Project: Automatic Geometry Theorem Proving

T. Recio, H. Sterk, M. P. Vélez

Universidad de Cantabria, Eindhoven University of Technology, Universidad Complutense de Madrid

1. Introduction

The aim of this project is to illustrate how the framework of polynomial rings and computational methods designed for them can be of help in proving (plane) geometry theorems. The idea is not original and there are already, even for the beginner, excellent references concerning this topic. In coherence with the "tapas" style of this book, we will recall just a few, tasty ones: for instance, the recent book by the founder of the modern approach to automatic geometry theorem proving, Wu Wen Tsun [Wu]; the textbook of Cox-Little-O'Shea [Cox-Little-O'Shea], that integrates one section on this material in a commutative algebra/algebraic geometry course, and the book of Chou [Chou], including an impressive collection of computed examples.

The primary motivation for our own contribution to this chapter has been the preparation of undergraduate classroom material for computer-aided commutative algebra courses that have been offered, since the middle eighties, in the University of Cantabria and, more recently, in the University Complutense of Madrid. Thus, the following pages should be regarded as a (thoroughly) elaborated version of teaching notes. The rationale of our didactical approach is that algebraic geometry examples improve students understanding of commutative algebra concepts and conversely. This fruitful relation can be further advanced if students are required to handle concrete problems to which both algebra and algebraic geometry apply.

Automatic Geometry Theorem Proving provides an interesting framework to accomplish this, since an elementary geometry problem has to be modeled into a commutative algebra statement, which will be, in turn, regarded as a property of an algebraic variety. In this way students develop the commutative algebra computational skills to "read", in terms of the geometry of affine varieties, the fate of elementary geometry statements. The didactical relevance is that, in the context of elementary geometry theorems, students have their own "a priori" intuition (although it can be wrong) about what is or what is not likely to happen. Confronting intuition with the actual behaviour of mathematical objects seems the key to significant learning.

On the other hand, given the complexity of current algorithms for ideal manipulation (see Arjeh's chapter? [make a better reference!!]) and the limited resources usually available in undergraduate mathematics laboratories, it is not straightforward to identify a collection of examples which can be successfully manipulated with scientific freeware, such as CoCoA[1], running over small machines. We hope that this chapter also shows how Automatic Geometry Theorem Proving satisfies this requirement.

As a consequence of the didactical origin of the chapter, our classroom presentation of the topic turned out to converge towards [Cox-Little-O'Shea] style, several years before its publication. This coincidence responds to the obvious fact that Gröbner basis are very likely to be introduced in most computationally oriented commutative algebra courses) and it is also due to a common explotation of Kapur's [Kapur] formulation. We thank the authors of the book [Cox-Little-O'Shea] for sending us an earlier draft of their manuscript. Some results below are directly taken from them, but we take full responsability for many deviations and interpretations. Besides, we have enlarged their presentation to include an amusing introduction to automatic *discovery* of theorems. In other words, we proclaim automatization not only for proving some given statement, but even for inventing such one!

2. Approaches to Automatic Geometry Theorem Proving

Although there are several possible approaches to Automatic Geometry Theorem Proving, the main steps are always similar:

1) *Algebraic formulation*: the translation of a geometry statement into algebraic equations.
2) *Proof:* the use of some decision procedure, in the model we are working with, to determine the validity of the theorem.
3) *Searching conditions:* the search for extra conditions if the theorem, as it was formulated originally, is false.

This project is organized around these three items; it is a tour along classical results from geometry, with an illustration of the peculiarities that may arise.

3. Algebraic Geometry Formulation

Let K be a field of characteristic 0, for instance the field of rational numbers $\mathbb{Q}$, and let L be an algebraically closed field containing K, for instance the field of complex numbers $\mathbb{C}$. We will restrict our attention to theorems which can be phrased in terms of polynomial equalities.

---

[1] CoCoA is scientific software, produced and freely distributed by Robbiano-Niesi-Capani, Università di Genoa, cocoa@dima.unige.it

Start by choosing an appropriate coordinate system. Variables $\mathsf{x} = (x_1, \ldots, x_d)$, used to describe coordinates of points or geometric magnitudes (distances, radius, ...) that can be chosen arbitrarily, are called *independent variables*[2]; variables $\mathsf{y} = (y_1, \ldots, y_r)$, used to describe points which satisfy certain equations in the independent ones because of the construction procedure, are called *dependent variables.* In this manner, various geometric statements such as incidence, parallelism, perpendicularity, distance, etc... can be turned into polynomial equations in the variables (x, y) with coefficients in K.

Example 3.1. $\mathsf{ab} \perp \mathsf{cd}$ translates into

$$(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0,$$

where $\mathsf{a} = (a_1, a_2)$ etc.

The midpoint of ab is described by the two equations

$$2u_1 = a_1 + b_1 \text{ and } 2u_2 = a_2 + b_2.$$

Here u_1, u_2 are dependent variables.

Exercise 3.1. Express the following conditions as polynomial equations.

1. The point a lies on a circle with center m and radius r.
2. The point a lies on the line bc through points b and c.
3. Lines ab and cd are parallel.
4. Points a, b and c are collinear, i.e. on one line.

Therefore, after adopting a coordinate system, the *hypotheses* of a theorem can be expressed as a set of polynomial equations, $h_1(\mathsf{x}, \mathsf{y}) = 0, \ldots, h_p(\mathsf{x}, \mathsf{y}) = 0$, and the *thesis* can also be expressed as a polynomial equation, $t(\mathsf{x}, \mathsf{y}) = 0$, where $h_1, \ldots, h_p, t \in K[\mathsf{x}, \mathsf{y}]$. Then a geometry theorem $\mathcal{T}$ is translated into

$$\boxed{\forall (\mathsf{x}, \mathsf{y}) \in L^n,\ h_1(\mathsf{x}, \mathsf{y}) = 0, \ldots, h_p(\mathsf{x}, \mathsf{y}) = 0 \Rightarrow t(\mathsf{x}, \mathsf{y}) = 0} \tag{3.1}$$

In terms of algebraic geometry, this is phrased as: the algebraic variety defined by $\{h_1 = 0, \ldots, h_p = 0\} \subset L^n$ must be contained in the variety $\{t = 0\}$. At this point we need to introduce some notation from algebraic geometry.

Given $f_1, \ldots, f_q \in K[\mathsf{x}, \mathsf{y}]$ we denote by $\mathcal{Z}(f_1, \ldots, f_q) \subset L^n$ the algebraic variety defined by $f_1, \ldots, f_q$ in L^n. Given an algebraic variety $Z \subset L^n$ we denote by $\mathcal{J}(Z)$ the ideal defined by Z in $K[\mathsf{x}, \mathsf{y}]$.

Definition 3.1. *Given a geometry theorem $\mathcal{T}$, we define the* hypotheses variety *H as the algebraic set $\mathcal{Z}(h_1, \ldots, h_p)$ and the* thesis variety *T as the algebraic set $\mathcal{Z}(t)$.*

[2] This is a subtle point to which we will come back in section 4.

Definition 3.2. *A theorem $\mathcal{T}$ is* geometrically true *if the hypotheses variety H is contained in the thesis variety T.*

The notion of being geometrically true is related to the ideal membership problem in the following way.

Theorem 3.1. *The following statements are equivalent:*

(a) Theorem $\mathcal{T}$ is geometrically true.
(b) $t \in \sqrt{(h_1, \ldots, h_p)}$.
(c) $1 \in (h_1, \ldots, h_p, tz - 1)K[\mathsf{x}, \mathsf{y}, z]$.

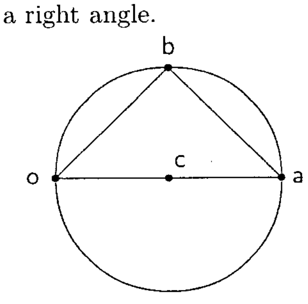
Exercise 3.2. Show that *(b)* and *(c)* are always equivalent, i.e. do not need the assumption that L be algebraically closed. Indicate where you use Hilbert's Nullstellensatz for proving the above theorem.

Item (c) of the theorem is suitable for the use of a computer algebra system, such as CoCoA. In CoCoA, $\mathsf{NormalForm}(f, (f_1, \ldots, f_q))$ computes the normal form of the polynomial f with respect to a Gröbner basis of the ideal generated by $\{f_1, \ldots, f_q\}$. Of course we have to select an ordering of the variables, but since we are only interested in deciding if the normal form is or is not 0 – and this is independent of the ordering – it makes sense to choose an ordering such as DegRevLex, which has the reputation of allowing faster computations. In conclusion, we have

$$\mathsf{NormalForm}(1, (h_1, \ldots, h_p, tz - 1)) \begin{cases} = 0 & \mathcal{T} \text{ is geometrically true} \\ \neq 0 & \mathcal{T} \text{ is not geometrically true} \end{cases}$$

It is important to remark that there is no unique algebraic formulation for a given geometric statement. When we talk about proving a theorem $\mathcal{T}$, we implicitly refer to the selected algebraic translation. In particular, it is often useful to choose formulations that reduce the number of variables appearing in the statement. Since most geometric properties are invariant under similarities, one can translate a given theorem to an equivalent statement in which one or several points are assigned numerical coordinates. Here is a simple but illustrative example.

Example 3.2. The angle subtended by a diameter of a circle from any point on the circumference is a right angle.



This statement concerns any circle and any point on it. But it is obvious that the theorem is true in general if and only if it is true for one concrete circle (since any two circles are similar and similarities preserve right angles). Thus we can fix (totally or partially) the given circle. Let us fix the center but not the radius. Take points $\mathsf{o} = (0,0)$, $\mathsf{a} = (2l, 0)$ and $\mathsf{b} = (u, v)$ such that the segment between o and a is a diameter of a circle and b belongs to this circle. Observe that $\{l, u, v\}$ are the variables, $\{l, u\}$ can be considered as independent and v can be considered as dependent on $\{l, u\}$ as it must satisfy the equation of the circle.

Hypothesis: b is on the circle centered at $(l, 0)$ with radius l translates into

$$h = h(l, u, v) = (u - l)^2 + v^2 - l^2 = 0.$$

Thesis: the angle $\widehat{\mathsf{oba}}$ is a right angle, i.e. $\mathsf{ba} \perp \mathsf{bo}$,

$$t = t(l, u, v) = u(u - 2l) + v^2.$$

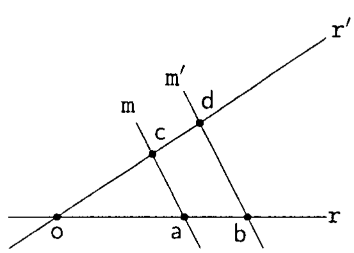
Thus we must check whether $\mathsf{NormalForm}(1, (h, tz-1)) = 0$, which is easily verified in CoCoa. Therefore, we have a theorem which is geometrically true. (Of course, the computation in this example is trivial, even by hand.)

Exercise 3.3. Describe hypotheses and theses in the following cases and show that the two statements are geometrically true.

1. In a right triangle oba with right angle at b, let p be the projection of b on oa. Then
$$\frac{|\mathsf{oa}|}{|\mathsf{ob}|} = \frac{|\mathsf{ob}|}{|\mathsf{op}|}.$$
2. Same situation as before. Then
$$\frac{|\mathsf{op}|}{|\mathsf{bp}|} = \frac{|\mathsf{bp}|}{|\mathsf{pa}|}.$$

It seems that we have found a nice way to prove geometry theorems. Unfortunately, there are well-known theorems which seem "false" using this method. For instance, Thales' Theorem turns out to be not geometrically true, according to this procedure, as the following example shows.

Example 3.3. (Thales' Theorem) Given two secant lines $\mathtt{r}$ and $\mathtt{r}'$, the triangles obtained by intersecting any two parallel lines $\mathtt{m}$ and $\mathtt{m}'$ with the two secants are similar.



Consider the x-axis as one of the secant lines and the line joining points $\mathsf{o} = (0,0)$ and $\mathsf{c} = (p,q)$ as the other one. Take points $\mathsf{a} = (l,0)$ and $\mathsf{b} = (s,0)$ on the x-axis and draw the line ac. Let $\mathsf{d} = (u,v)$ be the intersection of oc and the line parallel to ac passing through b.

Hypotheses: $\mathsf{d} \in \mathsf{oc}$: $h_1(l,s,p,q,u,v) = qu - pv = 0$

$\mathsf{ac} \parallel \mathsf{bd}$: $h_2(l,s,p,q,u,v) = q(u-s) - v(p-l) = 0$

Thesis: the ratios of the lengths of the corresponding sides of the two triangles oac and obd are equal, i.e.

$$\frac{|\mathsf{oa}|}{|\mathsf{ob}|} = \frac{|\mathsf{oc}|}{|\mathsf{od}|} = \frac{|\mathsf{ac}|}{|\mathsf{bd}|}$$

This is expressed by the following equations: $t_1 := (u^2+v^2)l^2 - s^2(p^2+q^2) = 0$, $t_2 := ((s-u)^2+v^2)l^2 - s^2((p-l)^2+q^2) = 0$

We must check that the hypotheses variety $\{h_1 = 0, h_2 = 0\}$ is contained in the zeroes of t_1 (resp. the zeroes of t_2). CoCoA's answer for thesis t_1 is negative:

```
Ring ( "ring name:" R ; "characteristic:" 0 ;
      "variables:" zuvpqsl ; "weights:" 1 , 1 , 1 , 1 , 1 , 1 , 1
      "ordering:" DEGREVLEX );
```

$$\mathsf{NormalForm}(1, \mathsf{Ideal}(-vp+uq, -vp+uq-qs+vl, -zp^2s^2 - zq^2s^2 + zu^2l^2 + zv^2l^2 - 1));$$

1

Therefore, the first thesis of this theorem is not geometrically true. Similarly, the second thesis is found to be not geometrically true.

This last example makes clear that our procedure to prove geometry theorems is not complete: if the answer is YES we can guarantee the statement's validity, but if the answer is NO the theorem can still be "true". This can happen because our algebraic formulation sometimes does not represent correctly the geometric construction we have in mind. Think for a moment how you would prove by hand Thales' theorem (Example 3.3): at a certain step it would be necessary to discard that some variables are equal to zero. Geometrically, this is equivalent to avoiding degenerate cases (for instance, in the above example, when point c is on the x-axis; or, in general, when a triangle reduces to a point or to a line, etc...). Nevertheless, these degenerate cases satisfy the algebraic hypotheses, but the theorem does not hold for all these cases. Let us deal with this problem.

Let $\mathcal{T}$ be a geometry theorem and suppose that it is not geometrically true. In the language of algebraic geometry this means that the thesis variety T does not contain the hypotheses variety H. But the validity of the theorem can be thought of as a *generic* matter in the following sense: it can happen that for some polynomial $g \in K[\mathrm{x}, \mathrm{y}]$, the set $H \setminus \{g = 0\}$ is contained in T. That is, it can happen that, upon removing some degenerate cases from the hypotheses variety, the thesis holds over the remaining configurations. Therefore, we propose a little change in formulation (3.1):

$$\boxed{\forall(\mathrm{x}, \mathrm{y}) \in L^n,\ h_1(\mathrm{x}, \mathrm{y}) = 0, \dots, h_p(\mathrm{x}, \mathrm{y}) = 0, g(\mathrm{x}, \mathrm{y}) \neq 0 \Rightarrow t(\mathrm{x}, \mathrm{y}) = 0} \tag{3.2}$$

Definition 3.3. *Let* $h_1, \dots, h_p, g, t \in K[\mathrm{x}, \mathrm{y}]$ *as above. We define the* hypotheses+condition variety H_g *as the algebraic set* $\mathcal{Z}(h_1, \dots, h_p, gk-1)$ *in* L^{n+1}, *where* k *is a new indeterminate.*

Definition 3.4. *Let* $h_1, \dots, h_p, g, t \in K[\mathrm{x}, \mathrm{y}]$. *A theorem formulated as in* (3.2) *is* geometrically true under the condition $g \neq 0$ *if the variety of hypotheses+condition* H_g *is contained in the thesis variety* $T = \mathcal{Z}(t) \subset L^{n+1}$.

Exercise 3.4. Show that the validity of a theorem under the condition $g \neq 0$ is equivalent to

$$t \in \sqrt{(h_1, \dots, h_p, gk - 1)}.$$

Prove that this last condition holds if and only if

$$1 \in (h_1, \dots, h_p, gk - 1, tz - 1)K[\mathrm{x}, \mathrm{y}, k, z],$$

where z is a new indeterminate.

Now let us go back to Example 3.3.

Exercise 3.5. (**Thales' Theorem** revisited) We proved above that, without any extra condition, Thales' Theorem is not geometrically true. Let us impose the first nondegeneracy condition that arises by considering the hypothesis: the line oc must be different from the x-axis (i.e. $q \neq 0$). Check that with this extra condition the theorem is true.

In the next paragraph we show how to look for such nondegeneracy conditions.

4. Searching for conditions

Notations remain as in the previous paragraph, i.e. $h_1, \dots, h_p$ describe the hypotheses and t the thesis for a geometry theorem.

Definition 4.1. *A* non-degeneracy condition *for a geometry theorem is a polynomial* $g \in K[\mathrm{x}, \mathrm{y}]$ *such that the theorem is geometrically true under the condition* $g \neq 0$.

Exercise 4.1. Prove that a polynomial $g \in K[\mathsf{x},\mathsf{y}]$ is a condition for a geometry theorem if and only if

$$g^l \in (h_1, \dots, h_p, tz - 1) \cap K[\mathsf{x},\mathsf{y}]$$

for some $l \geq 0$.

Remark 4.1. From the computational point of view, it is easier to search for conditions among the elements of the ideal

$$(h_1, \dots, h_p, tz - 1) \cap K[\mathsf{x},\mathsf{y}]$$

than in its radical. Radical computation is more difficult and less often implemented in computer algebra packages. Nevertheless, this restricted (for practical purposes) computation yields the same results from a geometric point of view, because $g \neq 0$ (over some point in affine space) if and only if $g^l \neq 0$.

The last remark motivates the following

Definition 4.2. *The ideal*

$$(h_1, \dots, h_p, tz - 1) \cap K[\mathsf{x},\mathsf{y}]$$

will be called the ideal of non-degeneracy conditions *for the given theorem* $\mathcal{T}$.

Definition 4.1 is too coarse and has some drawbacks. For example, note that

$$(h_1, \dots, h_p) \subset (h_1, \dots, h_p, gk - 1) \cap K[\mathsf{x},\mathsf{y}].$$

Thus, any set of hypotheses jointly with any thesis could turn into a valid geometry theorem just by finding a suitable condition g: it is enough to choose $g \in (h_1, \dots, h_p)$; then

$$(h_1, \dots, h_p, gk - 1) \cap K[\mathsf{x},\mathsf{y}] = K[\mathsf{x},\mathsf{y}]$$

and we have, by Exercise 3.4, the validity of any thesis under such condition $g \neq 0$.

Therefore, it is necessary to classify conditions in order to avoid pathological or redundant situations.

Definition 4.3. *Let $g \in k[\mathsf{x},\mathsf{y}]$ be a condition for a geometry theorem.*

(i) g is a trivial condition *if $g \in \sqrt{(h_1, \dots, h_p)}$.*
(ii) Otherwise, g is a nontrivial condition. *Moreover, these can be split in two further cases:*
- *a) g is* relevant *if $1 \notin (h_1, \dots, h_p, g)$.*
- *b) g is* irrelevant *if $1 \in (h_1, \dots, h_p, g)$.*

Remark 4.2. Geometrically, a trivial condition g means that $\{g = 0\}$ contains the hypotheses variety H. So the hypotheses+condition variety H_g is empty; therefore, from a logical point of view, any thesis t follows from H_g.

On the other hand, if a nontrivial condition g is irrelevant, the variety $\{g = 0\}$ does not intersect the hypotheses variety H. Thus, $H \subset \{g \neq 0\}$ and this means that condition $g \neq 0$ is already implicit in the hypotheses.

Although trivial or irrelevant conditions do not yield true conditions, they are important because it can happen that relevant conditions arise as combinations of other conditions, including trivial and irrelevant ones (i.e. such "odd" conditions could appear in a basis of the ideal of conditions (see Definition 4.3) or in a basis of its radical).

Exercise 4.2. Let $\mathcal{T}$ be a geometry theorem which is not geometrically true. Prove the following statements:

1. If the hypotheses ideal $\sqrt{(h_1, \ldots, h_p)}$ is prime, all conditions are trivial.
2. If there exist nontrivial conditions for $\mathcal{T}$, each nontrivial condition is relevant.
3. There are relevant conditions for $\mathcal{T}$ if and only if there are relevant conditions in any basis of the ideal of conditions of $\mathcal{T}$.

Remark 4.3. The computation of the ideal of conditions (see Definition 4.2) with CoCoA is done using the command

$$\mathsf{Elim}(z, \mathsf{Ideal}(h_1, \ldots, h_p, tz - 1))$$

that yields a Gröbner basis of $(h_1, \ldots, h_s, tz - 1) \cap K[\mathsf{x}, \mathsf{y}]$.

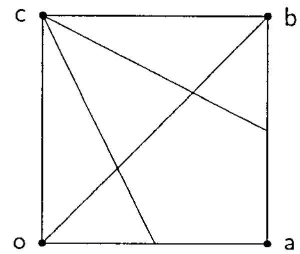
For each element of this basis we discard conditions g which are contained in the hypotheses ideal (i.e. trivial conditions) computing

$$\mathsf{NormalForm}(1, \mathsf{Ideal}(h_1, \ldots, h_p, gk - 1))$$

in $K[\mathsf{x}, \mathsf{y}, k]$. If it is 0 the condition is trivial. For nontrivial conditions, again using NormalForm, we detect relevant and irrelevant ones.

Exercise 4.3. *This theorem appears in the proposal School Mathematics in the 1990s (ed. Geoffrey Howson and Bryan Wilson, Cambridge University Press, Cambridge, 1986) of the International Commission on Mathematical Instruction, where the didactical impact of automatic theorem proving in elementary geometry is already mentioned.*

Let oabc be a square. Then the two lines connecting c with the midpoints of oa and ab, respectively, divide the diagonal ob into three segments of equal length.



1. Give a translation into a system of polynomial equations. Take $\mathsf{o} = (0,0), \mathsf{a} = (l,0), \mathsf{b} = (l,l), \mathsf{c} = (0,l)$
2. Show that the theorem is not geometrically true.
3. Is the hypotheses ideal prime? If not, can you find a decomposition as intersection of prime ideals?
4. Analyze trivial and non-trivial conditions in the basis of the ideal of conditions.
5. Find a nondegeneracy condition so that the theorem holds under this extra condition.

It is quite obvious that sets of the kind $H \setminus \{g = 0\}$ are Zariski open in the hypotheses variety H, but there are open sets which are not so easily described. Thus, our method for finding conditions yields specially simple open sets where the thesis holds. But, could it happen that the theorem holds over more complicated open sets and we could not tell just by finding conditions? The next exercise asks you to show this is not possible.

Exercise 4.4. Show that there is a nonempty Zariski open set in H where the thesis $t = 0$ holds if and only if there exists a nontrivial condition g such that

$$h_1 = 0, \dots, h_p = 0, g \neq 0 \Rightarrow t = 0$$

At this point we present the geometric interpretation of the ideal of all conditions for a given thesis. Roughly speaking, the zero set of such an ideal covers, quite tightly, the set of failures $\{t \neq 0\} \cap H$.

Exercise 4.5. Let $\{g_1, \dots, g_s\}$ be a basis of the ideal

$$(h_1, \dots, h_p, tz - 1) \cap K[\mathrm{x}, \mathrm{y}].$$

1. Prove that the algebraic set $\mathcal{Z}(g_1, \dots, g_s)$ is the Zariski closure of $\{t \neq 0\}$ in H.
2. Prove that, therefore, $\mathcal{Z}(g_1, \dots, g_s)$ does not contain a nonempty Zariski open of H contained in $\mathcal{Z}(t)$.
3. Show that there is a proper algebraic set (possibly empty) W on H such that

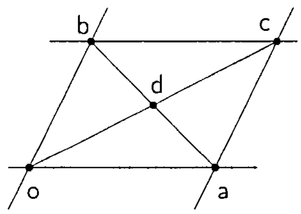
$$\mathcal{Z}(g_1, \dots, g_s) \setminus W \subset \{t \neq 0\} \text{ on } H.$$

For instance, in Exercise 4.3, the Zariski closure of $\{t \neq 0\} \cap H$ is equal to $\{l = 0\} \cap H$, the set of degenerate squares. The meaning of Exercise 4.5.1 and 4.5.2, is that $\{l = 0\}$ is a necessary condition for the thesis to fail over some point of H. On the other hand, Exercise 4.5.3, shows that it may not be a sufficient condition: there could be some values of $\{l = 0\} \cap H$ where the thesis holds, but such values are contained in a proper Zariski-closed set of $H \cap \{t = 0\}$. Intuitively speaking, we could think of the set $\mathcal{Z}(g_1, \dots, g_s)$ as the collection of truly degenerate cases; perhaps, a few of these cases still satisfy the theorem.

Exercise 4.6. Find, in Exercise 4.3, the set of points in $\{l = 0\} \cap H$ that satisfy the thesis.

But, as you can see in the next example, sometimes $\mathcal{Z}(g_1, \dots, g_s)$ contains all the "usual" cases.

Example 4.1. Suppose we want to prove the following statement: The center of a parallelogram is on one of its edges.



Consider the parallelogram with vertices o $= (0, 0)$, a $= (l, 0)$, b $= (r, s)$ and c $= (p, q)$. Let d $= (u, v)$ be the center of this parallelogram, i.e. the intersection of the diagonals. Here l, r, s are the independent variables.

Hypotheses:
- oa || bc : $h_1 := l(s - q) = 0$
- ob || ac : $h_2 := qr - s(p - l) = 0$
- d ∈ oc : $h_3 := uq - vp = 0$
- d ∈ ab : $h_4 := s(u - l) - v(r - l) = 0$

Thesis: d ∈ oa : $t := lv = 0$

```
Ring ( "ring name:" R ; "characteristic:" 0 ;
    "variables:" yzuvpqrsl ; "weights:" 1, 1 , 1 , 1 , 1 , 1 , 1 , 1 , 1
    "ordering:" DEGREVLEX );
NormalForm(1, Ideal(l(s − q), qr − s(p − l), uq − vp, s(u − l) − v(r − l),
                (lv)z − 1));
1
```

The theorem is not geometrically true. Let us look for some conditions.

$\mathsf{Elim}(z, \mathsf{Ideal}(l(s-q), qr - s(p-l), uq - vp, s(u-l) - v(r-l),$
$(lv)z - 1));$

$\mathsf{Ideal}(vp - 1/2rs - 1/2sl, vr - 1/2rs - vl + 1/2sl, q - s, ps - rs - sl,$
$pr - r^2 - pl + l^2, p^2 - r^2 - 2pl + l^2, up - 1/2r^2 - pl + 1/2l^2,$
$vs - 1/2s^2, us - 1/2rs - 1/2sl, ur - 1/2r^2 - ul + 1/2l^2);$

We choose the condition $ps - rs - sl$, which is nontrivial:

$\mathsf{NormalForm}(1, \mathsf{Ideal}(l(s-q), qr - s(p-l), uq - vp, s(u-l) - v(r-l),$
$(ps - rs - sl)z - 1));$

1

And we verify that the theorem is valid under this condition:

$\mathsf{NormalForm}(1, \mathsf{Ideal}(l(s-q), qr - s(p-l), uq - vp, s(u-l) - v(r-l),$
$(ps - rs - sl)z - 1), (vl)y - 1);$

0

What is the geometric meaning of this condition?

$$s(p - r - l) \neq 0 \;\Leftrightarrow\; s \neq 0 \text{ and } p - r - l \neq 0$$

Now $s \neq 0$ says that the parallelogram is not degenerated (i.e., it is not a point). But $p - l \neq r$ gives the condition:

> the length of the projection of the segment ob onto oa is different from the length of the projection of the segment ac onto oa.

Obviously, this condition holds only on "unusual" parallelograms. Our method requires further analysis, since we have proved a (false) "theorem" that holds over an open set of the variety of parallelograms!.

True non–degeneracy conditions

The last example shows that, in order to bridge the gap between a geometry theorem and standard geometric intuition, we need a finer analysis of the conditions yielding "degenerate cases". Note that when we first established the algebraic formulation of a geometry statement, we identified some variables as being independent. Now, we emphasize this fact by naming such variables *geometrically independent*, because they correspond to coordinates of points which can be freely chosen to draw a geometric sketch representing our statement. For instance, when drawing a parallelogram, three points can be thought as independent, but the fourth one is then totally determined. Again, for parallelograms having fixed vertices at $(0,0), (1,0)$, only the two coordinates of a third vertex should be considered as independent. And if we deal with parallelograms having one vertex at $(0,0)$ and another one over the x-axis (see Example 4.1), say, $(l, 0)$, then l plus the coordinates of a third vertex should be considered as independent. Thus the concept of geometrically independent variables is linked to the precise formulation we are considering for a given geometry theorem: for Example 4.1 the number of such variables could be 6, if we choose to deal with a general parallelogram; or 2, if we follow the simplest formulation with three fixed vertices (bearing in mind that our statement is invariant by similarities).

It seems natural to carry over the hypotheses variety the idea of geometrically independent variables. Namely, it "should" mean that no polynomial in these variables vanishes over the variety. Unfortunately, it can happen that a polynomial, only in the geometrically independent variables, vanishes over an open subset of the hypotheses variety: in that case, this subset consists of points corresponding to degenerate cases of our geometric statement. For instance the reader can check, working over Exercise 4.3, that points where $l = 0$ form an irreducible component of the hypotheses variety. Although l is, from the point of view of the geometric construction, the only independent variable, there is an open set of points with $l = 0$, namely, the complement of the remaining irreducible (hence closed) components of the same variety. Moreover, one can show that the hypotheses variety of Example 4.1 decomposes[3] as the union of the following irreducible algebraic sets:

$$\mathcal{Z}(l, qr - sp, -su + vr, -uq + vp) \cup \mathcal{Z}(p, -r + l, s, q) \cup \\ \mathcal{Z}(s, q,) \cup \mathcal{Z}(s - q, p - l - r, -2u + l + r, s - 2v).$$

Since the geometrically independent variables are l, r, s, we see that only the last component includes non degenerated parallelograms. But the condition we have found $s(p - r - l) \neq 0$ holds only only over an open set of the first algebraic component, which is degenerated. Since we do not want to establish theorems that are true just in degenerate cases, finding non-degeneracy conditions should focus on exhibiting an open set of points in the hypotheses variety where the geometrically independent variables remain algebraically independent, i.e. such that *no polynomial in these variables vanishes over the open set.* Therefore, the method to find non degeneracy conditions of the precedent section needs to be improved.

First of all, in the framework of commutative algebra, we have the following concept of independence:

Definition 4.4. *Let I be an ideal of the polynomial ring $K[x_1, \ldots, x_n]$. The variables $x_{i_1}, \ldots, x_{i_d} \in \{x_1, \ldots, x_n\}$ are* independent modulo the ideal I *if $I \cap K[x_{i_1}, \ldots, x_{i_d}] = (0)$.*

The dimension of an algebraic set $H \subset L^n$ (and of the ideal $\mathcal{J}(H)$) *is the number*

$$\begin{aligned} d &= \dim(H) = \dim(\mathcal{J}(H)) \\ &= \max\{r : \text{there are } r \text{ independent variables modulo } \mathcal{J}(H)\}. \end{aligned}$$

[3] Using CASA, a computer algebra package running on top of MAPLE, developed by Risc-Linz.

Note that the dimension of an ideal coincides with the dimension of its radical:

Lemma 4.1. *Let I be an ideal of $K[x_1, \dots, x_n]$. Denote by x' the variables $(x_{i_1}, \dots, x_{i_d})$. Then, $I \cap K[\mathrm{x}'] = (0)$ if and only if $\sqrt{I} \cap K[\mathrm{x}'] = (0)$.*

Proof. The only if part is trivial, because $I \subset \sqrt{I}$. Conversely, it suffices to prove that there is a prime ideal $\mathfrak{p}$ containing I such that $\mathfrak{p} \cap K[\mathrm{x}'] = (0)$, because each prime ideal $\mathfrak{p} \supset I$ contains $\sqrt{(I)}$.

Consider the set $\Sigma = \{A \supset I \text{ ideal} : A \cap K[\mathrm{x}'] = (0)\}$. The set Σ is not empty ($I \in \Sigma$) and it is inductive; therefore Zorn's lemma implies there is a maximal element $\mathfrak{p}$ in Σ. Let us prove that $\mathfrak{p}$ is prime: take a, b such that $ab \in \mathfrak{p}$ and $a, b \notin \mathfrak{p}$; then, as $\mathfrak{p}$ is maximal, $\mathfrak{p} + (a) \cap K[\mathrm{x}'] \neq (0)$ and $\mathfrak{p} + (b) \cap K[\mathrm{x}'] \neq (0)$. But, then $\mathfrak{p} \cap K[\mathrm{x}'] \neq (0)$, because $ab \in \mathfrak{p}$, which is impossible. Therefore, $\mathfrak{p}$ is prime.

Exercise 4.7. 1. Prove that a set x' of variables is independent modulo an ideal if and only if there is an isolated prime $\mathfrak{p}$ of this ideal such that x' is independent modulo $\mathfrak{p}$.
2. Prove that the dimension of an ideal agrees with the maximum dimension of its associated primes.

The following simple exercise shows that the concept of independent variables is quite tricky, at least when the ideal is not prime.

Exercise 4.8. Show that $\{x\}$ and $\{y, z\}$ are two maximal sets (with different cardinality) of independent variables modulo the ideal (xy, xz). Find the dimension of $\mathcal{Z}(xy, xz)$.

The connection between these algebraic results and the discussion at the beginning of this section is provided by the following easy:

Proposition 4.1. *Let x' be a set of variables, I an ideal of $K[x_1, \dots, x_n]$. The following statements are equivalent:*

(a) The set x' is independent modulo I,
(b) There is an open subset Γ of some irreducible component of the variety $\mathcal{Z}(I)$ such that no polynomial in the variables x' vanishes over every point of Γ,
(c) There is an open subset Ω of the variety $\mathcal{Z}(I)$ such that no polynomial in the variables x' vanishes over every point of Ω.

Proof. Assume x' is independent modulo I. Then x' is also independent modulo an isolated prime ideal of I. Since the irreducible components of $\mathcal{Z}(I)$ are the zeroes of the isolated primes of I, it is enough to remark that a polynomial vanishes over an open set of an irreducible variety if and only if it vanishes over the whole variety. This yields *(b)*. Now assume *(b)* holds. Then Γ is open in some irreducible component and is not, in general, open in $\mathcal{Z}(I)$. But since all open on irreducible varieties are Zariski dense, Γ contains a non empty open set Ω of $\mathcal{Z}(I)$ (intersecting Γ with the complement of the union of the remaining components). The implication *(c)* $\Rightarrow$ *(a)* is trivial.

Remark 4.4. There is a brute force way, using CoCoA, to check if some variables are independent (and therefore to find a set with maximum cardinal of independent variables). It is enough (but rather tiring!) to use repeatedly the CoCoA command Elim. There is also a direct way to find the dimension of an ideal, using Dim.

It goes without saying that the variables we choose as geometrically independent in the formulation of the geometry theorem should actually be a set of independent variables of $K[\mathrm{x}, \mathrm{y}]$ modulo the hypotheses ideal $\mathcal{J}(H)$, in order to guarantee that at least we have an open set of non degenerate cases in the hypotheses variety. Then we conclude that there are also some irreducible components of the variety enjoying this property. We will like, at least, that the thesis holds over an open set of these particular irreducible components and we do not care much about what happens with the other components, which are filled with degenerate cases. This motivates the following definition and proposition.

Definition 4.5. *A non zero polynomial $g \in K[\mathrm{x}, \mathrm{y}]$ is a* true non–degeneracy condition *for a geometry theorem $\mathcal{T}$ if $g \in K[\mathrm{x}]$, where $\mathrm{x} = (x_1, \ldots, x_d)$ is a set of geometrically independent variables over the hypotheses variety, and $\mathcal{T}$ is geometrically true under the condition $g \neq 0$.*

Notice that such conditions are always non trivial, since they belong neither to the ideal of the hypotheses variety (by definition of independent variables), hence (by Lemma 4.1) nor to its radical.

Proposition 4.2. *Using the same notations as in Definition 4.5. The following statements are equivalent:*

- *(a) There is a true nondegeneracy condition for a geometry theorem $\mathcal{T}$.*
- *(b) There is $g \in K[\mathrm{x}]$ such that $g \cdot t \in \sqrt{(h_1, \ldots, h_p)}$.*
- *(c) $I_c = (h_1, \ldots, h_p, tz - 1) \cap K[\mathrm{x}] \neq (0)$*
- *(d) t vanishes on all irreducible components of H where x is a geometrically independent set of variables.*

In this case, we say that the theorem is generically true.

Proof. We leave to the reader to show *(a)*, *(b)* and *(c)* are equivalent.

Now let us assume *(b)* and let H_i be an irreducible component of H where x is an independent set of variables. As there is $g \in K[\mathrm{x}]$ such that $g \cdot t \in \sqrt{(h_1, \ldots, h_p)} = \mathcal{J}(H)$, then $g \cdot t \in \mathcal{J}(H_i)$. And since $\mathcal{J}(H_i)$ is prime, g or t are in $\mathcal{J}(H_i)$. But $g \notin \mathcal{J}(H_i)$, because $\{x_1, \ldots, x_d\}$ are independent modulo $\mathcal{J}(H_i)$. Then $t \in \mathcal{J}(H_i)$.

Conversely, let be $H = H_1 \cup \ldots \cup H_r \cup H_1^* \cup \ldots \cup H_l^*$ the decomposition of H in irreducible components, labeled so that x is a set of independent variables over each H_i and it is not independent over each H_j^*. As $\{x_1, \ldots, x_d\}$ are dependent modulo $\mathcal{J}(H_j^*)$, for each $j = 1, \ldots, l$ there is $g_j \in K[\mathrm{x}]$ such that g_j vanishes on H_j^*. Take $g = g_1 \cdots g_l$ (if $l = 0$ choose $g = 1$), then $g \cdot t$ vanishes on H.

Remark 4.5. By 4.1, non–degeneracy conditions for a statement $\mathcal{T}$ are to be found in the elimination ideal (or, rather, in its radical, but we follow here the same simplification as in 3.1)

$$I_c = (h_1, \ldots, h_p, tz - 1) \cap K[x_1, \ldots, x_d]$$

Using CoCoA we can obtain a Gröbner basis of I_c by

$$\mathsf{Elim}(y_1..y_r, \mathsf{Ideal}(h_1, \ldots, h_p, tz - 1))$$

Then,

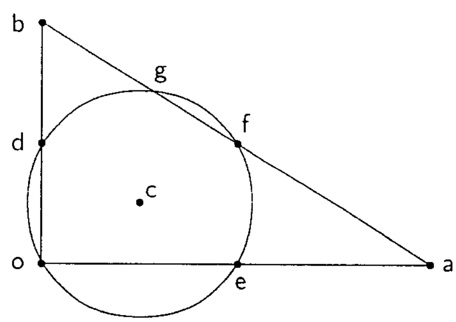
1) If $I_c = (0)$, by 4.2, we say that theorem $\mathcal{T}$ is *not generically true.* In terms of algebraic geometry this means that $\{t \neq 0\}$ holds over some "geometrically relevant" component of H. In most cases it also means that $\{t \neq 0\} \cap H$ has the same dimension as H (since degenerate components "should" have smaller dimension), but see Exercise 4.3.
2) If $I_c = (g_1, \ldots, g_s) \neq (0)$, then

$$h_1 = 0, \ldots, h_p = 0 \text{ and } (g_1 \neq 0, \text{ or } g_2 \neq 0, \ldots, \text{ or } g_s \neq 0) \;\Rightarrow\; t = 0$$

   We leave to the reader the task of finding, as in Exercises 4.4 and 4.5, the geometrical interpretation of the zero set $\mathcal{Z}(g_1, \ldots, g_s)$.

This analysis implies that Example 4.1 is not generically true since there are not conditions in the independent variables l, r, s.

Example 4.2. In any right triangle the circle passing through the midpoints of the sides also contains the feet of the three altitudes.



Consider the triangle of vertices $\mathsf{o} = (0,0)$, $\mathsf{a} = (2r, 0)$ and $\mathsf{b} = (0, 2s)$. Let be $\mathsf{d} = (0, s)$, $\mathsf{e} = (r, 0)$ and $\mathsf{f} = (r, s)$. Denote by $\mathsf{c} = (p, q)$ the center of the circle passing by the points d, e and f. Let be $\mathsf{g} = (u, v)$ the feet of the altitude from o. Remark that r, s are the geometrically independent variables.

Hypotheses: $|\mathsf{cd}| = |\mathsf{ce}|$: $h_1 = (r-p)^2 + q^2 - p^2 - (q-s)^2 = 0$
$|\mathsf{cd}| = |\mathsf{cf}|$: $h_2 = (r-p)^2 + (s-q)^2 - p^2 - (q-s)^2 = 0$
$\mathsf{g} \in \mathsf{ab}$: $h_3 = r(v-2s) + su = 0$
$\mathsf{og} \perp \mathsf{ab}$: $h_4 = ru - sv = 0$

Thesis: $|\mathsf{cd}| = |\mathsf{cg}|$: $t = (u-p)^2 + (v-q)^2 - p^2 - (q-s)^2 = 0$

Ring ("ring name:" R ; "characteristic:" 0 ;
"variables:" zuvpqrs ; "weights:" 1, 1 , 1 , 1 , 1 , 1 , 1
"ordering:" DEGREVLEX);

NormalForm$(1, \mathsf{Ideal}((r-p)^2 + q^2 - p^2 - (q-s)^2, r(v-2s) + su,$
$(r-p)^2 + (s-q)^2 - p^2 - (q-s)^2, ru - sv,$
$((u-p)^2 + (v-q)^2 - p^2 - (q-s)^2)z - 1));$

1

The theorem is not geometrically true. Thus, we look for true non–degeneracy conditions:

Elim$(z..q, \mathsf{Ideal}((r-p)^2 + q^2 - p^2 - (q-s)^2, (r-p)^2 + (s-q)^2 - p^2 - (q-s)^2,$
$r(v-2s) + su, ru - sv,$
$((u-p)^2 + (v-q)^2 - p^2 - (q-s)^2)z - 1));$

Ideal(s, r);

Therefore this theorem is generically true. It fails only for degenerate triangles, i.e. when $s = r = 0$.

Exercise 4.9 (Simson's Theorem). The pedal points (feet) of the altitudes drawn from an arbitrary point on a triangle's circumscribed circle to the three edges are collinear.

1. Let C be the circle with center $\mathsf{c} = (p, q)$ which is circumscribed in the triangle with vertices $\mathsf{o} = (0,0)$, $\mathsf{a} = (l, o)$ and $\mathsf{b} = (r, s)$. Set up equations describing hypotheses and thesis for the theorem.
2. Show that Simson's Theorem is generically true and derive a true non degeneracy condition for its validity. Phrase this as a condition on the sides of the triangle.

5. Searching for extra hypotheses

So far our method identifies a theorem's validity in nondegenerate cases. It discovers, essentially, statements that hold over open sets of the hypotheses variety. But unless one is very lucky (or clever) most properties that one states "at random" about a certain geometric setting will not be generally true. For instance, we could make a statement which is not true for general triangles, but does hold for a special kind of triangles. Therefore, for these theorems that are *not generically true*, our method has nothing to say (except that they are not true). Our next task is to find, if possible, extra hypotheses so that the resulting statement will be generically true over the new set of hypotheses (see [Recio-Velez] for a detailed account of this method).

As above we suppose that $\mathrm{x} = (x_1, \dots, x_d)$ is a distinguished set of geometrically independent variables on some hypotheses variety. The proof of the following statement is omitted, since it is very similar to 4.2.

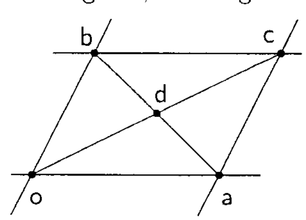
Proposition 5.1. *The following statements are equivalent:*

a) $(h_1, \dots, h_p, t) \cap K[\mathrm{x}] \neq (0)$.
b) t *vanishes on none of the irreducible components of the hypotheses variety* H *where the variables* x *are independent.*

In this case we say that the theorem is generically false.

Let us assume that a given thesis is generically false. Then, we simply add the thesis to the collection of hypotheses. Obviously, the thesis itself now contains the new hypothesis variety: the best hypothesis for a theorem is always its thesis!! Seriously speaking, this is too crude, since we do want only to allow new hypotheses which are expressible in terms of the independent geometric variables. Therefore, we eliminate the remaining variables from the new ideal of hypotheses, i.e. $(hypotheses, thesis)$. The vanishing of every element h' in the elimination ideal $(hypotheses. thesis) \cap K[independent\ variables]$ is a necessary condition for the theorem to hold. If the elimination ideal is not zero, we are sure that it is not contained in the radical of the old hypotheses ideal, since it would imply that the geometrically independent variables are not independent over any point of the variety, contradicting 4.1. Thus, in this case we end up with some strictly smaller hypotheses variety, that must now be analyzed via the standard procedure, searching for non-degeneracy conditions and so on. No guarantee that the new collection of hypotheses will yield a generically true theorem, but we can try...!

Example 5.1. In any parallelogram, the diagonals intersect at a right angle.



Consider a parallelogram as in example 4.1, of vertices $\mathsf{o} = (0,0)$, $\mathsf{a} = (l, 0)$, $\mathsf{b} = (r, s)$ and $\mathsf{c} = (p, q)$. Remark that the independent variables are r, s, l.

Hypotheses: oa || bc : $h_1 := l(s-q) = 0$
ob || ac : $h_2 := qr - s(p-l) = 0$

Thesis: oc ⊥ ab : $t := p(r-l) + qs = 0$

```
Ring ( "ring name:" R : "characteristic:" 0 ;
      "variables:" zpqrsl ; "weights:" 1, 1 , 1 , 1 , 1 , 1
      "ordering:" DEGREVLEX );
NormalForm(1, Ideal(l(s − q), qr − s(p − l), (p(r − l) + qs)z − 1));
 1
Elim(z..q, Ideal(l(s − q). rq − s(p − l), (p(r − l) + qs)z − 1));
 Ideal(0);
Elim(z..q, Ideal(l(s − q). rq − s(p − l), p(r − l) + qs));
 Ideal(r^2sl + s^3l − sl^3);
NormalForm(1, Ideal(l(s − q), rq − s(p − l), r^2sl + s^3l − sl^3, (p(r − l) + qs)z − 1))):
 1
Elim(z..q, Ideal(l(s − q). rq − s(p − l), r^2sl + s^3l − sl^3, (p(r − l) + qs)z − 1)));
 Ideal(sl);
```

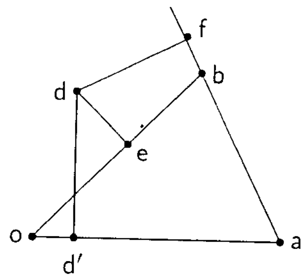
Thus, this theorem is generically false, but we have found a new hypothesis (in the third output above)

$$g = r^2sl + s^3l - sl^3 = sl(r^2 + s^2 - l^2)$$

which makes the theorem generically true (remark that adding this hypothesis implies a change in the set of independent variables: now they are just s, l). More specifically, we discover that *the theorem is true if the sides of the parallelogram are equal, namely, if $r^2 + s^2 = l^2$ (i.e. when it is a square or a rhomboide) and it is not degenerated: $sl \neq 0$.*

The next example shows how to discover the converse of Simson's theorem.

Example 5.2. We consider a triangle, and without loss of generality we assume the vertices have coordinates o $= (0,0)$, a $= (l,0)$, b $= (r,s)$; and let d $= (m,n)$ be an arbitrary point in the plane. Next we give coordinates to the feet of the perpendiculars traced from d to the three sides of the triangle: let them be e $= (v,w)$, f $= (t,u)$, d' $= (m,0)$. We conjecture that these three points are collinear.



This construction yields the following equations:

Hypotheses:	e ∈ ob :	$sv - rw = 0$
	ob ⊥ de :	$r(m - v) + s(n - w) = 0$
	f ∈ ab :	$s(t - l) - u(r - l) = 0$
	ab ⊥ df :	$(t - m)(r - l) + s(u - n) = 0$

Next we conjecture, in this situation, that points e, f, d′ are collinear (perhaps because they look like lying in a line, in the above figure); i.e. $(w - u)(m - t) + u(v - t) = 0$. Obviously, it turns out that

$$\text{NormalForm}(1, \text{Ideal}(s(t-l) - u(r-l), (t-m)(r-l) + s(u-n), sv - rw, \\ r(m-v) + s(n-w), z(w(t-m) - u(v-m)) - 1)) = 1$$

so the conjecture is not geometrically true. But it also happens that eliminating the slack variable z:

$$\text{Elim}(z, \text{Ideal}(s(t-l) - u(r-l), (t-m)(r-l) + s(u-n), sv - rw, \\ r(m-v) + s(n-w), z(w(t-m) - u(v-m)) - 1))$$

yields an ideal not contained in the radical of the hypotheses ideal

$$\text{Ideal}(s(t-l) - u(r-l), (t-m)(r-l) + s(u-n), sv - rw, r(m-v) + s(n-w))$$

so the conjecture holds over an open set of the hypotheses variety! Some extra computations confirm that such open set lies entirely in a degenerate locus of the hypotheses variety (namely, it is contained in the subset where $s = 0$). This is possible, as remarked above, because this hypotheses variety has components of dimension 6, while there are only 5 independent variables $mnrsl$, from a geometric point of view. On the other hand, if we eliminate the slack variable z plus the geometrically dependent variables v, w, t, u, we get the zero ideal, so the conjecture is not generically true over an open set of non-degenerate cases:

```
Elim(z..u, Ideal(s(t - l) - u(r - l), (t - m)(r - l) + s(u - n), sv - rw,
        r(m - v) + s(n - w), z(w(t - m) - u(v - m)) - 1));
  Ideal(0);
```

Now we start again, this time eliminating just all the geometrically dependent variables; i.e., from v to u in the set $vwtunmrsl$, on the ideal generated by the hypotheses plus the thesis:

$$\begin{aligned}&\mathsf{Elim}(v..u, \mathsf{Ideal}(s(t-l)-u(r-l),(t-m)(r-l)+s(u-n),sv-rw,\\&\quad r(m-v)+s(n-w),((w-u)(m-t)+u(v-t))));\\&\mathsf{Ideal}(nr^2s^2l-m^2s^3l-n^2s^3l+ns^4l-nrs^2l^2+ms^3l^2);\end{aligned}$$

This yields an extra hypothesis:

$$nr^2s^2l-m^2s^3l-n^2s^3l+ns^4l-nrs^2l^2+ms^3l^2=0$$

Now we observe that sl is a common factor, and its vanishing clearly corresponds to degenerate cases of "flat" triangles. After removing this factor, the equation: $nr^2s-m^2s^2-n^2s^2+ns^3-nrsl+ms^2l=0$ remains. Since for a given triangle the values of lrs will be fixed, the above equation should be only regarded on the mn variables. Then it is the equation of a circle, passing through the three vertices of the triangle. Thus our conjectural statement is not true in general, but it *could be true either if the triangle degenerates or the given point d is not arbitrary, but lies on the circle determined by the vertices of the triangle.* Over non–degenerated triangles the last condition is therefore necessary. It is easy to check that this condition is also sufficient (with some non degeneracy conditions). Indeed, as explained above, we add one extra variable z, and proceed to eliminate, in the ideal generated by all the hypotheses (old ones plus the newly discovered) and the thesis (multiplied by z and subtracting 1), all non-independent variables from $zvwtunmrsl$:

$$\begin{aligned}&\mathsf{Elim}(z..n, \mathsf{Ideal}(s(t-l)-u(r-l),(t-m)(r-l)+s(u-n),sv-rw,\\&\quad r(m-v)+s(n-w),nr^2s-m^2s^2-n^2s^2+ns^3-nrsl+ms^2l,\\&\quad ((w-u)(m-t)+u(v-t))z-1))\\&\mathsf{Ideal}(r^4+2r^2s^2+s^4-2r^3l-2rs^2l+r^2l^2+s^2l^2)\end{aligned}$$

Now this non-degeneracy condition is $(r^2+s^2)((r-l)^2+s^2)\neq 0$, which means –over the reals– that the triangle can not have coincident vertices. Thus we have, so to speak, rediscovered *Simson's Theorem* starting from a wrong assumption.

We finish with a couple of exercises on this technique of automatic discovery of theorems.

Exercise 5.1. In a triangle $\mathsf{a}=(b,0)$, $\mathsf{b}=(0,a)$, $\mathsf{c}=(1,0)$ consider a point $\mathsf{d}=(c,d)$ on the line ab, and the following lengths: the distance from d to ac $(=x)$, the distance from d to bc $(=y)$ and the length of the altitude from b to the opposite side $(=z)$. Then, the algebraic sum of two of these lengths is equal to the third one.

1. Denote by $\mathsf{e}=(u,v)$ the intersection point of bc with its perpendicular from d and let $\mathsf{f}=(c,0)$. Set up equations describing hypotheses and thesis (you must assign some signs to the lengths according to the position of d in ab.

2. Show that the theorem is generically false; add a new hypothesis. Can you describe the meaning of it?

Exercise 5.2. In a triangle, the orthocenter (intersection of heights), the baricenter (intersection of medians), the circumcenter and the incenter (center of the circle circumscribing, resp. inscribed in, the triangle) lie on a line.

1. Consider the triangle of vertices $\mathsf{a} = (-1, 0)$, $\mathsf{b} = (1, 0)$, $\mathsf{c} = (a, b)$. First prove that the following statement is generically true: the orthocenter $\mathsf{d} = (p, q)$, the circumcenter $\mathsf{e} = (u, v)$ and the baricenter $\mathsf{f} = (l, r)$ lie on a line.
2. Let us investigate now the statement: the incenter $\mathsf{g} = (s, w)$, the circumcenter $\mathsf{e} = (u, v)$ and the baricenter $\mathsf{f} = (l, r)$ lie in a line. The incenter $\mathsf{g} = (s, w)$ satisfies the property that it is the center of a circle of radius w: $(x - s)^2 + (y - w)^2 - w^2 = 0$ which is tangent to the edges of the triangle. Find the equations of this point by eliminating variables x, y in the equations of the circle of center (s, w) and radius w and in the equations giving the perpendicularity from a radius of the circle to ac (resp. bc): $b(x + 1) - (a + 1)y$ (resp. $b(x - 1) - (a - 1)y$).
3. Is the new theorem generically true or generically false?
4. Introduce an extra hypothesis expressing that the triangle be isosceles. Is the new theorem generically true or generically false?

References

[Chou] S.C. CHOU: Mechanical Geometry Theorem Proving, *D. Reidel*, 1987.
[Cox-Little-O'Shea] D. COX, J. LITTLE, D. O'SHEA: Ideals, Varieties and Algorithms, *Undergraduate Texts in Mathematics*, Springer-Verlag 1992.
[Kapur] D. KAPUR: Geometry Theorem Proving Using Hilbert's Nullstellensatz, in *Proc. of the 1986 Symposium on Symbolic and Algebraic Computation*, Ed. B.W, ACM Press, Waterloo 1986.
[Recio-Vélez] T. RECIO, M.P. VÉLEZ: Automatic discovery of theorems in elementary geometry, submitted to *Journal of Automated Reasoning*.
[Wu] W.T.WU: Mechanical Theorem Proving in Geometries, *Texts and Monographs in Symbolic Computation*, Springer-Verlag, Wien 1994.