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Voronoi Diagrams on orbifolds ☆

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Abstract

We present a method for computing Voronoi Diagrams for a relevant class of metric surfaces, namely all Euclidean and spherical two orbifolds. Since these surfaces are quotients of the Euclidean plane (sphere) by a discrete group of motions, the computation of Voronoi Diagram is reduced to the computation of this diagram for periodic sets of points on the Euclidean plane (sphere). This is accomplished by further reduction to the standard case of a finite set of points.

Keywords: Voronoi Diagram; Periodic plane Voronoi Diagram; Voronoi Diagram on an orbifold

1. Introduction

Given a collection of sites $S = \{P_i\colon 1 \leqslant i \leqslant n\}$ in the Euclidean plane E^2, the set of points closer to a point $P_i \in S$ than to any other point of S is a convex polygonal region $V(i)$ called the Voronoi region associated to P_i. The entire collection of Voronoi regions $V(i)$ for $1 \leqslant i \leqslant n$ gives a partition of the plane that is named the Voronoi diagram $\mathrm{Vor}_{E^2}(S)$ of S.

The Voronoi Diagram is a fundamental data structure in Computational Geometry, useful to solve many proximity problems. The problem of computing the Voronoi Diagram, initially considered for finite collections of sites in the Euclidean plane E^2, has been generalized in many directions that include, among others, Voronoi Diagrams on metric surfaces [3,6,9,11,13] (see [1] for a survey).

Several optimal algorithms exist to compute Euclidean Voronoi Diagrams for finite collections of sites on the plane [3,8,16], but only for some particular cases of curved surfaces embedded in the Euclidean space E^3 the problem has been solved, namely:
- on the Riemann sphere S^2 by Brown [3];
- on the surface of a cone, Dehne and Klein [6] generalize the planar sweepcircle technique of the plane to working on a cone.

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In this paper we study both computational and theoretical aspects of Voronoi Diagrams for a relevant class of metric surfaces that are the Euclidean and spherical two orbifolds. This class of surfaces includes, among many others, all the locally-Euclidean and locally-spherical surfaces (i.e., cylinders, Möbius bands, Klein bottles, flat toruses and projective planes) as they are all of them obtained as a quotient by a discrete group of motions.

Our method of computing Voronoi Diagrams on such surfaces uses mainly the fact that all these surfaces are isometrically covered by the Euclidean plane E^2 or the Riemann sphere S^2. Finite sets of points on the surface give rise to periodic point sets in the corresponding covering space which can be finite (this is always the case on the sphere) or infinite, depending on the discrete group of motions involved. Voronoi Diagrams for such periodic sets of points on the plane are proved to be computable (Section 3) using the algorithms that work for finite sets of points on the plane. This periodic Voronoi diagrams will be proved to be useful when computing the Voronoi diagrams on the surfaces (Section 4).

The paper is organized as follows. In Section 2 standard facts about discrete groups and two-orbifolds are introduced. In Section 3 we discuss an algorithm that computes Voronoi Diagrams for periodic set of points on the Euclidean plane. In Section 4, we present an algorithm for computing Voronoi Diagrams on the Euclidean and spherical two-orbifolds.

2. Discrete groups of motions and two orbifolds

Let M denote the Euclidean plane E^2 or the two-sphere S^2, the latter with the Riemannian metric inherited from E^3 (i.e., distance between two points P and Q is given as the infimum of the lenghts of all the paths on the sphere joining P and Q). S^2 with this metric is known as the Riemann sphere.

A *motion* of M is any bijection f from M onto M that preserves distances (i.e., $d(P,Q) = d(f(P), f(Q))$, $\forall P, Q \in M$). Under the composition of motions, the set of all motions of M is a group. Let Motions (M) be the full group of motions of M. A subgroup G of Motions (M) is called *discrete* if $\forall P \in M$, there exists a constant $c(P) > 0$ such that

$$\forall g \in G \text{ with } gP \neq P, \quad d(P, gP) \geqslant c(P),$$

where gP denotes the action of the motion g on point P and $d(P,Q)$ is the distance in M between points P and Q.

Note that the orbit of any point P under the action of a discrete group G, $GP = \{gP\colon\ g \in G\}$, is a closed and discrete subset of M. Any two points belonging to the same orbit will be called *equivalent* points.

The quotient space M/G, whose points are the orbits of points of M under the action of G on M, inherits a natural metric from the metric in M.

Definition 1. Distance $d(p,q)$ between two orbits p and q in M/G is defined as the distance between the sets p and q, that is, as the infimum (that is in fact the minimum) of the distances in M, between points $P \in p$ and $Q \in q$.

With this distance defined on M/G, it becomes a metric space [14].

In order to specify a point $p \in M/G$ we need only to know one point $P \in p$ in M, as the remaining points in the orbit p are all of them of the form gP for some $g \in G$. It is then useful, in order to handle the quotient space M/G, to determine some region in M which contains at least one point of each orbit and is as small as possible.

First we consider the case $M = E^2$.

A *fundamental domain* for a discrete group G of Euclidean motions is a convex and closed subset D_G of the Euclidean plane E^2 with nonempty interior and satisfying:
(1) D_G contains at least one point of each orbit;
(2) If there are equivalent points in D_G, then they lie on its boundary.

The orbit space E^2/G can be thought of as the surface T obtained from a fundamental domain D_G for G, by identifying or glueing together equivalent points in its boundary. This topological surface T is in one-to-one correspondence with E^2/G and the natural metric in this space defines, via the bijection, a metric on the surface T, so we have a topological surface T with a metric, isometric to the quotient space E^2/G. These metric surfaces are all connected and are called the *two dimensional Euclidean orbifolds* [14].

Example. Consider the group generated by a single translation of the plane. In Fig. 1 we have drawn the orbits of two points P and Q. Lenght of segment PQ gives the distance between the two orbits. A fundamental domain for this group is, for instance, the shadowed region in Fig. 1. The corresponding orbifold can be easily recognized as a cylinder.

There are as many of these surfaces as discrete groups of Euclidean motions exist. Because of this, let us recall the possible types of discrete groups of Euclidean motions of the plane [5,10,15] (due to a theorem of Bieberbach, this number is finite in any dimension [17]).

There are only a finite number of discrete groups of Euclidean motions, modulo conjugation in the affine group of Euclidean transformations (i.e., two groups G and G' are said to be conjugated if and only if there exists an affine bijection ψ from the plane onto the plane s.t. $G = \psi.G'.\psi^{-1}$). A discrete group is finite if and only if it contains no translation. Leonardo's Theorem [10] establishes that the only discrete and finite groups of Euclidean motions are the groups C_n and D_n, for n any natural number, defined as follows: the cyclic group C_n of order n consists of all the rotations leaving invariant a regular n-gon and the dihedral group D_n of order $2n$, of all the motions leaving invariant the same n-polygon.

If the group contains translations, then it is infinite. The groups that contain translations in only one direction are commonly called two-dimensional *frieze groups* and there are seven [4,10], again modulo conjugation in the affine group. The groups that contain translations in two linearly independent directions are often termed two dimensional *crystallographic groups* and there are seventeen, modulo

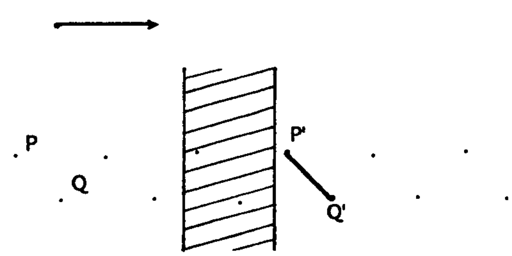

Fig. 1.

conjugation in the affine group. The history of this classification dates back to the late nineteenth century [4,5,10,15].

Among the surfaces obtained in this way from the plane, the most well-known are the cones (such that, cuting and unfolding the cone creates a planar circular sector of angle $2\pi/n$, G equals the group C_n), the cylinders (G is generated by one translation), the Möbius bands (G is generated by a glide reflection), the flat toruses (G generated by two independent translations), the Klein bottles, the pillows and the pillow-cases, all of them with the metric inherited from the Euclidean plane as stated in Definition 1.

Next we consider the case $M = S^2$.

The full group of motions of the Riemann sphere is the orthogonal group O(3), realized as the set of all rotations (around one axis passing through the center of S^2) and all rotary inversions (composition of a rotation with the central inversion).

A subgroup G of O(3) is discrete if and only if it is finite and so the problem of classification of the discrete subgroups of O(3) is the one of finding its finite subgroups. There are exactly fourteen of such groups, except for conjugation in O(3) [4].

The definition of fundamental domain for a discrete group G of motions of the Riemann sphere is the same as for the plane case, replacing the condition of being convex by spherically convex [11]. From a fundamental domain D_G for G, the corresponding surface is obtained, as before, by identifying equivalent points in its boundary.

The resulting topological surface is in one to one correspondence with the metric space S^2/G and inherits its metric, via this bijection. Then they become metric surfaces that are known as the *closed spherical two-orbifolds.*

Each of these surfaces is topologically a two-sphere, with two or three metrically singular points (i.e., the surface is a Riemannian 2-manifold of constant curvature equal to one except at a finite (2 or 3) number of singular points [12].

Althought in what follows an infinite collection of points can be involved, we will use here the following extended definition of Voronoi Diagram that applies to discrete collections of points in a Riemannian manifold [7].

Definition 2. If a discrete subset $S = \{P_i\colon\ i \in I\}$ of M is given, the Voronoi region $V_S(P_i)$ of point P_i with respect to the set S is defined as

$$V_S(P_i) = \{Q \in M\colon\ d(Q,P_i) < d(Q,P_k),\ \forall k \in I,\ k \neq i\}.$$

The following results will be used throughout the paper.

Proposition 1 (See [7,11] for a proof). *Let M be the Euclidean plane E^2 or the Riemann sphere S^2, G a discrete group of motions of M and $P \in M$ a point with trivial stabilizer* (*i.e., if $gP = P$ then $g = e$*) [2]. *Consider the orbit of P, $GP = \{gP\colon\ g \in G\}$. Then the topological closure in M of the Voronoi region of P with respect to the discrete set GP, $\mathrm{Cl}\, V_{GP}(P)$, equals the set*

$$\{Q \in M\colon\ d(Q,P) \leqslant d(Q,gP),\ \forall g \in G\}$$

and is a fundamental domain for G.

These types of fundamental domains will be referred to as *Dirichlet fundamentals domains.*

Given a discrete group G of motions of M, we define an *edge* E_g of $V_{GP}(P)$ as $E_g = \{R \in M\colon d(R,P) = d(R,gP) < d(R,hP),\ \forall h \in G - \{e,g\}\}$, whenever E_g is not empty.

Now it is well known [7] that the following result holds.

Proposition 2. *If G is a discrete group of motions of M, P a point with trivial stabilizer and $D_G = V_{GP}(P)$ a Dirichlet fundamental domain, then the set of motions $\{g \in G\colon\ E_g$ is an edge of $V_{GP}(P)\}$ is a generator system for G.*

3. The computation of Voronoi Diagrams for saturated sets of points in the plane

Given any finite collection S of points in E^2, $S = \{P_1, \ldots, P_n\}$, and any discrete group G of Euclidean motions, let the set GS be defined as the union of all orbits of points in S, that is

$$GS = \{gP_i\colon\ g \in G,\ P_i \in S\}.$$

This set GS is the *saturation* of S by the action of G.

The discrete group G can be given by a Dirichlet fundamental domain D_G of the form $\mathrm{Cl}\, V_{GP}(P)$ for some point P with trivial stabilizer and so its set of edges provide us with a generator system for G, according to Proposition 2.

Except for the cyclic or the dihedral groups, the rest of the discrete groups of Euclidean motions of the plane are infinite and therefore, most of the time the set GS is an infinite but discrete, subset of the plane. Although the existing algorithms to compute Voronoi diagrams deal only with finite collections of points, points in GS are regularly distributed and its Voronoi Diagram has then some kind of regularity that allows us to compute it by computing only the Voronoi Diagram of a certain finite subcollection of points in GS as stated by the following theorem.

Theorem 1. *Let G be a discrete group of Euclidean motions given by some generator system for G. Let D_G be a Dirichlet fundamental domain of the form $\mathrm{Cl}\, V_{GP}(P)$, for P a point in E^2 with trivial stabilizer.*

Suppose $S = \{P_1, \ldots, P_n\}$ is a subset of D_G and consider its saturation GS by the action of G. Then, there exists a finite subset S^* of GS such that S^* contains S and

$$\mathrm{Vor}(GS) = G\big(\mathrm{Vor}(S^*) \cap D_G\big).$$

The theorem is an easy conclusion of the following sequence of lemmas.

Lemma 1.

$$g_o V_{GS}(P_j) = V_{GS}(g_o P_j) \quad \forall g_o \in G \text{ and } \forall P_j \in S.$$

Lemma 2. *There exists a finite subset $G^* = \{g_1 = e, g_2, \ldots, g_m\}$ of G such that for every point $X \in D_G$ and for every point $Y \in E^2 - \bigcup_{j=1}^m g_j D_G$, there exists another point $Y^* \in \bigcup_{j=1}^m g_j D_G$ such that Y^* is equivalent to Y by G and $d(X,Y^*) < d(X,Y)$.*

As a consequence it happens that

$$V_{GS}(P_i) \subset \bigcup_{j=1}^{m} g_j D_G.$$

Lemma 3.

$$V_{GS}(P_i) = \bigcup_{j=1}^{m} g_j\big(V_{GS}(g_j^{-1}P_i) \cap D_G\big), \quad \forall i = 1, \ldots, n.$$

Let S^* be defined as $GS \cap (\bigcup_{j=1}^{m} g_j D_G)$. Then the following lemma holds.

Lemma 4. *If $Q \in S^*$ then $V_{GS}(Q) \cap D_G = V_{S*}(Q) \cap D_G$.*

Corollary.

$$\text{Vor}(GS) \cap D_G = \text{Vor}(S^*) \cap D_G.$$

Lemma 1 shows the regularity of Vor(GS): to compute Vor(GS) it is enough to compute only the Voronoi regions in Vor(GS) corresponding to points in D_G, that is of points $P_i \in S$, because any of the other regions is congruent with one of them, via an element $g \in G$.

Lemma 2 proves that points in D_G are metrically affected only by the points of GS lying in D_G or in a certain finite union of copies of D_G around D_G. As a consequence, only a finite subset S^* of GS has to be considered in order to compute the Voronoi region $V_{GS}(P_i)$ of one of the points $P_i \in D_G$. In fact, Lemma 3 allows to obtain each Voronoi region $V_{GS}(P_i)$ from the Voronoi regions of this finite set of points S^* in GS, even when restricted to the portion of these Voronoi regions that lies in D_G.

But because of Lemma 4, Voronoi Diagrams Vor(GS) and Vor(S^*) are equal when restricted to D_G.

Let us now show the proofs for the lemmas.

Proof of Lemma 1.

Let X be in $V_{GS}(P_j)$. This means that

$$d(X, P_j) < d(X, X'), \quad \forall X' \in GS - \{P_j\}. \tag{1}$$

As $d(gX, gX') = d(X, X')$, $\forall g \in G$, (1) is equivalent to

$$d(g_oX, g_oP_j) < d(g_oX, g_oX'), \quad \forall X' \in GS - \{P_j\}. \tag{1'}$$

But $\{g_oX'\colon\ X' \in GS - \{P_j\}\} = \{X''\colon\ X'' \in GS - \{g_oP_j\}\}$ and therefore (1′) can be rewritten as

$$d(g_oX, g_oP_j) < d(g_oX, X''), \quad \forall X'' \in GS - \{g_oP_j\},$$

meaning that g_oX belongs to $V_{GS}(g_oP_j)$. And so, $g_oV_{GS}(P_j) \subset V_{GS}(g_oP_j)$.

Conversely, if X is in $V_{GS}(g_oP_j)$, then

$$d(X, g_oP_j) < d(X, X'), \quad \forall X' \in GS - \{g_oP_j\}. \tag{2}$$

As before (2) is equivalent to

$$d(g_o^{-1}X, P_j) < d(g_o^{-1}X, g_o^{-1}X'), \quad \forall X' \in GS - \{g_oP_j\}. \tag{2'}$$

But $\{g_o^{-1}X'\colon\ X' \in GS - \{g_oP_j\}\} = \{X''\colon\ X'' \in GS - \{P_j\}\}$ and (2′) can be rewritten as

$$d(g_o^{-1}X, P_j) < d(g_o^{-1}X, X''), \quad \forall X'' \in GS - \{P_j\},$$

meaning that $g_o^{-1}X$ belongs to $V_{GS}(P_j)$, and so $X = g_o.g_o^{-1}X$ belongs to $g_oV_{GS}(P_j)$, proving finally that $V_{GS}(g_oP_j) \subset g_oV_{GS}(P_j)$. □

Proof of Lemma 2.

Case a. G contains no translation.

In this case G is finite and lemma holds for $G^* = G$.

Case b. G contains translations in two independent directions.

In this case the fundamental domain $D_G = \mathrm{Cl}\, V_{GP}(P)$ is bounded. Let k be its diameter. Distance from P to the boundary $\mathrm{Bd}\, V_{GP}(P)$ of $V_{GP}(P)$, considered as a function defined on $\mathrm{Bd}\, V_{GP}(P)$, attains its maximum L, as it is a continuous function defined on the compact set $\mathrm{Bd}\, V_{GP}(P)$.

Let K be $k+L$ and let $\mathrm{Cl}\, B_K(P)$ denote the closed ball of radius K centered at P. As the covering $\{gD_G\colon\ g \in G\}$ is locally finite, the compact $\mathrm{Cl}\, B_K(P)$ intersects gD_G only for a finite number $G^* = \{g_1 = e, \ldots, g_m\}$ of elements of G.

Now, if X belongs to D_G and Y is not contained in $\bigcup_{j=1}^m g_j D_G$, let Y^* be equivalent to Y and belonging to D_G (it always exists, as D_G is a fundamental domain). Then

$$d(X,Y^*) \leqslant k < d(X,Y)$$

and the lemma holds.

Case c. G contains translations in only one direction.

Let t be a translation of G whose corresponding vector has minimal length. In this case $D_G = \mathrm{Cl}\, V_{GP}(P)$ is unbounded, it extends to infinity in the perpendicular direction to t and it can be embedded in a closed band (i.e., a closed region between two parallel lines) of finite width, also perpendicular to t. Choose the width of the closed band as small as possible but large enough to contain $-tD_G$, D_G and tD_G. Call it B. This band B intersects gD_G only for a finite number $G^* = \{g_1 = e, g_2, \ldots, g_m\}$ of elements of G.

Now if X belongs to D_G and Y is not contained in $\bigcup_{j=1}^m g_j D_G$, consider the set T_Y of points equivalent to Y by any translation in G (i.e., $\{gY\colon\ g \in G$ and g is a translation$\}$). It is easy to see that the Voronoi diagram of such a set of points is an infinite collection of parallel and closed bands, equal to each other. At least three points in T_Y (one in $-tD_G$, one in D_G and one in tD_G) belong to B and because their corresponding Voronoi regions cover D_G, one of them, say Y^*, satisfies that

$$d(X,Y^*) < d(X,Y).$$

Without loss of generality, let us suppose that the set $G^* = \{g_1 = e, g_2, \ldots, g_m\}$ is saturated by the operation of taking inverses. If it is not so, just add them. We still have a finite subset of G that we call G^* again, such that Lemma 2 holds for it. □

Proof of Lemma 3. Let us first prove that the Voronoi region $V_{GS}(P_i)$, of P_i with respect to GS, is completely contained in the union $\bigcup_{j=1}^m g_j D_G$. As before we distinguish some cases.

Case a. G contains no translation.

In this case G is finite and $G^* = G$. Therefore $\bigcup_{j=1}^m g_j D_G$ is the whole plane and the lemma holds.

Case b. G contains translations in two independent directions.

Let Y be a point not contained in $\bigcup_{j=1}^m g_j D_G$. Then Y will belong to a certain gD_G, with $g \notin G^* = \{g_1 = e, g_2, \ldots, g_m\}$.

Note that $Y \in gD_G$ implies $g^{-1}Y \in D_G$ and so, both P_i and $g^{-1}Y$, belong to D_G. Therefore distance between them does not exceed the diameter k of D_G. This diameter k, in turn, is strictly smaller than distance from any point of D_G, for instance P_i, to any point outside $\bigcup_{j=1}^m g_j D_G$, say Y. This can be expressed as

$$d(gP_i, Y) = d(P_i, g^{-1}Y) < k < d(P_i, Y),$$

that means that Y does not belong to $V_{GS}(P_i)$.

Case c. G contains translations in only one direction.

Let Y be a point not belonging to $\bigcup_{j=1}^m g_j D_G$. Apply Lemma 2 (Case c) to $P_i \in D_G$ and $Y \in E^2 - \bigcup_{j=1}^m g_j D_G$, to find a point $Y^* = T_{nt}Y$, equivalent to Y by translation, and such that

$$d(P_i, Y^*) = d(P_i, T_{nt}Y) = d(T_{-nt}P_i, Y) < d(Y, P_i).$$

We conclude that Y cannot belong to $V_{GS}(P_i)$.

Recall that the more important consequence of Lemma 2 is that

$$V_{GS}(P_i) \subset \bigcup_{j=1}^m g_j D_G.$$

Now, using this fact and Lemma 1, proof of this lemma is straightforward:

$$V_{GS}(P_i) = \bigcup_{j=1}^m \left(V_{GS}(P_i) \cap g_j D_G\right) = \bigcup_{j=1}^m g_j\left(g_j^{-1} V_{GS}(P_i) \cap D_G\right) = \bigcup_{j=1}^m g_j\left(V_{GS}(g_j^{-1}P_i) \cap D_G\right). \square$$

Proof of Lemma 4. Let S^* be defined as $GS \cap (\bigcup_{j=1}^m g_j D_G)$ and let Q be a point in S^*. As S^* is a subset of GS, $V_{S^*}(Q)$ contains $V_{GS}(Q)$, implying that $V_{S^*}(Q) \cap D_G$ contains $V_{GS}(Q) \cap D_G$.

Conversely, let X be a point in $V_{S^*}(Q) \cap D_G$. This means that

$$d(X, Q) < d(X, Q'), \quad \forall Q' \in S^* - \{Q\}.$$

Due to Lemma 2, for any other point Q' in $GS - S^*$, there exists some $Q^* \in S^*$ equivalent to Q' and verifying

$$d(X, Q^*) < d(X, Q').$$

In conclusion we have that

$$d(X, Q) < d(X, Q'), \quad \forall Q' \in GS - \{Q\},$$

that means that X belongs to $V_{GS}(Q) \cap D_G$, proving that $V_{S^*}(Q) \cap D_G \subset V_{GS}(Q) \cap D_G$ and equality holds. □

Remark 1. From a quantitative point of view, it might be interesting to estimate bounds on the cardinality of the subset S^* as a function of the cardinality of the set S and the type of the group G. In fact, the size of S^* determines, after Theorem 1, the complexity of computing periodic Voronoi Diagrams.

In Lemma 4, we have constructed a set S^* whose cardinality is m times the cardinality of S, where m is a certain number of copies of the fundamental domain as required in Lemma 2.

Now, for every concrete realization of a discrete group (i.e., given the generator system and a Dirichlet fundamental domain for it), it is an easy metric problem to bound m. A case analysis yields that $m = 37$ is an upper bound for all possible realizations and all groups. This number is obtained

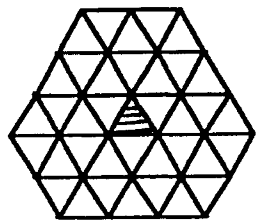

Fig. 2.

considering all first and second order adjacent copies of the fundamental domain (first order means adjacent to the fundamental domain; second order means adjacent to the adjacent ones). See Fig. 2. Note that this bound for m needs not to be the sharpest one for some groups.

Remark 2. Theorem 1 is also true for discrete groups of motions of the sphere, but since any discrete group on the sphere is finite, the set GS is always finite and the computation of $\mathrm{Vor}(GS)$ is not a problem. Nevertheless, the result in Theorem 1 can be used to reduce the number of points involved in the computation.

4. The computation of Voronoi Diagrams in the Euclidean and spherical 2-orbifolds

Now the general problem we want to solve can be stated as follows.

Problem. Given an Euclidean or spherical 2-orbifold M/G, where G is a discrete subgroup of the group Motions(M) and M denotes E^2 or S^2, and given a finite collection s of points in M/G, $s = \{p_i\colon 1 \leqslant i \leqslant n\}$, find the Voronoi Diagram $\mathrm{Vor}_{M/G}(s)$ of s in M/G.

We can suppose G is given by means of a Dirichlet fundamental domain D_G whose corresponding edges provide us with a generator system for G. Let $S = \{P_1, \ldots, P_n\} \subset D_G$ be such that $P_i \in p_i$, $\forall i = 1, \ldots, n$. In what follows, if x denotes a point (i.e., an orbit) in M/G, we will write X for any point in $M \cap D_G$ such that GX is the orbit x.

The algorithm. The Voronoi Diagram $\mathrm{Vor}_{M/G}(s)$ of s in M/G can be computed as follows.

Step 1. Compute $\mathrm{Vor}_M(GS)$ (if G is infinite, use Theorem 1).

Step 2. Remove the edges and vertices of $\mathrm{Vor}_M(GS)$ between regions of equivalent points and call $\mathrm{Vor}^*_M(GS)$ the resulting partition of M.

Step 3. Intersect $\mathrm{Vor}^*_M(GS)$ with D_G. Call it $\mathrm{Vor}_{D_G}(S)$.

Step 4. Identify equivalent points in $\mathrm{Vor}^*_M(GS) \cap D_G$.

Proof. $\mathrm{Vor}_{M/G}(s)$ is going to be computed from the periodic Voronoi diagram $\mathrm{Vor}_M(GS)$. This is possible due to the existing relation between the metric in M/G and the metric in M: distance $d(p,q)$ between two points (orbits) in M/G is the minimum of the distances between one point $P \in p$ and any point $Q \in q$.

What we are going to prove is that, when one point $X \in D_G$ belongs to the Voronoi region $V_{GS}(gP_i) \subset M$ for some $g \in G$ or when it belongs to an edge of $\mathrm{Vor}_M(GS)$ between regions of two equivalent points $g'P_i$ and $g''P_i$ or to a vertex of $\mathrm{Vor}_M(GS)$ between regions of three or more equivalent points, then the orbit x that this point X represents, considered as a point in M/G, necessarily belongs to the Voronoi region $V_s(p_i) \subset M/G$.

And conversely, if $x \in M/G$ is such that $x \in V_s(p_i)$, then its corresponding point $X \in D_G$ will belong to $V_{GS}(gP_i)$, for some $g \in G$, or it will be on an edge of $\mathrm{Vor}_M(GS)$ between regions of equivalent points or will be a vertex of $\mathrm{Vor}_M(GS)$ between regions of three or more equivalent points.

We represent the surface by means of a Dirichlet fundamental domain D_G, with maybe some identifications on its boundary (the orbifold M/G is in one-to-one correspondence with it) and give a partition $\mathrm{Vor}_{D_G}(S)$ of the fundamental domain such that, after identification, it corresponds exactly with the Voronoi diagram $\mathrm{Vor}_{M/G}(s)$.

Then all we have to prove is that the following two assertions are equivalent:

(a) $x \in V_s(p_i)$;

(b) $X \in V_{GS}(g_1P_i)$, for some $g_1 \in G$, or X belongs to an edge of $\mathrm{Vor}_M(GS)$ between regions of equivalent points, or X is a vertex of $\mathrm{Vor}_M(GS)$ between regions of three or more equivalent points.

(a) $\Rightarrow$ (b). Suppose $x \in V_s(p_i)$. This means that $d(x,p_i) < d(x,p_j)$, $\forall j = 1,\dots,n,\ j \neq i$.

Consider a representative $X \in D_G$ of x. Then, by definition of distance on M/G, there exist elements g_1, g_2 in G such that

$$d(x,p_i) = \min\{d(X,gP_i)\colon\ g \in G\} = d(X,g_1P_i) \leqslant d(X,gP_i) \quad \forall g \in G, \tag{3}$$

$$d(x,p_j) = \min\{d(X,gP_j)\colon\ g \in G\} = d(X,g_2P_j) \leqslant d(X,gP_j) \quad \forall g \in G. \tag{4}$$

Therefore we have that

$$d(x,p_i) = d(X,g_1P_i) < d(X,gP_j) \quad \forall j \neq i,\ \forall g \in G. \tag{*}$$

It can happen that in (3)

$$d(X,g_1P_i) < d(X,gP_i), \quad \forall g \in G - \{g_1\} \tag{3'}$$

or that

$$d(X,g_1P_i) = d(X,g'P_i) \quad \text{for some } g' \in G - \{g_1\}. \tag{4'}$$

If (3′) happens, (3′) together with (*) means that $X \in V_{GS}(g_1P_i)$.

If (4′) happens, (4′) together with (*) means that X belongs to an edge (if only one g') or is a vertex (if more than one g') of $\mathrm{Vor}_M(GS)$ between regions of equivalent points g_1P_i and $g'P_i$; and then (a) implies (b) is true.

(b) $\Rightarrow$ (a). Conversely, suppose (b) holds and let X belong to $x \cap D_G$. If $X \in V_{GS}(g_oP_i)$, for some $g_o \in G$, then clearly $d(x,p_i) < d(x,p_j)$, $\forall j \neq i$ and $x \in V_s(p_i)$. If X belongs to an edge between regions of equivalent points, then

$$d(X,g'P_i) = d(X,g''P_i) < d(X,Q)$$

for some g' and g'' in G and $\forall Q \in GS - \{g'P_i, g''P_i\}$.

Thus in particular

$$d(x,p_i) = d(X,g'P_i) = d(X,g''P_i) < \min\{d(X,gP_j)\colon\ g \in G\} = d(x,p_j), \quad \forall j \neq i.$$

A similar argument for the case of vertices proves that (a) holds. $\square$

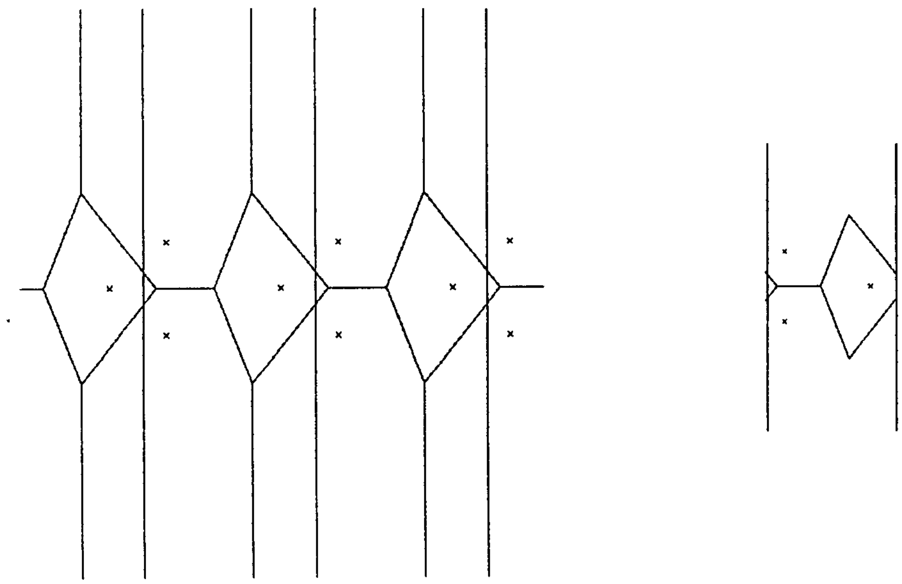

Fig. 3.

Remark 3. Note that Step 2 of the algorithm can produce bends, or vertices of degree two on the boundary of a Voronoi region in M/G. See Fig. 3 for an example for the surface of a cylinder.

Remark 4. The procedure can be easily adapted to compute Voronoi diagrams on the surface of a cone of arbitrarily angle α (defined as in Section 2), whenever 3α does not exceed 2π.

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