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Hi,

I'm sorry this has taken so long. I reviewed your paper before I left for Israel, but didn't have time to send you (and Bob Caviness) the basic letter. I am recommending the paper for acceptance as it is, although the final letter will come from Bob. The Jewish holidays have started here now, and combined with a general strike that is going on it has been difficult to get things going. My email address will change soon. I'll sent it to you as soon as it is organized. Thanks for your patience.

--Rich

Advances on the Simplification of Sine–Cosine Equations

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In this paper we contribute with several results to the approach initiated by Hommel and Kovács (well documented with applications in the recent book of the latter author Kovács (1993)) on the symbolic simplification of sine/cosine polynomials that arise, for instance, as determining equations for joint values in robotics inverse kinematic problem. We present, considering for the first time sine/cosine polynomials, fast algorithms for the functional decomposition and factorization problems, reducing the solving of such s - c equations to a sequence of lower degree ones. Moreover, we show that triangularization of a given sine/cosine equation provides a conceptual understanding of the conditions that yield extraneous roots in the half angle tangent substitution (and therefore that imply a reduction of the degree in the determining equation of a given s - c system).

1. Motivations and main contributions of the paper

By a *sine-cosine* equation we understand a polynomial equality $f(s, c) = 0$, with f in the quotient ring $K[s, c]/(s^2 + c^2 - 1)$, and where K is a field of characteristic zero (typically, a numerical field such as \mathbb{Q} or \mathbb{R} , or a field of parameters $\mathbb{Q}(d_1, \dots, d_m)$). Therefore, when we write $f(s, c)$ we consider, throughout this paper, that this expression is implicitly univariate in some unknown angle θ such that $s = \sin(\theta)$, $c = \cos(\theta)$. Our goal is finding methods for solving or simplifying equations of the sort $f(s, c) = 0$; and thus, equivalently, for solving or simplifying systems

$$\begin{aligned} f(s, c) &= 0, \\ s^2 + c^2 - 1 &= 0. \end{aligned}$$

1.1. INTEREST OF THE PROBLEM

Polynomial systems, where the variables are interpreted as trigonometric functions of unknown angles, are quite ubiquitous, arising, for instance, in electrical networking and in molecular kinematics. Here, our applications will be taken from the field of robot kinematics. Besides referring to the many situations described in the recent book of Kovács (1993), we will sketch, for the sake of being self-contained, a few examples of the role of *sine-cosine* systems in robotics:

EXAMPLE 1.1. Given a robot arm with six revolute joints, i.e. a $6R$ robot (see Figure 1), a typical problem is to find the values of the different joint angles (with respect to some standard way of measuring them) that place the tip (or hand) of the robot at some desired position and orientation.

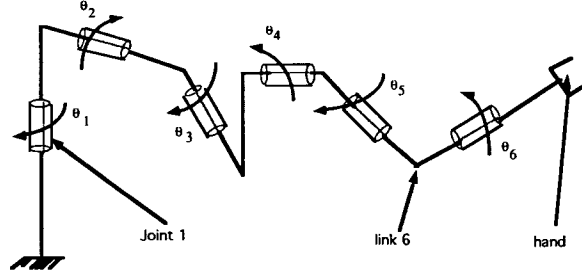


Figure 1.

This issue, known as the *inverse kinematics* problem, amounts to solving a polynomial system where the unknowns are the sines and cosines $\{s_i = \sin(\theta_i), c_i = \cos(\theta_i), i = 1, \dots, 6\}$ of the six joint angles $\{\theta_i, i = 1, \dots, 6\}$. The solution of such systems, for general robots, is quite involved, as remarked in the next example. But for robots of particular geometry the solution can be easier to achieve. For instance, if the robot is constructed so that the last three joint axes intersect at one point, the corresponding system essentially simplifies, since the robot has a sort of wrist (represented by a point where the three joints coincide, in Figure 2) that takes care of tip's orientation.

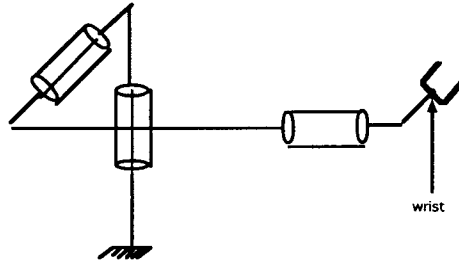


Figure 2.

Thus, instead of six unknown angles we are reduced to finding the first three (to position the wrist). The well known book of Craig (1989), contains a detailed exposition of this particular case. There it is shown that the system solution for the third joint angle can be expressed as

$$\frac{(r - k_3)^2}{4a_1^2} + \frac{(z - k_4)^2}{\sin(\alpha_1)^2} = k_1^2 + k_2^2$$

where k_1, k_2, k_3, k_4 are linear functions of s_3 and c_3 , a_1, α_1 are parameters describing robot's geometry (such as the length of the links or the relative angles between two consecutive joints) and z, r are some input data for the tip position. This gives, in general,

a fourth degree equation *determining* the angle θ_3 , since a second degree equation on $\sin(\theta)$ can be verified for up to four different values of θ . Without entering into details, it also happens that this kinematic system allows θ_1 and θ_2 to be linearly solved from θ_3 (the system contains equations, linear in sine and cosine of θ_2 , with coefficients polynomials in θ_3 ; and the same for θ_1 , that can be linearly expressed as a function of θ_2 and θ_3). What we want to emphasize here is that, because of this linearity, solving such system is essentially reduced to solving just one second degree sine-cosine equation.

Moreover it has been observed that imposing some geometric features on a robot of this kind (i.e. requiring that it is constructed so that the first three joints verify some relative to each other specific position) yields that the determining degree-four equation decomposes into two degree-two equations. For instance, when the first two joint axis intersect, i.e. if the robot parameter $a_1 = 0$, (see Figure 3), then we get a quadratic equation in θ_3 . See also Example 5.2 and Smith and Lipkin (1990) for a precise analysis.

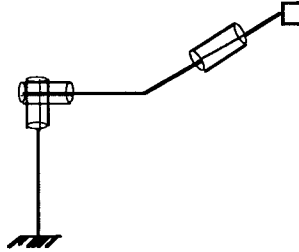


Figure 3.

EXAMPLE 1.2. After decades of research, a symbolic solution (though not in closed form) for the general $6R$ manipulator inverse kinematics system has been found (see Lee and Liang (1988a), (1988b), and Raghavan and Roth (1989)). By a clever elimination method it turns out that in this system θ_3 can be determined as the solution of a degree-16 polynomial in the tangent of $\theta_3/2$; then θ_1 and θ_2 are found by solving a system of sine-cosine polynomials, linear in these trigonometric functions, with coefficients in θ_3 . Of course, the *determining* degree-16 polynomial can be also expressed as a degree 8 polynomial in the sine and cosine of θ_3 . A mixed symbolic-numeric strategy for solving the $6R$ systems is presented in Canny and Manocha (1994), see below further comments on this. It has been already remarked (cf. Kovács and Hommel (1990)) that it could simplify greatly its solution if the determining equation could be solved by a sequence of lower degree equations. \square

EXAMPLE 1.3. The inverse kinematics problem of the robot ROMIN (see Gonzalez-Lopez and Recio (1993)) can be solved by many different methods, but it is specifically interesting since it is one of the few examples in which a new “lazy evaluation method” for solving systems of equations, the dynamic evaluation procedure (Duval (1990)), has been used.

Given a position (a, b, c) of the tip point P and the length of the links m, n (see Figure 4), the algebraic kinematic equations of the ROMIN are:

$$\begin{aligned} -s_1(mc_2 + nc_3) &= a, \\ c_1(mc_2 + nc_3) &= b, \end{aligned}$$

plus the trigonometric identities: $ms_2 + ns_3 = c$,
 $s_1^2 + c_1^2 = 1, s_2^2 + c_2^2 = 1, s_3^2 + c_3^2 = 1$.

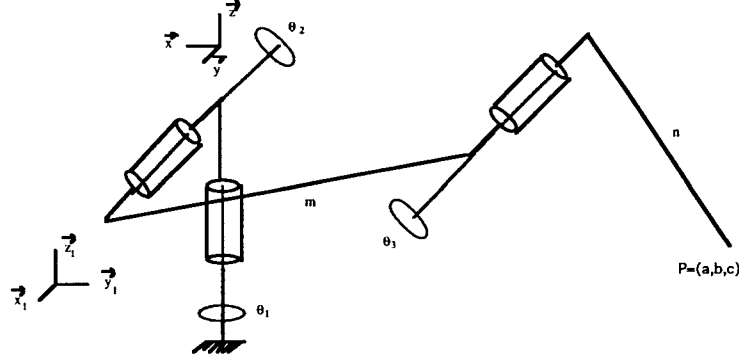


Figure 4.

Triangulation of this system yields:

$$\begin{aligned} & [ac + amc_2 + (b^2 + a^2)s_1 - am s_2, -cb - mbc_2 + mbs_2 + (b^2 + a^2)c_1, \\ & -n^2 + b^2 + a^2 - 2cmc_2 - m^2c_2^2 + n^2s_3^2 + 2m^2s_2c_2, ms_2 + nc_3 - c, \\ & -b^2 - a^2 + c^2 + 2cmc_2 + m^2s_2^2 + m^2c_2^2 - 2m^2s_2c_2 - 2cms_2]. \end{aligned}$$

Now, the last equation of this system, only in the angle θ_2 and the input parameters, can be rewritten as $g(h(s_2, c_2))$, where

$$g(x) = -a^2 + c^2 - b^2 + 2cmx + m^2x^2, \quad h(s_2, c_2) = c_2 - s_2.$$

This reduces solving a degree-two, sine-cosine equation, to an ordinary, degree-two, univariate polynomial equation $g = 0$, plus a linear, sine-cosine, equation $h = \rho$, for each root $g(\rho) = 0$. \square

Summarizing, in all the above examples, a polynomial system in the different joint angles is presented, describing the inverse kinematics problem of a whole robot class or of one concrete manipulator. The system unknowns are the sines and cosines of the joint angles, and they have to be solved as a function of the parameters describing the location of the robot hand. Roughly speaking, the solution is found by triangulating the system, i.e. deriving a sequence of equations such that the first one contains just one joint angle variable (the *determining* equation) and such that each of the following equations contains exactly one joint variable more than the preceeding ones. Replacing the solutions of the determining equation for the first variable in the second equation allows to find the solutions for the second variable, and so on. After this triangulation procedure, it is usually the case that the complexity of solving the system is concentrated just in solving the determining equation, since it has the highest degree. Thus, it is of primordial interest to simplify, when possible, such univariate sine-cosine equation.

Usually, the determining equation has coefficients that depend on parameters of two sorts: some correspond to the robot class under consideration (length of links, twist angles between joints, offset distances) while others describe generically the position of the end effector or hand (*pose* parameters). Therefore, the natural goal is to analyze symbolically

this equation, finding relations among the robot class parameters such that, when satisfied, the determining equation for the joint variables can be easily solved for any position of the end effector. For instance, in the Example 1.1. above, the fact that the wrist's three axes intersect is expressed by making zero some robot class parameters, yielding that the determining equation has lower degree (four) than in the totally general $6R$ case (sixteen). Of course, one wants to proceed in the other direction: i.e. detecting, first, potential simplifications of the general equation and, then, finding geometric conditions leading to them. This kind of analysis could lead to the design of industrially interesting robots, since the availability of simple methods to solve the determining equations is a typical requirement in practical situations.

But even working with one concrete robot (such as in Example 1.3, giving m, n specific values), in which class parameters have assigned numerical values, one is still interested in the symbolic manipulation of the determining equation. In fact, then its coefficients involve the pose parameters and it could be the case that the equation $f(s, c) = 0$ factorizes or decomposes symbolically, i.e. that there are lower degree polynomials $g(x)$ and $h(s, c)$, such that $f(s, c) = g(h(s, c))$ modulo $s^2 + c^2 - 1$, for all values of these parameters. Then the roots of $f = 0$ will be the roots of $h = \rho$, for all roots ρ of $g = 0$. In this situation we believe that the numerical approach for solving $f = 0$ benefits substantially from reducing its degree, even at the cost of increasing the number of equations to be solved. For instance, it seems better to solve four, degree-4 equations, or one of degree 8 and eight of degree two, than a single one of degree 16. Roughly speaking, finding all roots of a degree- n equation has a time complexity of about n^2 operations with any standard procedure. If the equation decomposes into a sequence of composition factors of degree, say, n_1, n_2, \dots, n_r , such that $n_1 n_2 \dots n_r = n$, will give, instead, a $n_1(n_1 + n_2(n_2 + \dots + n_{(r-1)}(n_{(r-1)} + n_r^2) \dots))$ complexity, applying iteratively the above solving procedure. In a balanced situation, in which every factor is approximately of "the r -th root of n " degree, the cost is bounded by $r n n^{\frac{1}{r}} = r n^{\frac{r+1}{r}}$.

It must be reckoned that this last conclusion does not take into consideration the problem of numerical stability or numerical conditioning of the involved equations. It seems hard to decide whether a well conditioned equation could turn, by performing some decomposition or other kind of simplification procedures, into solving lower degree, but poorly conditioned ones. In Canny and Manocha (1994), an efficient symbolic-numeric method for solving the general $6R$ manipulator is presented that converts root finding procedures into eigenvalue computations of numerical companion matrices. It has the advantage that the numerical approach to eigenvalues is well understood and that fast algorithms are available. We ignore if there is an operation on the companion matrices that corresponds to the decomposition of the determining equation. Nevertheless, it must be said that our aim is to study sine-cosine equations in full generality, and not just those that appear in robotics. Moreover, even in this case, we are more interested in the symbolic simplification as a way to guide robot design than on the efficient solution of the determining equation of a specific robot, after replacing the class and pose parameters by numerical values, as in Canny and Manocha (1994). Still, we think that automatically finding all possible simplifications for a general $6R$ problem is a challenging, non-trivial task for the algorithms we will propose.

Therefore, in the following we will concentrate on symbolic methods that reduce, by different means, the solution of sine-cosine polynomial equations to (perhaps) several ones of lower degree. The immediate antecedent of our work is the series of recent

books and papers Kovács and Hommel (1990), Kovács and Hommel (1992), Kovács and Hommel (1993), Kovács and Hommel (1993), Kovács (1991), Kovács (1993). As they do, we will highlight two kinds of possible simplification procedures: factorization and decomposition.

1.2. FACTORIZATION VERSUS DECOMPOSITION

Probably the more natural approach to simplification is that of factoring a given s-c equation f modulo $s^2 + c^2 - 1$. Although $K[s, c]/(s^2 + c^2 - 1)$ is not a unique factorization domain, we can still look for lower degree factors of f . As a by product of our work in the half angle tangent substitution, we are able to present (see section 3) a complete factorization algorithm for sine-cosine polynomials over fields that do not contain the square root of -1 (as compared to Kovács and Hommel (1990), Kovács and Hommel (1992) where only necessary conditions are given). If instead of working with one equation we deal with a system of equations, such as in inverse kinematics, the concept of factoring has to be generalized. The corresponding notion in commutative algebra is to decompose the ideal generated by the polynomials in the system into primary components (see Atiyah and MacDonald (1969)). If multiplicity of solutions is not the main concern in solving the system, we can go further, considering the prime ideals associated to the primary components as the counterpart to the irreducible factors of the one-equation case. In this way, the solutions of the given system can be obtained as the union of the zero sets of the prime ideals (like the roots of $f = 0$ are the union of all zeroes of the irreducible factors of f). Of course, if the given system already generates a prime ideal, then no simplification can be attained by this procedure.

Moreover, it is often the case that only real solutions are relevant (such as in robotics; here “real” has a concrete meaning assuming the coefficients of the equations are included in the real field). In this case one should consider, first, the ideal of all polynomials vanishing over the set of real solutions of the given system. Such ideal is called the *real radical* of the system (see the basic reference of Bochnak, Coste and Roy (1986) concerning ideals of polynomial equations over the reals). Then this real radical should be decomposed into real prime ideals. Again, the real zero set of the given system would be, then, the union of the real zero set of these primes. There are algorithms to perform all these computations (Becker and Neuhaus (1993)). Conceptually speaking, this is the simplest possible way of describing the real zeroes of a system by means of prime ideals: it is the analogous to throwing away, in some equation, those real irreducible factors that do not have any real root, retaining only linear factors.

The reason we do not enter into the details of the approach, is that it was conjectured in Kovács (1991) and shown in Gonzalez-Lopez and Recio (1994), Gonzalez-Lopez and Recio (1993) that neither prime decomposition nor real radication consideration will provide essential simplifications to the kinematic equations arising from most general categories of robots ($6R$, Stewart platform, etc.). Even worst, the same happens to any specialized version (i.e. giving numerical values to the class parameters) of these classes. In other words, the ideals generated by inverse kinematic equations are already prime and real radical, both considered with numerical coefficients (i.e. evaluating the class parameters) or in a purely symbolic setting. In fact, it is reasonable to expect that ideals corresponding to generic robots are unsimplifiable: for instance, a similar statement in the context of bivariate homogeneous decomposition (see below) appears in von zur Gathen

and Weiss (1995). But the remarkable property here is that, for whatever numerical values, the specialized ideal remains also unsimplifiable.

Therefore we can say that, at least in robotics, factorization does not play an important role towards simplification, although it could be so in the many other instances in which sine/cosine polynomials are involved.

On the other hand, as pointed out in the above examples, for specific values of the robot geometrical parameters it is possible to attain functional decomposability. Kovács (1993) presented a collection of well documented applications of this approach to concrete robots, and we direct the reader there in order to have a panorama of the power of this tool. Roughly speaking, a function $f(x, y)$ can be called *decomposable* if there is some polynomial $g(z)$ in a new variable z and some other function $h(x, y)$, such that $f(x, y) = g(h(x, y))$. The natural notion of decomposability for s - c polynomials $f(s, c)$ states, therefore, the existence of a standard polynomial $g(x)$ and of another s - c polynomial $h(s, c)$, such that $f(s, c) = g(h(s, c))$ modulo $s^2 + c^2 - 1$. As in the case of factorization, we look for composition factors which are simpler than the given polynomial (see section 5 for precise definitions). Advanced methods for the decomposition of ordinary multivariate polynomials and rational functions (see Gutierrez (1991), Alonso, Gutierrez and Recio (1995a)) cannot be directly applied to kinematics, as shown in the next example.

EXAMPLE 1.4. The polynomial $f(x, y) = -63y^2 + 60yx - 8y - 20x + 78$ can not be written as the composition of two polynomials $g(x)$ and $h(x, y)$ such that: $f(x, y) = g(h(x, y))$, but $f(s, c) = g(h(s, c))$ modulo $s^2 + c^2 - 1$, where $g(x) = 3x^2 - 4x + 3$ and $h(s, c) = 2c + 5s$.

□

Therefore, s - c decomposability seems the correct notion to understand several simplification situations in robotics. It is not only that this kind of decomposition yields simplification, but also that it goes the other way.

EXAMPLE 1.5. Given a general second degree s - c polynomial:

$$f(s, c) = A_{11}c^2 + 2A_{12}cs + 2A_{13}c + 2A_{22}s^2 + A_{33}.$$

We obtain its normal form:

$$NF(f(s, c)) = Ac^2 + Bcs + Cc + Ds + E$$

where $A = A_{11} - A_{22}$, $B = 2A_{12}$, $C = 2A_{13}$, $D = 2A_{23}$, $E = A_{33} + A_{22}$.

Then, Smith-Lipkin condition (see Duffly and Lipkin (1985), Smith and Lipkin (1990) and recall notions of Example 1.1.) for the geometric simplification of the 6R manipulator with the three last axes intersecting is that the coefficients of the its second-degree s - c determining equation satisfy

$$2CDA - B(C^2 - D^2) = 0.$$

And it is straightforward to see that this is equivalent to the condition for decomposability of the given s - c polynomial, i.e. we can find coefficients M, N, T, L, R, Q such that:

$$Ac^2 + Bcs + Cc + Ds + E = M(Lc + Rs + Q)^2 + N(Lc + Rs + Q) + T$$

modulo $s^2 + c^2 - 1$, if and only if

$$2CDA - B(C^2 - D^2) = 0. \square$$

The idea of considering algorithms for the s - c decomposition problem has already been studied in the work of Kovács and Hommel (1992) and Kovács and Hommel (1993), but their algorithms require an exponential number of field operations in the input degree; even if their last paper reduces the complexity by magnitudes and the authors state that it satisfies all needs in kinematics, it is still exponential. We also must mention in this context the recent work of von zur Gathen and Weiss (1995), about Bivariate Homogeneous Decomposition (BHD): a homogenous bivariate decomposition of a univariate polynomial $f(t)$ is of the form $f(t) = g(h(t), k(t))$ with polynomials $g(x, y)$, $h(t)$, $k(t)$, where $g(x, y)$ is a bivariate and homogenous. The authors present an algorithm for finding such decompositions, but it is also of exponential time complexity in the input degree.

Such BHD decompositions are of interest in kinematics, since s - c polynomials can be converted, via the tangent half angle substitution, into a t -polynomial (see section 3 for definitions and notation), where t is the tangent of $\theta/2$. Now, suppose that a quartic monic polynomial $F(t)$ has a bivariate homogeneous decomposition:

$$F(t) = G(H(t), J(t)),$$

with $G(x, y)$, $H(t)$, $J(t)$ quadratic polynomials. This allows us to find the four roots of $F(t)$ by factoring $G(x, y)$ as $G(x, y) = (x - \alpha_1 y)(x - \alpha_2 y)$, and then finding the two roots of $H(t) - \alpha_i J(t)$, for each $i \in 1, 2$. So, in this case we have reduced the problem of computing the roots of one quartic polynomial to computing roots of three quadratic polynomials. It is easy to see that if a s - c polynomial $f(s, c)$ is decomposable, then the associated univariate t -polynomial $T(f)$ has a bivariate homogeneous decomposition, but not conversely. Thus it could seem, in principle, that BHD decomposition is a finer tool in robotics than s - c decomposition. Nevertheless, there is a serious limitation for efficient robotic applications to t -polynomials of degree bigger than six, because the BHD decomposition algorithm requires factorization procedures over algebraic extensions of the field $K(t)$. On the other hand, we do not know concrete examples in robotics, where the determining equation has a BHD decomposition, but not a decomposition in the s - c sense.

The s - c decomposition method we will present in section 5 has a low polynomial time complexity in the input degree; therefore we can decompose, easily, s - c polynomials of degree 16 with a small machine such as a Macintosh Centris (see below subsection 5.4). As compared with the previous algorithms Kovács and Hommel (1992), Kovács and Hommel (1993), our procedure does not require factorizing polynomials; instead, the more difficult step is solving a linear system of equations. Moreover, if one allows enlarging the coefficient field (searching for the “irrational” decompositions, in the terminology of Kovács and Hommel (1993)) our method proceeds exactly as in the simpler case. These results have already been announced at the P_{RO}Motion (Planning Robot Motion) workshop, see Recio (1994).

1.3. GRÖBNER BASIS AND MINIMAL POLYNOMIAL

There are in this paper two other contributions to the simplification of sine-cosine polynomials. First of all, since the work of Buchberger (1989), it has been of theoretical

interest the use of Gröbner basis algorithms (Cox, Little and O'Shea (1992) for a survey on basic facts on Gröbner basis) and methods in order to obtain the triangulation of the collection of kinematic equations with respect to the set of joint variables (therefore, in theory, allowing the solution of the inverse kinematics problem). It is also well known that the complexity (in terms of time but also in terms of the size of the involved coefficients) for computing such triangular basis is usually quite high and prevents the use of this method in most practical situations. We have been able to find (see section 2) specific formulae that describe the Gröbner basis, for the pure lexicographic ordering, of the system:

$$\begin{cases} f(s, c) = 0, \\ s^2 + c^2 - 1 = 0. \end{cases}$$

Such basis is described in terms of the coefficients of $f(s, c)$ and is valid over any field. In particular, the basis gives (when the ordering $s > c$ is selected, but it will be similar otherwise) the minimal polynomial satisfied by $\cos(\theta)$, and –in general– the (linear in the variable s) equation giving, for every value of $\cos(\theta)$ that is a root of the minimal equation, the value of $\sin(\theta)$.

The interest of having such explicit formulae for solving the given equation is (obviously, appart from the fact that one does not need anymore to perform the Gröbner basis or resultant computation) twofold:

- there is, a priori, a control on the size of the coefficients of the Gröbner basis; in particular, for the minimal cosine polynomial, they are bounded by the square of the given coefficients of the sine-cosine polynomial. It is just in the s -linear equation where coefficient size grows, but following a well studied pattern in computer algebra (the size of coefficients in the extended GCD algorithm), see Loos (1982), Gonzalez-Vega (1989).
- there is a possibility of simplifying (factoring, decomposing) the minimal cosine polynomial, even when the given s - c polynomial does not allow such simplification. See Examples 2.2 and 2.3. Clearly, the tools developed in section 2 apply to solving s - c equations, see subsection 2.2, example 2.5.

1.4. EXTRANEOUS FACTORS

A classical way of dealing with sine-cosine equations is to introduce the substitution $\sin(\theta) = \frac{2t}{1+t^2}$, $\cos(\theta) = \frac{1-t^2}{1+t^2}$, where t is the tangent of $\theta/2$, solving for t the resulting rational expression. It turns out that, sometimes, a power of $1+t^2$ can be cancelled out in this expression. This seems irrelevant when dealing with just one equation, but it is not so when we make such substitution in a system of equations (as in the elimination process to solve the general 6R): the possibility of cancelling a factor of this form might appear at later stages of the elimination procedure, or it can give rise to “false” (i.e. *extraneous*) roots in the determining equation for a different variable. Looking for values of the robot class parameters such that the evaluated system has extraneous factors is a way to determine conditions that yield simpler systems (by cancelling factors out). Our work here explores simplification methods linked with the half angle tangent substitution (existence of solution to the so called positive and negative control systems) as introduced in Kovács and Hommel (1993). In that paper it is analyzed how to detect a priori, i.e. before performing the substitution and before performing any elimination procedure, by

means of the above controls, the presence of extraneous factors. We will show how our results on Gröbner basis gives a better conceptual insight into this problem and also some actual improvements, see section 4. Moreover, in subsection 2.3, the classical issue of *cocircularity* (see Mourrain (1996)) is related to the existence of extraneous factors.

2. Gröbner basis and minimal polynomial of a s-c polynomial

2.1. MINIMAL POLYNOMIAL

Let $f(s, c)$ be a sine-cosine polynomial with coefficients in a field K . We will choose to write $f(s, c)$ in *normal* or canonical form: i.e. replacing s^2 by $(1 - c^2)$ as much as possible. The result is, then, a polynomial of the form $A + Bs$, where A and B are polynomials in c only. We remark that if the total degree (as a two-variable polynomial) of $f(s, c)$ is n , then there are up to $2n$ values of the angle θ (when properly counted) satisfying the equation.

Given a s-c polynomial $f(s, c)$ of normal form, $A + Bs$, let us consider the monic polynomial on the variable c only, of minimum degree, contained in the ideal I generated by $(f(s, c), s^2 + c^2 - 1)$. It is clear that this polynomial appears in the Gröbner basis of the ideal I with respect to the lex ordering with $s > c$, since otherwise it could not be reduced to zero. On the other hand, this polynomial is not exactly the resultant of $A + Bs$ and $s^2 + c^2 - 1$ with respect to s . In fact, it is easy to see that the resultant is $A^2 - (1 - c^2)B^2$; for instance, $\text{Resultant}_s(c^2 + sc, s^2 + c^2 - 1) = c^2(2c^2 - 1)$, but $c(2c^2 - 1)$ is in the ideal and has lower degree.

PROPOSITION 2.1. *The minimum degree univariate polynomial in the variable c , contained in the ideal $I = (f(s, c), s^2 + c^2 - 1)$, is the monic polynomial associated to $P = G(A'^2 - (1 - c^2)B'^2)$, where G is the greatest common divisor of A, B in $K[c]$, $A' = \frac{A}{G}$ and $B' = \frac{B}{G}$.*

PROOF. Is clear that P belongs I , since $I = (A + Bs, s^2 + c^2 - 1)$, $P = G(A' + sB')(A' - sB')$ modulo $s^2 + c^2 - 1$, and the product of the first two factors of the last expression gives $A + Bs$. Now suppose that Q is a polynomial only in c , belonging to the ideal. Then Q is a combination of $A + Bs = G(A' + B's)$ and of $s^2 + c^2 - 1$, say $Q = L(s, c)G(A' + B's) + M(s, c)(s^2 + c^2 - 1)$. Next, we express $L(s, c)$ in normal form as $C + Ds$. Thus,

$$Q = G(A'C + (1 - c^2)B'D + s(A'D + B'C)) + M'(s, c)(s^2 + c^2 - 1)$$

after collecting multiples of $s^2 + c^2 - 1$ in a new polynomial M' . Due to the uniqueness of normal forms we conclude that $A'D + B'C = 0$ and $Q = G(A'C + (1 - c^2)B'D)$. Next suppose A' and B' are not zero. Then $D = -\frac{B'C}{A'}$ and, being A' prime with B' , it must divide C . Replacing this value of D in $Q = G(A'C + (1 - c^2)B'D)$ we get $Q = G(A'C - \frac{(1 - c^2)B'^2C}{A'})$. Call $H = \frac{C}{A'}$ (a polynomial, since division is exact here). Finally we obtain $Q = G(A'^2H - (1 - c^2)B'^2H)$ and, therefore, Q is a multiple of P and this is the minimal degree polynomial. On the other hand, if A (or B) is zero, then $G = B$ (respectively, $G = A$), $A' = 0$ (respec. $B' = 0$), and $B' = 1$ (respec. $A' = 1$). Then, when $A' = 0$, we get from $A'D + B'C = 0$ that $C = 0$ (as B is then not zero). Thus $Q = G(1 - c^2)B'D$, which is, again, a multiple of P (when A' is zero). If B' is zero, then $D = 0$, and $Q = GA'C$, also multiple of P when B is zero. \square

It follows that this minimal polynomial has coefficients of size, roughly, as the square of the coefficients in the given s - c polynomial, $f(s, c)$. Moreover we remark that the above proof yields a similar but slightly modified conclusion (since not every polynomial is associated with a monic one), when considering $A + Bs$ with coefficients in a unique factorization domain, such as a polynomial ring, say, $A + Bs \in \mathbb{Q}[X_1, \dots, X_n][s, c]$.

EXAMPLE 2.1. Take, over $\mathbb{Q}[d][s, c]$ the polynomial $A + Bs$ where $A = c - 5, B = \frac{3}{5}d$. Then the minimal polynomial is obtained directly by elimination (using some symbolic computation package) as the only generator of the ideal $(A + Bs, s^2 + c^2 - 1) \cap \mathbb{Q}[d][c]$:

$$\text{Ideal}(d^2c^2 - d^2 + \frac{25}{9}c^2 - \frac{250}{9}c + \frac{625}{9});$$

Now we check that it agrees with our expected result. First, we see that $\gcd(A, B)$ is 1 and then we compute

$$P(c) = A^2 + (1 - c^2)B^2 = -\frac{9}{55}d^2c^2 + \frac{9}{25}d^2 + c^2 - 10c + 25$$

which coincides with the previous polynomial up to a constant factor. \square

EXAMPLE 2.2. On the other hand, this polynomial $P(c)$ may be is “easy” to simplify while $f(s, c)$ is not. Let us take $f(s, c) = 2c^2 + 3c - 2sc - 7s + 1$. We can check, with the methods of sections 3 and 5, that it is irreducible and indecomposable modulo $s^2 + c^2 - 1$. But the minimal polynomial $P(c) = 4c^4 + 20c^3 + 29c^2 - 11c - 24$ can be factorized over the rational numbers:

$$P(c) = (c + 1)(4c^3 + 16c^2 + 13c - 24). \square$$

EXAMPLE 2.3. Now we take $f(s, c) = c^6 + c^4 - 2c^3s + 1$, that is irreducible and indecomposable modulo $s^2 + c^2 - 1$ (using again the techniques of sections 3 and 5). But the minimal polynomial $P(c) = c^{12} + 2c^{10} + 5c^8 - 2c^6 + 2c^4 + 1$ can be decomposed as:

$$d^6 + 2d^5 + 5d^4 - 2d^3 + 2d^2 + 1, d = c^2. \square$$

2.2. GRÖBNER BASIS

In this section we want to compute a Gröbner basis, using the lexicographic order with $s > c$, of the ideal $I = (f(s, c), s^2 + c^2 - 1)$, for a given sine-cosine polynomial $f(s, c) \in K[s, c]$. As in Proposition 2.1., let $A + Bs$ be the normal form of f and let G be the greatest common divisor of A, B in $K[c]$, $A' = \frac{A}{G}$, $B' = \frac{B}{G}$ and $P(c) = G(A'^2 - (1 - c^2)B'^2)$. Moreover, let M, N be the cofactors of A, B in the extended \gcd computation, i.e. such that $MA + NB = G$. Denote by $L(s, c) = sG + NA + MB(1 - c^2)$. Then:

PROPOSITION 2.2. *A Gröbner basis of $I = (f(s, c), s^2 + c^2 - 1)$ is $\{P(c), L(s, c), s^2 + c^2 - 1\}$, if $G \neq 1$; otherwise it is just $\{P(c), L(s, c)\}$.*

PROOF. First we will show that L is in the ideal I . In fact, in the given ideal it holds that $A = -sB$, and thus:

$$NA = -sNB \text{ and } sMA = -s^2MB \text{ modulo } s^2 + c^2 - 1.$$

Adding the two equalities, we get that $sG + NA + (1 - c^2)MB$ is in the ideal. We have already checked, in Proposition 2.1, that $P(c)$ is also there. Next we prove that the leading monomial of every polynomial in I is generated by the leading monomials of the proposed base: $\{s^2, \text{lt}(P(c)), s \cdot \text{lt}(G(c))\}$ (where lt indicates the leading monomial with

respect the lexical ordering) if $G \neq 1$; and by $\{\text{lt}(P(c)), s \cdot \text{lt}(G(c))\}$, otherwise. In the first case, if a polynomial $g(s, c)$ is in the ideal and has a monomial involving s to a power greater or equal than two, clearly it is a multiple of s^2 ; if it has no monomials in s , then it must be a multiple of the minimum polynomial $P(c)$; finally, if it is of the form $g = sR(c) + Q(c)$, then $sR + Q$ must be a multiple of $A + Bs$, modulo $(s^2 + c^2 - 1)$. Thus $sR + Q = (C + Ds)(A + Bs)$ modulo $(s^2 + c^2 - 1)$, and by the uniqueness of canonical forms, it follows that $R = AD + BC$, so it must be a multiple of the $\gcd(A, B)$, i.e. of G . The case $G = 1$ is trivial. \square

Notice that the above basis can be not reduced. For instance, the reduced Gröbner basis of $(c + 1, s^2 + c^2 - 1)$ is $\{P(c) = c + 1, s^2\}$, but our computation yields $\{P = c + 1, L = s(c + 1), s^2 + c^2 - 1\}$. In general, this occurs only in quite special simple cases and the reduced base is easy to obtain. It must be also remarked that the size (degree, length of coefficients) of the so called Bezout coefficients, M and N , are bounded by well known expressions (polynomial in the size of A and B , see Loos (1982), Gonzalez-Vega (1989)). The following example shows the apparently uncontrolled coefficient growth when computing the Gröbner base of the ideal of a s - c polynomial.

EXAMPLE 2.4. Consider the numerical s - c polynomial $f(s, c) = -177749s - 806874c + 1362294c^2 - 926688c^3 - 31867c^4 + 414950c^5 - 237970c^6 + 54210c^7 - 4216c^8 - 2688c^7s + 5655c^6s + 96696c^5s - 557135c^4s + 1264056c^3s - 1438004c^2s + 809864cs + 176343$.

Here $G = 1$. Using the lex ordering $s > c$, the Gröbner base of the ideal $(f(s, c), s^2 + c^2 - 1)$, computed directly by Maple, is:

$$\begin{aligned} & [8960484792403227914520620347912751702649098085405193090369s - \\ & 2167575234857741070651125343593466791525520566904959412159683 + \\ & 11925514418075970523023979176438424373661553324630394807760142c - \\ & 3834673857320878379743098402916105974896269992952124653414213c^2 - \\ & 152366386329875047709842890575128803576581647127989533607438180c^3 + \\ & 664134269628435066636515307207119107647992765824715482222407767c^4 - \\ & 1569664522659950342027442761475369976394963224765110445040189560c^5 + \\ & 2511991719469051437241684365098866462365145865432926213382755055c^6 - \\ & 2929470292353050068265825368822154445420671610603075312869874440c^7 + \\ & 2567382700752785146217081820671985813028114322749517572866287250c^8 - \\ & 1709770986600820233777532775393297583275293649586213030717469500c^9 + \\ & 863622566611078340412619638718259063061293447519152955661075625c^{10} - \\ & 326270525085033751863962707769682461230113379313042967314687500c^{11} + \\ & 89514624550553249487563931502735111683577332769224144785546875c^{12} - \\ & 16879644546590135526268815260825560065264521109707592445000000c^{13} + \\ & 1960983930449447913557380244499793639053790340965970375000000c^{14} - \\ & 106106416655237515520572167269768379560042953491524000000000c^{15}, \\ & -497853352 + 3331868708c - 3984863927c^2 - 34589411352c^3 + 193791269772c^4 - \\ & 533397801792c^5 + 976942396828c^6 - 1302962510900c^7 + 1315151818514c^8 - 1021659798700c^9 + \\ & 613378624075c^{10} - 282939548500c^{11} + 98628515625c^{12} - 25180375000c^{13} + 4450203125c^{14} - \\ & 487500000c^{15} + 25000000c^{16}]. \end{aligned}$$

Of course, the first polynomial corresponds to L and the second one to the minimal polynomial P . \square

In general, given a numerical s - c polynomial $f(s, c) = A + Bs$, the system $\{f = 0, s^2 + c^2 - 1 = 0\}$ has twice as many solutions (properly counted) as the degree of f . A common way of solving such system is to rewrite $s = \sin(\theta)$ and $c = \cos(\theta)$ as rational functions of the tangent of the half angle $\theta/2$, and to consider the univariate polynomial in the numerator of the resulting expression. This implies a lot of computations and some extra problems due to the potential cancellation of factors (see section 4). It is easier to solve the system using the polynomials in its Gröbner basis. Roughly, the idea is as follows. First, we notice that the equation $A + Bs$ can be considered as a curve in the s - c plane. This curve decomposes as a product of lines parallel to the s -axis (see Figure 5) and this product is equal to the $G = \gcd(A, B)$. For each line, the intersection with the unit circle $s^2 + c^2 - 1$ yields two values of s . Removing the common factor G in $A + Bs$ gives a curve which intersects the unit circle in some points, all having for every different c -coordinate, just one value of the s -coordinate. Thus, the corresponding value of s can be linearly solved.

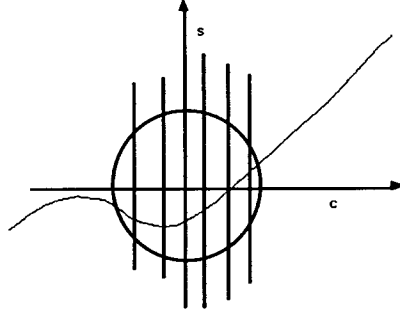


Figure 5.

Formally speaking, we see that the minimal polynomial $P(c) = G(A'^2 - (1 - c^2)B'^2)$ has degree equal $2\deg(f) - \deg(G)$. For each root ρ of $P(c) = 0$ such that $G(\rho) \neq 0$, $B(\rho)$ must be also different from zero, since $B(\rho) = 0$ and $P(\rho) = 0$ imply $A(\rho) = 0$ and thus, having ρ as a common root of A and B , $G(\rho)$ should be zero. So if $G(\rho) \neq 0$, the value of s can be obtained from the equation $A(\rho) + B(\rho)s = 0$, that gives only one value of s . It is important to remark that such values of s and c automatically verify $s^2 + c^2 - 1 = 0$, since the following identity holds:

$$P = GB'^2(s^2 + c^2 - 1) + G(A' + B's)(A' - B's). \quad (2.1)$$

In this way we can obtain $2(\deg(f) - \deg(G))$ roots of the system. To get the value of s corresponding to ρ , alternatively, we can solve $L(s, \rho) = 0$ directly for s , since

$$L(s, c) = sG + NA + MB(1 - c^2) = G(s + NA' + MB'(1 - c^2)),$$

and thus, when $G(\rho) \neq 0$,

$$s = (\rho^2 - 1)M(\rho)B'(\rho) - N(\rho)A'(\rho).$$

We claim that solutions $P(\rho) = 0$, with $G(\rho) \neq 0$, and the corresponding value for the sine $s = (\rho^2 - 1)M(\rho)B'(\rho) - N(\rho)A'(\rho)$, automatically verify both $A + sB = 0$ and $s^2 + c^2 - 1 = 0$. In fact,

$$A + Bs = A + B(-(1 - c^2)MB' - NA') = A + (-(1 - c^2)MB'^2G - NA'B'G).$$

Now, using that $P(\rho) = 0$, we replace $-(1 - c^2)MB'^2G$ by $-MA'^2G$ in the last equality, finally getting

$$A + Bs = A + (-MA'^2G - NA'B'G) = -A'G(-1 + A'M + B'N) = 0.$$

The claim follows using both this expression and identity (2.1) above.

On the other hand, when $G(\rho) = 0$, we obtain two values of s solving, directly, the equation $(s^2 + c^2 - 1) = 0$. Thus we find, in this way, the remaining $2\deg(G)$ values of the angle θ verifying the system.

EXAMPLE 2.5. Let us take the numerical s - c polynomial $f(s, c) = c^6 - 10c^4 + c^5s - 12c^3s + 25c^2 + 35sc + 3c^3 - 15c + 3c^2 - 21s = A + sB = c^6 - 10c^4 + 25c^2 + 3c^3 - 15c + s(c^5 - 12c^3 + 35c + 3c^2 - 21)$.

We have to compute $G = \text{GCD}(A, B) = c^3 - 5c + 3$, so the minimal polynomial is $P(c) = G(A'^2 - (1 - c^2)B'^2) = (c^3 - 5c + 3)(2c^6 - 25c^4 + 88c^2 - 49)$.

The zeroes of this polynomial, such that $G \neq 0$, are:

$$\begin{aligned} & -2.439730614 - 0.2075778378\sqrt{-1}, \\ & -2.439730614 + 0.2075778378\sqrt{-1}, \\ & -0.8255944410, \\ & 0.8255944410, \\ & 2.439730614 - 0.2075778378\sqrt{-1}, \\ & 2.439730614 + 0.2075778378\sqrt{-1}. \end{aligned}$$

For each zero, the value of s is obtained from $f(s, c) = A + sB = 0$:

$$\begin{aligned} & (0.2273749746 - 2.227306968\sqrt{-1}, -2.439730614 - 0.2075778378\sqrt{-1}), \\ & (0.2273749746 + 2.227306968\sqrt{-1}, -2.439730614 + 0.2075778378\sqrt{-1}), \\ & (0.5642639624, -0.8255944410), \\ & (-0.5642639614, 0.8255944410), \\ & (-0.2273750280 - 2.227306978\sqrt{-1}, 2.439730614 - 0.2075778378\sqrt{-1}), \\ & (-0.2273750280 + 2.227306978\sqrt{-1}, 2.439730614 + 0.2075778378\sqrt{-1}). \end{aligned}$$

The roots of the polynomial $G(c) = 0$ are:

$$-2.490863615, 0.6566204310, 1.834243184.$$

For each root, the two values of s are obtained from $s^2 + c^2 - 1$:

$$\begin{aligned} & (-2.281315750\sqrt{-1}, -2.490863615), (2.281315750\sqrt{-1}, -2.490863615), \\ & (-0.7542211941, 0.6566204310), (0.7542211941, 0.6566204310), \\ & (-1.537676188\sqrt{-1}, 1.834243184), (1.537676188\sqrt{-1}, 1.834243184). \end{aligned}$$

Thus, we have found $2 \cdot \deg(f) = 2 \cdot 6 = 12$ solutions in total, although there are only 4 real solutions. On the other hand, computing the polynomial associated to $f(s, c)$ in the tangent of the half angle substitution seems, clearly, much more involved.

2.3. PARAMETER SPECIALIZATION

It is often the case in kinematics that the coefficients of $f(s, c)$ are given in a domain with parametric coefficients, say $\mathbb{Q}[d_1, \dots, d_m]$. We write then $f(s, c) = f(d_1, \dots, d_m; s, c)$ to

remark this fact. Rather than solving the sine-cosine equation over some extension of the quotient field $\mathbb{Q}(d_1, \dots, d_m)$, one is interested in studying the solution of the specialized systems, i.e. those obtained by (partially) replacing the parameters $\{d_1, \dots, d_m\}$ for real numerical values $\{d_1^0, \dots, d_m^0\}$. As the previous paragraphs show, the structure of the minimal polynomial $P(c)$ is relevant for solving $f(s, c) = 0$. Unfortunately, Gröbner bases do not specialize well: it is not true, in general, that the specialization of the minimal polynomial for $f(d_1, \dots, d_m; s, c)$ agrees with the minimal polynomial of the specialized system $f(d_1^0, \dots, d_m^0; s, c)$. Still some analysis of this situation is possible.

It is clear that the canonical form $A(d_1, \dots, d_m; c) + sB(d_1, \dots, d_m; c)$ of $f(d_1, \dots, d_m; s, c)$ specializes to the canonical form of $f(d_1^0, \dots, d_m^0; s, c)$, since it is obtained rewriting $s^2 = 1 - c^2$. Next, let $G(d_1, \dots, d_m; c) = \gcd(A(d_1, \dots, d_m; c), B(d_1, \dots, d_m; c))$, where the \gcd is computed in $\mathbb{Q}(d_1, \dots, d_m)[c]$. Then:

$$G(d_1, \dots, d_m; c) \gcd(\text{cont}(A), \text{cont}(B)) = G'((d_1, \dots, d_m; c))$$

where cont denotes the content of a polynomial in c with coefficients in $\mathbb{Q}[d_1, \dots, d_m]$ and G' is the $\gcd(A, B) \in \mathbb{Q}[d_1, \dots, d_m][c]$. Now $G'(d_1, \dots, d_m; c)$ divides $A(d_1, \dots, d_m; c)$ and $B(d_1, \dots, d_m; c)$; therefore $G'(d_1^0, \dots, d_m^0; c)$ divides $A(d_1^0, \dots, d_m^0; c)$ and $B(d_1^0, \dots, d_m^0; c)$ when $A(d_1^0, \dots, d_m^0; c) + sB(d_1^0, \dots, d_m^0; c)$ is not zero (the interesting case), hence it divides the $\gcd(A(d_1^0, \dots, d_m^0; c), B(d_1^0, \dots, d_m^0; c))$. It follows that specializing the minimal polynomial for $A(d_1, \dots, d_m; c) + sB(d_1, \dots, d_m; c)$,

$$\frac{A^2(d_1^0, \dots, d_m^0; c) - (1 - c^2)B^2(d_1^0, \dots, d_m^0; c)}{G'(d_1^0, \dots, d_m^0; c)}$$

one gets just a multiple of the minimal polynomial of the specialized system

$$A(d_1^0, \dots, d_m^0; c) + sB(d_1^0, \dots, d_m^0; c).$$

It follows that the degree of this minimal polynomial can be lower than the one of the general case if and only if the numerator $A^2(d_1^0, \dots, d_m^0; c) - (1 - c^2)B^2(d_1^0, \dots, d_m^0; c)$ has lower degree or if $G'(d_1^0, \dots, d_m^0; c)$ strictly divides $\gcd(A(d_1^0, \dots, d_m^0; c), B(d_1^0, \dots, d_m^0; c))$. Let be n the degree of $A(d_1, \dots, d_m; c) + sB(d_1, \dots, d_m; c)$ as a polynomial in s, c and suppose that for some numerical values, the coefficients of the terms of highest degree in c of $A^2(d_1, \dots, d_m; c) - (1 - c^2)B^2(d_1, \dots, d_m; c)$ vanish. Then,

$$(\text{coeff}(c^n))^2 + (\text{coeff}(sc^{n-1}))^2 = 0 \quad (2.2)$$

where $\text{coeff}(c^n)$, etc ..., denotes the coefficient of the c^n term in $A + sB$, etc.... When we consider only real numerical values of the parameters, this condition is equivalent to

$$\text{coeff}(c^n) = 0 \quad \text{AND} \quad \text{coeff}(sc^{n-1}) = 0. \quad (2.3)$$

Obviously, this condition is equivalent to lowering the total degree of $A + sB$ as a s - c polynomial. If we allow complex values for the parameters, the condition (2.2) is equivalent to

$$\text{coeff}(c^n) + \sqrt{-1} \cdot \text{coeff}(sc^{n-1}) = 0, \quad (2.4)$$

OR

$$\text{coeff}(c^n) - \sqrt{-1} \cdot \text{coeff}(sc^{n-1}) = 0. \quad (2.5)$$

Therefore, under these conditions, the specialized system has lower degree than in the parametrized case.

The above results can be interpreted also from the point of view of cocircularity. As in Mourrain (1996) or Merlet and Lazard (1994), the *cocircularity* of a two-variable equation is the minimum of the multiplicity of the curve at the two cyclic points of infinity: i.e. the projective points $(1, \sqrt{-1}, 0), (1, -\sqrt{-1}, 0)$, that are the points at infinity of a projective circle. If we consider our parametrized equation $A + Bs$ as a curve in the variables s - c , the fact that for specific values of the parameters this curve passes through one of the cyclic points is exactly equivalent to the vanishing of one of the equations (2.4) or (2.5): thus (2.3) is the condition for having multiplicity at least one at both points. It is shown in the above mentioned papers that cocircularity lowers the degree of intersection of the curve $A + Bs$ with the curve $s^2 + c^2 - 1$: Mourrain (1996) shows that the “number of common points properly counted” (i.e. our solutions to the s - c equation) is less or equal than:

$$\deg(A + Bs) \cdot \deg(s^2 + c^2 - 1) - 2 \cdot \text{cocircularity}(A + Bs) \cdot \text{cocircularity}(s^2 + c^2 - 1).$$

Now, the cocircularity of the circle is always 1 and the cocircularity of the s - c curve is at least one if (2.4) and (2.5) simultaneously hold. Moreover, the formula above holds with equality if the two curves have no other common points at infinity, the multiplicity at cyclic points is the same for both and they are not tangent at the cyclic points. It is easy to see that both $A + Bs$ and $s^2 + c^2 - 1$ have no other common points at infinity; moreover the multiplicity of the circle at the cyclic points is one and with tangent $s = 0$ in the affine plane that contains such points with $c = 1$. Moreover, one can compute the multiplicity and tangents of $A + Bs$ at the cyclic points as a function of the different coefficients of this polynomial.

3. Factorization of s - c equations

In this section we deal with the problem of factoring (over an orderable field) a given sine-cosine polynomial $f(s, c)$, modulo $s^2 + c^2 - 1$. We will denote, sometimes, its normal form $A + Bs$ by $NF(f(s, c))$ or $NF(f)$. Since our aim is simplifying the solution of a sine-cosine equation, we might assume that the given polynomial is already in normal form and that no polynomials in c can be factored out from f (as this is trivially attained by an univariate *gcd* computation). Next we remark that $f = gh$ modulo $s^2 + c^2 - 1$, implies $f = NF(g)NF(h)$ modulo $s^2 + c^2 - 1$. Moreover, in this section *we deal with an orderable coefficient field such as \mathbb{Q} , $\mathbb{Q}(d_1, \dots, d_m)$, \mathbb{R} , etc ...* (more specifically, all we need is that -1 is not a square in the field), as it is the case in most applications. Then the equality $f = gh$ modulo $s^2 + c^2 - 1$, among normal form polynomials, implies $\deg(f) = \deg(g) + \deg(h)$ (see Lemma 3.1. below). Thus, factoring f over an orderable field essentially means finding polynomials $g(s, c), h(s, c)$, already in normal form and verifying $f = gh$ modulo $s^2 + c^2 - 1$, plus the conditions: $\deg(f) > \deg(g)$ and $\deg(f) > \deg(h)$, in order to avoid trivial factorizations.

3.1. THE DEFECT

In what follows it plays an important role the well known relation between the trigonometric functions sine and cosine of θ and the tangent of $\theta/2$. The parametrization:

$$s = \frac{2t}{1+t^2}, \quad c = \frac{1-t^2}{1+t^2} \quad (3.1)$$

covers, for finite values of t , the whole unit circle except the point $(-1, 0)$. Thus the values $s = 0$ or $c = -1$ have to be studied with some care.

DEFINITION 3.1. *The defect of an s - c polynomial, $def(f)$ is the maximum power of $(c+1)$ that divides $NF(f(s, c))$.*

Given a polynomial $f(s, c)$, after performing the above substitution of s and c by t -rational functions and clearing denominators we obtain a polynomial $T(f)$ in the variable t (the associated t -polynomial to $f(s, c)$). Then it is easy to prove the following facts:

- (i) $T(f)$ has no $(t^2 + 1)$ as a factor (by construction).
- (ii) The closest integer bigger or equal to $\frac{\deg(T(f))}{2} = \lceil \frac{\deg(T(f))}{2} \rceil$, is equal to

$$\deg(NF(f(s, c))) - def(f(s, c)).$$

That is:

$$2 \cdot \deg(NF(f)) - 2 \cdot def(f) = \deg(T(f))$$

or

$$2 \cdot \deg(NF(f)) - 2 \cdot def(f) - 1 = \deg(T(f)).$$

- (iii) The degree of $T(f)$ is odd if and only if $c + 1$ is a factor in A to a larger power than in B (for instance, if $A = 0$ and $B \neq 0$).

Conversely, if we start with a t -polynomial $T(f)$ without $(1 + t^2)$ factor, and we divide $T(f)$ by $(1 + t^2)$ to the power $\lceil \frac{\deg(T(f))}{2} \rceil$, and we perform the inverse substitution

$$t = \frac{1-c}{s}, \quad (3.2)$$

we obtain a s - c polynomial in normal form and without defect, such that the given $T(f)$ is the associated t -polynomial. Moreover, if we divide by $(1 + t^2)$ to the power $\lceil \frac{\deg(T(f))}{2} \rceil$ plus some natural number r , we obtain a s - c polynomial of defect exactly r . Thus there is a non-injective mapping from normal form s - c polynomials to t -polynomials not divisible by $(1 + t^2)$, since dividing the given s - c polynomial by a power of $(c + 1)$ (when possible) has no effect in the corresponding t -polynomial.

The following lemma will be very useful.

LEMMA 3.1. *Let f, g, h be normal form polynomials over an orderable field, such that $f = gh$, modulo $(s^2 + c^2 - 1)$. Then:*

- (i) $\deg(f) = \deg(g) + \deg(h)$. Therefore, the constants are the only multiplicative units in $K[s, c]/(s^2 + c^2 - 1)$.
- (ii) $T(f) = T(g)T(h)$.

- (iii) $\text{def}(f) = \text{def}(g) + \text{def}(h)$, except when both $T(g)$ and $T(h)$ are of odd degree, and in this case, $\text{def}(f) = \text{def}(g) + \text{def}(h) + 1$.

PROOF. (i) Assume

$$g = g_n c^n + g_{n-1} c^{n-1} s + \dots,$$

$$h = h_m c^m + h_{m-1} c^{m-1} s + \dots,$$

are normal forms of degree n, m , respectively, where g_n, g_{n-1} represent the coefficients of c^n and $c^{n-1}s$ in g , and so on. Then, the normal form of gh is $(g_n h_m - g_{n-1} h_{m-1}) c^{n+m} + (g_n h_{m-1} + g_{n-1} h_m) c^{n+m-1} s + \dots$. Now we observe that cancelling both coefficients of total degree $n + m$ implies

$$g_n h_m - g_{n-1} h_{m-1} = 0, g_n h_{m-1} + g_{n-1} h_m = 0.$$

Since we have assumed our polynomials to be of degree n, m , neither both g_n, g_{n-1} nor h_m, h_{m-1} can be zero. But the homogeneous system (in the variables h_m, h_{m-1}) has $g_n^2 + g_{n-1}^2$ as determinant. Then this system has no non-zero solution over a field where -1 is not a square. Contradiction.

(ii) In fact, performing the substitution of (3.1) in $f = gh$, modulo $(s^2 + c^2 - 1)$, we get that the product of $T(g)$ and $T(h)$ does not divide $(1 + t^2)$, since the ground field does not contain i ; therefore, the product of the numerators $(T(g), T(h))$ and denominators (powers of $1 + t^2$) of the irreducible rational fractions associated to g and to h gives already an irreducible fraction, ie. with numerator equal to $T(f)$.

(iii) This is easy, considering the above two items and the equalities linking the degree of a polynomial, its defect and the degree of the associated t -polynomial. \square

The equality $1 = (\sqrt{-1}c - s)(\sqrt{-1}c + s)$ modulo $s^2 + c^2 - 1$ shows that the above lemma fails if the coefficient field contains $\text{sqr}t{-1}$. Even so, it is easy to observe that these units are, essentially, the only ones in such cases. This remark allows us to extend, with some modifications, the factorization procedure below to arbitrary fields.

3.2. FACTORIZATION

As stated in the introduction to this section, in order to factor over an orderable field a given s - c polynomial, modulo $s^2 + c^2 - 1$, we can assume that the given polynomial is in normal form and has no c -factors, in particular, that it has no defect. Moreover, we only look for factors that are also in normal form.

PROPOSITION 3.1. *Under the above conditions, if there are normal form polynomials g, h such that $f = gh$, modulo $(s^2 + c^2 - 1)$, and $\text{deg}(f) > \text{deg}(g), \text{deg}(f) > \text{deg}(h)$, then $T(f) = T(g)T(h)$ and $\text{deg}(T(f)) > \text{deg}(T(g)), \text{deg}(T(f)) > \text{deg}(T(h))$ (i.e. $T(f)$ is not irreducible as a univariate polynomial in t). Moreover, if $\text{deg}(T(f))$ is even, then it can not happen that $\text{deg}(T(g)), \text{deg}(T(h))$ are both odd.*

PROOF. Let $f = A + Bs, g = C + Ds, h = M + Ns$. Then $A = CM + DN(1 - c^2)$ and $B = CN + DM$, by uniqueness of canonical forms. It follows that also g and h have not defect, since if, say, g has defect, then C and D will be divisible by $(1 + c)$ and so will A and B , by the above relations. Considering the associated t -polynomials

$T(f), T(g), T(h)$, we now by the above lemma that $T(f) = T(g)T(h)$ and thus that $\deg(T(f)) = \deg(T(g)) + \deg(T(h))$. But $\deg(T(f)) = 2\deg(f)$ or $\deg(T(f)) = 2\deg(f) - 1$, and the same alternative holds for the other polynomials.

Now, if $\deg(T(f)) = 2\deg(f)$, it could happen that $\deg(T(g))$ and $\deg(T(h))$ are, respectively $2\deg(g)$, $2\deg(h)$ or $2\deg(g) - 1$ and $2\deg(h) - 1$. In the first case we easily conclude that $\deg(f) > \deg(g), \deg(f) > \deg(h)$ implies $\deg(T(f)) > \deg(T(g)), \deg(T(f)) > \deg(T(h))$ and we are done. If both $\deg(T(g)), \deg(T(h))$ are odd, then by lemma 3.1., the defect of f cannot be zero (since it is at least 1 more than the sum of the defects of g and h), against the assumption that f has no defect.

If $\deg(T(f)) = 2\deg(f) - 1$, then we must have, say, $\deg(T(g)) = 2\deg(g) - 1$ and $\deg(T(h)) = 2\deg(h)$. Again, $\deg(T(f)) > \deg(T(g)), \deg(T(f)) > \deg(T(h))$ since $\deg(T(f)) \neq \deg(T(h))$ for parity reasons and $\deg(T(f)) > \deg(T(g))$ because $\deg(f) > \deg(g)$. \square

PROPOSITION 3.2. *Conversely, the existence of a proper factorization of $T(f) = GH$, allows to recover a proper factorization of f , except when $\deg(T(f))$ is even and both $\deg(G), \deg(H)$ are odd.*

PROOF. We must distinguish two cases:

(a) All $\deg(T(f)), \deg(G), \deg(H)$ are even.

Here $\deg(f) = \frac{\deg(T(f))}{2}$. Since $\deg(T(f)) = \deg(G) + \deg(H)$, dividing $T(f)$ by

$$(1 + t^2)^{\frac{\deg(T(f))}{2}}$$

is the same as dividing by

$$(1 + t^2)^{\frac{\deg(G)}{2}} (1 + t^2)^{\frac{\deg(H)}{2}}.$$

Thus G and H are converted, by the inverse substitution, into normal form, defectless factors g, h of f . Let us show that they verify $\deg(f) > \deg(g), \deg(f) > \deg(h)$. We have that $\deg(G)$ and $\deg(H)$ are, respectively, equal to $2\deg(g), 2\deg(h)$. Since the factorization of $T(f)$ is proper, $\deg(T(f)) > \deg(G), \deg(H)$, and this directly yields $\deg(f) > \deg(g), \deg(f) > \deg(h)$.

(b) degree of Tf is odd.

In this case the degrees of G and H must be odd and even or conversely. Assume the first is odd. Dividing $T(f)$ by:

$$(1 + t^2)^{\frac{\deg(T(f))+1}{2}}$$

is the same as dividing by:

$$(1 + t^2)^{\frac{\deg(G)+1}{2}} (1 + t^2)^{\frac{\deg(H)}{2}}.$$

Thus G and H are converted, by the inverse substitution, into defectless factors g, h of f . Let us show that they verify $\deg(f) > \deg(g), \deg(f) > \deg(h)$. We have that $\deg(T(f)), \deg(G)$ and $\deg(H)$ are, respectively, equal to $2\deg(f) - 1, 2\deg(g) - 1, 2\deg(h)$. Since $\deg(T(f)) > \deg(G), \deg(H)$, it directly yields $\deg(f) > \deg(g), \deg(f) > \deg(h)$. \square

We have seen that the existence of a factorization of f into lower degree factors is equivalent to the existence of a factorization of $T(f)$ into lower degree factors, not both of odd

degree. Moreover, if $T(f)$ has only a factorization into two odd degree polynomials G, H , then it follows that f has no factorization. Still a simplification can be attained in some cases. We must divide Tf by $(1+t^2)^{\frac{\deg(T(f))+2}{2}}$, i.e. by $(1+t^2)^{\frac{\deg(G)+1}{2}}(1+t^2)^{\frac{\deg(H)+1}{2}}$ to obtain a factorization of $(c+1)f(s, c)$ via the s - c polynomials g, h associated to G and H . Remark none of these factors will be a multiple of $c+1$, because there exists no defect in the associated s - c polynomials to G and H . Since f can not be factorized, the best we can hope is to factorize $(c+1)f$. Moreover, because of the odd degrees of G and H , we see that the two factors g, h are of the form: $(c+1)^k X(c) + sY(c)$ and $Y(-1) \neq 0$. Therefore both have the root $c = -1, s = 0$ and all the remaining roots will be roots of $f(s, c) = 0$. Thus solving $f = 0$ can be replaced by solving $gh = 0$. In this case $\deg(f) = \deg(g) + \deg(h) - 1$. If degree of, say H , is one, the factorization yields no real advantage for solving the s - c equation, since $\deg(h) = 1, \deg(g) = \deg(f)$.

EXAMPLE 3.1. We consider the following irreducible polynomial $f(s, c) \in Q[s, c]$:

$$-3/2c^3 - 7/2sc^2 + 7/4c^2 - 5sc + 9/2c - s + 5/4.$$

After performing the tangent half angle substitution (3.1) and clearing denominators, we obtain the associated t -polynomial $T(f)$:

$$t^5 - 7t^4 + 10t^3 + 11t^2 - 19t + 6.$$

Now, we factor $T(f) = GH$, where

$$G = t^2 - 5t + 3, \quad H = t^3 - 2t^2 - 3t + 2.$$

We consider the rational functions (case b):

$$\frac{G}{1+t^2}, \quad \frac{H}{(1+t^2)^2}$$

and we perform the inverse substitution (3.2) to these rational functions, yielding:

$$m(s, c) = c - 5/2s + 2, \quad n(s, c) = c^2 + c - sc - 1/2s.$$

We finally obtain a factorization modulo the circle:

$$f(s, c) = (c - 5/2s + 2)(c^2 + c - sc - 1/2s) + (-5/2c - 5/4)(s^2 + c^2 - 1). \quad \square$$

EXAMPLE 3.2. Now, we consider the following polynomial $f(s, c)$:

$$6c^4 - 36sc^3 - 24c^3 + 52c^2 - 104sc^2 - 92sc + 56c + 6 - 24s.$$

The associated t -polynomial $T(f)$ is:

$$32t^8 - 32t^4 + 96t^3 - 160t^6 + 160t^2 - 512t + 32t^5 + 96.$$

Now, we factor $T(f) = GH$, where

$$G = 4t^3 - 20t + 4, \quad H = 24 - 8t + 8t^5.$$

The degrees of G and H are both odd and there is no other factorization, so the best we can hope is to factorize a multiple of $f(s, c)$ of the form $(c + 1)f(s, c)$. In the same way, we have to consider the rational functions :

$$\frac{G}{(1+t^2)^2}, \quad \frac{H}{(1+t^2)^3}$$

and we perform the inverse substitution (3.2) to these rational functions, yielding:

$$m(s, c) = c^2 - 6sc + 2c - 4s + 1, \quad n(s, c) = 3c^3 + 9c^2 + 9c - 4sc + 3.$$

We finally obtain a factorization of $(c + 1)f(s, c)$ modulo the circle:

$$f(s, c)(c+1) = 2(c^2 - 6sc + 2c - 4s + 1)(3c^3 + 9c^2 + 9c - 4sc + 3) + (-48c^2 - 32c)(s^2 + c^2 - 1).$$

4. Simplification by extraneous factors

4.1. CASE OF ONE EQUATION

Let us go back to the equations in subsection 2.3, (2.3), (2.4) and (2.5), involving some coefficients of a s - c polynomial in normal form $f(s, c) = A(c) + B(c)s$. As in that subsection, let us assume these coefficients are polynomials in several parameters (robot class and pose parameters, as explained in section 1), i.e. $A(c), B(c) \in Q[d_1, \dots, d_m][c]$. In 2.3. we stated conditions that the coefficients should verify in order to lower the degree of the minimal polynomial of the evaluated s - c polynomial. Here we are going to obtain a different interpretation in connection with the associated t polynomial, introduced in section 3.1. Suppose that for some numerical values of the geometric or robot class parameters, one of the above equations (say (2.4)) is identically zero for all pose parameters. Then one knows that for such concrete geometric parameters the s - c equation will have less solutions than expected. This implies that if we perform in the unevaluated s - c equation the half angle tangent substitution (see 3.1 (3.1)) and then we evaluate the parameters for the numerical values, some “extraneous” factor $(t + \sqrt{-1})$ or $(t - \sqrt{-1})$ occurs in the numerator of the evaluated expression, since the number of solutions is reduced. The conclusion is that there can be, as shown in the example below, simplifications without necessarily involving degree two extraneous factors (of course, this involves complex solutions of (2.4), but such solutions can appear quite naturally as in Kovács and Hommel (1993), example 2.1, where $d = 5\sqrt{-1}/3$). This possibility is somehow overlooked in previous work (see Kovács and Hommel (1993)), where such simplification is always linked with the existence of factors of the kind $1 + t^2$. This possibility seems, also, not regarded by Mourrain’s formula (see Mourrain (1996)), which always diminishes by even quantities the number of solutions of the s - c equation that are lost due to co-circularity contributions. Obviously, when we restrict to real solutions of (2.4), if there are any, we will have then an even number reduction of the minimal equation degree, since both leading coefficients of $A(c)$ and $B(c)$ will be zero for these values of the parameters. Therefore, in the real case, the contribution of 2.3.(2.4) is that $1 + t^2$ appears as a factor in the numerator after the t -substitution. A similar explanation arises from co-circularity conditions.

EXAMPLE 4.1. Consider the s - c polynomial: $f(s, c) = c^4d + 2c^4 - d^2sc^3 - 5dsc^3 - 6sc^3 - 2c^2 + c^3 - 5c + s - 3$;

The associated t -polynomial $T(f)$ is:

$$-7-4dt^2+6dt^4-4dt^6+dt^8+6d^2t^3-6d^2t^5+d-2d^2t-10dt-10t-32t^2-2t^4-8t^6+t^8+42t^3-30t^5+14t^7+2d^2t^7+30dt^3-30dt^5+10dt^7.$$

In general, the minimal polynomial of the system $\{f(s, c) = 0, s^2 + c^2 - 1 = 0\}$ has degree 8, $8 + 30c + 38c^2 + (2d^2 + 10d + 26)c^3 - (-6d - 18)c^4 - (-2d^2 - 20d - 36)c^5 - (-d^4 - 10d^3 - 37d^2 - 64d - 43)c^6 + (2d + 4)c^7 + (d^4 + 10d^3 + 38d^2 + 64d + 40)c^8$, but for some values of d , satisfying some of the equations (2.3), (2.4) and (2.5) the corresponding evaluated $f(d)$ has at most 7 zeroes. Solving each of these equations we obtain the values $d = -2$ (root of (2.3)), $d = -3 + \sqrt{-1}$ (root of (2.4)) and $d = -3 - \sqrt{-1}$ (root of (2.5)).

For instance, if we take $d = -3 + \sqrt{-1}$, then $f(-3 + \sqrt{-1}) = -c^4 + \sqrt{-1}c^4 + sc^3 + \sqrt{-1}sc^3 - 2c^2 + c^3 - 5c + s - 3$, and the minimal polynomial has only 7 zeroes: $8 + 30c + 38c^2 + 12c^3 - 2\sqrt{-1}c^3 - 6\sqrt{-1}c^4 + 8c^5 - 8\sqrt{-1}c^5 + 5c^6 - 6\sqrt{-1}c^6 - 2c^7 + 2\sqrt{-1}c^7$.

Moreover, it can be checked that this corresponds with the presence of a factor $(t + \sqrt{-1})$ in the evaluated t -polynomial, that factors as:

$$\left(\frac{-2+\sqrt{-1}}{5}\right)(5t^7 - 2t^6 - \sqrt{-1}t^6 - 13t^5 + 6\sqrt{-1}t^5 - 12t^4 - 11\sqrt{-1}t^4 + 35t^3 + 20\sqrt{-1}t^3 + 14t^2 - 23\sqrt{-1}t^2 + 13t + 14\sqrt{-1}t + 8 - 21\sqrt{-1})(t + \sqrt{-1}).$$

For $d = -2$, we have $f(-2) = c^3 - 2c^2 - 5c + s - 3$ and the minimal polynomial is: $8 + 30c + 38c^2 + 14c^3 - 6c^4 - 4c^5 + c^6$. Again, we check that this corresponds with the presence of a factor $1 + t^2$ in the t -polynomial: $-(1 + t^2)(t^6 - 2t^5 - t^4 - 4t^3 + 15t^2 - 2t + 9)$.

□

In summary, the two equations (2.4) and (2.5) constitute the positive and negative control equations of Kovács and Hommel (1993), but, contrary as stated there (section 3.1, fourth paragraph up from the end), both equations do not need to be satisfied simultaneously, but alternatively, in order to have a “simplification” of the degree of the resulting s - c polynomial, when the solutions of such equations are replaced in the given s - c polynomial. Of course the main objection to Kovács and Hommel (1993) and to our comments here is that no complex values of parameters are usually involved in robotic problems; therefore, control equations should be better replaced by the more natural system (2.3), which yields the same real roots.

In fact, this remark makes sense for one s - c equation with parametric coefficients. But if we extend this analysis to several s - c parametric equations, searching for conditions that lead to a simpler system solution, we will have to check whether there is a common root for all (say, positive) control equations (2.4) derived from each of the equations of the system. We will see that such common root, even if complex, is just an indicator of simplification and has not a physical interpretation in terms of the robot parameters.

4.2. CASE OF SEVERAL EQUATIONS

Let us consider this apparently innocent system:

$$\begin{aligned} \frac{3}{5}sd + c - 5 &= 0, \\ s - \frac{3}{5}cd + 2 &= 0. \end{aligned}$$

We see that the positive control system (2.4)

$$1 + \frac{3}{5}d\sqrt{-1} = 0,$$

$$-\frac{3}{5}d + \sqrt{-1} = 0,$$

for both equations has as common root $d = \frac{5\sqrt{-1}}{3}$ and that, analogously, the negative (2.5) control has as common solution $d = -\frac{5\sqrt{-1}}{3}$.

This implies that modulo $s^2 + c^2 - 1$, and for the value, say, $d = \frac{5\sqrt{-1}}{3}$, each of the equations of the given system yields a minimal cosine equation of degree one less than expected (it “should” have been (formally) of degree two, but it is of degree one), of course, with complex coefficients. But one does not need to evaluate d to obtain such simplification. In fact, the $f(s, c)$ equation obtained eliminating d in the ideal $(\frac{3}{5}sd + c - 5, s - \frac{3}{5}cd + 2, s^2 + c^2 - 1)$:

$$\begin{aligned} \text{Elim}(d, \text{Ideal}(\frac{3}{5}sd + c - 5, s - \frac{3}{5}cd + 2, s^2 + c^2 - 1)) = \\ \text{Ideal}(s - \frac{5}{2}c + \frac{1}{2}, c^2 - \frac{10}{29}c - \frac{3}{29}); \end{aligned}$$

has degree lower than expected: it “should” have been of degree two in s - c , therefore of degree four in c ... We remark that the complex value of d that is hidden behind this simplification does not imply that complex numbers or values of the parameter are involved in the usual solving of the above system, nor if we eliminate it with respect to “ d ”:

$$\begin{aligned} \text{Elim}(s..c, \text{Ideal}(\frac{3}{5}sd + c - 5, s - \frac{3}{5}cd + 2, s^2 + c^2 - 1)) = \\ \text{Ideal}(d^2 - \frac{700}{9}). \end{aligned}$$

Since this behaviour requires only the existence of solutions for the positive (or the negative) control system, and do not involve the solutions themselves, in the case of systems (as it is usual in robotics) where, besides the parameters, the coefficients only consist of real numbers, the satisfiability of a system 2.3.(2.4) implies –by conjugation– the same for the other system 2.3. (2.5), and thus we only need to check one of them.

This analysis can be also explained via the half angle tangent substitution as in Kovács and Hommel (1993). Essentially, it involves the following argument: the fact that both the positive and negative control systems have (separately) a common root for all equations of the given system in s - c , implies that both $t = \sqrt{-1}$ and $t = -\sqrt{-1}$ are roots of each of the equations of the associated t -system, after parameter evaluation in the corresponding common root (a root for the positive control and another one, perhaps, for the negative control system). Of course, the common roots of the control systems are, in general, complex values of the parameters. But it also implies that if we eliminate all the parameters in the associated t -system, a factor $(1+t^2)$ appears in the resulting equation in t alone, by well known properties of elimination ideals. This elimination makes the difference with the one variable case, since the elimination procedure does not yield complex coefficients. Thus, converting this t -equation back to the s - c form gives lower than expected degree, but no complex coefficients.

A similar argument can be made without the detour to the half angle substitution: we homogenize the given system with respect to the s - c variables and then look for some solutions at infinity in the circle $s^2 + c^2 - 1$. This implies looking for values of the parameters that satisfy the system for the values $(1, \sqrt{-1}, 0)$ and $(1, -\sqrt{-1}, 0)$ where the first coordinate is the c -value, the second is the s -value and the last one is the value of the homogenizing variable. If there are such values of the parameters, then we know that in the system there are roots of the s - c elimination equation which lie at infinity when intersected with the circle. Clearly, a system without this property would have more affine

roots and correspondingly a greater degree. Of course this is an argument at geometric level and we should be sure that the algebraic elimination reflects well the geometric properties for the geometric object defined by the system (for instance, that the given system defines a radical ideal). But the same prevention holds for the argument of Kovács and Hommel (1993): the fact that some roots $\sqrt{-1}$ and $-\sqrt{-1}$ appear as solution, does not immediately imply that they are eliminated if we merely apply algebraic elimination on the given system. Fortunately, there are results as in Gonzalez-Lopez and Recio (1994) that show that in general this is the case for robotic kinematic systems.

EXAMPLE 4.2. Consider the system of section 2.2 in Kovács and Hommel (1993):

$$I = \text{Ideal}(10s_2s_1 - 4dc_1 + 7,$$

$$4ds_1 + 5c_1 - 1,$$

$$s_2^2 - ds_1 - \frac{5}{2}s_2c_1,$$

$$s_1^2 + c_1^2 - 1);$$

$$\text{Elim}(d..c_2, I) =$$

$$\text{Ideal}(s_1^2 + c_1^2 - 1,$$

$$s_1c_1^3 - \frac{2}{7}c_1^4 - \frac{4}{5}s_1c_1^2 + \frac{29}{35}c_1^3 - \frac{21}{25}s_1c_1 + \frac{71}{175}c_1^2 - \frac{5}{7}c_1 - 73/175,$$

$$c_1^5 - \frac{296}{265}c_1^4 - \frac{791}{1325}s_1c_1^2 - \frac{1787}{1325}c_1^3 + \frac{1288}{6625}s_1c_1 + \frac{7641}{6625}c_1^2 +$$

$$\frac{511}{1325}s_1 + \frac{26}{53}c_1 - \frac{1533}{6625}).$$

We obtain a degree 5, c_1 polynomial, lower than the expected degree 6. Next we suppose we had not performed this elimination and we are going to discover beforehand this simplification property. Let us consider the equations in the system as polynomials in s_1c_1 , and let us write the following conditions for the vanishing of the highest terms in each equation, using, say, the control system $\text{coeff}(c_1^n) + \sqrt{-1}\text{coeff}(s_1c_1^{n-1}) = 0$

yielding the equations

$$J = \text{Ideal}(10\sqrt{-1}s_2 - 4d,$$

$$4\sqrt{-1}d + 5,$$

$$-\sqrt{-1}d - \frac{5}{2}s_2);$$

having as Gröbner basis:

$$[\sqrt{-1} - \frac{4}{5}d,$$

$$s_2 - 1/2,$$

$$d^2 + \frac{25}{16}].$$

Thus we see that the system has the solutions $d = \frac{\pm 5\sqrt{-1}}{4}$ and, therefore, that it will simplify.

The same behaviour appears considering the other control system (but we do not really need to check it):

$$J = \text{Ideal}(-10\sqrt{-1}s_2 - 4d,$$

$$-4\sqrt{-1}d + 5,$$

$$\sqrt{-1}d - \frac{5}{2}s_2);$$

Gröbner basis:

$$[\sqrt{-1} + \frac{4}{5}d,$$

$$\begin{bmatrix} s_2 - \frac{1}{2}, \\ d^2 + \frac{25}{16} \end{bmatrix};$$

Next let us do the same analysis for a slightly modified system, just replacing d by $d + 1$ in one of the equations:

$$\begin{aligned} I = & \text{Ideal}(10s_2s_1 - 4dc_1 - 4c_1 + 7, \\ & 4ds_1 + 5c_1 - 1, \\ & s_2^2 - ds_1 - \frac{5}{2}s_2c_1, \\ & s_1^2 + c_1^2 - 1); \end{aligned}$$

$$\text{Elim}(d, c_2, I) =$$

$$\begin{aligned} & \text{Ideal}(s_1^2 + c_1^2 - 1, \\ & s_1c_1^4 - \frac{51}{20}s_1c_1^3 + \frac{9}{50}c_1^4 + \frac{14}{25}s_1c_1^2 - \frac{33}{100}c_1^3 + \frac{147}{100}s_1c_1 - \frac{39}{100}c_1^2 + \\ & \frac{13}{100}c_1 + \frac{73}{100}, \\ & c_1^6 - \frac{51}{20}c_1^5 - \frac{129}{1000}s_1c_1^3 - \frac{1019}{2500}c_1^4 + \frac{1227}{2500}s_1c_1^2 + \frac{19803}{5000}c_1^3 + \\ & \frac{673}{5000}s_1c_1 - \frac{3151}{5000}c_1^2 - \frac{73}{100}s_1 - \frac{7233}{5000}c_1 + \frac{657}{5000}). \end{aligned}$$

Here the degree in c_1 is 6 and not 5 as in the previous system. Let us see what happens with the control system equations in this case:

$$\begin{aligned} J = & \text{Ideal}(-4(d+1) + 10s_2\sqrt{-1}, \\ & 5 + 4\sqrt{-1}d, \\ & -\frac{5}{2}s_2 - \sqrt{-1}d) \end{aligned}$$

$$\begin{aligned} J = & \text{Ideal}(10\sqrt{-1}s_2 - 4d - 4, \\ & 4\sqrt{-1}d + 5, \\ & -\sqrt{-1}d - \frac{5}{2}s_2); \end{aligned}$$

Gröbner basis:

$$\begin{bmatrix} 1 \end{bmatrix}.$$

Thus, in this case there is no common solution for the control system equation. \square

This analysis is called in Kovács and Hommel (1993) *the simplification method by detecting "extraneous" factors*, because the fact that a factor $(1 + t^2)$ appears in the converted system by means of the half angle substitution is "extraneous" to the affine solving of such system. The authors remark also that such factor, easy to identify directly, can be hidden if the system is solved in some other variables: i.e. if instead of solving with respect to s - c we solve with respect to some other joint variable, then it could happen that, again, some factors of the determining equation of this joint angle –looking absolutely different from $(1 + t^2)$ – are linked to the extraneous factor in s - c . This means that other values of, say, angle s_2 - c_2 (or length d of a prismatic joint), correspond to roots of the angle s - c at infinity, and should be therefore simplified. But this is a dangerous way of reasoning: such values could also be linked to other finite values of s - c , and therefore removing them we are losing correct solutions (configurations of the robot) of the system. Therefore we should be warned about eliminating simultaneously all the induced extraneous factors without previously checking about its consequences. See the example below.

In the above Example 4.2, we check that the solution $s_2 = \frac{1}{2}$ is, in fact, not extraneous since, besides appearing linked with the cyclic s_1 - c_1 points, we also have the (non extraneous) system solution:

$$d = \frac{5}{4}, \quad s_2 = \frac{1}{2}, \quad c_2 = \frac{\sqrt{3}}{2}, \quad s_1 = -\frac{3}{5}, \quad c_1 = \frac{4}{5}.$$

4.3. RABINOWISTCH'S TRICK

Finally, we must observe that directly applying the half angle substitution it is quite possible to eliminate extraneous roots by adding to the system the equation $(1+t^2)y-1$; this method (Rabinowistch's trick) is not much more costly than the direct elimination procedure: it was not studied in Kovács and Hommel (1993) since it was considered as complicated as obtaining general ideal quotients. Recent methods for deciding this specific problem appear in Alonso, Gutierrez and Recio (1995b) and Licciardi and Mora (1994), since it is linked with the implicitization problem of parametric curves and varieties. The next example shows the direct application of this observation.

EXAMPLE 4.3. Let us consider the system of the Example 4.2 :

$$I = \text{Ideal}(f_1 = 10s_2s_1 - 4dc_1 + 7,$$

$$f_2 = 4ds_1 + 5c_1 - 1,$$

$$f_3 = s_2^2 - ds_1 - \frac{5}{2}s_2c_1).$$

First s_1 and c_1 are converted into t -polynomials and we consider the system of the t -polynomials:

$$Tf_1 = 20s_2t - 4d + 4dt^2 + 7 + 7t^2,$$

$$Tf_2 = 8td + 4 - 6t^2,$$

$$Tf_3 = 2s_2^2 + 2s_2^2t^2 - 5s_2 + 5s_2t^2 - 4td.$$

We solve the above system:

$$\text{Gröbner basis } ([Tf_1, Tf_2, Tf_3], [d, s_2, t], \text{plex}) =$$

$$[8d - 9t^9 - 139t - 214t^3 - 28 - 336t^2 + 46t^5 + 70t^4 + 336t^6 + 122t^7 - 42t^8, -90 + 1936t^2 - 398t^4 + 287t + 1638t^3 + 350t^5 - 2108t^6 - 938t^7 + 276t^8 + 80s_2 + 63t^9, 4 + 133t^2 + 214t^4 + 28t + 336t^3 - 70t^5 - 46t^6 - 336t^7 - 122t^8 + 42t^9 + 9t^{10}].$$

The unique polynomial involving t in the above Gröbner basis is:

$$4 + 133t^2 + 214t^4 + 28t + 336t^3 - 70t^5 - 46t^6 - 336t^7 - 122t^8 + 42t^9 + 9t^{10}.$$

Factoring this polynomial, we get the extraneous factor $1+t^2$:

$$(3t+1)(t+2)(1+t^2)(3t^6+7t^5-62t^4+14t^3+37t^2+7t+2).$$

On the other hand, applying Rabinowitsch's trick, we have to compute the following Gröbner basis:

$$\text{Gröbner basis}([Tf_1, Tf_2, Tf_3, (1+t^2)y-1], [y, d, s_2, t], \text{plex}) =$$

$$[5000y - 4996 + 56t + 5321t^2 + 1155t^3 - 3080t^4 - 938t^5 + 303t^6 + 63t^7, 8d - 9t^7 - 135t - 85t^3 - 28 - 308t^2 + 131t^5 + 378t^4 - 42t^6,$$

$$80s_2 + 63t^7 - 82 + 1998t^2 - 2384t^4 + 315t + 1351t^3 - 1001t^5 + 276t^6, \\ 4 + 28t + 129t^2 + 308t^3 + 85t^4 - 378t^5 - 131t^6 + 42t^7 + 9t^8].$$

Now, the unique polynomial on t is:

$$4 + 28t + 129t^2 + 308t^3 + 85t^4 - 378t^5 - 131t^6 + 42t^7 + 9t^8.$$

Factoring this polynomial we have :

$$(3t + 1)(t + 2)(3t^6 + 7t^5 - 62t^4 + 14t^3 + 37t^2 + 7t + 2).$$

So, we have eliminated the extraneous factor $1 + t^2$.

5. Fast functional decomposition of s - c equations

5.1. FOUNDATIONS AND NOTATIONS

Given a sine-cosine polynomial $f(s, c)$, if it decomposes as $f(s, c) = g(h(s, c))$ modulo $s^2 + c^2 - 1$, for some univariate polynomial $g(x)$ and some bivariate polynomial $h(s, c)$, then it is clear that the same equality applies replacing f, h by its normal forms. Thus, in this section we will study decomposition procedures assuming f is given in normal form and we will look for normal form composition factors h . Since our goal is to simplify solving sine-cosine equations, we search for factors such that the degree of h is strictly smaller than the degree of f . As in section 3, we will assume that $\sqrt{-1} \notin K$, the coefficient field, to prevent some complications. Summarizing, the sine-cosine polynomial decomposition problem can be stated as follows:

DEFINITION 5.1. *Given a bivariate, normal form, s - c polynomial $f(s, c)$ in the polynomial ring $K[s, c]$, we will say that $f(s, c)$ is decomposable modulo the circle if there exist a univariate polynomial $g(x) \in K[x]$ and a bivariate normal form polynomial $h(s, c) \in K[s, c]$ with $\deg(h(s, c)) < \deg(f(s, c))$ such that:*

$$f(s, c) = g(h(s, c)) \text{ modulo } s^2 + c^2 - 1.$$

Therefore, the decomposition problem for a given f is to decide if such g, h exist and, in the affirmative case, to find them.

From a computational point of view, it is important to know what is the relevance of extending the coefficient field regarding the existence of decomposition. It is well known (see Gutierrez (1991)) that an ordinary bivariate polynomial $f(x, y) \in K[x, y]$ is indecomposable over a field K if and only if it is indecomposable over any extension of K . We label such property saying that ordinary polynomial decomposition is a rational problem. On this issue the ordinary rational function decomposition problem differs from ordinary polynomial decomposition (see Alonso, Gutierrez and Recio (1995a)) as well as the sine-cosine polynomial decomposition problem:

EXAMPLE 5.1. Let us consider the numerical s - c polynomial, $f(s, c) = 2c^2 + cs + 1$ with coefficients over the rational numbers field. We can check with the SCDECPOL Algorithm (see subsection 5.2.2), that it is indecomposable modulo the circle over the rational number field \mathbb{Q} . But it can be written as a composition, modulo the circle, of polynomials with coefficients over an algebraic extension \mathbb{Q} . In fact, take:

$$g(x) = \frac{x^2}{-4 + 2\sqrt{5}} + \frac{13 - 6\sqrt{5}}{4 - 2\sqrt{5}}, \quad h(s, c) = c + (-2 + \sqrt{5})s.$$

Then $f(s, c) = g(h(s, c))$ modulo $s^2 + c^2 - 1$. \square

This implies that enlarging the coefficient field might yield to finding new decompositions. We will go back to this issue in subsection 5.2.2, showing that our algorithm gives decompositions over any field extension of K where computations are possible (even if $\sqrt{-1}$ is in this larger field). Another important aspect that we must consider is the concept of equivalent decompositions. The idea is to consider as equivalent those decompositions that are related via the identity $x = (\frac{ax+b}{a} - b/a) = (x/a - b/a) \circ (ax + b)$ (where \circ denotes functional composition) for some $a \neq 0, b \in K$. Remark that, for all g, h as above, this identity implies

$$g(x) \circ h(s, c) = g(x) \circ x \circ h(s, c) = g(x) \circ (x/a - b/a) \circ (ax + b) \circ h(s, c)$$

and collecting the first and the last two (composition) factors in the last equality an apparently different decomposition arises.

DEFINITION 5.2. *Two decompositions of a s - c polynomial $f(s, c)$, say:*

$$f(s, c) = NF(g_1(h_1(s, c))) = NF(g_2(h_2(s, c)))$$

are called equivalent if there is a non constant linear transformation $u(x) = ax + b \in K[x]$ such that: $ah_1(s, c) + b = h_2(s, c)$ and $g_1(x/a - b/a) = g_2(x)$, ie. $g_2(x) = g_1(x) \circ (x/a - b/a)$ and $h_2(s, c) = (ax + b) \circ (h_1(s, c))$.

It is well known (see Gutierrez (1991)) that if an ordinary polynomial $f(x, y)$ has two decompositions $f(x, y) = g_1(h_1(x, y)) = g_2(h_2(x, y))$ with $\deg(h_1(x, y)) = \deg(h_2(x, y))$, then they are equivalent. Again, neither this result holds for the ordinary rational function decomposition (see Alonso, Gutierrez and Recio (1995a)), nor for the sine-cosine polynomial decomposition, as illustrated by the following example:

EXAMPLE 5.2. Let us consider the numerical s - c polynomial:

$$f(s, c) = 8c^5s - 8c^3s - 6cs - 12c^4 + 12c^2 + 1.$$

We have two decompositions, $f(s, c) = NF(g_i(h_i(s, c)))$, $i = 1, 2$, where $g_1(x) = 8 - 12x^2 + 6x^4 - x^6$, $h_1(s, c) = c + s$ and $g_2(x) = x^6$, $h_2(s, c) = c - s$. So, we have two decompositions of $f(s, c)$ with $\deg(h_1(s, c)) = \deg(h_2(s, c)) = 1$, but they are non-equivalent (by direct checking). \square

The most interesting decomposition in kinematics is when the composition factor $h(s, c)$ has the smallest possible degree. On the other hand, finding all decompositions may be interesting for different applications. For instance, our method can compute all non-equivalent decompositions, even irrational (in the sense of Example 5.1) decompositions for real coefficients sine-cosine polynomials.

EXAMPLE 5.3. The polynomial $f(s, c) = 8c^5s - 8c^3s - 6cs - 12c^4 + 12c^2 + 1$ has exactly the following non-equivalent decompositions $f(s, c) = NF(g_i(h_i(s, c)))$, $i = 1, 2, 3$, where

$$\begin{aligned}
[g_1(x) &= 8 - 12x^2 + 6x^4 - x^6, & h_1(s, c) &= c + s] \\
[g_1(x) &= x^6, & h_1(s, c) &= c - s] \\
[g_2(x) &= 1 - 6x + 12x^2 - 8x^3, & h_2(c, s) &= cs] \\
[g_3(x) &= 4x^2, & h_3(s, c) &= c^3 + c^2s - 3/2c + 1/2s]. \quad \square
\end{aligned}$$

Besides these general observations, some normal forms of special polynomials turn out to be fundamental in our approach.

DEFINITION 5.3. *We define by recurrence the following polynomials in some new indeterminate Z and with coefficients over the integer numbers:*

$$\begin{aligned}
A_0(Z) &= 1, B_0(Z) = 0 \\
A_m(Z) &= A_{m-1}(Z) - ZB_{m-1}(Z) \\
B_m(Z) &= ZA_{m-1}(Z) + B_{m-1}(Z). \quad \square
\end{aligned}$$

Now it is easy to prove the following basic properties:

LEMMA 5.1. *For every positive integer m , we have:*

- (i) $A_m(Z)^2 + B_m(Z)^2 = (1 + Z^2)^m$
- (ii) $A_m(Z) + iB_m(Z) = (1 + iZ)^m$, where $\sqrt{-1} = i$.
- (iii)

$$A_m(Z) = \frac{(1 + iZ)^m + (1 - iZ)^m}{2}.$$

$$B_m(Z) = \frac{(1 + iZ)^m - (1 - iZ)^m}{2i}.$$

- (iv) If m is an even natural number, then $\deg(A_m(Z)) = m$ and $\deg(B_m(Z)) = m - 1$. If m is an odd natural number, then $\deg(A_m(Z)) = m - 1$ and $\deg(B_m(Z)) = m$.
- (v) The rational functions $G_m(Z) = \frac{B_m(Z)}{A_m(Z)}$ are conjugate of power polynomials, that is: $G_m(Z) = w \circ Z^m \circ w^{-1}$ (symbol \circ for rational function composition), where w is the linear fraction $w = \frac{-iZ+i}{1+Z}$ and $w^{-1} = \frac{i-Z}{Z+i} = \frac{1+iZ}{1-iZ}$.
- (vi) All roots of the polynomials $A_m(Z) = 0$ and $B_m(Z) = 0$ are real. Moreover, they do not have a common root.

PROOF. The proof of the two first items can be done by induction on m . Item (iii) follows by solving the system given by (i) and (ii). Item (iv) is then immediate. Item (v) is immediate, the most difficult part is to remark the structure of this fraction, as stated. For item (vi), in the case of $A_m(Z)$, we consider $\frac{(1+iZ)^m}{(1-iZ)^m} = -1$ and then we remark that the conformal mapping $\frac{(1+iZ)}{(1-iZ)}$ sends points from the real line onto the unit circle in the plane $R^2 = C$. For B_m is analogous. Finally, item (i) implies that the imaginary unit is the only possible common root of A_m, B_m , but since all their roots are real they do not have common roots. \square

Next, we establish the relation between the normal form of a sine/cosine polynomial with the polynomials $A_m(Z)$ and $B_m(Z)$, defined above :

LEMMA 5.2. *Let Z_0 be an arbitrary element of an extension of the field K , m a positive integer number, then we have:*

- (i) $NF(c + Z_0 s)^m = A_m(Z_0)c^m + B_m(Z_0)c^{m-1}s + D(s, c)$ with $m > \deg(D(s, c))$.
- (ii) If m is odd, say $m = 2k + 1$, $NF(s^m) = (1 - c^2)^k s$.
- (iii) If m is even, say $m = 2k$, $NF(s^m) = (1 - c^2)^k$.

PROOF. It is straightforward, by induction on m , remarking that $NF(c + Z_0 s)^m = NF[(c + Z_0 s)NF(c + Z_0 s)^{m-1}]$. \square

One might deal in a unified way with the above items, by homogenizing the whole situation, i.e. showing that

$$NF(Y_0 c + Z_0 s)^m = A_m(Y_0, Z_0)c^m + B_m(Y_0, Z_0)c^{m-1}s + D(s, c)$$

with $m > \deg(D(s, c))$ etc. . . where Y_0, Z_0 are elements in some extension of K and where $A_m(Y, Z), B_m(Y, Z)$ represent the homogenization (as polynomials of formal degree m) of A_m, B_m via the new variable Y . That is, $A_m(Y, Z) = Y A_{m-1}(Y, Z) - Z B_{m-1}(Y, Z)$, $B_m(Y, Z) = Z A_{m-1}(Y, Z) + Y B_{m-1}(Y, Z)$. This way of thinking, although it makes notation more complicated, is particularly useful in the following

LEMMA 5.3. *Let $g(x) = g_t x^t + \dots + g_0$, a polynomial of degree t and let $h(s, c) = h_r c^r + h_{r-1} c^{r-1} s + \dots$ a normal form s - c polynomial of degree r . Then $NF(g(x) \circ h(s, c))$ has degree rt .*

PROOF. Clearly, the higher degree terms of $NF(g(x) \circ h(s, c))$ are the higher degree terms of the normal form of $(h_r c^r + h_{r-1} c^{r-1} s)^t = c^{(r-1)t} (h_r c + h_{r-1} s)^t$. By the Lemma above, the normal form of $(h_r c + h_{r-1} s)^t$ is $A_r(h_r, h_{r-1})c^t + B_r(h_r, h_{r-1})c^{t-1}s + D(s, c)$ with $t > \deg(D(s, c))$. But $A_r(h_r, h_{r-1}), B_r(h_r, h_{r-1})$ can not be simultaneously zero. In fact $A_r(Y_0, Z_0)^2 + B_r(Y_0, Z_0)^2 = (Y_0^2 + Z_0^2)^r$ (homogeneous version of Lemma 5.1.), thus if $A_r(h_r, h_{r-1}) = 0, B_r(h_r, h_{r-1}) = 0$ then $(h_r^2 + h_{r-1}^2) = 0$. But there are not non trivial zeroes of A_r, B_r verifying this condition (see 5.1.). \square

LEMMA 5.4. (i) $NF(g_1(x) \circ h(s, c)) = NF(g_2(x) \circ h(s, c))$ implies $g_1 = g_2$ (right cancellation property)

(ii) Two decompositions $f(s, c) = NF(g_1(h_1(s, c))) = NF(g_2(h_2(s, c)))$ are equivalent if and only if $K[h_1] = K[h_2]$ as subalgebras of $K[s, c]/(s^2 + c^2 - 1)$.

PROOF. Item (i) proceeds by considering that $NF(g_1(x) \circ h(s, c)) = NF(g_2(x) \circ h(s, c))$ implies $NF((g_1(x) - g_2(x)) \circ h(s, c)) = 0$. Then, by lemma 5.3. $(g_1(x) - g_2(x))$ must have degree zero, and it follows it must be zero. For item (ii), if the two decompositions are equivalent, by definition, it is trivial that the two subalgebras are equal. Conversely, if $K[h_1] = K[h_2]$, then there polynomials $p_1(x), p_2(x)$ such that $h_1 = p_1(h_2), h_2 = p_2(h_1)$ modulo $s^2 + c^2 - 1$. Then $h_1 = (p_1 \circ p_2)(h_1)$ and $h_2 = (p_2 \circ p_1)(h_2)$ in the circle. This implies, by the Lemma, that the degree of both $p_1 \circ p_2$ and $p_2 \circ p_1$ is one. Therefore p_1, p_2 are linear and inverse to each other. \square

5.2. THE DECOMPOSITION ALGORITHM

The general techniques developed for solving decomposition problems all tend to divide it into two parts: given the s - c normal form polynomial $f(s, c)$, then

- first, to compute “right”, normal form, candidates $h(s, c)$ such there is a decomposition $f(s, c) = g(h(s, c))$, modulo $s^2 + c^2 - 1$,
- second, to determine the corresponding “left” component $g(x)$.
- finally, to verify that $f(s, c) = g(h(s, c))$ modulo $s^2 + c^2 - 1$.

As always the hard part is finding $h(s, c)$, because to determine $g(x)$ from $f(s, c)$ and $h(s, c)$ the most direct way is to explicitly solve a linear system of equations in the indeterminate coefficients of $g(x)$ (with unique solution if it exists, because of the right cancellation property). This works since for a given h the degree of the potential g 's is bounded; therefore we will not detail this point.

NOTATIONS 5.1. Let K be a field that does not contain $\sqrt{-1}$. Let $f(s, c) \in K[s, c]$ be a s - c polynomial in normal form:

$$f(s, c) = F_n(s, c) + F_{n-1}(s, c) + \dots + F_1(s, c) + F_0(s, c)$$

where $F_i(s, c)$ are homogenous polynomials of degree i of the form

$$F_i(s, c) = f_{i,0}c^i + f_{i-1,1}c^{i-1}s$$

and, where $f_{i,j} \in K$, $F_n(s, c) \neq 0$ and n is the total degree of the polynomial $f(s, c)$. Now, after the Lemma 5.3., for every positive r divisor of n , (so there exists a positive integer t with $n = rt$), we consider the possible, normal form, candidates $h(s, c)$ of degree r :

$$h(s, c) = H_r(s, c) + H_{r-1}(s, c) + \dots + H_1(s, c) + H_0(s, c),$$

where $H_i(s, c) = h_{i,0}c^i + h_{i-1,1}c^{i-1}s$ and $H_r(s, c) \neq 0$.

We would like to distinguish two possible cases:

- (a) $h_{r,0} \neq 0$.
- (b) $h_{r,0} = 0$.

In the first one we can restrict to decompositions of $f(s, c)$ such that

$$h_{r,0} = 1 \text{ and } H_0 = 0.$$

In fact, if $f(s, c) = NF(g(h(s, c)))$, we can consider the linear transformation $l(x) = h_{r,0}x + h_{0,0}$ and we can define: $g(l(x)) = g'(x)$, $h'(s, c) = l^{-1}(h(s, c))$, where $l^{-1}(x) = h_{r,0}^{-1}x - h_{r,0}^{-1}h_{0,0}$, yielding an equivalent decomposition of the $f(s, c)$ with:

$$h'(s, c) = c^r + Zc^{r-1}s + H'_{r-1}(s, c) + \dots + H'_1(s, c)$$

where $H'_i(s, c) = H_i(s, c)h_{r,0}^{-1}$ and $Z = h_{r-1,1}h_{r,0}^{-1}$. We will say that $h'(s, c)$ is a *zero symmetric, monic polynomial on c of degree r* .

Likewise, in the second case, we consider decompositions of $f(s, c)$ such that the right factor $h'(s, c)$ is a *zero symmetric, monic polynomial on c and s* of degree r .

Finally, to include both cases, we will say that the normal form polynomial $h(s, c)$ is *normed* if and only if it is zero symmetric and monic polynomial on c or on c and s . \square

5.2.1. DETERMINING CANDIDATES

Let $f(s, c) \in K[s, c]$ be a normal form s - c polynomial of total degree n :

$$f(s, c) = F_n(s, c) + F_{n-1}(s, c) + \dots + F_1(s, c) + F_0(s, c)$$

where $F_i(s, c)$ as above. Let r be a divisor of n , so $n = rt$, and we are looking for a normed polynomial $h(s, c)$ of degree r and for a univariate polynomial $g(x)$ of degree t such that $f(s, c) = g(h(s, c))$ modulo $s^2 + c^2 - 1$. So, we have to compute the coefficients $h_{i,j}$ and g_k where,

$$g(X) = g_t X^t + g_{t-1} X^{t-1} + \dots + g_1 X + g_0$$

so that the following equality holds:

$$\begin{aligned} f(s, c) &= NF(g_t h(s, c)^t + g_{t-1} h(s, c)^{t-1} + \dots + g_1 h(s, c) + g_0) = \\ &g_t NF(h(s, c)^t) + g_{t-1} NF(h(s, c)^{t-1}) + \dots + g_1 (NFh(s, c)) + g_0. \end{aligned}$$

By Lemma 5.3., it is easy to see that the degree of the second and latter terms in the above expression is at most $r(t-1) = rt - r = n - r$. This fact suggests to define –like in the ordinary polynomial case– the concept of approximate roots.

DEFINITION 5.4. *Let $f(s, c) \in K[s, c]$ be a normal form polynomial of degree $n = rt$. A normal form, degree r , normed polynomial $h(s, c) \in L[s, c]$, where L is an extension field of K , is a t -th approximate root of $f(s, c)$ if there exists $\alpha \in L$ such that $\deg(f(s, c) - \alpha NF(h(s, c)^t)) \leq n - r$. \square*

Trivially, if $g(x)$ is a univariate polynomial and $h(s, c)$ is a normed polynomial, of degrees t and r , respectively, then $h(s, c)$ is a t -th approximate root of $NF(g(h(s, c)))$ (since its degree will be rt , by Lemma 5.3.). The key of the method for decomposing sine-cosine equations is the following result which shows the existence of t -th approximate roots and how to compute them. As consequence we will obtain the calculation of $f(s, c)$ -right candidates:

PROPOSITION 5.1. *With the above notation, there are exactly t , normed t -th approximate roots $h(s, c) \in \bar{K}[s, c]$ of $f(s, c)$, where \bar{K} is the algebraic closure of K . More precisely:*

- (i) *If $f_{n,0}f_{n-1,1} \neq 0$, then there are exactly t , monic on c , polynomials, which are t -th approximate roots of $f(s, c)$.*

- (ii) If $f_{n,0} = 0$ and t is an even number, then there are exactly t , monic on c , polynomials which are t -th approximate roots of $f(s, c)$.
- (iii) If $f_{n,0} = 0$ and t is an odd number, then there are exactly $t - 1$, monic on c , and one, monic on c and s , polynomials which are t -th approximate roots of $f(s, c)$.
- (iv) If $f_{n-1,1} = 0$ and t is an odd number, then there are exactly t , monic on c , polynomials which are t -th approximate roots of $f(s, c)$.
- (v) If $f_{n-1,1} = 0$ and t is an even number, then there are exactly $t - 1$, monic on c , and one monic on c and s , polynomials which are t -th approximate roots of $f(s, c)$.

PROOF. We consider normed polynomials $h(s, c)$ of degree r , with indeterminate coefficients, and an indeterminate g_t , and we impose that the terms of degree strictly higher than $r(t - 1)$ cancel out in the equation:

$$f(s, c) = g_t NF(H_r + \dots + H_1)^t =$$

$$g_t [NF((H_r)^t) + \binom{t}{1} NF[(H_r)^{t-1}(H_{r-1} + \dots + H_1)] + \binom{t}{2} NF[(H_r)^{t-2}(H_{r-1} + \dots + H_1)^2] + \dots]$$

Now, the degree of the second and latter terms in the above equation is at most $r(t - 1) + (r - 1) = rt - 1 < rt = n$. Thus, we can consider the equation

$$f(s, c) = g_t NF(H_r)^t, \quad (5.1)$$

and we determine H_r so that the terms of degree $n = rt$ in this equation cancel each other, that is :

$$\deg(f(s, c) - g_t NF(H_r)^t) < n.$$

At this point, we have to distinguish two possibilities for the normed polynomial h :

- (a) h is monic on c .
- (b) h is monic on c and s .

(a) In the first case, assume that $H_r = c^r + Z_0 c^{r-1} s$. Then we have to compute values of Z_0 and $g_t \neq 0$ in some extension of K , so that the terms of degree rt cancel each other in the equation (5.1). But

$$NF((H_r)^t) = NF(c^r + Z_0 c^{r-1} s)^t = c^{(r-1)t} [A_t(Z_0) c^t + B_t(Z_0) c^{t-1} s + D(s, c)] \text{ with } t > \deg(D(s, c)). \quad (\text{by Lemma 5.2.-(i)})$$

Thus we have,

$$NF((H_r)^t) = A_t(Z_0) c^{rt} + B_t(Z_0) c^{rt-1} s + Q(s, c) \text{ with } rt > \deg(Q(s, c))$$

and

$$f(s, c) - g_t NF((H_r)^t) = (f_{n,0} - g_t A_t(Z_0)) c^n + (f_{n-1,1} - g_t B_t(Z_0)) c^{n-1} s + \dots$$

Therefore, the values we are looking for are the solutions of the following system in Z, g_t :

$$\begin{aligned} f_{n,0} - g_t A_t(Z) &= 0, \\ f_{n-1,1} - g_t B_t(Z) &= 0. \end{aligned}$$

We remark that $f_{n,0} \neq 0$ or $f_{n-1,1} \neq 0$, therefore no solution of the system makes simultaneously $A_t(Z) = B_t(Z) = 0$; therefore the values of Z satisfying the system are exactly the roots (on some finite extension of K) of the resultant polynomial $M(Z)$:

$$M(Z) = f_{n,0} B_t(Z) - f_{n-1,1} A_t(Z)$$

and then we obtain the corresponding values of g_t substituting this value of Z and solving some of the above equations.

Next, we will show inductively that the remaining terms $H_i(s, c)$ of $NF(h(s, c))$ can be computed by solving a linear system with unique solution. Suppose we have found H_r, \dots, H_{i+1} verifying:

$$\deg(f(s, c) - NF(H_r + \dots + H_{i+1})^t) < r(t-1) + (i+1).$$

Then, to find H_i , we arrange the equation

$$f(s, c) = g_t NF(H_r + \dots + H_1)^t$$

as

$$\begin{aligned} f(s, c) &= g_t (NF(H_r + \dots + H_1))^t = \\ g_t [&NF(H_r + \dots + H_i)^t + \binom{t}{1} NF[(H_r + \dots + H_i)^{t-1} (H_{i-1} + \dots + H_1)] + \\ &+ \binom{t}{2} NF[(H_r + \dots + H_i)^{t-2} (H_{i-1} + \dots + H_1)^2] + \dots] \end{aligned}$$

and we realize that the degree of the second and higher terms is at most $r(t-1) + (i-1) = rt - (r - (i-1))$. Therefore to cancel the terms of degree $rt, rt-1, \dots, rt - (r-i) = r(t-1) + i$ it is enough to study the equation:

$$\deg(f(s, c) - NF(H_r + \dots + H_i)^t) < r(t-1) + i$$

Now, we are looking for $H_i = h_{i,0}c^i + h_{i-1,1}c^{i-1}s$. We have,

$$NF(H_r + \dots + H_i)^t =$$

$$NF((H_r + \dots + H_{i+1})^t) + \binom{t}{1} NF[(H_r + \dots + H_{i+1})^{t-1} (h_{i,0}c^i + h_{i-1,1}c^{i-1}s + \dots)] + \dots$$

The degree of the 3-rd and latter terms is at most $r(t-1) + i - 1$, so in order to cancel the terms of degree $r(t-1) + i$ we only need to consider the equation: $f(s, c) =$

$$g_t [NF((H_r + \dots + H_{i+1})^t) + \binom{t}{1} [NF((H_r + \dots + H_{i+1})^{t-1} (h_{i,0}c^i + h_{i-1,1}c^{i-1}s))]]$$

and to determine $h_{i,0}$ and $h_{i-1,1}$ so that terms of degree $r(t-1) + i$ cancel each other. Operating in the above equation, taking the normal form using again Lemma 5.2-(i), we have that the terms of degree $r(t-1) + i$ involving $h_{i,0}, h_{i-1,1}$ are:

$$[h_{i,0}A_{t-1}(Z_0) - h_{i-1,1}B_{t-1}(Z_0)]c^{r(t-1)+i} + [h_{i,0}B_{t-1}(Z_0) + h_{i-1,1}A_{t-1}(Z_0)]c^{r(t-1)+i-1}s.$$

Now, in order that these terms cancel, we have to solve the following kind of linear algebraic system:

$$\begin{aligned} h_{i,0}A_{t-1}(Z_0) - h_{i-1,1}B_{t-1}(Z_0) &= \alpha. \\ h_{i,0}B_{t-1}(Z_0) + h_{i-1,1}A_{t-1}(Z_0) &= \beta. \end{aligned}$$

where α, β and Z_0 are all known, in fact : α involves $f_{r(t-1)+i,0}$ and previously computed terms of $H_j, j \geq i+1$; the same applies to $\beta = f_{r(t-1)+i-1,1} + \dots$, and Z_0 is a previously determined zero of the polynomial $M(Z)$. The determinant of the above linear system is $A_{t-1}(Z_0)^2 + B_{t-1}(Z_0)^2 = (1 + Z_0^2)^{t-1}$ (see Lemma 5.1.(i)). But $(1 + Z_0^2)^{t-1} \neq 0$ because if, say, i is a root of $M(Z)$ then $\frac{f_{n,0}}{f_{n-1,1}}$ is an imaginary number, against the assumption that $\sqrt{-1} \notin K$. Thus, there exists a unique solution.

(b) In the second case, we have that $H_r = c^{r-1}s$ and we must check if the equation holds:

$$\deg(f(s, c) - g_t N F(H_r)^t) = \deg(f(s, c) - g_t N F(c^{t(r-1)} s^t)) < n.$$

We analyze separately two subcases: t odd and t even.

(b-1) If t is an odd number, then the above equation holds if and only if $f_{n,0} = 0$. In this case $g_t = f_{n-1,1}$, (see Lemma 5.2-(ii)). The remaining terms $H_i, i = r-1, \dots, 1$ can be computed as above, yielding a unique solution.

(b-2) If t is an even number, then the above equation holds if and only if $f_{n-1,1} = 0$, in this case $g_t = f_{n,0}$, (see Lemma 5.2-(iii)). The remaining terms $H_i, i = r-1, \dots, 1$ can be computed like in the above situation.

Finally, in order to know how many t -th approximate roots has $f(s, c)$ has, we divide the counting into several cases, (corresponding to the different items in the Proposition):

- (i) If $f_{n,0}f_{n-1,1} \neq 0$, then the polynomial $M(Z)$ has degree t in the variable Z and all its roots are simple (by Lemma 5.1-(v)). So, there are exactly t elements, $Z \in \bar{K}$ such that $M(Z) = 0$. For each root of the above polynomial, we find $Z_0 = h_{r-1,1}$ and g_t and the procedure above yields there are exactly t polynomials $H_r \in \bar{K}[s, c]$ which are t -th approximate roots and monic polynomials on c .
- (ii) If $f_{n,0} = 0$ and t is an even number, then $M(Z) = f_{n-1,1}A_t(Z)$ and $\deg(M_t(Z)) = t$, (see Lemma 5.1-(iv)), and there are t simple roots of the polynomial $M(Z)$ and, consequently, we obtain the claim.
- (iii) If $f_{n,0} = 0$ and t is an odd number, then $\deg(M_t(Z)) = t-1$ (see Lemma 5.1-(iv)) and there are exactly $t-1$ roots of the polynomial $M(Z)$ and $t-1$ monic on c polynomials which are t -th approximate roots; and one monic on c and s polynomial t -th approximate root, (see (b-1)).
- (iv) If $f_{n-1,1} = 0$ and t is an odd number, we have $M(Z) = f_{n,0}B_t(Z)$ and $\deg(M(Z)) = t$ (see Lemma 5.1-(iv)). So, $f(s, c)$ has t monic polynomials on c that are t -th approximate roots.

- (v) If $f_{n-1,1} = 0$ and t is an even number, we have $M(Z) = f_{n,0}B_t(Z)$ and $\deg(M(Z)) = t - 1$ (see Lemma 5.1-(iv)). So, $f(s, c)$ has $t - 1$ monic polynomials on c that are t -th approximate roots; and one root which is monic on c and s .

Thus, the proof is complete. \square

5.2.2. THE ALGORITHM SCDECPOL

The basic structure of the algorithm is first to compute candidates and then to check if they have a left component. We now give the details:

- M1.** Compute the normal form of $f(s, c)$, $n = \deg(NF(f(s, c)))$.
- M2.** Set $L = \{\}$. For each divisor k of n , perform **M3**, **M4**, **M5**.
- M3.** Compute normal form candidates $h(s, c)$ of degree k .
- M4.** Check if there is $g(x)$ of degree t , $n = tk$ such that:
 $NF(f(s, c)) = g(h(s, c))$ modulo $s^2 + c^2 - 1$.
- M5.** If yes $L = L \cup \{g(x), NF(h(s, c))\}$.

The algorithm determines all possible non-equivalent decompositions of a sine-cosine polynomial. From the computational point of view, the hardest step of the algorithm is determining candidates, in step **M3**, (see Proposition 5.1). In order to compute candidates monic on c , we have to find a root of the polynomial $M(Z)$ in the variable Z ,

$$M(Z) = f_{n,0}B_t(Z) - f_{n-1,1}A_t(Z).$$

We would distinguish two cases: $f_{n,0}f_{n-1,1} \neq 0$ and $f_{n,0} = 0$ or $f_{n-1,1} = 0$.

For the first case, we have, by Lemma 5.1.(v):

$$G_t(Z) = \frac{B_t(Z)}{A_t(Z)} = \frac{f_{n-1,1}}{f_{n-1,1}} = w \circ Z^t \circ w^{-1}$$

then,

$$Z^t \circ w^{-1} = w^{-1} \circ \frac{f_{n-1,1}}{f_{n,0}}, \text{ where } w = \frac{-iZ+i}{1+Z} \text{ and } w^{-1} = \frac{i-Z}{Z+i} = \frac{1+iZ}{1-iZ}$$

Thus we only have to compute the t -th roots of $w^{-1}(\frac{f_{n-1,1}}{f_{n,0}})$ and to apply w to these roots.

For the second case, we have to compute a root Z of the polynomial $A_t(Z)$ or a root of the polynomial $B_t(Z)$, again by Lemma 5.1. (iii), we have:

$A_t(Z) = 0$ implies:

$$\frac{(1+iZ)^t}{(1-iZ)^t} = -1$$

$B_t(Z) = 0$ implies:

$$\frac{(1+iZ)^t}{(1-iZ)^t} = 1.$$

In other words, we have reduced the main problem to computing the t -th roots of an element in the field $K(i)$. In many practical cases we have precomputed formulae or

methods for extracting such t -th roots, such as when K is a subfield of the real numbers or a parametric field $Q(d_1, \dots, d_m)$ (Gutierrez (1991), Trager and Yun (1976)). Regardless, the algorithm can decompose sine-cosine polynomials over any field extension of K where we can find roots of $M(Z) = 0$, a very simple polynomial in $K[Z]$. If the ground field K is a subfield of the real numbers, since w^{-1} takes real numbers to the unit circle (in the complex plane) and w takes the unit circle to the real line, then the algorithm determines all "irrational" decompositions of $f(s, c)$ (see Example 5.1. and Example 5.8. below), and we can also conclude that in this case all decompositions must be real. In this particular case, we only have to apply the formula for the t -th root of a complex number.

5.3. THE COMPLEXITY

For a field K , we denote by $R_K(t)$ the number of field operations to extract a t -th root in $K(i)$ and $M_K(s)$ the number of field operations in $K(i)$ that takes to solve a linear system of s equations with s variables. The number of arithmetic operations in step M1 it is clearly linear in n (degree of f). Step M2 requires to know how many divisors n has, i.e. $O(n^\delta)$, where δ is any arbitrary small positive real number. Step M3 it is devoted to compute candidates. The total complexity of this step is $O(n^\delta(M_K(n) + R_K(n)))$. Finally, step M4 it is dedicated to checking if there exists $g(x)$ by solving a linear system on the indeterminate coefficients of $g(x)$ (with unique solution if it exists). Therefore, the complete algorithm can be performed within $O(n^\delta(M_K(n) + R_K(n)))$ number operations in the field K . For instance, if K is a subfield of the complex numbers, then no finding root method is required and the complexity of computing candidates it is dominated by solving a linear system equations. So the time bound for this step is $O(n^3)$ arithmetic operations. and, therefore, the complete algorithm can be performed with $O(n^3)$ number operations in the field K .

5.4. IMPLEMENTATION AND EXAMPLES OF THE ALGORITHM

We briefly discuss our experience gained from implementing the decomposition algorithm for an s - c polynomial on MAPLE V. The implementation of all steps of the algorithm is straightforward. For robotic applications, we would like to distinguish two cases: monic candidates on c and monic candidates on c and s .

For the first case, we have to find a t -th root of a multivariate polynomial involving the coefficients $f_{n,0}$, and $f_{n-1,1}$ which are rational functions that generally contain parameters. The most interesting decomposition in kinematics is when the candidate has degree one; in this particular case, the polynomial $M(Z)$ must have degree n in the variable Z . Our implementation on MAPLE decomposes the multivariate polynomial and then looks for power polynomials in the parameters. Most of the time is spent on this operation; may be a more subtle t -th root finding approach could have an even better performance (see Trager and Yun (1976) and Zippel (1993)). If our candidate is of degree greater than one, then in practice, most of the time will be spent on the determination of the "left component", i.e. solving linear systems.

If we are looking for monic candidates on c and s , in practice, most of the time is also spent on the determination of the "left components" $g(x)$.

Our procedure SCDECPOL has as input a s - c polynomial $f(s, c)$ and outputs the list of lists $[g(x), h(s, c)]$ if $f(s, c)$ has the decomposition $NF(f(s, c)) = g(NF(h(s, c)))$ modulo the circle; otherwise SCDECPOL returns the empty list.

The authors were able to decompose s - c polynomials of degree 8 with 100 digit coefficients highly complex terms within 20 seconds of CPU-time on an Apple Macintosh Centris 650, using MAPLE V. Therefore, we think that the algorithm now can be a useful tool in kinematics and we believe it is reasonable to adopt the decomposition test in solving the kinematic equations.

EXAMPLE 5.6. The s - c polynomial $f(s, c)$ below (see also Kovács and Hommel (1992)) contains two formal parameters a and p . The parameter a could be some link parameter a_i and p may represent some position parameter:

$$f(s, c) = 18972c^4 - c^3(5840a + 1752p) - c^2(44892 + 3168a^2 + 2016ap) + c(6120a + 48a^3 + 1764p + 48a^2p) + 26019 + 2058ap + 18a^3p + s[(-12096c^3 - c^2(12880a + 3864p) + c(13608 + 924a^2 + 588ap) + 4116p + 336a^2p + 15400a + 336a^3) + 3384a^2 + 13a^4]$$

In this case $K = Q(a, p)$, and actually, all coefficients belong to the polynomial ring $Z[a, p]$.

We apply the procedure SCDECPOL to $f(s, c)$:

$$\text{SCDECPOL}(f(s, c)) =$$

$$[13a^4 + 18a^3p + 150a^2 + (48a^3 + 240a + 48a^2p)x + (42ap + 66a^2 + 90)x^2 + (40a + 12p)x^3 + 9x^4, \quad c + 7s]$$

time 2.36 words 65128 . □

EXAMPLE 5.7. Let us consider the following numerical s - c polynomial $f(s, c)$ with coefficients over rational number field $K = Q$:

$$\begin{aligned} & -4270526070s + 61774930046775c + 669964656943644s^2c + \\ & 2327784495058306561186364871s^2c^2 - 4460854486596s^2 - \\ & 933425855495016661185c^2 - 9691328376007048326sc^2 - \\ & 107280460040652354123360s^3c + 81180793210769381823600131204888151c^4 \\ & + 171619854300504720c^3 + 24168288660858033955475184s^3c^2 \\ & - 1814889107572207526765722824s^3c^3 - 22448245171778614340725571593128sc^3 \\ & + 129056539685518305sc - 349604600320565643596072385240c^3s^2 + \\ & 13126609881140465919861837965178c^4s^2 - 98733455500 + 1854085472287462512s^4 \\ & - 4176282369914618086019232c^5s^3 + 45429059202423915535644349908c^4s^3 - \\ & 464906620050728393546958993120c^5s + 1685729287411107879583398465933300c^4s + \\ & 110934304585807567036032c^6s^2 - 2413452111312631432049207712c^5s^2 \\ & - 29851831981936756779881579532000c^5 - 252282079514824702801920c^7 + \\ & 5798078855747039232c^8 + 4116422604870988090525123200c^6 - \\ & 1309662394966451054592c^7s + 42738874657310635576932096c^6s + \\ & 62731934734335222034728s^4c^2 + 58958321656159136627605287s^4c^4 \\ & - 3140517828291004571049864s^4c^3 - 556920985034480917536s^4c \end{aligned}$$

Applying our SCDECPOL code to the above s - c polynomial yields:

$$\text{SCDECPOL}(f(s, c)) =$$

$$[-98733455500 - 5678975160x - 7888514716224x^2 + 5798078855747039232x^4, \quad c^2 - 567859/10056sc - 109387465/10056c + 3781/5028s]$$

time 7.46 words 236423 . \square

EXAMPLE 5.8. Let us consider the following numerical s - c polynomial $f(s, c)$, also with coefficients over the rational number field $K = \mathbb{Q}$:

$$1055c^2 + 935sc + 2341.$$

Now, we apply our ASCDECPOL code, which uses the implementation of roots of unity in MAPLE V:

ASCDECPOL($f(s, c)$) =

$$\left[\left[\left(-\frac{935}{2} \right) \left(\frac{211}{187} + \frac{\sqrt{79490}}{187} \right) x^2 + \left(-\frac{935}{2} \right) \left(\frac{211}{187} + \frac{\sqrt{79490}}{187} \right) x + 2341, \quad c - \left(\frac{211}{187} + \frac{\sqrt{79490}}{187} \right) s \right], \right.$$

$$\left. \left[\left(-\frac{935}{2} \right) \left(\frac{211}{187} - \frac{\sqrt{79490}}{187} \right) x^2 + \left(-\frac{935}{2} \right) \left(\frac{211}{187} - \frac{\sqrt{79490}}{187} \right) x + 3396, \quad c - \left(\frac{422}{187} - \frac{211}{187} - \frac{\sqrt{79490}}{187} \right) s \right] \right].$$

 time 12.33 words 991168 . \square

6. Conclusions

Several methods for simplifying sine-cosine systems solving have been analyzed. Some, such as the extraneous root analysis, can be performed before triangularization of the given system; some, such as the SCDECPOL algorithm or the factorization are carried over the highest degree equation in one joint angle, obtained after triangularization. We have shown how, even with parametric coefficients, decomposition can be performed on low polynomial time. The relation of the different simplification methods with previous work (BHD decompositions, Gröbner basis, co-circularity, geometric conditions for simplification, half angle substitution methods) has been carefully stated. We have remarked on points of the preceding arguments and tools, and completed them in many instances with more conceptual insight.

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