TRIANGULATIONS OF CYCLIC POLYTOPES AND HIGHER BRUHAT ORDERS

D-KOZU JÖRG RAMBAU

Abstract. Recently Edelman and Reiner suggested two poset structures $\mathscr{S}_1(n, d)$ and $\mathscr{S}_2(n, d)$ on the set of all triangulations of the cyclic d-polytope $C(n, d)$ with n vertices. Both posets are generalizations of the well-studied Tamari lattice. While $\mathscr{S}_2(n, d)$ is bounded by definition, the same is not obvious for $\mathscr{S}_1(n, d)$. In the paper by Edelman and Reiner the bounds of $\mathscr{S}_2(n, d)$ were also confirmed for $\mathscr{S}_1(n, d)$ whenever $d \leqslant 5$, leaving the general case as a conjecture.

In this paper their conjecture is answered in the affirmative for all d, using several new functorial constructions. Moreover, a structure theorem is presented, stating that the elements of $\mathscr{S}_1(n, d+1)$ are in one-to-one correspondence to certain equivalence classes of maximal chains of $\mathscr{S}_1(n, d)$. By similar methods it is proved that all triangulations of cyclic polytopes are shellable. In order to clarify the connection between $\mathscr{S}_1(n, d)$ and the higher Bruhat order $\mathscr{B}(n-2, d-1)$ of Manin and Schechtman, we define an order-preserving map from $\mathscr{B}(n-2, d-1)$ to $\mathscr{S}_1(n, d)$, thereby concretizing a result by Kapranov and Voevodsky in the theory of ordered n-categories.

§1. *Introduction.* In this paper we examine the structure of the *first higher Stasheff–Tamari order* $\mathscr{S}_1(n, d)$ on the set of *all triangulations of the cyclic polytope* $C(n, d)$ (definitions below), introduced by Edelman and Reiner [**7**]. It turns out that it is similarly structured as the higher Bruhat order $\mathscr{B}(n-2, d-1)$ of Manin and Schechtman [**15**]; in particular it is bounded.

Given a *triangulation* of the convex hull of a finite point configuration in Euclidean d-space that is not satisfying a certain quality measure, can one find a better, or even the best triangulation (with respect to this measure) by performing a finite sequence of (computationally cheap) local transformations? A necessary condition for the latter case is that any possible triangulation is accessible by this kind of transformation. In particular, a repeatedly posed question in combinatorial and computational geometry (see for example Billera, Kapranov and Sturmfels [**4**], Edelsbrunner [**8**, Open Problem 8], and Joe [**11**, Conjecture 1]) is whether or not any two triangulations of (the convex hull of) a given finite point configuration in Euclidean space of dimension d can be connected by a sequence of *bistellar operations*.

For $d=2$ the answer is affirmative, as is for the restriction to *regular triangulations* (by the work of Gelfand, Kapranov and Zelevinsky [**9**]). For $d \geqslant 3$ and general triangulations, however, the problem is open in spite of many attacks in this direction.

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Professor B. J. Harris,
Department of Mathematical Sciences,
Northern Illinois University,
DeKalb, Illinois 60115-2888,
U.S.A.

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Similar problems attained attention in several fields of pure mathematics, thereby leading to remarkable new concepts, such as the *secondary polytope* defined by Gelfand, Kapranov and Zelevinsky [**9**], further studied by Billera, Filliman and Sturmfels [**2**] and Billera, Gelfand and Sturmfels [**3**]. The theoretical question behind this all is the following: Has the set of *all* triangulations of a point configuration a well-behaved *global* structure with respect to local transformations? A far-reaching generalization of this question to *restricted polyhedral subdivisions* was recently answered in the negative by Rambau and Ziegler [**16**].

The cyclic d-polytope $C(n, d)$ with n vertices appears on the scene as a combinatorially well-understood natural generalization of (convex) n-gons to higher dimensions. The triangulations of an n-gon form the extensively studied Tamari lattice—which one is definitely willing to consider as a good-natured structure in this context. (For a historical background on Tamari lattices and their different combinatorial interpretations we refer to the paper by Edelman and Reiner [**7**] and references given there.) The natural question now is which properties of the Tamari lattices survive in higher dimensions.

Since in general dimensions there are many non-regular triangulations of cyclic polytopes (see Billera, Gelfand and Sturmfels [**3**] and de Loera, Hoşten, Santos and Sturmfels [**14**]) it is not *a priori* clear that the set of all triangulations of the cyclic polytope $C(n, d)$ is well-behaved. In the paper by Edelman and Reiner [**7**] two poset structures $\mathscr{S}_1(n, d)$ and $\mathscr{S}_2(n, d)$ are defined on this set, both generalizing the Tamari lattice and hence quite interesting from a purely combinatorial point of view. In the following we sketch their definitions.

The triangulations of the cyclic polytope $C(n, d)$ are in one-to-one correspondence to the piecewise linear sections from $C(n, d)$ into $C(n, d+1)$, according to the projection from $C(n, d+1)$ onto $C(n, d)$ that deletes the last coordinate. Edelman and Reiner [**7**] suggest two partial orders on all piecewise linear sections, and hence on the set of all triangulations of $C(n, d)$.

The *first higher Stasheff–Tamari order* $\mathscr{S}_1(n, d)$ is defined by a covering relation between two sections if exactly one $(d+1)$-simplex fits between them in $C(n, d+1)$; the section that contains the upper facets of this simplex is defined to be greater than the other one. This corresponds to an *increasing bistellar flip* that replaces the lower facets of the $(d+1)$-simplex by the upper facets. Thus we get a purely *combinatorial* description of this poset in terms of local transformations. The *second higher Stasheff–Tamari order* $\mathscr{S}_2(n, d)$ is defined *geometrically via* pointwise comparison of the heights of the sections.

While $\mathscr{S}_2(n, d)$ has a unique minimal element $\mathscr{F}^l(n, d+1)$ (the set of *lower* facets of $C(n, d+1)$) and a unique maximal element $\mathscr{F}^u(n, d+1)$ (the set of *upper* facets of $C(n, d+1)$), the same is not obvious for $\mathscr{S}_1(n, d)$. On the other hand, the local structure of $\mathscr{S}_1(n, d)$ is clear by definition while the covering relations in $\mathscr{S}_2(n, d)$ are *a priori* unknown.

This motivated the following conjectures and results by Edelman and Reiner.

For even d, both $\mathscr{S}_1(n, d)$ and $\mathscr{S}_2(n, d)$ are self-dual [**7**, Prop. 2.11, true in general].

$\mathscr{S}_1(n, d)$ coincides with $\mathscr{S}_2(n, d)$ [**7**, Conj. 2.6, true for $d \leqslant 3$].

$\mathcal{F}^{l}(n, d+1)$ is the unique minimal element of $\mathcal{S}_1(n, d)$ [7, Conj. 2.7a, true for $d \leqslant 5$].

$\mathcal{F}^{u}(n, d+1)$ is the unique maximal element of $\mathcal{S}_1(n, d)$ [7, Conj. 2.7b, true for $d \leqslant 4$].

Any two triangulations of $C(n, d)$ are connected by a sequence of bistellar operations [7, Conj. 2.8, true for $d \leqslant 5$].

$\mathcal{S}_1(n, d)$ respectively $\mathcal{S}_2(n, d)$ is a lattice [7, Conj. 2.13, true for $d \leqslant 3$].

In any interval of $\mathcal{S}_1(n, d)$ respectively $\mathcal{S}_2(n, d)$ distinct subsets of coatoms have different meets [7, Conj. 2.14, true for $d \leqslant 3$].

Our main Theorem answers their Conjectures 2.7a, 2.7b and 2.8 affirmatively and points out the connections between the triangulation posets in different dimensions. Its proof is completed in Section 5 and Section 6, using the functorial constructions in Section 3, which we consider as interesting in their own right.

THEOREM 1.1 (Main Result).

(i) *For all n and all $d<n$ the first higher Stasheff–Tamari order $\mathcal{S}_1(n, d)$ is bounded. The unique minimal element is the set $\mathcal{F}^{l}(n, d+1)$ of lower facets, the unique maximal element is the set $\mathcal{F}^{u}(n, d+1)$ of upper facets of $C(n, d+1)$.*

(ii) *The elements of $\mathcal{S}_1(n, d+1)$ are in one-to-one correspondence with the equivalence classes of maximal chains in $\mathcal{S}_1(n, d)$ under the following equivalence relation: Two maximal chains are equivalent if they differ only by a permutation of their increasing bistellar operations.*

(iii) *Two maximal chains in $\mathcal{S}_1(n, d)$ are equivalent if, and only if, they differ by a sequence of interchanges of consecutive bistellar operations that correspond to non-adjacent $(d+1)$-simplices in $C(n, d+1)$.*

(iv) *All triangulations of cyclic polytopes without new vertices are shellable.*

The following list of implications demonstrates the quantitative consequences of the main Theorem and the constructions provided in Section 3.

COROLLARY 1.2. *For all n and all $d<n$ the following hold.*

(i) *For odd d, $\mathcal{S}_1(n, d)$ is a ranked poset with rank function*

$$r(T) = \#\mathcal{F}^{l}(n, d+1) - \#T \qquad \text{for} \quad \text{all} \qquad T \in \mathcal{S}_1(n, d).$$

(ii) *The number of simplices in a triangulation of $C(n, d)$ lies between the number $\binom{n-\lfloor d/2\rfloor-1}{\lfloor d/2\rfloor}$ of upper facets and the number $\binom{n-\lfloor (d+1)/2\rfloor}{\lfloor (d+1)/2\rfloor}$ of lower facets of $C(n, d+1)$. In particular, for even d all triangulations of $C(n, d)$ consist of $\binom{n-d/2-1}{d/2}$ simplices. (That the latter fact is actually true for all weakly neighbourly polytopes, was proved by Bayer [1]).*

(iii) *The length of a maximal chain in $\mathcal{S}_1(n, d)$ lies between the number $\binom{n-\lfloor (d+1)/2\rfloor-1}{\lfloor (d+1)/2\rfloor}$ of upper facets and the number $\binom{n-\lfloor d/2\rfloor-1}{\lfloor d/2\rfloor+1}$ of lower facets of $C(n, d+2)$. In particular, for odd d the length of any maximal chain in $\mathcal{S}_1(n, d)$ equals $\binom{n-(d+1)/2-1}{(d+1)/2}$.*

(iv) *For even d the diameter of the Hasse-diagram of $\mathcal{S}_1(n, d)$ is between $\binom{n-d/2-2}{d/2}$ and twice this value. For odd d it is equal to $\binom{n-(d+1)/2-1}{(d+1)/2}$.*

Theorem 1.1 points out a similarity to the structure of the higher Bruhat order $\mathscr{B}(n-2, d-1)$, a certain generalization of the weak Bruhat order on the symmetric group, defined by Manin and Schechtman **[15]** (see also Ziegler **[17]**). Previously, Kapranov and Voevodsky **[12]** reported the existence of an order-preserving surjection from $\mathscr{B}(n-2, d-1)$ onto a poset structure on the set of all triangulations of $C(n, d)$ that is inherited by a certain ordered n-category. Unfortunately, it is not clear whether their poset structure is equivalent to $\mathscr{S}_1(n, d)$. This led us to the investigations in Section 8 where we present an explicit order-preserving map $\mathscr{T}$ from $\mathscr{B}(n-2, d-1)$ to $\mathscr{S}_1(n, d)$ that should help to get a more concrete idea of the connections between higher Bruhat orders and higher Stasheff–Tamari orders. Furthermore, we relate some of the functorial constructions for higher Bruhat orders to similar constructions for higher Stasheff–Tamari orders.

In Section 7 we will recall the main definitions and results in the framework of higher Bruhat orders. Additionally, we answer a question posed by Ziegler **[17]** on the existence of an order-preserving embedding of $\mathscr{B}(n, k)$ into $\mathscr{B}(n+1, k+1)$ affirmatively.

The following three problems concerning the higher Stasheff–Tamari orders remain open.

Is $\mathscr{S}_1(n, d)$ equal to $\mathscr{S}_2(n, d)$?

Is $\mathscr{S}_1(n, d)$ or $\mathscr{S}_2(n, d)$ a lattice?

Is $\mathscr{T}$ surjective; in particular is $\mathscr{T}$ the map suggested by Kapranov and Voevodsky?

Throughout this paper the following notation is used.

For a set L and "$<_l$" a linear order on L, we denote by $L_{<_l}$ the set L linearly ordered with "$<_l$".

Numbers in brackets $(i_1, \ldots, i_n)$ denote the set $\{i_1, \ldots, i_n\}_<$ which is linearly ordered with $i_v < i_{v+1}$ for $v = 1, \ldots, n-1$.

Let L be a set. For a subset $S \subseteq L$ let $\complement S = \complement_L S$ be the complement $L \setminus S$ of S in L.

For a set L and two sets K and K' of subsets of L such that $S \cap S' = \varnothing$ for all $S \in K$ and $S' \in K'$ let $K * K' = \{S \cup S' : S \in K, S' \in K'\}$ be the join of K and K'.

For a set K of subsets of L and $S_0 \in K$ the *deletion of* S_0 *from* K is the set $K \setminus S_0 = \{S \in K : S \cap S_0 = \varnothing\}$, and the *contraction of* S_0 *in* K is the set $K/S_0 = \{S \setminus S_0 : S \in K, S \supseteq S_0\}$.

For integers $a < b$ the interval $[a, b]$ is the set $\{a, a+1, \ldots, b-1, b\}$ and $]a, b[$ is the set $\{a+1, \ldots, b-1\}$,

$[n]$ denotes the interval $[1, n]$, and $]n[$ is the interval $]1, n[$.

§2. *A combinatorial framework for triangulations.* In this section we present a combinatorial concept of triangulations that is similar to that of de Loera **[6]**. Dealing with vertex labels when investigating triangulations is formally justified by the following considerations that are closely related to the theory of abstract simplicial complexes.

Definition 2.1. Let $\mathscr{L}$ be a finite set, the *label set*. A *combinatorial d-simplex in* $\mathscr{L}$ is a $(d+1)$-element-subset S of $\mathscr{L}$. Its $(k+1)$-subsets are called *k-faces of S*, and its d-subsets *facets of S*.

If $l\colon \mathscr{L}\to\mathbb{R}^N$ is an injective function with $l(\mathscr{L})=\mathscr{A}$, and $S\subset\mathscr{L}$ is a combinatorial d-simplex corresponding to affinely independent points then the convex hull $\sigma=\operatorname{conv} l(S)$ of $l(S)$ is *the geometric d-simplex with vertex set* $\operatorname{vert}\sigma = l(S)$ *and label set* $\operatorname{lab}(\sigma)=S$ *with respect to l*, the *labelling function*.

A *combinatorial simplicial complex in* $\mathscr{L}$ is a set K of combinatorial simplices in $\mathscr{L}$. Its k-simplices are the k-faces of its elements. (That is, we identify the usual abstract simplicial complexes with their sets of inclusion-maximal faces.) A set Δ of geometric simplices σ with the property that the set $\{\operatorname{lab}(\sigma)\colon \sigma\in\Delta\}$ of label sets is a combinatorial simplicial complex, and that

$$\operatorname{conv}(\operatorname{vert}\sigma\cap\operatorname{vert}\tau)=\sigma\cap\tau \qquad \text{for all} \quad \sigma,\,\tau\in\Delta,$$

is a *geometric simplicial complex*.

A combinatorial simplicial complex K' is a *combinatorial subcomplex* of K if all simplices of K' are faces of simplices in K. A *geometric subcomplex* is defined analogously.

For a combinatorial simplicial complex K in $\mathscr{L}$ and a combinatorial simplex S_0 in $\mathscr{L}$ the *combinatorial link of* S_0 *in* K is defined as

$$\operatorname{lk}_K(S_0)=\{S\setminus S_0 : S\in K,\, S_0\subseteq S\};$$

the *combinatorial star of* S_0 *in* K is defined by

$$\operatorname{st}_K(S_0)=\{S\in K\colon S_0\subseteq S\},$$

and *the combinatorial antistar of* S_0 *in* K is the complex

$$\operatorname{ast}_K(S_0)=\{S\in K\colon S\cap S_0=\varnothing\}.$$

If K is a combinatorial simplicial complex in $\mathscr{L}$ and S_0 is a combinatorial simplex in $\mathscr{L}'$ where $\mathscr{L}$ and $\mathscr{L}'$ are disjoint then *the combinatorial join of K and* S_0 is the complex

$$K * S_0=\{S\cup S_0 : S\in K\}.$$

The convex hull $\operatorname{conv}\mathscr{A}$ of $\mathscr{A}$ is a *d-polytope* if the affine hull of $\mathscr{A}$ is $\mathbb{R}^d$. For $\mathscr{A}'\subset\mathscr{A}$ the polytope $\operatorname{conv}\mathscr{A}'$ is a *facet of* $\operatorname{conv}\mathscr{A}$, if $\operatorname{conv}\mathscr{A}'$ is the $(d-1)$-dimensional intersection of $\mathscr{A}$ with a hyperplane H such that one closed half-space defined by H contains $\operatorname{conv}\mathscr{A}$. In this case the label set $\operatorname{lab}(\mathscr{A}')$ is a *combinatorial facet of l.* Note that the set of facets of a *simplicial* polytope (all facets are simplices) forms a simplicial complex.

If $Z=(Z^+,Z^-)$ is a pair of disjoint inclusion minimal subsets Z^+ and Z^- of $\mathscr{L}$ with the property

$$\operatorname{conv} l(Z^+)\cap\operatorname{conv} l(Z^-)\neq\varnothing$$

then Z is called a *minimal combinatorial dependence in* l, or—for short—a *circuit* of l. The set $\operatorname{supp}(Z) = Z^+ \cup Z^-$ is the *support of* Z.

The triple $\mathscr{P}(l) = (\mathscr{L}, \mathscr{F}_l, \mathscr{Z}_l)$, where $\mathscr{Z}_l$ denotes the set of all circuits of l, and $\mathscr{F}_l$ is the set of all combinatorial facets of l, is *the combinatorial polytope of* l.

If Δ a geometric simplicial complex with vertices in $\mathscr{A}$ such that $\operatorname{conv} \mathscr{A} = \bigcup_{\sigma \in \Delta} \sigma$ then Δ is called a *triangulation of* $\mathscr{A}$. In this case the set T of label sets of the simplices in Δ is a *combinatorial triangulation of* $\mathscr{P}(l)$.

We will sometimes call the geometric objects *geometric interpretations* of the corresponding combinatorial ones, which themselves are said to be *combinatorial models* for their geometric counterparts.

A combinatorial, label-based handling of triangulations is made possible by the following proposition. We present a complete elementary proof because this characterization is fundamental for this paper.

PROPOSITION 2.2. *Let* $\mathscr{L}$ *be a finite set and let* $l\colon \mathscr{L} \to \mathbb{R}^d$ *be injective with* $l(\mathscr{L}) = \mathscr{A}$. *Furthermore, let* $\mathscr{P}(l) = (\mathscr{L}, \mathscr{F}_l, \mathscr{Z}_l)$ *be the combinatorial polytope of* l. *A non-empty subset* T *of the* $(d+1)$*-subsets of* $\mathscr{L}$ *is a combinatorial triangulation of* l *if, and only if,*

(UP) *for all* $S \in T$ *and all facets* F *of* S *either* F *is contained in some* $F' \in \mathscr{F}_l$, *or there is another simplex* $S' \in T$ *such that* $S' \supset F$ (Union-Property), *and*

(IP) *there is no circuit* $Z \in \mathscr{Z}_l$ *with* $Z^+ \subset S$ *and* $Z^- \subset S'$ *for combinatorial simplices* $S, S' \in T$ (Intersection-Property).

Proof. We first prove that (UP) and (IP) are necessary. Let T be a combinatorial triangulation with respect to some geometric triangulation Δ of the point set $\mathscr{A}$ given by $l\colon \mathscr{L} \to \mathbb{R}^d$. Assume there is a combinatorial facet F of some combinatorial d-simplex S in T that is not contained in some F' in $\mathscr{F}_l$, such that there is no other combinatorial d-simplex in T containing F. Then the corresponding $(d-1)$-simplex $\tau = \operatorname{conv} l(F)$ is contained in only one simplex $\sigma = \operatorname{conv} l(S)$ of Δ.

Let H be a supporting hyperplane of τ such that its closed positive halfspace H^+ contains σ. Let q_τ be the barycentre of τ. Because τ is not a facet of $P = \operatorname{conv}(\mathscr{A})$ there is a point x_0 in P lying in the open negative halfspace $\operatorname{rel\,int}(H^-)$. Connect q_τ and x_0 by a segment I. This segment is completely contained in P since P is convex.

Δ is a triangulation. Hence, there must be at least one d-simplex σ_{x_0} that contains x_0. Either σ_{x_0} contains q_τ or not. If it does then σ_{x_0} must contain the complete $(d-1)$-simplex τ as a facet since q_τ lies in the relative interior of τ and the intersection of τ and σ_{x_0} must be a face of both. But this is a contradiction.

If σ_{x_0} does not contain q_τ then the segment I intersects the boundary of σ_{x_0} in a point q_{x_0}. Consider the mid-point x_1 of q_τ and q_{x_0} on I. This point is neither contained in τ nor in σ_{x_0}. Since I lies completely in P there must be a new d-simplex σ_{x_1} in Δ containing x_1. This procedure shows either a contradiction as above or an infinite sequence of d-simplices in Δ, which is a contradiction, too. Hence, Property (UP) is necessary.

For the necessity of Property (IP) assume that there are combinatorial d-simplices S and S' in T and a circuit $Z=(Z^+, Z^-)$ in $\mathscr{Z}(n, d)$ such that Z^+ is contained in S and Z^- is contained in S'. Then by the definition of circuits

$$\operatorname{conv} l(Z^+) \cap \operatorname{conv} l(Z^-) \neq \emptyset,$$

and their minimality implies that there are geometric simplices in Δ, namely $\operatorname{conv} l(Z^+)$ and $\operatorname{conv} l(Z^-)$ the relative interiors of which intersect, a contradiction. Hence, Property (IP) is necessary as well.

Let T be a collection of $(d+1)$-subsets of $\mathscr{L}$ (that is, $T \subseteq \binom{\mathscr{L}}{d+1}$) satisfying (UP) and (IP). Then T gives rise to a set of geometric simplices $\Delta = \{\operatorname{conv} l(S) : S \in T\}$. We must show that every point in P lies in at least one d-simplex σ in Δ and that for every pair of simplices σ and σ' we have $\operatorname{conv}(\operatorname{vert} \sigma \cap \operatorname{vert} \sigma') = \sigma \cap \sigma'$.

Let x be an arbitrary point in P. Since T is non-empty we find a combinatorial d-simplex S_0 in T. Hence there is a simplex $\sigma_0 = l(S_0)$ in Δ. Consider a segment I from an inner point x_0 of σ_0 to x that does not meet any $(d-2)$-simplex of Δ. Such a line exists because of the concept of general position. This segment is completely contained in P and meets exactly one facet τ of σ_0 unless $x \in \operatorname{rel\,int}(\sigma_0)$. If this intersection point q_τ equals x then we are done. Otherwise this facet is not a facet of P because then q_τ is an interior point of I and I is contained in P. Hence the label set F of τ is not contained in any element of $\mathscr{F}_l$, and we find another combinatorial d-simplex S_1 in T containing F corresponding to a geometric d-simplex σ_1 containing τ. The segment I meets the interior of σ_1 because of the general position property of I. Choose an arbitrary point x_1 in $I \cap \operatorname{rel\,int}(\sigma_1)$. Note that the distance between x_1 and x is strictly smaller than the distance between x_0 and x. Therefore, by repeating this procedure we will reach a d-simplex σ_r lying in Δ and containing x.

Now assume that there are geometric d-simplices σ and σ' in Δ with label sets S respectively S' in T and $\operatorname{conv}(\operatorname{vert} \sigma \cap \operatorname{vert} \sigma') \subset \sigma \cap \sigma'$. Since $\sigma \supseteq \sigma \cap \sigma'$ and $\sigma' \supseteq \sigma \cap \sigma'$ there are inclusion-minimal faces τ of σ and τ' of σ' with $\operatorname{conv}(\operatorname{vert} \tau \cap \operatorname{vert} \tau') \supseteq \sigma \cap \sigma'$. From the minimality assumption we get $\operatorname{rel\,int}(\tau) \cap \operatorname{rel\,int}(\tau') \neq \emptyset$, hence by Radon's Theorem there are minimal, vertex-disjoint faces ρ of τ and ρ' of τ' with $\operatorname{rel\,int}(\rho) \cap \operatorname{rel\,int}(\rho') \neq \emptyset$. Set $Z^+ = \operatorname{lab}(\rho)$ and $Z^- = \operatorname{lab}(\rho')$. Then Z^+ and Z^- are disjoint and $\operatorname{conv}(l(Z^+)) \cap \operatorname{conv}(l(Z^-)) \neq \emptyset$. Hence (Z^+, Z^-) lies in $\mathscr{Z}_l$, and Z^+ is contained in S and Z^- is contained in S', but this contradicts the assumption that T has Property (IP).

Pairs of simplices with property (IP) are called *admissible*.

§3. *Cyclic polytopes.* In this section we recall the basic definitions and theorems related to cyclic polytopes in a combinatorial language.

Definition 3.1. Let $\mathscr{L}$ be a linearly ordered set, and let $t: \mathscr{L} \to \mathbb{R}$, $i \mapsto t_i$ be a strictly monotone function.

The *d-dimensional cyclic polytope* $C(\mathscr{L}, d, t)$ *labelled by* $\mathscr{L}$, *parametrized by* t is the convex hull of the points $v_d(t_1), \ldots, v_d(t_n)$ with

$$v_d(x) = (x, x^2, \ldots, x^d) \in \mathbb{R}^d.$$

For simplicity we set $C(n, d, t) = C([n], d, t)$.

The main reason for the fact that triangulations of cyclic polytopes can be treated effectively in a purely combinatorial way are the following well-known properties that follow from the special structure of Vandermonde-determinants.

The first one—Gale's famous Evenness Criterion—characterizes the set $\mathscr{F}_{v_d \circ t}$ of all combinatorial facets of $C(\mathscr{L}, d, t)$. The following notion allows us to state that criterion in a compact way.

Definition 3.2. Let L be a linearly ordered set and S a subset of L. An element $s_0 \in \complement S$ is an *even gap* in S if $\#\{s \in S: s > s_0\}$ is even, otherwise it is an *odd gap*.

THEOREM 3.3 (Gale's Evenness Criterion [**10**]). *An ordered subset F of the vertex set of the cyclic polytope $C(\mathscr{L}, d, t)$ is a facet if, and only if, between any two vertices not in F there is an even number of vertices in F. Equivalently, F is a facet of $C(\mathscr{L}, d, t)$ if, and only if, either all gaps in F are even or all gaps in F are odd.*

The second one describes the form of those sets of vertices of $C(\mathscr{L}, d, t)$ the convex hulls of which intersect in the relative interior of both. Hence this determines $\mathscr{F}_{v_d \circ t}$.

THEOREM 3.4. [**5**]. *The circuits of $C(\mathscr{L}, d, t)$ are the alternating $(d+2)$-subsets of $\mathscr{L}$, i.e., the pairs (Z^o, Z^e) and (Z^e, Z^o), where Z^o is the set of odd elements $(z_1, z_3, z_5, \ldots)$, and Z^e is the set of even elements $(z_2, z_4, z_6, \ldots)$ of $Z = (z_1, \ldots, z_{d+2})$.*

The combinatorial polytopes $\mathscr{P}(v_d \circ t)$ are identical for all t because the strictly monotone function t does not affect the assertions of these criteria. This means that the combinatorial study of triangulations of cyclic polytopes with any parametrization is equivalent to the investigation of combinatorial triangulations of the combinatorial polytopes $\mathscr{P}(v_d \circ t)$.

Definition 3.5. The combinatorial polytope $C(\mathscr{L}, d) = \mathscr{P}(v_d \circ t)$ of $v_d \circ t$: $\mathscr{L} \to \mathbb{R}^d$ is called *the cyclic d-polytope with vertices labelled by* $\mathscr{L}$. The set of its combinatorial facets is denoted by $\mathscr{F}(\mathscr{L}, d)$, the set of its circuits is written as $\mathscr{Z}(\mathscr{L}, d)$. Those combinatorial facets with only odd gaps are the *upper facets* the set of which is denoted by $\mathscr{F}^u(\mathscr{L}, d)$, those with only even gaps are the *lower facets* of $C(\mathscr{L}, d)$, denoted by $\mathscr{F}^l(\mathscr{L}, d)$.

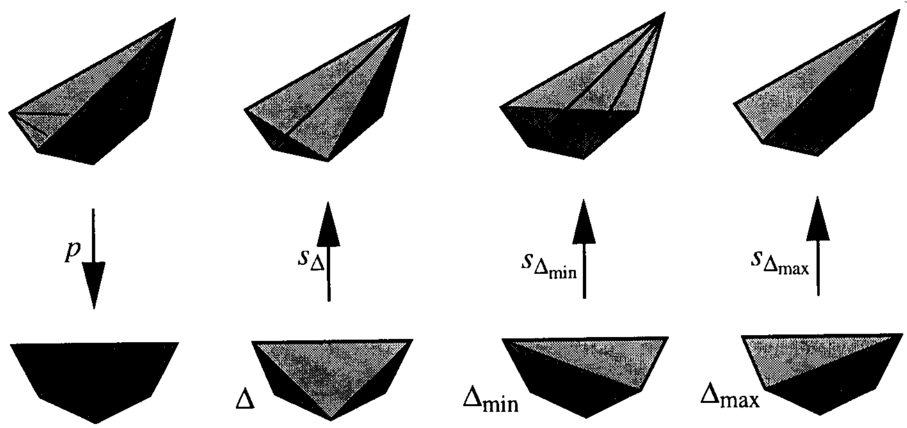

Figure 1. The canonical projection p: $C(5, 3) \to C(5, 2)$ and sections corresponding to triangulations of $C(5, 2)$.

The set of circuits Z with maximal element z_{d+2} in Z^+ is denoted by $\mathscr{Z}^+(n, d)$, the set of circuits having their maximal element in Z^- is written as $\mathscr{Z}^-(n, d)$. The cyclic polytope labelled by $[n]$ is denoted by $C(n, d)$.

Note that in odd dimensions there are polytopes that have the same face lattice as $C(n, d, t)$ but a different circuit structure (see **[5]**); this leads to completely different triangulations.

Remark 3.6 (Geometric Meaning, see Fig. 1). Consider for some strictly monotone t: $[n] \to \mathbb{R}$ the projection

$$p = p(n, d): \begin{cases} \qquad C(n, d+1, t) \to C(n, d, t), \\ (x_1, \ldots, x_d, x_{d+1}) \mapsto (x_1, \ldots, x_d). \end{cases}$$

Moreover, consider for some geometric triangulation Δ of $C(n, d, t)$ the unique piecewise linear section (linear on each simplex $\sigma \in \Delta$)

$$s_\Delta: \begin{cases} C(n, d, t) \to C(n, d+1, t), \\ \qquad \sigma \overset{\text{linear}}{\mapsto} \operatorname{conv}(v_{d+1} \circ t(\operatorname{lab}(\sigma))), \qquad \forall \sigma \in \Delta. \end{cases}$$

Then any triangulation Δ of $C(n, d, t)$ can be recovered from its *characteristic* section s_Δ.

The upper facets $\mathscr{F}^u(n, d+1)$ of $C(n, d+1)$ are the sets of those facets of $C(n, d+1, t)$ that can be seen from a point in $\mathbb{R}^{d+1}$ with very large positive $(d+1)$-st coordinate (geometric upper facets of $C(n, d+1, t)$), the lower facets $\mathscr{F}^u(n, d+1)$ label the sets of those facets of $C(n, d+1, t)$ that can be seen from a point in $\mathbb{R}^{d+1}$ with very large negative $(d+1)$-st coordinate (geometric lower facets of $C(n, d+1, t)$). The geometric upper (respectively lower) facets project down to $C(n, d, t)$ without overlapping. Therefore their projections define geometric triangulations of $C(n, d, t)$.

The support supp (Z) of any circuit $Z = (Z^+, Z^-)$ in $C(n, d)$ corresponds to the label set of a unique $(d+1)$-simplex in $C(n, d+1, t)$ where its set of geometric upper facets belongs to the elements of the star of the positive part Z^+ in supp (Z), and its set of geometric lower facets corresponds to the elements of the star of the negative part Z^- in supp (Z).

LEMMA 3.7 (Elementary Facts).

(i) $\mathscr{F}^l(n, d+1)$ *and* $\mathscr{F}^u(n, d+1)$ *are combinatorial triangulations of the cyclic polytope* $C(n, d)$.

(ii) *Every facet in* $\mathscr{F}^u(n, d)$ *contains* n.

(iii) *If a pair of simplices* S_1 *and* S_2 *is not admissible then there exists a circuit in* $\mathscr{Z}(n, d)$ *with maximal element* $z_{d+2} = \max(S_1 \cup S_2)$.

(iv) *If a* $(d-1)$*-simplex* F *is the common facet of the admissible pair* (S_1, S_2) *then* $S_1 \backslash F$ *lies in an odd gap of* F *and* $S_2 \backslash F$ *lies in an even gap of* F, *or vice versa.*

Remark 3.8. The circuits of $C(n, d)$ can be visualized in a table that consists of columns numbered from 1 to n and rows corresponding to Z^+ and Z^-, where a star "$*$" in column i and row Z^ε means that $i \in Z^\varepsilon$, $\varepsilon \in \{+, -\}$. The stars can then be connected by a zig-zag-path with $(d+2)$ nodes. For example, if $n=6$, $d=3$, and $Z=((1, 3, 5), (2, 4))$ we get the table

	1	2	3	4	5	6
Z^+	*		*		*	
Z^-		*		*		

If the rows are filled with stars corresponding to two simplices then these two simplices are admissible if, and only if, each zig-zag-path connects at most $(d+1)$ stars. For instance if $n=6$, $d=3$, $S=(1, 3, 4, 5)$, and $S'=(2, 3, 4, 6)$ the table looks as follows.

	1	2	3	4	5	6
S	*		*	*	*	
S'		*	*	*		*

The reader will easily find a zig-zag-path connecting even $6 > d+2$ stars, showing that S, S' is not an admissible pair.

Obviously all $C(\mathscr{L}, d)$ with $\#\mathscr{L} = n$ are isomorphic to $C(n, d)$. From now on we are exclusively dealing with combinatorial triangulations of $C(n, d)$, and we will leave out the "combinatorial" attribute whenever this is not confusing.

The following Propositions—consequences of Theorems 3.3 and 3.4—relate cyclic polytopes with different parameters. We use the notation $F = (f_1, \ldots, f_d)$ for $F \in \mathscr{F}(n, d)$ and $Z = (z_1, \ldots, z_{d+2})$ for $Z \in \mathscr{Z}(n, d)$.

PROPOSITION 3.9 (Functorial Facet Properties).

$$\mathscr{F}^u(n+1, d+1) = \mathscr{F}^l(n, d) * \{n+1\},$$
$$\mathscr{F}^l(n+1, d+1) = \mathscr{F}^u(n, d) * \{n+1\} \cup \{F \backslash n \cup \{j, j+1\} : F \in \mathscr{F}^u(n, d), j \in \,]f_{d-1}, n[\},$$
$$\mathscr{F}^u(n-1, d-1) = \mathrm{lk}_{\mathscr{F}^l(n,d)}(n),$$
$$\mathscr{F}^l(n-1, d-1) = \mathrm{lk}_{\mathscr{F}^u(n,d)}(n),$$
$$\mathscr{F}^u(n-1, d) = \mathrm{ast}_{\mathrm{lk}_{\mathscr{F}^u(n,d)}(n)}(n-1) * \{n-1\},$$
$$\mathscr{F}^l(n-1, d) = \mathrm{ast}_{\mathscr{F}^l(n,d)}(n).$$

PROPOSITION 3.10 (Functorial Circuit Properties).

$$\mathscr{Z}^+(n+1, d+1) = \{(Z^+ \cup \{j\}, Z^-) : (Z^+, Z^-) \in \mathscr{Z}^-(n, d), j > z_{d+2}\},$$
$$\mathscr{Z}^-(n+1, d+1) = \{(Z^+, Z^- \cup \{j\}) : (Z^+, Z^-) \in \mathscr{Z}^+(n, d), j > z_{d+2}\},$$
$$\mathscr{Z}^+(n-1, d-1) = \{(Z^+, Z^- \backslash z_{d+2}) : (Z^+, Z^-) \in \mathscr{Z}^-(n, d)\},$$
$$\mathscr{Z}^-(n-1, d-1) = \{(Z^+ \backslash z_{d+2}, Z^-) : (Z^+, Z^-) \in \mathscr{Z}^+(n, d)\},$$
$$\mathscr{Z}^+(n-1, d) = \{(Z^+, Z^-) \in \mathscr{Z}^+(n, d) : n \notin \mathrm{supp}\,(Z)\},$$
$$\mathscr{Z}^-(n-1, d) = \{(Z^+, Z^-) \in \mathscr{Z}^-(n, d) : n \notin \mathrm{supp}\,(Z)\}.$$

The following proposition is the combinatorial description for the geometric connection provided by the projection $p(n, d)$ between $(d+1)$-simplices in $C(n, d, t)$ and the minimal affine dependences in $C(n, d, t)$.

PROPOSITION 3.11 (Functorial Circuit-Facet-Relations). *For* $Z \in \mathscr{Z}^+(n, d)$ *and* supp (Z) *considered as a simplicial complex we have*

$$\mathrm{st}_{\mathrm{supp}\,(Z)}(Z^+) = \mathscr{F}^u(\mathrm{supp}\,(Z), d+1),$$
$$\mathrm{st}_{\mathrm{supp}\,(Z)}(Z^-) = \mathscr{F}^l(\mathrm{supp}\,(Z), d+1).$$

§4. *Special triangulations of cyclic polytopes.* In this section we show nice functorial constructions of triangulations of cyclic polytopes.

Definition 4.1. For a set T of $(d+1)$-subsets of $[n]$ define

$$\hat{T} = T * \{n+1\} \cup \{S \backslash s_{d+1} \cup \{j, j+1\} : S = (s_1, \ldots, s_{d+1}) \in T, j \in \,]s_d, s_{d+1}[\} \quad \text{(extension)}$$
$$T/n = \mathrm{lk}_T(n), \quad \text{(contraction)}$$
$$T \backslash n = \mathrm{ast}_T(n) \cup \mathrm{ast}_{\mathrm{lk}_T(n)}(n-1) * \{n-1\}. \quad \text{(deletion)}$$

THEOREM 4.2. *Let $T \in S(n, d)$. Then the following hold:*
(i) *$\hat{T}$ is a triangulation of $C(n+1, d+1)$;*
(ii) *T/n is a triangulation of $C(n-1, d-1)$;*
(iii) *$T\backslash n$ is a triangulation of $C(n-1, d)$.*

Proof. For each assertion we verify the Union-Property (UP) and the Intersection-Property (IP) of Proposition 2.2. Recall that we have to show—roughly speaking—that

all simplices are pairwise admissible, and that

each facet of a simplex is either a facet of the cyclic polytope or appears in at least one other simplex.

The reader may get a picture from the proof by inspecting the tables suggested in Remark 3.8, using that circuits correspond to zig-zag-paths and facets to sets with only even or only odd gaps.

Part (ii) is true because the link of a triangulation of any polytope at some vertex triangulates the corresponding vertex figure, and for cyclic polytopes this vertex figure is cyclic with the correct parameters. This follows from Propositions 3.9 and 3.10 and well-known properties of vertex figures (see, *e.g.*, Grünbaum [**10**]).

The proof of (UP) (i). The following abbreviations are used:

$$A = T * \{n+1\},$$
$$B = \{S \backslash s_{d+1} \cup \{j, j+1\} : S \in T, j \in \,]s_d, s_{d+1}[\}.$$

Let $F = (f_1, \dots, f_{d+1})$ be a facet of a simplex S in $A \backslash \mathscr{F}(n+1, d+1)$.

The case $f_{d+1} = n+1$. By Proposition 3.9, $F \backslash n+1 \notin \mathscr{F}(n, d)$ because otherwise $(F \backslash n+1) \cup \{n+1\}$ is a facet of $C(n+1, d+1)$. Since T has the Union-Property there must be a simplex $F' \in T$ with $F \backslash n+1 \subset F'$ and $F' \neq F$. Hence

$$F \subset \underbrace{F' \cup n+1}_{\substack{\neq S \\ \text{since } F' \neq F}} \in \hat{T}.$$

The case $F \in T$, $f_{d+1} - f_d > 1$. Then

$$F \subset \underbrace{F \backslash f_{d+1} \cup \{f_{d+1} - 1, f_{d+1}\}}_{\substack{\neq S \\ \text{since } n+1 \in S}} \in \hat{T}.$$

The case $F \in T$, $f_{d+1} - f_d = 1$. By Proposition 3.9, $F \backslash f_d \notin \mathscr{F}(n, d)$ because either f_{d+1} is an inner singleton in $F \backslash f_d$ or $f_{d+1} = n$ with the consequence that $(F \backslash f_d) \backslash n \cup \{n-1, n\} = F$ is a facet of $C(n+1, d+1)$. The Union-Property in T leads to the existence of a simplex $F' = (f'_1, \dots, f'_{d+1})$ in T with $F \backslash f_d \subset F'$ and $F' \neq F$. The Intersection-Property in T implies either

$$f'_{d+1} = f_{d+1}, \qquad f'_{d-1} = f_d, \tag{$*$}$$

or that

$$f'_{d+1} > f_{d+1}, \qquad f'_{d-1} = f_{d-1}. \tag{$**$}$$

(Compare Lemma 3.7(iv).)

Table 1. The expansion of F' in T.

	1 .... n							
$F^{(1)}=F'$	....	f'_{d-1}	...	f'_d	...	f'_{d+1}	....	
$F^{(2)}$	....	$f^{(2)}_{d-1}$	....	$f^{(2)}_d$	...	$f^{(2)}_{d+1}$	....	
$\vdots$			$\vdots$					
$F^{(r-1)}$	....	$f^{(r-1)}_{d-1}$	....	$f^{(r-1)}_d$	...	$f^{(r-1)}_{d+1}$	...	
$F^{(r)}=F''$	...	f''_k	...	f''_d	....	f''_{d+1}	...	

In the first case $(*)$ we get

$$F \subset \underbrace{F' \setminus f'_{d+1} \cup \{f_{d+1}-1,\ f_{d+1}\}}_{\substack{\neq S \\ \text{since } n+1 \in S}} \in \hat{T}.$$

In the second case $(**)$ we know that $F' \setminus f'_d \notin \mathscr{F}(n, d)$. Performing the same steps for $F' \setminus f'_d$ yields a finite sequence $F' = F^{(1)}, F^{(2)}, \ldots, F^{(r)} = F''$ (where $F^{(\mu)} = (f_1^{(\mu)}, \ldots, f_{d+1}^{(\mu)})$ for $\mu \in \{1, \ldots, r\}$) of simplices in T with

$$f''_{d+1} = f_{d+1}^{(r-1)} > f_{d+1}^{(r-2)} > \ldots > f_{d+1},$$

$$f''_d = f_{d-1}^{(r-1)} = f_{d-1}^{(r-2)} = \ldots = f_{d-1} < f_d = f_{d+1} - 1,$$

where at step (r) we end up in case $(*)$ because case $(**)$ can occur at most $n - f_{d+1}$ times. This leads to

$$F \subset \underbrace{F'' \setminus f''_{d+1} \cup \{f_{d+1}-1,\ f_{d+1}\}}_{\substack{\neq S \\ \text{since } n+1 \in S}} \in \hat{T}.$$

For further use we refer to this sequence as the *expansion of* F'.

Now let $F = (f_1, \ldots, f_{d+1})$ be a facet of the simplex $S = G \setminus g_{d+1} \cup \{j, j+1\}$ in B, such that F is not a facet of $C(n+1, d+1)$, with $G = (g_1, \ldots, g_{d+1})$ in T.

The case $F = S \setminus j+1, j = f_{d+1} > f_d + 1 = g_d + 1$. Then

$$F \subset \underbrace{G \setminus g_{d+1} \cup \{j-1, j\}}_{\substack{\neq S \\ \text{since } j+1 \in S}} \in \hat{T}.$$

The case $F = S \setminus j+1, j = f_{d+1} = f_d + 1 = g_d + 1$. Then we proceed as follows, $G \setminus g_d$ is not in $\mathscr{F}(n, d)$. Hence there is another simplex $G' = (g'_1, \ldots, g'_{d+1})$ in T with $G \setminus g_d \subset G'$. Consider the expansion $G' = G^{(1)}, G^{(2)}, \ldots, G^{(r)} = G''$ of G'. We have

$$g''_{d+1} \geqslant f_{d+1}, \qquad g''_d = f_{d-1} < f_d = f_{d+1} - 1,$$

and therefore

$$F \subset \underbrace{G'' \setminus g''_{d+1} \cup \{f_{d+1}-1,\ f_{d+1}\}}_{\substack{\neq S \\ \text{since } j+1 \in S,\ j+1 > f_{d+1}}} \in \hat{T}.$$

Table 2. The compression of H' in T.

	1 .... j .... n											
$H^{(1)}=H'$	...	h'_{d-1}	....					h'_d	...	h'_{d+1}	...	
$H^{(2)}$	...	$h^{(2)}_{d-1}$	....				$h^{(2)}_d$	...	$h^{(2)}_{d+1}$	....		
$\vdots$						$\vdots$						
$H^{(r-1)}$	...	$h^{(r-1)}_{d-1}$	...	$h^{(r-1)}_d$	...	$h^{(r-1)}_{d+1}$	....					
$H^{(r)}=H''$	...	h''_{d-1}	...	h''_d	...	h''_{d+1}	....					

The case $F=S\setminus j, j+1<g_{d+1}$. Then

$$F\subset \underbrace{G\setminus g_{d+1}\cup\{j+1,j+2\}}_{\substack{\neq S\\ \text{since } j\in S,\, j>g_d}}\in\hat{T}.$$

The case $F=S\setminus j, j+1=g_{d+1}$. Then

$$F=G\subset \underbrace{G\cup\{n+1\}}_{\substack{\neq S\\ \text{since } n+1\notin S}}\in\hat{T}.$$

The case $F=S\setminus g_i$, $1\leqslant i\leqslant d$. In this case $G\setminus g_i$ is not in $\mathscr{F}(n,d)$ because otherwise $(G\setminus g_i)\setminus g_{d+1}\cup\{j,j+1\}=F$ is a facet of $C(n+1,d+1)$ by Proposition 3.9. Hence we find a simplex $H=(h_1,\dots,h_{d+1})$ in T with $G\setminus g_i\in H$ and $H\neq G$.

(*) If $h_{d+1}=g_{d+1}$ and $h_d<j$ then

$$F\subset \underbrace{H\setminus h_{d+1}\cup\{j,j+1\}}_{\substack{\neq S\\ \text{since } H\neq G,\, h_{d+1}=g_{d+1}}}\in\hat{T}.$$

(**) If $h_{d+1}=g_{d+1}$ and $h_d=j$ then either $h_{d+1}=j+1$ and thus $F=H$, or $h_{d+1}>j+1$, whence

$$F=H\subset \underbrace{H\cup\{n+1\}}_{\substack{\neq S\\ \text{since } n+1\notin S}}\in\hat{T} \qquad \text{in the first case,}$$

$$F\subset \underbrace{H\setminus h_{d+1}\cup\{j+1,j+2\}}_{\substack{\neq S\\ \text{since } j+2\notin S}}\in\hat{T} \qquad \text{in the second case.}$$

(***) If $h_{d+1}=g_{d+1}$ and $h_d>j$ then $h_d>g_d+1$ and hence $h_d-h_{d-1}>g_d+1-g_d=1$. Therefore $H\setminus h_{d+1}$ is not in $\mathscr{F}(n,d)$ because h_d is an inner singleton. This implies that there is a simplex $H'=(h'_1,\dots,h'_{d+1})$ in T with $H\setminus h_{d+1}\subset H'$. The Intersection-Property in T leads to

$$h'_{d+1}=h_d>j, \qquad h'_d<h_d.$$

Performing the above step with H' instead of H induces a finite sequence (the *compression* of H') $H'=H^{(1)}, H^{(2)},\dots,H^{(r)}=H''$ where for H'' case (*) or case (**) must occur because the d-th element decreases monotonely. Then

$$h''_d\leqslant j<h^{(r-1)}_d=h''_{d+1},$$

and the constructions in (*) and (**) work with H'' instead of H as well.

(****) If $h_{d+1} > g_{d+1}$ then $H \backslash h_d$ is not a facet of $C(n, d)$, *i.e.*, we find a simplex $H' = (h'_1, \dots, h'_{d+1})$ in T with $H \backslash h_d \subset H'$ and $H' \neq H$, and we can finish the proof by using the expansion of H'.

The proof of (IP) (i). We must show that any pair of simplices (R, S) with $R = (r_1, \dots, r_{d+2})$ and $S = (s_1, \dots, s_{d+2})$ in $\hat{T}$ is admissible. Without loss of generality $\max(R \cup S) \in R$. There are three different cases:

The case $R \in A, S \in A$. It is well-known that a pyramid over a simplicial complex is again a simplicial complex, *i.e.*, it has the Intersection-Property.

The case $R \in B, S \in B$. There exist $R' = (r'_1, \dots, r'_{d+1})$ and $S' = (s'_1, \dots, s'_{d+2})$ in T such that

$$R = R' \backslash r'_{d+1} \cup \{j, j+1\}, \qquad r'_d < j < r'_{d+1},$$
$$S = S' \backslash s'_{d+1} \cup \{k, k+1\}, \qquad s'_d < k < s'_{d+1}.$$

Without loss of generality, $j \geqslant k$. Assume (R, S) is not admissible. Then, by Lemma 3.7, there exists a circuit $Z \in \mathscr{Z}^+(n+1, d+1)$ with $\operatorname{supp}(Z) = (z_1, \dots, z_{d+3})$ and

$$Z^+ \subset R, \qquad Z^- \subset S, \qquad z_{d+3} = r_{d+2} = j+1.$$

From Proposition 3.10 it follows that $Z' = (Z^+ \backslash z_{d+3}, Z^-)$ is a circuit in $\mathscr{Z}^-(n, d)$ with

$$(Z')^+ \subset R \backslash \{j+1\}, \qquad (Z')^- \subset S, \qquad z'_{d+2} \leqslant k+1 \leqslant s'_{d+1}.$$

Hence $z'_{d+1} < j$ and $z'_d < k$. Therefore

$$(Z')^+ \subset R', \qquad (Z')^- \backslash z'_{d+2} \cup s'_{d+1} \subset S'.$$

But then

$$Z'' = (\underbrace{(Z')^+}_{\subset R'}, \underbrace{(Z')^- \backslash z'_{d+2} \cup s'_{d+1}}_{\subset S'})$$

is a circuit in $\mathscr{Z}^-(n, d)$ showing that (R', S') is not admissible and yielding a contradiction.

The case $R \in A$, $S \in B$. There exist $R' = (r'_1, \dots, r'_{d+2})$ and $S' = (s'_1, \dots, s'_{d+2})$ in T with

$$R = R' \cup \{n+1\},$$
$$S = S' \backslash s'_{d+1} \cup \{k, k+1\}, \qquad s'_d < k < s'_{d+1}.$$

Assume again that (R, S) is not admissible. Let $Z \in \mathscr{Z}^+(n+1, d+1)$ be a circuit with $\operatorname{supp}(Z) = (z_1, \dots, z_{d+3})$ such that

$$Z^+ \subset R, \qquad Z^- \subset S, \qquad z_{d+3} = r_{d+2} = n+1.$$

Then

$$Z' = (\underbrace{Z^+ \backslash n+1}_{\subset R'}, \underbrace{Z^- \backslash z_{d+2} \cup s'_{d+1}}_{\subset S'})$$

is a circuit in $\mathscr{Z}^-(n, d)$ showing that (R', S') is not admissible and giving a contradiction.

The proof of (UP) (iii). In order to simplify notation we set

$$A = \mathrm{ast}_T(n),$$
$$B = \mathrm{ast}_{\mathrm{lk}_T(n)}(n-1) * \{n-1\}.$$

We bring some known facts into a useful form.

(a) Let F be a facet of $C(n-1, d-1)$ that does not contain $n-1$. Then $(F, n-1)$ is a facet of $C(n-1, d)$.

(b) Let F be a facet of $C(n, d)$ that does not contain n then F is a facet of $C(n-1, d)$.

(c) $\mathrm{st}_T(n) \cup \mathrm{as}_T(n) = T$, $\mathrm{st}_T(n) \cap \mathrm{ast}_T(n) = \mathrm{lk}_T(n)$.

Because of (c) all boundary facets of A are contained in $\mathrm{lk}_T(n)$ or are facets of $C(n, d)$ that do not contain n. Then by (b) all boundary facets of A that are not facets of $C(n-1, d)$ are contained in $\mathrm{lk}_T(n)$. Now let F be an element of $\mathrm{lk}_T(n)$ but not a facet of $C(n-1, d)$. If $n-1 \notin F$ then $(F, n-1) \in T\backslash n$. If $n-1 \in F$ then by (a) we know that $F\backslash(n-1)$ is not a facet of $C(n, d)/n$. Hence there is a simplex S in $\mathrm{ast}_{\mathrm{lk}_T(n)}(n-1)$ that contains $F\backslash(n-1)$ and therefore $F \subset (S, n-1) \in T\backslash n$.

Now let F be a facet in B that is not in $\mathscr{F}(n-1, d)$. If $n-1 \notin F$, then F is contained in $\mathrm{ast}_{\mathrm{lk}_T(n)}(n-1)$ and there must be a simplex in A containing F since there is such a simplex for all elements of $\mathrm{lk}_T(n)$ by (c). If $n-1 \in F$, then—by (a)—$F\backslash(n-1)$ is not a facet of $\mathrm{lk}_T(n)$. Hence there must be a simplex S in $\mathrm{ast}_{\mathrm{lk}_T(n)}(n-1)$ containing $F\backslash(n-1)$ and therefore the simplex $(S, n-1)$ is in B and contains F, which completes the proof.

The proof of (IP) (iii). The simplices in A are pairwise admissible because they are part of T, the simplices in B are pairwise admissible because B is a pyramid over a set of admissible simplices. Therefore assume there are $S_1 \in A$ and $S_2 \in B$ and a circuit Z with $Z^+ \subseteq S_1$ and $Z^- \subseteq S_2$, where $n-1 \in S_2$ by definition. If $n-1 \notin Z^-$ then $S_2' = S_2\backslash(n-1) \cup n$ and S_1 are not admissible either, contradiction because S_1 and S_2' are in T. But if we replace $n-1$ by n in Z then we get a circuit Z' that again shows that S_1 and S_2' are not admissible.

COROLLARY 4.3. *Any triangulation of the cyclic d-polytope $C(n, d)$ with n vertices induces:*

a canonical triangulation $\hat{T}$ of $C(n+1, d+1)$ containing T as the link of $n+1$;

a canonical triangulation T/n of $C(n-1, d-1)$ which is the link of n;

a canonical triangulation $T\backslash n$ of $C(n-1, d)$ containing the antistar of n as a subcomplex; and

a canonical triangulation δT defined as $\hat{T}\backslash n+1$ of $C(n, d+1)$ containing T as a subcomplex.

Remark 4.4. All these constructions—except for the link—are specific for cyclic polytopes and are incorrect for some more general polytopes.

In order to demonstrate that triangulating cyclic polytopes is nevertheless non-trivial, we provide an example showing that they are not *greedily triangulable*.

Example 4.5. Let $n=8$, $d=5$ and

$$S_1=(3,4,5,6,7,8),$$
$$S_2=(1,2,3,6,7,8),$$
$$S_3=(1,2,3,4,5,6).$$

Every pair of these simplices is admissible.

However, consider the facet $F=(1,3,6,7,8)$ of S_2: it is not a facet of $C(8,5)$. Hence, in any triangulation T of $C(8,5)$ that contains S_1, S_2 and S_3 there must be a simplex S' containing F. But all three possibilities for such a simplex produce non-admissible pairs. Therefore there is no such triangulation. Hence, one can get stuck while triangulating a cyclic polytope.

§5. *The higher Stasheff-Tamari orders.* In this section we describe the notion of increasing bistellar flips (as suggested by Edelman and Reiner [7]) in terms of our set-up. This leads to a combinatorial definition of the first higher Stasheff-Tamari order $\mathscr{S}_1(n,d)$. In contrast to this, the geometric definition of the second higher Stasheff-Tamari order $\mathscr{S}_2(n,d,t)$ is related to a geometric interpretation $\mathscr{S}_1(n,d,t)$ of $\mathscr{S}_1(n,d)$. Specific properties of cyclic polytopes lead to a simple proof of Theorem 1.1.

The set of all triangulations of $C(n,d)$, respectively $C(n,d,t)$, is denoted by $S(n,d)$, respectively $S(n,d,t)$.

Definition 5.1. An *increasing (bistellar) flip set in* $T\in S(n,d)$ is a simplex $\tilde{S}\in\binom{[n]}{d+2}$ with the property that the set of simplices $\mathscr{F}^l(\tilde{S},d+1)$ is a subset of T.

For all $(d+2)$-subsets $\tilde{S}$ of $[n]$ we have the *increasing flip function of* $\tilde{S}$

$$\text{flip}_{\tilde{S}}:\begin{cases} S(n,d)\to S(n,d),\\ \qquad T\mapsto\begin{cases} T\setminus\mathscr{F}^l(\tilde{S},d+1)\cup\mathscr{F}^u(\tilde{S},d+1) & \text{if } \mathscr{F}^l(\tilde{S},d+1)\subseteq T\\ T & \text{otherwise.}\end{cases}\end{cases}$$

Remark 5.2. By Proposition 3.11 this definition is equivalent to the notion of directed bistellar operations in Edelman and Reiner [7].

Remark 5.3 (Geometric Meaning, see Fig. 2). Let $t: [n] \to \mathbb{R}$ be strictly monotone. Let Δ be a geometric triangulation of $C(n, d, t)$ labelled by T, and Δ' geometric triangulation of $C(n, d, t)$ defined by the labels of $\text{flip}_{\tilde{S}}(T)$ for some increasing flip $\tilde{S}$ in $T \in S(n, d)$. Then the geometric lower facets of the $(d+1)$-simplex $\tilde{\sigma} = v_{d+1} \circ t(\tilde{S})$ in $C(n, d+1, t)$ defined by $\tilde{S}$ are contained in the piecewise linear section s_Δ, the geometric upper facets lie in $s_{\Delta'}$, and elsewhere the sections coincide.

Definition 5.4 (Edelman and Reiner [7]). The *first higher Stasheff–Tamari order on* $S(n, d)$ is defined *via*

$$T_1 \leqslant_1 T_2 \iff T_2 = \text{flip}_{\tilde{S}_r} \circ \ldots \circ \text{flip}_{\tilde{S}_1}(T_1)$$

for some sequence $(\tilde{S}_1, \ldots, \tilde{S}_r)$ in $\binom{[n]}{d+2}$. The set of all triangulations of $C(n, d)$ with this partial order is denoted by $\mathscr{S}_1(n, d)$.

The *second higher Stasheff–Tamari order on* $S(n, d, t)$ is defined *via*

$$\Delta_1 \leqslant_2 \Delta_2 \iff s_{\Delta_1}(x)_{d+1} \leqslant s_{\Delta_2}(x)_{d+1} \quad \text{for all} \quad x \in C(n, d, t),$$

that is, s_{Δ_1} *lifts* $C(n, d)$ *weakly lower than* s_{Δ_2}. It is written as $\mathscr{S}_2(n, d, t)$.

Remark 5.5. The triangulation $\mathscr{F}^l(n, d+1)$ is locally minimal, the triangulation $\mathscr{F}^u(n, d+1)$ is locally maximal in $\mathscr{S}_1(n, d, t)$.

Moreover, $\mathscr{F}^l(n, d+1)$ represents the unique (hence global) minimal element, and $\mathscr{F}^u(n, d+1)$ to the unique maximal element of $\mathscr{S}_2(n, d, t)$ for all strictly monotone $t: [n] \to \mathbb{R}$.

Edelman and Reiner [7, Conjecture 2.6] conjectured that $\mathscr{S}_1(n, d)$ is the correct combinatorial model for $\mathscr{S}_2(n, d)$, that is, $\mathscr{S}_2(n, d, t)$ coincides with $\mathscr{S}_1(n, d, t)$ for all strictly monotone $t: [n] \to \mathbb{R}$. Theorem 1.1 shows that, at least the maximal and minimal elements of both partial orders coincide.

In order to prove this, we introduce in the following for all T in $S(n, d)$ a partial order on the set of their simplices. In this context the notion of the parity of "gaps" in linearly ordered sets of Definition 3.2 is again useful.

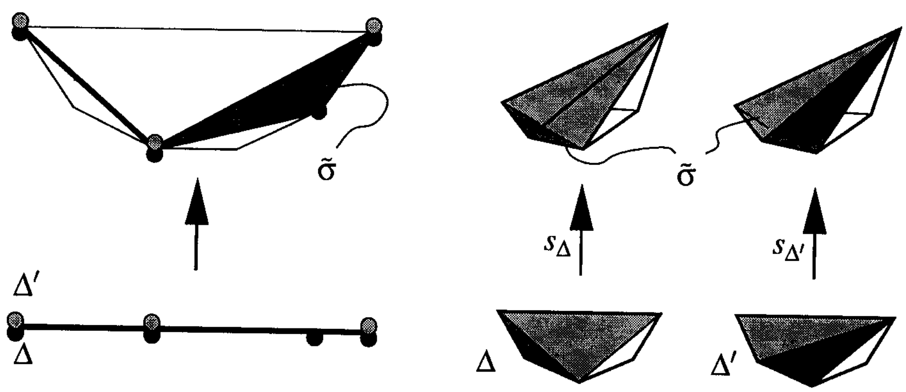

Figure 2. Increasing flips in $\mathscr{S}_1(6, 1)$ respectively $\mathscr{S}_1(5, 2)$.

DEFINITION 5.6. *To each* $S \in \binom{[n]}{d+1}$ we assign a unique string by

$$\Gamma: \begin{cases} \binom{[n]}{d+1} \to \{o, *, e\}^n \\ S \mapsto (\gamma_1, \ldots, \gamma_n), \\ \qquad \text{with } \gamma_i = \begin{cases} e & \text{if } i \notin S \text{ and } \#\{j \in S : j > i\} \quad \text{even}, \\ * & \text{if } i \in S, \\ o & \text{if } i \notin S \text{ and } \#\{j \in S : j > i\} \quad \text{odd}. \end{cases} \end{cases}$$

(Here the letter "e" denotes an *even gap*, the letter "o" an *odd gap* in S, while "$*$" corresponds to an element of S.)

Let "$\prec_{(o*e)}$" be the lexicographic order on $\binom{[n]}{d+1}$ induced by Γ and the linear order of letters "$o \prec_{(o*e)} * \prec_{(o*e)} e$".

Definition 5.7. For S_1 and S_2 in $T \in S(n, d)$ with $\#(S_1 \cup S_2) = d+2$ define the relation

$$S_1 \prec S_2 \Leftrightarrow S_1 \cap S_2 \in \mathscr{F}^u(S_1, d) \cap \mathscr{F}^l(S_2, d).$$

Moving from one simplex of a triangulation to an adjacent one can either be considered as moving an element or moving a gap of the support.

LEMMA 5.8. *Let* $T \in S(n, d)$ *and* $S_1, S_2 \in T$ *with* $S_1 \prec S_2$. *Set* $S_{12} = S_1 \cap S_2$, $S_1 \setminus S_{12} = i_1$, *and* $S_2 \setminus S_{12} = i_2$.

(1) *If* i_2 *is an even gap in* S_1 *then* i_1 *is an even gap in* S_2 *and* $i_1 < i_2$, *that is,* "$\prec$" *moves even gaps to the left.*

(2) *If* i_2 *is an odd gap in* S_1 *then* i_1 *is an odd gap in* S_2 *and* $i_1 > i_2$, *that is,* "$\prec$" *moves odd gaps to the right.*

(3) *A gap changes parity if, and only if, it lies between* i_1 *and* i_2.

Proof. The assumptions imply that S_2 is obtained from S_1 by deleting an odd element i_1 from S_1 and adding an even gap $i_2 \notin S_1$ to S_{12}, or equivalently, the gap i_2 moves to position i_1.

If $i_1 < i_2$ then i_2 is an even gap in S_1, and i_1 is an even gap in S_2, *i.e.*, the even gap at i_2 moves to the left. If $i_2 < i_1$ then i_2 is an odd gap in S_1, and i_1 is an odd gap in S_2, *i.e.*, the odd gap at i_2 moves to the right.

The third assertion is true because for any label $i \notin \{_1, i_2\}$ not between i_1 and i_2 the number of elements to the right stays constant.

COROLLARY 5.9. *The transitive closure of* "$\prec$" *is a partial order on the set of all d-simplices in* $\binom{[n]}{d+1}$. *A d-simplex S is minimal if, and only if, all of its lower facets are contained in* $\mathscr{F}^l(n, d)$; *it is maximal if, and only if, all of its upper facets are in* $\mathscr{F}^u(n, d)$.

Proof. By Lemma 5.8 we have that

$$S_1 \prec S_2 \Rightarrow S_1 \prec_{(o*e)} S_2.$$

Hence "$\prec$" is acyclic, thus defining a partial order.

Remark 5.10 (Geometric Meaning). Let Δ be a triangulation of $C(n, d, t)$. Corollary 5.9 tells us that the repeated transition from one simplex $\sigma \in \Delta$ to an adjacent one docking from below does not create any cycles.

One cannot expect a similar property for triangulations of general polytopes, as is shown by the strongly non-regular triangulation of the *twisted capped prism* in Lee [**13**].

Now the following proposition can be proved by combining combinatorial and geometric facts.

PROPOSITION 5.11. *Let $T \in S(n, d) \setminus \mathscr{F}^u(n, d+1)$ and $\tilde{T} \in S(n, d+1)$ such that T is a subcomplex of $\tilde{T}$. Then there is a $(d+1)$-simplex $\tilde{S} \in \tilde{T}$ that defines an increasing flip in T.*

Similarly, if $T \in S(n, d) \setminus \mathscr{F}^l(n, d+1)$ there is a $(d+1)$-simplex that defines a decreasing flip in T.

Proof. Choose a simplex S in $T \setminus (\mathscr{F}^u(n, d+1) \cap T)$. Since S is not an upper facet of $C(n, d+1)$, condition (UP) for $\tilde{T}$ implies that there must be a simplex $\tilde{S}$ in $\tilde{T}$ containing S as a *lower* facet. (Either S is a lower facet of $C(n, d)$, and hence a lower facet of a simplex in $\tilde{T}$, or S lies in two different simplices of $\tilde{T}$, and not both of them can simultaneously contain S as an upper facet because of (IP).)

We now choose a geometric interpretation by fixing $t: [n] \to \mathbb{R}$, strictly monotone. This gives rise to geometric interpretations $C(n, d, t)$ of $C(n, d)$, $C(n, d+1, t)$ of $C(n, d+1)$, $\tilde{\Delta}$ of $\tilde{T}$, Δ of T, and $\tilde{\sigma}$ of $\tilde{S}$. Because T is a subcomplex of $\tilde{T}$ we know that its piecewise linear section s_Δ is a subcomplex of $\tilde{\Delta}$. But then $\tilde{\sigma}$ lies weakly above the section s_Δ because at least one of its lower facets, namely $s_\Delta(\sigma)$, is contained in s_Δ.

If there exists a lower facet $F_l \in \mathscr{F}^l(\tilde{S}, d+1)$ of $\tilde{S}$ that is not contained in T then either F_l is a lower facet of $C(n, d+1)$—which is impossible because between the geometric interpretation σ' of F_l and the lower facets of $C(n, d+1, t)$ lies the section s_Δ— or there is a simplex $\tilde{S}' \in \tilde{T}$ with $F_l \subset \tilde{S}'$ and $\tilde{S}' \prec \tilde{S}$, the geometric intepretation of which is still lying weakly above the section. By continuing this process we will—by Corollary 5.9—end up with a simplex $\tilde{S}'' \in \tilde{T}$ with $\mathscr{F}^l(\tilde{S}'', d+1) \subseteq T$ (see Fig. 3). The decreasing flip can be found analogously.

We know that all geometric interpretations have the same combinatorial structure, thus the proof is complete.

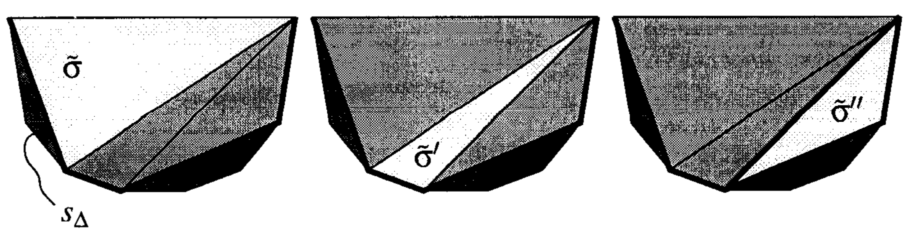

Figure 3. Finding an increasing flip in $\mathscr{S}_1(8, 1)$.

The special form of the increasing (decreasing) flips in Proposition 5.11 leads to the following result.

COROLLARY 5.12. *Let $\tilde{T}$ be a triangulation of $C(n, d+1)$. Then every linear extension "$\prec_l$" of "$\prec$" on $\tilde{T}$ defines a maximal chain in $\mathcal{S}_1(n, d)$ via*

$$\mathcal{F}^l(n, d+1) = T_0 \overset{\tilde{S}_1}{\lessdot} T_1 \overset{\tilde{S}_2}{\lessdot} \dots \overset{\tilde{S}_{r-1}}{\lessdot} T_{r-1} \overset{\tilde{S}_r}{\lessdot} T_r = \mathcal{F}^u(n, d+1),$$

where

$$\tilde{T} = \{\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_r\}, \qquad \tilde{S}_1 \prec_l \tilde{S}_2 \prec_l \dots \prec_l \tilde{S}_r.$$

Proof of Theorem 1.1. In order to prove (i) we show that any triangulation of $C(n, d)$ is on a chain from $\mathcal{F}^l(n, d+1)$ to $\mathcal{F}^u(n, d+1)$. Let T be an arbitrary triangulation of $C(n, d)$. Then, by Theorem 4.3, δT is a triangulation of $C(n, d+1)$ containing T as a subcomplex. Thus, by Proposition 5.11 and induction, we can connect T to $\mathcal{F}^u(n, d+1)$ by a sequence of increasing flips (compare Fig. 3), and to $\mathcal{F}^l(n, d+1)$ by a sequence of decreasing flips, which implies the assertion.

For the proof of (ii) observe that, by the definition of increasing bistellar flips, any chain

$$c: \mathcal{F}^l(n, d+1) \overset{\tilde{S}_1}{<} \dots \overset{\tilde{S}_r}{<} \mathcal{F}^u(n, d+1)$$

from $\mathcal{F}^l(n, d+1)$ to $\mathcal{F}^u(n, d+1)$ defines a triangulation T_c of $C(n, d+1)$ *via*

$$T_c = \{\tilde{S}_1, \dots, \tilde{S}_r\},$$

hence factoring out the order of c. For the converse, let $\tilde{T}$ be an arbitrary triangulation of $C(n, d+1)$. Then, by Corollary 5.12,

$$c_{\tilde{T}} = T_{\prec_{(o*c)}}$$

is a chain in $\mathcal{S}_1(n, d)$ from $\mathcal{F}^l(n, d+1)$ to $\mathcal{F}^u(n, d+1)$.

Part (iii) follows directly from Corollary 5.12.

The central roles of the triangulations $\hat{T}$, T/n, $T\backslash n$, $\delta(T)$ are underlined by the following additional results.

LEMMA 5.13 (Functorial Flip Properties). *If $\tilde{S}$ is an increasing flip from T to T' then*

$$(\tilde{S})\hat{}_{\prec_l} = \{\tilde{S}\backslash \tilde{s}_{d+2} \cup \{j, j+1\} : \tilde{s}_{d+1} < j < \tilde{s}_{d+2}\}_{\prec_l}$$

is a decreasing flip sequence from $\hat{T}$ to $\hat{T}'$,

$$(\tilde{S}/n) = \begin{cases} (\tilde{S}\backslash\{n\}), & \text{if } n \in \tilde{S}, \\ (\,) & \text{otherwise,} \end{cases}$$

is an increasing flip from T/n to T'/n,

$$(\tilde{S}\backslash n)=\begin{cases}(\tilde{S}) & \text{if } n\notin\tilde{S},\\ (\tilde{S}\backslash\{n\}\cup\{n-1\}) & \text{if } n\in\tilde{S},\ n-1\notin\tilde{S},\\ (\,) & \text{otherwise},\end{cases}$$

is a decreasing flip sequence from $T\backslash n$ to $T'\backslash n$, where "$\prec_l$" is any linear extension of "$\prec$."

PROPOSITION 5.14 (Functorial Order Properties).

(i) *The map*

$$\hat{\cdot}:\begin{cases}\mathscr{S}_1(n,d)\to\mathscr{S}_1(n+1,d+1),\\ \qquad T\mapsto\hat{T},\end{cases}$$

is order-reversing.

(ii) *The map*

$$\cdot/n:\begin{cases}\mathscr{S}_1(n,d)\to\mathscr{S}_1(n-1,d-1),\\ \qquad T\mapsto T/n,\end{cases}$$

is order-reversing.

(iii) *The map*

$$\cdot\backslash n:\begin{cases}\mathscr{S}_1(n,d)\to\mathscr{S}_1(n-1,d),\\ \qquad T\mapsto T\backslash n,\end{cases}$$

is order-preserving.

(iv) *The map*

$$\delta:\begin{cases}\mathscr{S}_1(n,d)\to\mathscr{S}_1(n,d+1),\\ \qquad T\mapsto\delta(T),\end{cases}$$

is order-reversing.

COROLLARY 5.15. *Every chain in $\mathscr{S}_1(n,d)$ corresponding to a flip sequence $(\tilde{T})=(\tilde{S}_1,\ldots,\tilde{S}_r)$ gives rise to flip sequences*

(i) $(\tilde{T})^{\wedge}_{\prec_l}$ *in $\mathscr{S}_1(n+1,d+1)$,*

(ii) $(\tilde{T}/n)_{\prec_l}$ *in $\mathscr{S}_1(n-1,d-1)$,*

(iii) $(\tilde{T}\backslash n)_{\prec_l}$ *in $\mathscr{S}_1(n-1,d)$, and*

(iv) $\delta(\tilde{T})_{\prec_l}$ *in $\mathscr{S}_1(n,d+1)$.*

§6. *All triangulations of cyclic polytopes are shellable.* In this section we present another application of the partial order property of the simplices in a triangulation of a cyclic polytope, namely that all triangulations (without new vertices) of a cyclic polytope are shellable. This fact is far from trivial because there exists for example a non-shellable triangulation of a convex polytope with all vertices in convex position, namely a perturbed version of Rudin's non-shellable tetrahedron (see **[18]**).

THEOREM 6.1. *All $T \in \mathscr{S}_1(n, d)$ are shellable. A shelling order on the simplices of T is given by first shelling the star of n in T corresponding to a shelling order on the link of n in T, and then shelling the rest of T according to a reversed linear extension of "$\prec$," for example "$\succ_{(o*e)}$."*

The rest of this section is devoted to the proof of Theorem 6.1, which implies Theorem 1.1(iv). We start with some lemmata that are intuitively plausible at once when one considers the geometric interpretations of the objects under consideration. With the results of Section 3, however, we have tools at hand that provide more security.

LEMMA 6.2. *Let S be a d-simplex in $\binom{[n]}{d+1}$. G is the intersection of lower facets of S if, and only if, S is of the form*

$$S = (G_{\#(S\setminus G)}, s_{\#(S\setminus G)}, \ldots, \underbrace{G_2}_{\text{odd}}, s_2, \underbrace{G_1}_{\text{odd}}, s_1),$$

$$G = G_1 \cup G_2 \cup \ldots \cup G_{\#(S\setminus G)}, \qquad s_1, s_2, \ldots, s_{\#(S\setminus G)} \in S\setminus G.$$

G is the intersection of upper facets of S if, and only if, S is of the form

$$S = (G_{\#(S\setminus G)}, s_{\#(S\setminus G)}, \underbrace{G_{\#(S\setminus G)-1}}_{\text{odd}}, \ldots, s_2, \underbrace{G_1}_{\text{odd}}, s_1, \underbrace{G_0}_{\text{odd}}),$$

$$G = G_0 \cup G_1 \cup \ldots \cup G_{\#(S\setminus G)}, \qquad s_1, s_2, \ldots, s_{\#(S\setminus G)} \in S\setminus G.$$

Proof. If there were two elements in $S\setminus G$ separated by an even number of elements in G then leaving them out separately in S would produce gaps of different parity. From this the claim follows.

LEMMA 6.3. *Let $S \in \binom{[n]}{d+1}$. If G is the intersection of lower (resp. upper) facets of S then G is not contained in any upper (resp. lower) facet of $C(n, d)$.*

Proof. If $d = 1$, everything is clear. Let G be the intersection of lower facets $F^{(1)}, \ldots, F^{(r)}$ of $S = (s_1, \ldots, s_{d+1})$. Then all $F^{(i)}$ contain only even gaps, in particular they contain $s_d < s_{d+1} \leqslant n$. Assume G is contained in some upper facet $F = (f_1, \ldots, f_d)$ of $C(n, d)$. Then F contains s_d and n. Consider $F' = F\setminus\{n\}$, a lower facet of $C(n-1, d-1)$, and the $(d-1)$-simplex $S' = S\setminus\{s_{d+1}\}$. The sets

$$F^{(i)}\setminus\{f^{(i)}_{d-1}, f^{(i)}_d\} \cup \{s_d\}, \qquad i = 1, \ldots, r,$$

are upper facets of S', and their intersection is contained in F' (because $s_d \in F'$), contradiction by the following paragraph with d replaced by $d-1$ and induction.

If G is the intersection of upper facets $F^{(1)}, \ldots, F^{(r)}$ of $S = (s_1, \ldots, s_{d+1})$ then all $F^{(i)}$ contain only odd gaps, in particular they contain s_{d+1}. Assume G is contained in some lower facet $F = (f_1, \ldots, f_d)$ of $C(n, d)$. Then F contains s_{d+1} as well, so $f_d \geqslant s_{d+1}$. Therefore we may assume, without loss of generality, that $n = f_d$. Since F contains only even gaps we have $f_{d-1} = f_d - 1$. Consider

$F' = F \setminus \{f_{d-1}, f_d\} \cup \{n-1\} = F \setminus n$ which is an upper facet of $C(n-1, d-1)$. The sets

$$F^{(i)} \setminus \{s_{d+1}\}, \qquad i = 1, \ldots, r,$$

are lower facets of the $(d-1)$-simplex $S' = S \setminus \{s_{d+1}\}$, and their intersection is contained in F' (because $n \geqslant s_{d+1}$), contradiction by the previous paragraph with d replaced by $d-1$ and induction.

LEMMA 6.4. *Let $T \in \mathscr{S}_1(n, d)$ and $S_1 \neq S_2 \in T$. If $S_1 \cap S_2$ is not contained in any upper (resp. lower) facet of S_1, i.e., is the intersection of lower (resp. upper) facets of S_1, then it is contained in some upper (resp. lower) facet of S_2.*

Proof. Assume $S_1 \cap S_2$ is the intersection of lower (resp. upper) facets of S_1 and also the intersection of lower (resp. upper) facets of S_2. We show by induction that $S_1 \cup S_2$ contains the support of a circuit Z in $\mathscr{X}(n, d)$ with $S_1 \subseteq Z^+$ and $S_2 \subseteq Z^-$. If $d = 1$ everything is clear.

Both S_1 and S_2 are of the form given in Lemma 6.2, in particular we may set $S_1 = (S_1', s_1)$ and $S_2 = (S_2', s_2)$ with $s_1 > s_2'$ for all $s_2' \in S_2'$ and $s_2 > s_1'$ for all $s_1' \in S_1'$. If $S_1' = S_2'$ then $s_1 \neq s_2$ and (S_1', s_1, s_2) supports a circuit Z in $\mathscr{X}(n, d)$ with $S_1 \subseteq Z^+$ and $S_2 \subseteq Z^-$, showing that (S_1, S_2) is not admissible, contradiction. If $S_1' \neq S_2'$ then $S_1' \cap S_2'$ is the intersection of upper or the intersection of lower facets of S_1', and the same for S_2', by Lemma 6.2. Hence by the induction hypothesis, $S_1' \cup S_2'$ contains the support of a circuit Z' in $\mathscr{X}(n-1, d-1)$ with $Z^+ \subseteq S_1'$ and $Z^- \subseteq S_2'$. Without loss of generality, $z_{d+1} = \max(S_1' \cup S_2') \in S_1'$. Then $Z = (Z^+, Z^- \cup \{s_2\})$ is a circuit (recall that $s_2 > s_1'$ for all $s_1' \in S_1'$) in $\mathscr{X}(n, d)$ proving that (S_1, S_2) is not admissible giving a contradiction.

Definition 6.5. Let $T \in \mathscr{S}_1(n, d)$ and $T' \subseteq T$. An upper (resp. lower) facet F of a simplex $S \in T'$ is a *free upper (resp. lower) facet of T'* if F is neither a facet of $C(n, d)$ nor a facet of some other simplex in T'.

PROPOSITION 6.6. *Let $T \in \mathscr{S}_1(n, d)$ and $T' \subseteq T$ such that T' contains no free upper facet. Then the intersection of T' with any simplex $S \in T \setminus T'$ having all upper facets in T' equals the union of the upper facets of S.*

Proof. Assume that $G \subseteq S \cap T'$ is not contained in any upper facet of S. Let S' be a simplex in T' with $G \subseteq S'$ that is maximal with respect to "$\prec$." By Lemma 6.4, G is contained in some upper facet F' of S' that is not a facet of $C(n, d)$ by Lemma 6.3. Thus there is a simplex S'' with $S'' \neq S'$ and $F' \subset S''$. But then $S'' \succ S'$, a contradiction to the maximality of S'.

LEMMA 6.7. *Let $T \in \mathscr{S}_1(n, d)$. Then $\mathrm{st}_T(n)$ contains no free upper facets.*

Proof. Every simplex in $\mathrm{st}_T(n)$ contains n, thus any upper facet in $\mathrm{st}_T(n)$ contains n, so it cannot be contained in a simplex outside $\mathrm{st}_T(n)$.

Proof of Theorem 6.1. A triangulation of $C(n, 1)$ is just a dissection of an interval, thus shellable. Let $d > 1$ and $T \in \mathscr{S}_1(n, d)$. Assume all triangulations of $C(n-1, d-1)$ are proven to be shellable. Then we know that $\mathrm{lk}_T(n) \in \mathcal{S}_1(n-1, d-1)$ is shellable. Let $(\mathrm{lk}_T(n))_{<_{d-1}}$ denote a shelling order of $\mathrm{lk}_T(n)$. Then $(\mathrm{st}_T(n))_{<_{d-1}}$ is a canonical shelling order of $\mathrm{st}_T(n)$.

Now pick any linear extension "$\succ_l$" of "$\succ$." We claim that

$$(\mathrm{ast}_T(n))_{\succ_l} = (S^{(1)}, \ldots, S^{(r)})$$

completes $(\mathrm{st}_T(n))_{<_{d-1}}$ to a shelling order on T. Let $T^{(0)} = \mathrm{st}_T(n)$ and

$$T^{(i)} = \mathrm{st}_T(n) \cup S^{(1)} \cup \ldots \cup S^{(i)}, \qquad i = 1, \ldots, r.$$

We know by Lemma 6.7 that there is no free upper facet in $T^{(0)} = \mathrm{st}_T(n)$. Since "$\succ_l$" extends "$\succ$," all upper facets of $S^{(i)}$ are contained in $T^{(i-1)}$ for all $i = 1, \ldots, r$. Hence, there are no free upper facets in $T^{(i)}$ for all $i = 1, \ldots, r$. Thus, by Proposition 6.6, the intersection of $S^{(i)}$ and $T^{(i-1)}$ is indeed the union of these upper facets, in particular pure of dimension $d-1$ for all $i = 1, \ldots, r$, which proves the Theorem by induction.

§7. *Higher Bruhat orders.* In this section we recall the basic definitions and theorems in the framework of higher Bruhat orders and answer a question of Ziegler [**17**]. Let $\mathscr{L}$ be a linearly ordered finite set. The reader may consider $\mathscr{L}$ as the set $[n]$, without loss of generality.

Definition 7.1 (Manin and Schechtman [**15**], Ziegler [**17**]).

For some $(k+1)$-subset $P = (p_1, \ldots, p_{k+1})$ of $\mathscr{L}$ the set of its k-subsets

$$\mathscr{P} = \binom{P}{k} = \{P \backslash p_\nu : \nu = 1, \ldots, k+1\}$$

is a *k-packet of* $\mathscr{L}$. It is naturally ordered by $P\backslash p_\nu < P \backslash p_\mu \Leftrightarrow \mu < \nu$, the *lexicographic order*.

An ordering α of $\binom{\mathscr{L}}{k}$ is *admissible* if the elements of any $(k+1)$-packet appear in lexicographic or in reverse-lexicographic order. Two orderings α and α' are *equivalent* if they differ by a sequence of interchanges of two neighbours that do not lie in a common packet.

The inversion set inv (α) of an admissible ordering α is the set of all $(k+1)$-subsets of $\mathscr{L}$ the k-subsets of which appear in reverse-lexicographic order in α.

A set U of $(k+1)$-subsets of $\mathscr{L}$ is *consistent* if its intersection with any $(k+1)$-packet $\mathscr{P}$ of $\mathscr{L}$ is a beginning or an ending segment of $\mathscr{P}$ with respect to the lexicographic order on $\mathscr{P}$.

The set of all equivalence classes of admissible orders of $\binom{\mathscr{L}}{k}$ partially ordered by single-step-inclusion of inversion sets—that is, $[\alpha] \leqslant [\alpha']$ if, and only if,

$$\mathrm{inv}\,(\alpha) = U_1 \subset U_2 \subset \ldots \subset U_K = \mathrm{inv}\,(\alpha'),$$

with $\# U_\nu \backslash U_{\nu-1} = 1$ and all U_ν are admissible—is *the higher Bruhat order* $\mathscr{B}(\mathscr{L}, k)$, where $\mathscr{B}(n, k)$ denotes $\mathscr{B}([n], k)$.

For an inversion set $U \in \mathscr{B}(\mathscr{L}, k)$ define

$$\partial U = \left\{ I \in \binom{\mathscr{L}}{k+2} : I \backslash i_1 \notin U, I \backslash i_{k+2} \in U \right\}.$$

The structure of $\mathscr{B}(\mathscr{L}, k)$ does of course only depend on the cardinality of $\mathscr{L}$, but the general setting leads to some advantages in the notation of functorial constructions. For simplicity, however, we switch now to $\mathscr{B}(n, k)$.

THEOREM 7.2 (Manin and Schechtman [**15**], Ziegler [**17**]). *The higher Bruhat order $\mathscr{B}(n, k)$ is a ranked poset with rank function $r(U) = \# U$. Moreover, it has a unique minimal element $\hat{0}_{n,k} = \varnothing$ and a unique maximal element $\hat{1}_{n,k} = \binom{[n]}{k+1}$.*

The following Theorem gives a more geometric insight into the structure of higher Bruhat orders.

THEOREM 7.3 (Ziegler [**17**]). *The higher Bruhat order $\mathscr{B}(n, k)$ is isomorphic to*

(1) *the set of all consistent sets U of $(k+1)$-subsets of $[n]$ with single-step-inclusion-order,*
(2) *the set of (equivalence classes of) extensions of the cyclic hyperplane arrangement $X^{n,n-k-1}$ by a new pseudo-hyperplane in general position, partially ordered by single-step-inclusion of the sets of vertices on "the negative side", and to*
(3) *the set of maximal chains of inversion sets in $\mathscr{B}(n, k-1)$—corresponding to orders of k-sets—modulo equivalence of admissible orders.*

The following notations for deletion and contraction in $\mathscr{B}(n, k)$ provide intuition *via* the corresponding notions in $X^{n,n-k-1}$.

Definition 7.4. For $U \in \mathscr{B}(n, k)$ define

$$U/n = \{I \backslash n : n \in I, I \in U\}, \qquad \text{(contraction)}$$

$$U \backslash n = \{I \in U : n \notin I\}. \qquad \text{(deletion)}$$

In order to construct inversion sets in $\mathscr{B}(n+1, k+1)$ from inversion sets in $\mathscr{B}(n, k)$ and in $\mathscr{B}(n, k+1)$ the following Theorem is useful.

THEOREM 7.5 (Ziegler [**17**]). *Let U be an inversion set in $\mathscr{B}(n, k)$ and V be an inversion set in $\mathscr{B}(n, k+1)$. Then $U' = V \cup U * (n+1)$ is consistent if, and only if,*

$$\partial U \subseteq V \quad \textit{and} \quad \partial \complement U \subseteq \complement V.$$

Corollary 7.6. *The following maps from $\mathscr{B}(n, k)$ to $\mathscr{B}(n+1, k+1)$ are injective*:

$$U \mapsto \tilde{U} = U * (n+1) \cup \partial U, \qquad (extension)$$

$$U \mapsto \hat{U} = U * (n+1) \cup \delta(U) = U * (n+1) \cup (U \backslash n)^\wedge, \qquad (expansion)$$

where $\delta(U)$ *is defined as*

$$\delta(U) = \left\{ I \in \binom{[n]}{k+2} : I \backslash i_{k+2} \in U \right\}.$$

The extension is not order-preserving in general. But the following definition yields a canonical single-step-inclusion order for the expansion of U from an arbitrary single-step-inclusion order of U.

Definition 7.7. For some $U \in \mathscr{B}(n, k)$ with a given single-step-inclusion-order $\Omega(U) = (\Omega(U'), I)$ define the following order $\hat{\Omega}$. For $n = k+1$ start with

$$\hat{\Omega}(\{[n]\}^\wedge) = ([n+1])$$

corresponding to $\Omega(\{[n]\}) = ([n])$ in $\mathscr{B}(n, k)$. If $n > k+1$ and $\hat{\Omega}(\hat{U}')$ is already constructed then define

$$\hat{\Omega}(\hat{U}) = (\hat{\Omega}(\hat{U}'), \hat{\Omega}(\partial I), I \cup \{n+1\}, \hat{\Omega}(\delta I \backslash \partial I)),$$

where the orders on ∂I and $\delta I \backslash \partial I$ are given recursively by restriction of $\hat{\Omega}((U \backslash n)^\wedge)$.

Proposition 7.8. *For all* $U \in \mathscr{B}(n, k)$ *and all single-step-inclusion orders* Ω *of* U *the order* $\hat{\Omega}$ *is a single-step-inclusion order of the expansion* $\hat{U}$ *of* U *in* $\mathscr{B}(n+1, k)$.

Proof. The following properties make sure that no cycles are produced:

$$\delta(U) \backslash n = \delta(U \backslash n),$$

$$\partial(U) \backslash n = \partial(U \backslash n).$$

At each single-step-inclusion step all packets in $\mathscr{B}(n, k+1)$ are consistent by induction. From the remaining packets only those containing $I \cup \{n+1\}$ are involved.

If $n \notin I$ then the order increases just by $I \cup \{n\}$ which is consistent because Ω is a single-step-inclusion order of U and $\hat{U}'$ is already ordered consistently.

Let n be in I. For all packets $\mathscr{P}$ containing $I \cup \{n+1\}$ either $\mathscr{P}/n+1$ is completely contained in U or only I meets U. In the first case the only element $P \backslash a'$ of $\mathscr{P} \backslash n+1$ comes before $I \cup \{n+1\}$ in $\hat{\Omega}$, in the second case $I \cup \{n+1\}$ is positioned after $P \backslash n+1$ in $\hat{\Omega}$; both cases lead to consistent orders on $\mathscr{P}$.

From this we derive the promised result.

THEOREM 7.9. *The expansion*

$$\hat{}: \begin{cases} \mathscr{B}(n,k) \to \mathscr{B}(n+1,k+1), \\ U \mapsto \hat{U}, \end{cases}$$

is an order-preserving embedding that maps $\hat{0}_{n,k}$ *to* $\hat{0}_{n+1,k+1}$ *and* $\hat{1}_{n,k}$ *to* $\hat{1}_{n+1,k+1}$.

§8. *The connection between* $\mathscr{B}(n-2,d-1)$ *and* $\mathscr{S}_1(n,d)$. In this section we present an order-preserving map from the higher Bruhat order $\mathscr{B}(]n[,d-1) \cong \mathscr{B}(n-2,d-1)$ to the poset $\mathscr{S}_1(n,d)$ of all triangulations of $C(n,d)$. This map is obtained by two different constructions, each of them providing complementary parts of the properties claimed. It is not quite clear whether this map coincides with the map suggested by Kapranov and Voevodsky **[12]**.

We start with some additional specific properties of triangulations of cyclic polytopes.

LEMMA 8.1. *Let* $T \in \mathscr{S}_1(n,d)$. Then for each (d−1)-subset $(s_2,\ldots,s_d)$ *there is at most one simplex* $S \in T$ *with* $S=(s_1,s_2,\ldots,s_d,s_{d+1})$ *for some* $s_1<s_2$ *and some* $s_{d+1}>s_d$.

Proof. Assume there were $S \neq S' \in T$ and

$$S=(s_1,s_2,\ldots,s_d,s_{d+1}),$$
$$S'=(s'_1,s_2,\ldots,s_d,s'_{d+1}).$$

Either $s_1 \neq s'_1$ or $s_{d+1} \neq s'_{d+1}$. If $s_1<s'_1$ then define

$$Z=\begin{cases} (s_1,s'_1,\ldots,s_d,s'_{d+1}) & \text{if } d \text{ even}, \\ (s_1,s'_1,\ldots,s_d,s_{d+1}) & \text{if } d \text{ odd}. \end{cases}$$

In any case $Z^+ \subseteq S$ and $Z^- \subseteq S'$.

The cases $s_1>s'_1$, $s_{d+1}<s'_{d+1}$, and $s_{d+1}>s'_{d+1}$ are analogous.

Definition 8.2. For $S=(s_1,\ldots,s_{d+1}) \in T \in \mathscr{S}_1(n,d)$ let $X_S=(s_2,\ldots,s_d)$ be *the central set of S*. The number $l_S=s_1$ is called *the left boundary*, the number $r_S=s_{d+1}$ *the right boundary* of X_S in T.

Since there are no multiple central sets in triangulations of cyclic polytopes we have the following simple representation.

COROLLARY 8.3. *Any triangulation T of C(n, d) is determined by its set of central sets and their boundaries.*

LEMMA 8.4. *In every triangulation T of C(n, d) every interval of length* $(d-1)$ *in* $[2,n-1]$ *appears as a central set of some simplex* $S \in T$.

Proof. Here is a proof for d odd. Let T be in $\mathscr{S}_1(n, d)$ and I an interval of length $d-1$. From Gale's evenness criterion it follows that I is contained in exactly two facets of $C(n, d)$, namely $(1, I)$ and (I, n). Therefore, there must be a simplex S_1 in the triangulation T containing $(1, I)$.

If $S_1 = (1, I, r)$ we are done. Otherwise $S_1 = (1, l_1, I)$. Because (l_1, I) is not a facet of $C(n, d)$ there must be another simplex $S_2 \in T$ with $(l_1, I) \subseteq S_2$. If $S_2 = (l_1, I, r)$ we are done. Otherwise proceed as above. Because of Lemma 8.1 at each step $l_i < l_{i+1}$. Hence there must be a k and an r such that the simplex $S_k = (l_{k-1}, I, r)$ is in T.

The case d even is analogous where the corresponding facets of $C(n, d)$ are $(i_1 - 1, I)$ and $(I, i_d + 1)$ and the sequence of the l_k is decreasing.

We start now to construct a map by defining a natural family of functions on $\mathscr{S}_1(n, d)$.

Definition 8.5. For an element $I = (i_1, \ldots, i_d) \in \binom{]n[}{d}$ define the map

$$\mathrm{flip}_I : \begin{cases} \mathscr{S}_1(n, d) \to \mathscr{S}_1(n, d), \\ \qquad T \mapsto \begin{cases} \mathrm{flip}_{(l,I,r)}(T), & \text{if } (l, I, r) \text{ is an increasing flip,} \\ T, & \text{otherwise.} \end{cases} \end{cases}$$

For an inversion set $U \in \mathscr{B}(]n[, d-1)$ let $\Omega(U) = (I_i)_{i=1,\ldots,\#U}$ be a single-step-inclusion-order of the elements of U, *i.e.*, $\bigcup_{i=1}^{K} I_i$ is consistent for all $K = 1, \ldots, \#U$. The *flip-map* $\mathscr{T}_{\mathrm{flip}}$ is now defined as follows:

$$\mathscr{T}_{\mathrm{flip}} : \begin{cases} \mathscr{B}(]n[, d-1) \to \mathscr{S}_1(n, d), \\ \qquad U \mapsto \mathrm{flip}_{I_{\#U}} \circ \ldots \circ \mathrm{flip}_{I_1}(\mathscr{F}'(n, d)). \end{cases}$$

Remark 8.6. At this point it is not obvious that this definition is independent of the special order $\Omega(U) = (I_i)_{i=1,\ldots,\#U}$ of U. Up to now we only know that $\mathscr{T}_{\mathrm{flip}}$ maps each pair $(U, \Omega(U))$ to a triangulation in $\mathscr{S}_1(n, d)$, where $U \in \mathscr{B}(]n[, d-1)$ and $\Omega(U)$ is a single-step inclusion order of its elements. It is order-preserving in the sense that if $U < U'$ and $\Omega(U), \Omega(U')$ are corresponding single-step inclusion orders with the property that $\Omega(U)$ is an initial segment of $\Omega(U')$, then $\mathscr{T}_{\mathrm{flip}}(U, \Omega(U)) < \mathscr{T}_{\mathrm{flip}}(U', \Omega(U'))$.

Definition 8.7. For $i \in I \in \binom{]n[}{d}$ define *the index of i in I* as

$$\mathrm{ind}_I(i) = k \qquad \text{if} \qquad I = (i_1, \ldots, i = i_k, \ldots, i_d).$$

Definition 8.8. For an inversion set $U \in \mathscr{B}(]n[, d-1)$ define the *central set of U* as

$$X_U = \left\{ \begin{array}{lll} X=(x_1, \ldots, x_{d-1}) \in \binom{]n[}{d-1} : & X \cup j \in U & \forall j \in [n] \backslash X: \\ & & x_1 < j < x_{d-1}, \\ & & d - \operatorname{ind}_{X \cup j}(j) \text{ even}, \\ & X \cup j \notin U & \forall j \in [n] \backslash X: \\ & & x_1 < j < x_{d-1}, \\ & & d - \operatorname{ind}_{X \cup j}(j) \text{ odd}. \end{array} \right\}$$

Definition 8.9. For an inversion set $U \in \mathscr{B}(]n[, d-1)$ define the *left boundary function of U* as

$$\lambda_U : \begin{cases} X_U \to [n], \\ X \mapsto \begin{cases} \max\{\in [n] : (l, X) \notin U\} & \text{for } d \text{ odd}, \\ \max\{\in [n] : (l, X) \in U\} & \text{for } d \text{ even}, \end{cases} \end{cases}$$

and the *right boundary function of U* as

$$\rho_U : \begin{cases} X_U \to [n], \\ X \mapsto \min\{r \in [n] : (X, r) \notin U\}, \end{cases}$$

with the additional notation

$$\min(\varnothing) = n \qquad \text{and} \qquad \max(\varnothing) = 1.$$

Definition 8.10. Now define the *direct map* $\mathscr{T}_{\text{dir}}$ as

$$\mathscr{T}_{\text{dir}} : \begin{cases} \mathscr{B}(]n[, d-1) \to \mathscr{S}_1(n, d), \\ \qquad U \mapsto \{(\lambda_U(X), X, \rho_U(X)) : X \in X_U\}. \end{cases}$$

Remark 8.11. Here it is neither obvious that $\mathscr{T}_{\text{dir}}(U)$ is indeed a triangulation nor that the map is order-preserving, but it is uniquely defined.

PROPOSITION 8.12. *Let U and $U' = U \cup \{I\}$ be inversion sets in $\mathscr{B}(]n[, d-1)$. Define the following two properties for some $i_k \in I$, $1 < k < d-1$.*

Property A: *$I \backslash i_k \in X_U$ but $I \backslash i_k \notin X_{U'}$.*
Property B: *$I \backslash i_k \notin X_U$ but $I \backslash i_k \in X_{U'}$.*

Then the following hold:

(i) *If i_k has Property A then all $i_m \in I$ with $m \equiv k$ mod 2 have Property A as well,*
(ii) *If i_k has Property B then all $i_m \in I$ with $m \equiv k$ mod 2 have Property B as well.*

Proof. From Definition 8.8 we know that

i_k has Property A if, and only if, $d-k$ is odd and i_k has Property C, namely $\mathrm{ind}_{I\backslash i_k \cup j}(j)$ is even for all $j \notin I$ with $i_1 < j < i_{d-1}$ and $I \backslash i_k \cup j \in U$, and $\mathrm{ind}_{I\backslash i_k \cup j}(j)$ is odd for all $j \notin I$ with $i_1 < j < i_{d-1}$ and $I \backslash i_k \cup j \notin U$,

i_k has Property B if, and only if, $d-k$ is even and i_k has Property C.

In the sequel we will show that Property C for i_k induces Property C for all $i_m \in I$.

Assume $i_k \in I$ has Property C. Let $j \notin I$, $i_1 < j < i_d$ be arbitrary. (If there is no such j the proof is finished.) Consider the inversion $J = (I \cup j) \backslash i_k$. From Property C we know that J has Property D, namely

$$J \subseteq \begin{cases} U & \text{if } \mathrm{ind}_J(j) \text{ even,} \\ \bar{U} & \text{if } \mathrm{ind}_J(j) \text{ odd.} \end{cases}$$

Now we investigate the d-packet $P = I \cup J$. Because both U and U' are consistent, the complete segment that starts at a neighbour of $I = P \backslash j$ and contains $J = P \backslash i_k$ must have property D as well as J, and the complementary segment must have exactly the contrapositive property $\bar{D}$. That means by parsing the packet P from one end to the other "having property D" switches at $I = P \backslash j$.

In other words, $I \backslash i_m \cup j \in U$ if, and only if, $I \backslash i_k \cup j \in U$ for all i_m lying on the same side of j as i_k in P and $I \backslash i_m \cup j \in U$ if, and only if, $I \backslash i_k \cup j \notin U$ for all i_m lying on the opposite side of j as i_k.

Additionally, if m is congruent k modulo 2 then $\mathrm{ind}_{I\backslash i_m \cup j}(j)$ is congruent to $\mathrm{ind}_{I\backslash i_k \cup j}(j)$ modulo 2 if, and only if, i_m lies on the same side of j as i_k in P, but—since j was arbitrary—this means that i_m has Property C.

Remark 8.13. The above Proposition roughly states that for $I \backslash i_m$ "being contained in the central set of U" for all possible m only depends on whether I is in U—not on whether some inversion $I \backslash i_m \cup j$ is in U—whenever this is correct for one m.

PROPOSITION 8.14. *Let U and U' be as above. Then the following hold for all $1 < l < i_1$ and $i_{d-1} < r < n$:*

$$(l, I \backslash i_k) \in U \Leftrightarrow (l, I \backslash i_m) \in U \quad \text{for all} \quad m \equiv k \bmod 2,$$

$$(I \backslash i_k, r) \in U \Leftrightarrow (I \backslash i_m, r) \in U \quad \text{for all} \quad m \equiv k \bmod 2.$$

Proof. The proof is analogous to the proof of Proposition 8.12 with j replaced by l, r.

THEOREM 8.15. *The maps $\mathscr{T}_{\mathrm{flip}}$ and $\mathscr{T}_{\mathrm{dir}}$ coincide.*

Proof. We will show that $\mathscr{T}_{\mathrm{flip}}(U) = \mathscr{T}_{\mathrm{dir}}(U)$ for all $U \in \mathscr{B}(]n[, d-1)$. Because $\mathscr{B}(]n[, d-1)$ has a unique minimal element $\varnothing$ we can proceed by induction on $\#U$.

The proof for $U=\varnothing$ is a simple computation. Therefore we assume that the claim is true for some inversion set U and we will show that then the claim is also true for all consistent $U'=U\cup\{I\}$.

It remains to check the following points.

(1) If $\mathcal{T}_{\mathrm{dir}}(U')\neq\mathcal{T}_{\mathrm{dir}}(U)$ then there exist $1\leqslant l<i_1$ and $i_d<r\leqslant n$ such that (l, I, r) is an increasing flip in $\mathcal{T}_{\mathrm{flip}}(U)=\mathcal{T}_{\mathrm{dir}}(U)$, and

(2) if the $(d+2)$-set (l, I, r) is an increasing flip in $\mathcal{T}_{\mathrm{flip}}(U)=\mathcal{T}_{\mathrm{dir}}(U)$ then $\mathcal{T}_{\mathrm{dir}}(U')=\mathrm{flip}_I\,\mathcal{T}_{\mathrm{dir}}(U)$.

From Proposition 8.12 it follows that the assertions (1) and (2) are correct as far as the central sets of U or U', resp., are concerned.

From Proposition 8.14 and the corresponding definitions in 8.9 we get that in the situations of both (1) and (2) the left and right boundary functions are constant on the sets $I\backslash i_k$ with $1<k<d-1$, *i.e.*, there exist l and r with $1<l<i_1$ and $i_{d-1}<r<n$ such that

$$\lambda_U(I\backslash i_k)=l, \qquad \rho_U(I\backslash i_k)=r.$$

Moreover, it follows that

$$\lambda_U(I\backslash i_1)=\begin{cases} i_1 & \text{for } d \text{ odd,}\\ l & \text{for } d \text{ even,}\end{cases} \qquad \rho_U(I\backslash i_1)=r,$$

$$\lambda_U(I\backslash i_{d-1})=l, \qquad \rho_U(I\backslash i_{d-1})=i_{d-1}.$$

After having added I to the inversion set U we have

$$\lambda_{U'}(I\backslash i_k)=\begin{cases} l & \text{for } d \text{ odd,}\\ i_1 & \text{for } d \text{ even,}\end{cases} \qquad \rho_{U'}(I\backslash i_k)=r,$$

$$\lambda_{U'}(I\backslash i_1)=l, \qquad \rho_{U'}(I\backslash i_1)=r,$$

$$\lambda_{U'}(I\backslash i_{d-1})=l, \qquad \rho_{U'}(I\backslash i_{d-1})=r.$$

With this the proof of Theorem 8.15 is complete.

COROLLARY 8.16. *The map*

$$\mathcal{T}=\mathcal{T}_{\mathrm{flip}}=\mathcal{T}_{\mathrm{dir}}$$

is well-defined and order-preserving.

We finish the paper by stating—as a bonus track without a proof—the following connections between the constructions of this paper.

PROPOSITION 8.17 (Functorial Relations).

$$\mathcal{T}(\hat{U})=(\mathcal{T}(U))^\wedge,$$
$$\mathcal{T}(U\backslash n-1)=\mathcal{T}(U)\backslash n,$$
$$\mathcal{T}(\delta U)=\delta\mathcal{T}(U).$$

The analogous property for the link does not hold in general!

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Dr. J. Rambau,
Konrad-Zuse-Zentrum für
Informationstechnik,
Takustr. 7,
D-14195 Berlin,
Germany.
E-mail: rambau@zib.de

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