tilde{F} = \frac{7}{2}y_1^2 + 23y_1y_2 + \frac{7}{2}y_2^2 - 9y_1y_3 - 9y_2y_3 + \frac{23}{2}y_3^2 - 9y_1y_4 - 9y_2y_4 + 7y_3y_4 + \frac{23}{2}y_4^2$$

which is positive on Δ_4. The minimum of $\tilde{F}$ on Δ_4 is $3/10$ and $L(\tilde{F}) = 23/2$. Hence the bound in Theorem 1 is 38. This means that that $(y_1 + y_2 + y_3 + y_4)^{38}\tilde{F}$ must have positive coefficients. Expanding, and plugging in the λ_i's, we could then obtain an explicit representation for f.

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