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Reconsidering Algorithms for Real Parametric Curves

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Abstract. Complex parametric curves have been subject of a symbolic algorithm approach in recent years. In this paper we analyze the theoretical applicability of some of these algorithms to the real parametric curve case. In particular, we show how several results are valid both over the real and the complex numbers, as they hold equivalently over a real curve and its complexification. Therefore, the standard algorithms for the complex case can be applied to obtain real answers in the real case. A second issue in our paper is the study of the very different behaviour of the real parametric mapping and we characterize here the properties of being (almost) injective or surjective.

Keywords: Real parametric curves, Parametrizations, Algorithms, CAD.

1 Introduction

Recently several algorithms have been proposed for dealing with parametric curves: given a system of polynomial equations with coefficients over a computable subfield of the complex numbers (for example, with rational coefficients) we know symbolic algorithms (for instance, computing via Groebner basis the dimension of the ideal generated by these equations, cf. [19], or [14]) to decide whether the solution set of the polynomial system is actually a curve, in the geometrical sense of the word, over the complex affine space. Also one can find algorithmically the irreducible components of the curve (see [13]) by computing its primary decomposition. Moreover, for an irreducible complex curve, in [12] there are algorithms describing the implicit equation of a plane curve whose function field is birational to the function field of the given curve. For plane curves we have algorithmic procedures (see [1]) to compute its genus and, therefore, to decide whether a curve is parametrizable by rational functions. Finding explicit parametrizations for plane curves of genus zero is accomplished in [24]. An algorithmic criterion for the existence of a polynomial

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parametrization is given in [3] and [20]. Simplifying parametric equations of a curve is done in [23] and [11]. Conversely, implicitization procedures appear in [18] or [8].

A standard reference for basic algebraic geometry terminology and concepts is the book of Abhyankar [3].

Though many of the above mentioned algorithms have, at this moment, only theoretical interest because of their high complexity, their natural field of potential application is in Computer Aided Design, where parametric varieties play a relevent role. Now, most of the curves treated in CAD are *real curves*, in the sense of one-dimensional solution sets of real polynomial equations over the real affine space. But the majority of the above mentioned algorithms assume, either implicitly or explicitly, facts and methods that are valid, in principle, only over algebraically closed fields. For instance, it is known that computing the genus involves–even for a curve given by a polynomial with real coefficients–operations with the complex points of the curve; and then that having genus zero implies being rationally parametrizable over the complex field. Is it the case that if the curve is defined over the reals and has genus zero then it has a real parametrization? Are the current algorithms for simplifying rational parametrizations of curves also dependent on the field of coefficients? Finally, let us mention that the image of any parametrization mapping, for complex values of the parameter, covers – with a finite number of exceptions – the points of the whole complex curve; and that this mapping is almost everywhere *1–1* if and only if the parametrization is *faithful* (i.e. if the function field generated over the complex field by the parametrization is exactly $C(T)$ where C is the complex numbers field and T is one transcedental over C). But both assertions are clearly false over the reals (think of the image of the mapping (T^2, T^4) or (T^3, T^6) over the parabola). What can we say about the real parametrization mapping? A given parametrization is called *quasi-polynomial* if the function field generated by the parametrization contains at least one non-constant polynomial; for instance the field $C(t-(1/t))$ is not *quasi-polynomial*. The interest in detecting quasi-polynomial parameterizations is that, given one of these parametrizations, there are especially fast algorithms to obtain a faithful one, see [16].

Therefore, we feel there is a need to reconsider from a real point of view some standard facts for parametric curves, either providing proofs of their validity or suitably adapting them to the case of real curves. In particular we shall consider the following three algorithmic questions:

- When does a real curve have a real rational parametrization i.e. a parametrization with rational functions that are the quotient of polynomials with real coefficients? When does it have a real polynomial parametrization?
- When are the points of a real parametrizable curve (with a finite number of exceptions) in a one to one correspondence with the real values of the parameter? When are these points covered by the image of the parametrization, for real values of the parameter?
- Being faithful or quasi-polynomial or making faithful one given parametrization by changing the parameter, is it independent of the base field under consideration?

This work will use freely some simple facts from real computational algebraic geometry, both from the standard reference [5] and from the more specialized survey [17].

2 Genus and Reality

Here we present the answer to the first of the above questions. The Lemma below seems to be well known to specialists, but we state it here in a way that is parallel to the classical Lüroth's theorem. A simple proof appears in the treatise of Chevalley "Introduction to the theory of algebraic functions of one variable" pag. 23, [6].

Lemma 2.1 (real version of L uroth's theorem). *If K is a formally real field extension of the real numbers field R of transcendence degree one and K is contained in $C(T)$, where T is a transcendental over C, then K is isomorphic to $R(T)$.*

Next let us introduce some precise definitions concerning real curves. By a *real curve* we will understand a collection of points in real affine space which is the solution set of a finite system of real polynomial equations, such that this collection of points is of topological dimension one. In the particular case of plane curves this definition is equivalent to the more intuitive one of an infinite collection of points which is the solution set of a non zero real polynomial. This definition excludes considering the zero set of $x^2 + y^2 = 0$ or of $x^2 + y^2 = -1$ as real curves (although they are plane curves from the complex point of view).

Given a real curve, its *complexification* is the smallest complex curve containing the real curve. In the proof of the proposition below we will use some algebraic facts for real curves and their complexifications that can be consulted in [10]. The *genus* of the real curve is by definition the genus of its complexification. Moreover we will need a concept of *real parametrizable curve*: this applies to a real curve that is the real Zariski closure of the image set of a non constant real rational function mapping. We remark that this definition agrees with the usual one in the complex case if we replace everywhere in the above definition "real" by "complex". Although these concepts might seem rather technical, they correspond to the standard ideas of what everybody will admit as curve and parametrizable curve in a real setting.

Proposition 2.2. *A real curve has a real parametrization if and only if it has genus zero; i.e. if and only if it has a complex parametrization.*

Proof. It follows easily from the definition that if a real curve has a real parametrization then it is irreducible and its function field K is contained in $R(T)$. Then the complexification of the real curve has function field $K[i]$, and therefore it is contained in $R(T)[i] = C(T)$. Thus the curve has genus zero and a complex parametrization.

Conversely, if the real curve has genus zero then, by definition, it is given by an irreducible real ideal I in $R[X_1, X_2, \ldots, X_n]$ such that the irreducible complex curve also given by I in $C[X_1, X_2, \ldots, X_n]$ (the complexification) is of genus zero. Let K be the function field of the real curve. Such a field is formally real and $K[i]$ is the function field of the complexification of the curve. Being of genus zero and, therefore, rational over the complex field, we have $K[i] \subseteq C(T)$. By the lemma above it turns out that the real curve is also real rational. □

Remarks 2.3. It follows from the proposition that the algorithmic methods to decide if a curve is parametrizable are also valid over real curves. If one starts with just an ideal in $R[X_1, \ldots, X_n]$ and wants to know if it is the ideal of a parametrizable real curve, then the standard procedure over the complex field involves determining that the ideal is prime and of dimension 1; and then finding the genus of a birationally equivalent plane curve. The procedure can be carried over to the reals but one must moreover check that the plane curve is actually a real curve, i.e. that it is not something like $x^2 + y^2 + 1 = 0$. Algorithmic methods for solving this question, or the more general of computing the real radical, can be consulted in [4].

Proposition 2.4. *A real curve is parametrizable by real polynomials if and only if its complexification (given by the same ideal) is parametrizable by complex polynomials.*

Proof. As mentioned in the introduction, in [3] and [20] it is given a criterion for a complex curve of genus zero to have a polynomial parameterization: it must have one and only one place at infinity. Now the complex points at infinity of the complexification of a real curve come in conjugate pairs; thus, if this complexification has a complex parametrization by complex polynomials it must have one and only one point at infinity and it must be real. Likewise, the complex branches at the real points of the complexification of a real curve must come in conjugate pairs; thus if there is only one place at the point of infinity it must be a real place. But this implies, again by another criterion of Abhyankar (same reference as above), that the curve has a real polynomial parametrization, as it has one and only one *real place* at infinity.

Remarks 2.5.

2.5.1. On the other hand, if V is a plane curve defined over the field Q of rationals, then it may happen that V has a parametrization with real rational functions but has no parametrization with rational coefficients. Consider, for instance, the circle defined by the equation $x^2 = y^2 - 3 = 0$, that has not any rational point (this can be checked consider the equation $a^2 + b^2 = 3c^2$ and showing that has no integer solution with a, b, c relatively prime; reduce the equation modulo 3 and discuss cases) and therefore no parametrization by rational functions with rational coefficients can exist.
2.5.2. If V is a real curve, then there are algorithmic procedures to test if the curve has a real polynomial parametrization. We can use anyone of Abhyankar's criteria (both the real or the complex polynomial criteria are equivalent, after the above proposition). His criterion requires to determine algorithmically some properties of the places (branches) of the curve at infinity. Abhyankar mentions he does not know an algorithm to check this condition. Now, the works of [7] and [9] (see also [17]), imply that one can handle algorithmically the computation and the symbolic representation of the coordinates of the singular points; and of the infinitely near points; and the coefficients of the terms of the Puiseux expansion of the branches of a curve at a singular point, up to the degree required to determine whether the branch is real or not, or the multiplicity of the point. Thus Abhyankar's criteria can be algorithmically tested.

3 The Real Parametrization Mapping

Complex parametrizations of curves, considered as mappings from the complex line C to C^2 have a good behaviour: all points of a plane rational curve, except a finite number of them, are the image of a value of the parameter. Moreover, again except for a finite set of points, the map from the set of parameter values to the points of the curve is d-to-one, where d is the degree of the algebraic extension of the rational function field over the field generated by the parametrization. Finally it is obvious that a rational parametrization is defined except for a finite number of parameter values.

These results are often assumed when we are working in the real case, as they are basic requirements for drawing a picture of a real curve by means of a parametrization mapping. Unfortunately not all of these properties remain true. In fact, the image of the parametrization of the parabola $x = t^2$, $y = t^4$ does not reach all points of the parabola, but rather is a kind of semi-parabola. Other counterexample could be the parametrization $x = t^3$, $y = t^6$ which is 3-to-1 in the complex case but 1-to-1 in the real case, although the degree of the algebraic extension is 3. In order to understand the "real" situation we need to introduce a definition that generalizes the kind of sets that can arise as images of a parametrization.

Definition 3.1. A subset W of R^n is a *semialgebraic set* if there are polynomials in n variables $f_{i,j}$, $g_{i,j}$, $i \in I$, $j \in J$, finite, such that

$$W = \bigcup_{i \in I} W_i, \quad \text{where}$$

$$W_i = \{\underline{x} \in R^n / f_{i,j}(\underline{x}) = 0, \ldots, g_{i,j}(\underline{x}) > 0, \quad j \in J\}$$

The situation in the real case is as follows:

Proposition 3.2. *Let $f_1(x_1, \ldots, x_n) = 0, \ldots, f_r(x_1, \ldots, x_n) = 0$ be a set of equations defining a basis of the real ideal of a real parametrizable curve, and let $\phi_1(T), \ldots, \phi_n(T)$ be real rational functions, not all constant, such that $f_1(\phi_1(T), \ldots, \phi_n(T)) \equiv 0, \ldots, f_r(\phi_1(T), \ldots, \phi_n(T)) \equiv 0$. Then:*

- *the image of R by the parametrization mapping is a semialgebraic subset of the real curve, whose real Zariski closure is the given curve.*
- *if the parametrization is almost injective (i.e. for all points of the curve, except for a finite number of them, the number of inverse images under the parametrization mapping is at most one) then the image of the real line by the parametrization mapping will cover the whole real curve except for a finite set of points (i.e. it will be almost surjective).*

Proof. For the first point we see that the image set is semialgebraic using Tarski Principle (see [5]). Moreover if the Zariski closure of this set were a proper subset of the irreducible curve, then it would be a finite number of points and this will imply that the rational functions are constant, in contradiction with the definition of rational curve.

The second point follows if one proves that the image set contains all points of the curve except a finite number. By the classical case, for almost all points of the curve the number of (complex) inverse images under the parametrization is equal to the degree d of the field extension given by the parametrization. The non real values of the parameter corresponding to a given point of the curve must be pairwise conjugate and, then, the number of real values of the parameter must be congruent to d modulo 2, for almost all points. By the injectivity hypothesis, this number is 1 over almost all the real image. This implies that d is odd; therefore the number of real values of the parameter over almost all points of the curve is always odd, i.e. at least one.

Remark 3.3. These results emphasize the particular relevance of injective parametrizations in the real case, as it guarantees that (almost) the whole curve is obtained under the parametrization mapping. We will study in more detail this situation in the following paragraph.

4 Real Faithful and Quasi-Polynomial Parametrizations

In the complex case, one to one parametrizations are precisely those that are *faithful*, i.e. such that $K(\phi_1(T), \ldots, \phi_n(T)) = K(T)$, as mentioned in the introduction. We remark that this fact can be algorithmically tested, see [2, 21, 16]. If the parametrization is not faithful, then a change of parameter is needed to simplify the parametrization; this change can be computed (cf. [23] or constructive versions of Luroth's theorem as in [22]). But it could happen, in principle, that the change of parameter were field dependent. Let us prove that this situation does not happen:

Lemma 4.1. *Let $\phi_1(T), \ldots, \phi_n(T) \in K(T)$ and let $L \supseteq K$ a field extension. Then if $h(T) \in K(T)$, we have $K(\phi_1, \ldots, \phi_n) = K(h)$ if and only if $L(\phi_1, \ldots, \phi_n) = L(h)$.*

Proof. $\Rightarrow$) Clearly $K(\phi_1, \ldots, \phi_n) = K(h)$ implies $L(\phi_1, \ldots, \phi_n) = L(h)$. $\Leftarrow$) Let $L(\phi_1, \ldots, \phi_n) = L(h)$, $h \in K(T)$. Now by Luroth $K(\phi_1, \ldots, \phi_n) = K(h')$ with $h' \in K(T)$. By the other implication, then $L(\phi_1, \ldots, \phi_n) = L(h')$. Therefore $L(h') = L(h)$. Thus $h' = (ah + b)/(ch + d)$, $ad - cb \neq 0$, $a, b, c, d \in L$. We claim that we can take a, b, c, d in K verifying the same equality. In fact let $h = h_1/h_2$, $h' = h'_1/h'_2$ be the irreducible form of h and h'. We may assume $\deg(h_1) \neq \deg(h_2)$, since if $\deg(h_1) = \deg(h_2)$ we can choose $\alpha \in K$ such that $\deg(h_2) \neq \deg(h_1 - \alpha h_2)$ and $L(h) = L((h_1 - \alpha h_2)/h_2)$.

From $h'_1/h'_2 = (a(h_1/h_2) + b)/(c(h_1/h_2) + d) = (ah_1 + bh_2)/(ch_1 + dh_2)$, we deduce that $ah_1 + bh_2$ and $ch_1 + dh_2$ are relatively prime since, by hypothesis, h_1, h_2 are also relatively prime. Therefore there is a constant $\lambda \in L$ such that $h'_1 = \lambda a h_1 + \lambda b h_2$ and $h'_2 = \lambda c h_1 + \lambda d h_2$. Identifying coefficients and since the degrees of h_1 and h_2 are different, we conclude that $\lambda a, \lambda b, \lambda c, \lambda d \in K$, concluding the proof of thelemma. □

Proposition 4.2. *A real parametrization $(\phi_1(T), \ldots, \phi_n(T)) \in R(T)^n$ is real faithful if and only if it is complex faithful (and therefore we speak only of faithful parametrizations): i.e. $R(\phi_1(T), \ldots, \phi_n(T)) = R(T)$ if and only if $C(\phi_1(T), \ldots, \phi_n(T)) = C(T)$. Moreover, if there is a change of parameter $U = h(T)$ with $h(T) \in C(T)$, such that $C(\phi_1(T), \ldots, \phi_n(T)) = C(U)$ then we can construct an automorphism σ of the field $C(T)$ such that $h'(T) = \sigma(h(T)) \in R(T)$; and $U' = h'(T)$ verifies $R(\phi_1(T), \ldots, \phi_n(T)) = R(U')$.*

Proof. The first statement of the proposition follows considering in the above Lemma $h(T) = T$. For the second statement, if $C(\phi_1(T), \ldots, \phi_n(T)) = C(h(T))$ for some $h(T) \in C(T)$, then let $h'(T) \in R(T)$ be such that $R(\phi_1(T), \ldots, \phi_n(T)) = R(h'(T))$. It follows from the Lemma that $C(h'(T)) = C(h(T))$; identifying coefficients as in the above proof we can construct an automorphism $\sigma = (aT + b)/(cT + d)$, $ad - cb \neq 0$, $a, b, c, d \in C$ sending $h(T)$ to $h'(T)$. □

Proposition 4.3. *A real parametrization $R(\phi_1(T), \ldots, \phi_n(T)) = R(h(T))$ is almost injective if and only if there is for every real number k, at most one real solution for the equation $h(T) = k$.*

Likewise, a real parametrization is almost surjective if and only if for every real k, $h(T) = k$ has at least one real solution (taking $h(T)$, as in the lemma, such that the degree of the numerator is strictly greater than the degree of the denominator).

Proof. Assume the equation $h(T) = k$ has at most one real solution for every real k. Then changing the parameter $h(T) = U$ we obtain a real faithful parametrization in U. Being real faithful it will be also complex faithful and therefore, by the classical case, is almost one to one (even in the complexification of the curve). Let $t, t' \in R$ be two values of the parameter giving the same point of the real curve. Then, except for a finite number of points, $h(t)$ and $h(t')$ must be equal to the same value k of U. By hypothesis, this implies $t = t'$. Conversely, if the parametrization ϕ is almost injective and it is not faithful, then by changing the parameter as above we know that the parametrization ϕ in $U = h(T)$ is faithful. For almost all values k of U, the solutions of the equation $h(T) = k$ correspond with values of the parameter T giving points of the curve where ϕ is injective. Therefore, it follows that $h(T) = k$ has at most one real solution except, possibly, for a finite number of values of k. Let us show that the same holds for all values of k. In fact if we display the graph of the rational function $Y = h(T)$ and, for some k, the line $Y = k$ intersects the graph in more than one point, then for every value k' close enough to k, the line $Y = k'$ will have the same property (analyzing the behaviour of the tangent to the graph in the intersection points). Thus having more than one real root in the equation $h(T) = k$, for one value of k, implies the same for an infinite number of $k's$. Thus we conclude that $h(T) = k$ has always at most one real root.

The second assertion can be proved along the same lines. As above we will end the proof by showing that the condition: "for almost all values of k, $h(T) = k$ has at least one real root", implies the same property for all values of k. As stated in the proposition, $h(T)$ must be chosen, without loss of generality, among generators of the field $K(h(T))$ having the additional condition that the numerator has bigger degree than the denominator. Under such condition, the graph of the rational function has not horizontal asymptotes and we conclude that, if for a value of k, the line $Y = k$ does not intersect the graph, then the same happens for an infinite number of $k's$. □

Remark 4.4. Both conditions on the equation $h(T) = k$ can be tested algorithmically using the parametrized Sturm theorem, (or Sturm-Habitch sequences cf. [15]), and deciding if a set of algebraic inequalities in one variable is empty or not. When $h(T)$ is a polynomial the condition about having for every k at most one real root is equivalent to "$h(T)$ is of odd degree and the curve $y = h(T)$ does not have any real maximum or minimum"; likewise, the condition "$h(T)$ is of odd degree" will be equivalent to having, for every k, at least one real root.

We have mentioned in the introduction that quasi-polynomial parametrizations are computationally interesting since in this case it is easier (almost linear in the degree of the parametrization) to obtain the corresponding faithful parametrization. Finally, let us show that the lemma above allows us to prove the following:

Proposition 4.5. *Given a real parametrization of a real curve, its function field contains a real polynomial if and only if the function field of the complexification has also a complex polynomial.*

Proof. Suppose the complex function field of the complexification of the curve contains a polynomial. Then it is easy to prove (see [22]) that the function field is also generated over C by a polynomial $h(T)$, maybe with complex coefficients, which can be taken monic and without constant term. Let the real function field of the curve be equal to $R(h'(T))$, where $h'(T) = h'_1(T)/h'_2(T)$ is a rational function with real coefficients. As in the lemma, we have $C(h(T)) = C(h'(T))$ and therefore $h' = (ah + b)/(ch + d)$, where a, b, c, $d \in C$ and $ad - bc \neq 0$. Then $h'_1 = \lambda ah + \lambda b$, and $h'_2 = \lambda ch + \lambda d$, with some $\lambda \in C$. Identifying coefficients we obtain that $\lambda a, \lambda b, \lambda c, \lambda d \in R$. Thus h' is equal to h composed with a real automorphism; conversely h will be the composition of h' with a real automorphism, and therefore h will be real. By the lemma, this real polynomial will be in the real field of the curve. The other direction is trivial. □

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