ABOUT VORONOI DIAGRAMS FOR STRICTLY CONVEX DISTANCES *

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ABSTRACT

In this paper we study the topology and geometry of the family of bisectors for a convex distance d (Sections 1 and 2). Good properties for the Voronoi regions are derived (Section 4 and 5). Although in this case bisectors are homeomorphic to lines, pairs of them can exist intersecting infinitely many times (Section 3). This leads to the conclusion that convex distances are not always nice in the sense of Klein and Wood[4]. We prove also that d-balls whose boundary is given by finitely many algebraic conditions produces nice distances (Section 3).

0. Introduction

Voronoi Diagrams in the plane for distances different from the Euclidean one have been considered in several papers[1,2,3,4,5].

Lee[1] has considered this problem for the class of all the L_p-distances for $1 \leq p \leq \infty$, and after studying the behaviour of bisectors, he describes an algorithm, generalizing the standard divide and conquer approach, to construct the Voronoi diagram.

Chew and Drysdale[2] consider the same problem for the more general class of convex distance functions. They propose also the divide and conquer scheme, but do not prove why essential parts of their algorithm, like contour scan during the merge phase, can be applied to convex distance functions as it does to the Euclidean distance.

Klein[3] provides details about a divide and conquer algorithm that works for the class of *nice* distances in the plane. A distance d is *nice* if the following four properties hold: (i) d induces the usual topology. (ii) The d-circles are bounded with respect to the Euclidean distance. (iii) d verifies the *between* condition, i. e. given any two distinct points A and C, there exists a point B, different from A and C and such that: $d(A,C) = d(A,B) + d(B,C)$. (iv) Bisectors are closed sets, homeomorphic to the interval $(0,1)$ and halve the plane in two unbounded regions; moreover it is required that the intersection of any two bisectors has a finite number of connected components.

Whereas the first three properties are fulfilled by every convex distance, it remains open wheather property (iv) always holds.

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In this paper we provide the first detailed investigation of the class of *strictly convex distances*, their bisectors and their Voronoi regions.

First we show that any two d-circles (with respect to a strictly convex distance d) intersect at most twice (Th.1). Second we prove that for each bisector $Bi(P,Q)$, there exist an homeomorphism mapping the plane onto the plane that sends $Bi(P,Q)$ to a line (Th.2). As a consequence of the former result, any two *associated* bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect at most once and, if they do, they cross transversally (Th.7). This is the reason why the merge step works here as for the Euclidean distance .

But bisectors of a strictly convex distance can behave quite differently from the straight lines of Euclidean distance. For example, bisectors do not always have an asymptotic line (Th.4). Moreover, there do exist pairs of bisectors $Bi(P,Q)$ and $Bi(R,S)$ that intersect at an infinite number of points (Section 3). Therefore, strictly convex distances do not in general fulfill property (iv) in the definition of nice distances. However, we show that this problem does not occur if the d-circles are semialgebraic (Th.5). This is, in fact, the case for all L_p-distances and also for most practical applications.

In conclusion in this paper we are able to close some major gaps in the existing literature concerning Voronoi diagrams for convex distances.

1. Good Properties of d-circles

A *strictly convex distance* on the plane is the one induced by any norm and such that the boundary of the unit ball defined by this distance contains no three collinear points. The closure of the unit ball under a strictly convex distance can be characterized as being a compact and strictly convex subset K of the plane that contains the origin as an interior point and that is symmetrical with respect to it. Conversely, any such set K can serve to define a normed distance in such a way that the set K is the closure of the unit ball for this distance[6]. The *distance induced by* K, between two points P and Q, is measured as follows: translate K so that it is centered at P and call it K_P. Let Z be the unique point of intersection of the half line from P through Q, with the boundary $Bd\,K_P$ of K_P. Distance between P and Q is the quotient of the Euclidean distances between P and Q and P and Z.

Strictly convex distances verifies the *strong triangle inequality*, i. e.: P does not belong to the closed segment $[X,Y]$ if and only if $d(X,Y) < d(X,P)+d(P,Y)$[5]. Moreover, given two points P and Q, there exists a unique *midpoint* which is the Euclidean midpoint of P and Q[5]. (Given two points P and Q on the plane, a point R is a *midpoint* of P and Q for the distance d if and only if $d(P,R) = d(R,Q) = \frac{1}{2}d(P,Q)$).

Next we present some key properties that are verified by d-circles of a strictly convex distance.

Let d be a strictly convex distance on the plane and let P and Q be any two distinct points. The *bisector* $Bi(P,Q)$ of P and Q with respect to the distance d is defined as $Bi(P,Q) = \{X \in R^2 : d(P,X) = d(Q,X)\}$. The *d-circle of centre P and* *radius* r, $C_d(P,r)$ is defined as $C_d(P,r) = \{X \in R^2 : d(P,X) = r\}$ and equals the boundary $Bd\,B_d(P,r)$ of the open d-ball $B_d(P,r)$ centered at P and of radius r.

Theorem 1. ***If d is a strictly convex distance on the plane then any two d-circles intersect at most in two points. As a consequence, given three points on the plane, there exist at most one d-circle containing them.***

Proof. (See [7]) Let C_1 and C_2 be any two d-circles. Let T and L be the common outer tangents to C_1 and C_2, and t_1, d_1, and t_2, d_2 be their corresponding points of intersections with C_1 and C_2, as shown in Fig. 1. Assume that T and L intersect at some point c (the case where T and L are parallel can be dealt in the same way). Let A_i, B_i, $i = 1,2$, denote the open arcs of C_i between t_i and d_i such that A_1 is on the same side as A_2 and closer to c. As A_i, B_i, for $i = 1,2$ are strictly convex and pairwise homothetical we have: $A_1 \cap A_2 = \emptyset$, $A_1 \cap B_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Thus $C_1 \cap C_2 = A_2 \cap B_1$. The rays from c through C_1 impose the same ordering on A_2 and on B_1; let p (and q) denote the topmost (respectively the bottommost) point in $A_2 \cap B_1$. Since the open line segment (p,q) is contained in the interior of both C_1 and C_2, it separates the arc segment A_2 from B_1 avoiding a third intersection. Hence C_1 and C_2 intersect at most twice.

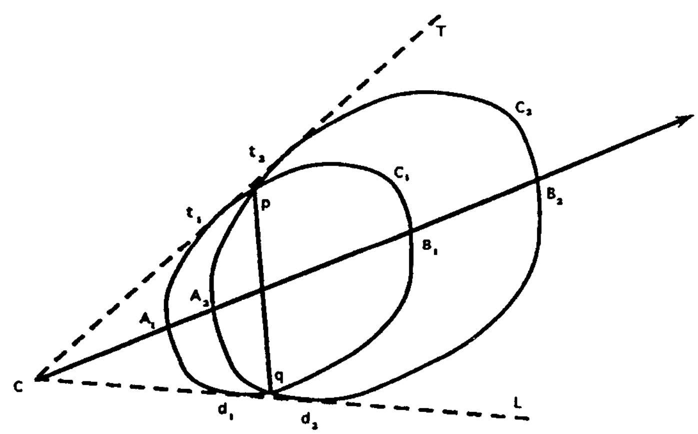

Fig. 1

For the proof of the last assertion in the theorem suppose that there exist two d-circles containing the three given points. This two d-circles would have those three points in common, contradicting the above conclusion. ⊔

2. Shape and Geometry of Bisectors.

Theorem 2. *(topological structure of bisectors). Bisectors for a given strictly convex distance are simple curves that divide the plane in two unbounded regions.*

Moreover, there exists an homeomorphism from the plane onto the plane, that sends a line onto the bisector.

Proof. A point X is in the bisector $Bi(P,Q)$ of P and Q if and only if X belongs to the intersection of two, equal radii, d-circles centered in P and Q respectively. Two such d-circles intersect at most in two points (cf. Th. 1) each of them belonging to one of the half planes L^+ and L^- that line PQ determines.

Because of the central symmetry of the d-circles, it suffices to study $Bi(P,Q)$ in one of those half planes, say L^+, as $Bi(P,Q)$ in L^- is obtained from $Bi(P,Q) \cap L^+$ via an angle π turn, centered in the midpoint O of P and Q.

Note that $O = C_d(P,e) \cap C_d(Q,e)$, where e is equal to $d(P,Q)/2$. So O belongs to $Bi(P,Q)$ and e is the smallest radius for which the d-circles do intersect.

Increasing the radius of this two equal radii d-circles centered in P and Q allows to parametrize $Bi(P,Q)$ defining $f : R \to Bi(P,Q)$ as follows:

$$f(t) = C_d(P,(|t|+1)e) \cap C_d(Q,(|t|+1)e) \cap L^{sign(t)}$$

where *sign(t)* means the sign of parameter t. Function f is continuous, bijective and $\lim_{t\to\infty} \|f(t)\| = \infty$ (see [5] for a detailed proof).

Compactifying both R and $Bi(P,Q)$ with a point at infinity and defining the image of the point of infinity of R to be the point of infinity of $Bi(P,Q)$, we obtain a continuous and bijective function from a compact space to a Hausdorff one, that is, then, an homeomorphism.

Via Schoenflies theorem from general topology, we extend the homeomorphism to the whole plane, yielding the desired result. ⊔

Now we are going to give more geometric information about bisectors. Let us introduce some notation. In what follows suppose a strictly convex distance d on the plane is given. Let us call C the unit d-circle. Given any two points P and Q on the plane, let m be the slope of the line determined by the center of C and the point S of contact of one supporting line of C parallel to the line PQ. There is no loss of generality in supposing that line PQ is horizontal and that the midpoint between P and Q is the origin O. Let $r_m(P)$ and $r_m(Q)$ be the lines of slope m through the points P and Q respectively. These two parallel lines determine a *band* of finite width between them.

Theorem 3. *$Bi(P,Q)$ is contained in the band determined by $r_m(P)$ and $r_m(Q)$ and is symmetrical with respect to the midpoint of P and Q. As a consequence, if $f(t) = (x(t), y(t))$ is the parametrization of $Bi(P,Q)$ in Theorem 2 then:*

$$\lim_{t\to\infty} \frac{y(t)}{x(t)} = m$$

Proof. Any point of the bisector $Bi(P,Q)$ is a point of intersection of two equal radii d-circles, centered at P and Q respectively. The one centered at Q is obtained from the one centered at P via a translation of its center of vector PQ. These two d-circles either don't intersect, are tangent at the midpoint of P and Q, or are the endpoints of a chord of slope m which, evidently, separates P from Q. Symmetry of the bisectors follows from symmetry of the d-circles with respect to their centers and the first assertion is proved.

To prove the consequence note that:

$$\lim_{t\to\infty} |y(t) - mx(t)| < k$$

where k is half of the vertical width of the band, unless $m = \infty$. In this case $x(t)$ remains bounded while $y(t) \to \infty$. ⊔

We conclude that the *asymptotic direction* of $Bi(P,Q)$ is m. But this doesn't mean at all that an asymptotic line must exist for $Bi(P,Q)$ and even if it exists we only know its slope but not its exact situation. The following theorem gives a necessary and sufficient condition for an asymptotic line for $Bi(P,Q)$ to exist.

Let us introduce first some more notation. Given two distinct points P and Q, consider the d-circle centered in the midpoint of P and Q and passing through P and Q. There is no loss of generality in supposing that this d-circle is the unit circle C and that PQ is horizontal (changing the reference system and scaling if necessary) so the origin O is the midpoint between P and Q. As before let S be the point of contact of the supporting line of C parallel to the line PQ. Note that point S is the highest point in C. Chords $c(h)$ of C parallel to line PQ (i. e. horizontal) at distance h from S are divided in two segments $c_1(h)$ and $c_2(h)$ by the line OS. Let $s_1(h)$ and $s_2(h)$ be their respective lenghts (See Fig.2).

In what follows let us assume that P, Q, C, S, $s_1(h)$ and $s_2(h)$ are as described. With this notation the existence of an asymptotic line for $Bi(P,Q)$ is characterized as follows.

Theorem 4. *A necessary and sufficient condition in order that an asymptotic line for $Bi(P,Q)$ exists is the existence of the following limit:*

$$\lim_{h\to 0} \frac{s_1(h)}{s_2(h)} = l.$$

If limit l exists then the asymptotic line is the one having slope m and passing through a point $T \in [P,Q]$ such that:

$$\frac{\| P - T \|}{\| T - Q \|} = l$$

Proof. Note first that points in $Bi(P,Q)$ are also characterized as being the centers of increasing d-circles containing both P and Q, and that the bigger the radii, the farther with respect to O the centers.

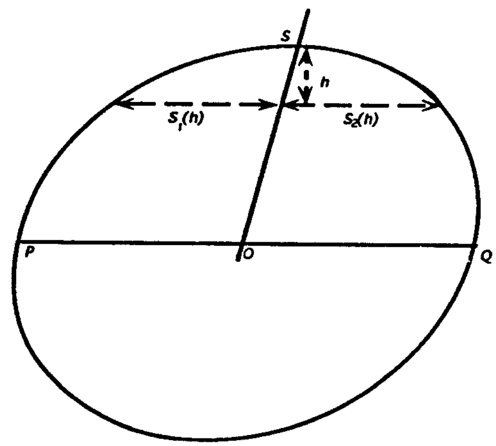

Fig. 2

Let $C(t)$ be a d-circle of radius t through P and Q and let $O(t) = (x(t), y(t))$ its center.

We know this center is inside the band determined by $r_m(P)$ and $r_m(Q)$ and we are interested in determining the evolution of its *position* as $t \to \infty$, where by *position* $P(t)$ of a center $O(t)$ we mean the ratio of the distances from that center to the sides $r_m(P)$ and $r_m(Q)$ of the band in the PQ direction.

The key is to observe that $O(t) = (x(t), y(t))$ is on the line of slope m through the highest point $S(t)$ of $C(t)$, as O is on the line of slope m through the highest point S of C, and $C(t)$ and C are homothetic (See Fig.3).

The position $P(t)$ of $O(t) = (x(t), y(t))$ is determined by the ratio of the lenghts of the segments that line $S(t)O(t)$ intercepts in the chord PQ of $C(t)$ (See again Fig. 3).

Obviously an asymptotic line for $Bi(P, Q)$ will exist if and only if it exists

$$\lim_{t \to \infty} P(t)$$

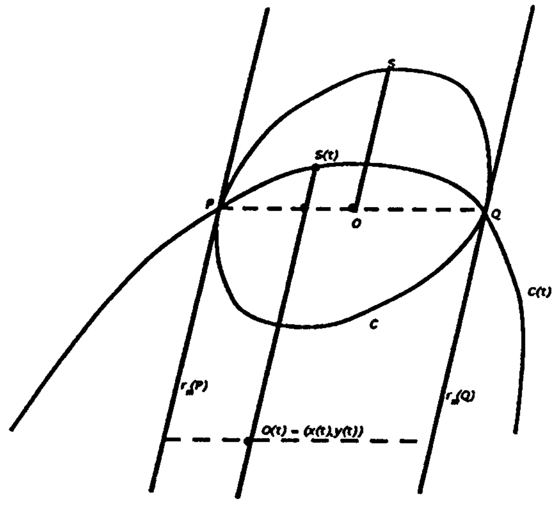

Fig. 3

and in this case this limit indicates where is the asymptotic line situated inside the band .

In order to calculate $P(t)$ in $C(t)$, we apply an homothecy which transforms $C(t)$ in C, doing the calculation in C . This is possible because the ratio between lenghts remains invariant through an homothecy. This homothecy takes chord PQ of $C(t)$ to a chord of C parallel to PQ and at a certain distance h from S. It is easy to see that as $t \to \infty$, $h \to 0$ and then:

$$\lim_{t\to\infty} P(t) = \lim_{h\to 0} \frac{s_1(h)}{s_2(h)}.$$

Note that if $l = 0$ then the asymptotic line is $r_m(P)$ and that if $l = \infty$ the asymptotic line is $r_m(Q)$. ⊔

Remark 1. If an asymptotic line for $Bi(P,Q)$ doesn't exist, then if l_1 (respectively l_2) is: $\underline{\lim}_{h\to 0} \frac{s_1(h)}{s_2(h)}$ (respectively $\overline{\lim}_{h\to 0} \frac{s_1(h)}{s_2(h)}$) we have that the bisector $Bi(P,Q)$ approachs infinitely many times the line l_1 (respectively l_2) of slope m and intersecting segment PQ in a point T (respectively T') such that: $\frac{\|P-T\|}{\|T-Q\|} = l_1$ (*respectively* $\frac{\|P-T'\|}{\|T'-Q\|} = l_2$). This situation implies that $Bi(P,Q)$ must have infinite *inflection points* inside the band determined by the lines l_1 and l_2. Here, by *inflection points* we mean a point in the curve $Bi(P,Q)$ through which there is no line that leaves the curve in one of the halfplanes determined by the line.

Remark 2. Let C and S be as before. Suposse that $f : (x_0 - \delta, x_0 + \delta) \to R$ is a function such that $f(x_0) = S$ and whose graph equals C in some neighbourhood of S. Note that as f is continuous and strictly convex, the lateral derivatives $f'_+(x_0)$ and $f'_-(x_0)$ always exist. Moreover, as $S = f(x_0)$ is a relative maximum for f, it follows that $f'_+(x_0) \leq 0$ and $f'_-(x_0) \geq 0$.

If curve C is not differentiable at S, i. e. if $f'_+(x_0) \neq f'_-(x_0)$, the next Proposition, whose proof is given in the Appendix, assures that limit l always exists and explicitly gives its value. Let P, Q, C, S, $s_1(h)$, $s_2(h)$ and f as before.

Proposition 1. *If curve C is not differentiable at S, i. e. $f'_+(x_0)$ and $f'_-(x_0)$ are different, then l exists and takes the value:*

$$l = \lim_{h \to 0} \frac{s_1(h)}{s_2(h)} = \frac{1/f'_-(x_0) - 1/m}{1/m - 1/f'_+(x_0)}$$

where m is the slope of line OS, unless $m = \infty$ but then:

$$l = \frac{-f'_+(x_0)}{f'_-(x_0)}.$$

If curve C is differentiable at S, then $f'_+(x_0) = f'_-(x_0) = 0$. It is possible, in this case of differentiability of C at S, that limit l does not exist. But if limit l does exist then the following Proposition, whose proof is also given in the Appendix, can simplify its calculation.

Proposition 2. *If curve C is differentiable at S, then the line OS, which determines the segments $c_1(h)$ and $c_2(h)$, of lenghts $s_1(h)$ and $s_2(h)$ respectively, on the chord $c(h)$ of C, can be replaced by the perpendicular through S to PQ without changing the value of the limit l (see Fig. 4), i. e.*

$$\lim_{h \to 0} \frac{s_1(h)}{s_2(h)} = \lim_{h \to 0} \frac{p_1(h)}{p_2(h)}.$$

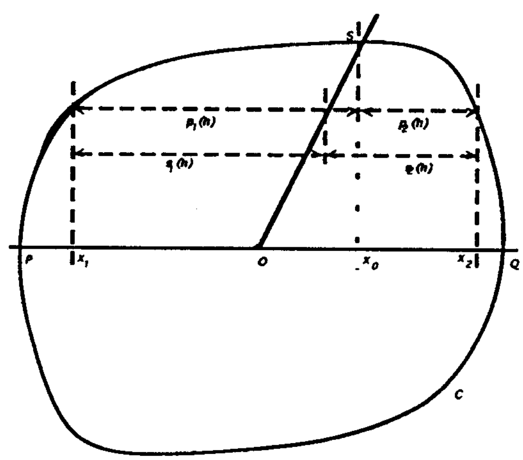

Fig. 4

The next example consists on a strictly convex distance d such that any pair of points P and Q, lying on an horizontal line, have a bisector $Bi(P,Q)$ with an asymptotic line not centered in the band determined by the lines $r_m(P)$ and $r_m(Q)$. In the calculation of limit l we will make explicit use of Proposition 2.

Example. Consider two arcs of Euclidean circles of radius r and R, with $r < R$, the second centered in the origin $(0,0)$ and the first somewhere between the origin $(0,0)$ and point $(0,R)$, so that these two arcs have a common tangent at point $(0,R)$. As indicated in Fig.5, they can be considered as part of a unit d-circle C whose highest point is $S = (0,R)$.

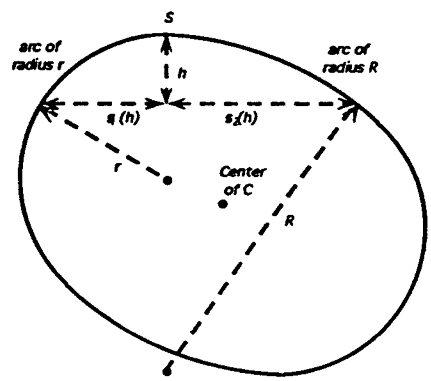

Fig. 5

Let us calculate limit l in this case, in order to study the existence of an asymptotic line for bisectors of pairs of horizontal points for such a distance:

$$\lim_{h\to 0}\frac{s_1(h)}{s_2(h)} = \lim_{h\to 0}\frac{\sqrt{r^2-(r-h)^2}}{\sqrt{R^2-(R-h)^2}} = \sqrt{\lim_{h\to 0}\frac{2rh-h^2}{2Rh-h^2}} = \sqrt{\lim_{h\to 0}\frac{2r-h}{2R-h}} = \sqrt{\frac{r}{R}}$$

We conclude that for a pair of horizontal points P and Q and for the distance d induced by such a C, an asymptotic line for $Bi(P,Q)$ exists that is parallel to the line passing through the center O (of C) and S, and such that its point T of intersection with segment PQ verifies the relation:

$$\frac{\| P-T \|}{\| T-Q \|} = \sqrt{\frac{r}{R}}$$

In this case $f'(x_0) = 0$ and $f''_+(x_o) \neq f''_-(x_0)$. Then, as $1/r = f''(x_0)/(1+f'(x_0))^{3/2}$ where r is the radius of curvature, it follows that :

$$\sqrt{\frac{r}{R}} = \sqrt{\frac{f''_+(x_0)}{f''_-(x_0)}}$$

The result in this example suggests that maybe we could, in some cases, eliminate the direct computation of the limit l if we know the second derivatives of curve C at S, in view of the relation between the radii of curvature of curve C at S and the second derivatives of function f at x_0. Moreover limit l can be calculated from the knowledge of the derivatives of the curve C at S, as the following Proposition (whose proof is included in the Appendix), establishes.

Proposition 3. *If curve C is differentiable of order p at S and the following two conditions hold:*

(i) $f'(x_0) = f''(x_0) = ... = f^p(x_0) = 0$.

(ii) $f_+^{p+1}(x_0)$ and $f_-^{p+1}(x_0)$ are distinct.

Then:

$$l = \sqrt[p+1]{\frac{f_+^{p+1}(x_0)}{f_-^{p+1}(x_0)}}.$$

3. Nice and not so Nice Distances

Klein and Wood[4] gave no examples of "nice" distances different from the Euclidean one, possibly because in the definition of "niceness" some conditions appear which are difficult to stablish. Note that strictly convex distances verify trivially (i) and (ii) in the definition of a nice distance. (iii) is also verified as strictly convex distances are additive along lines[5]. First part of (iv) follows from Theorem 2. As we will see in this Section, pairs of bisectors that intersect infinitely many times, do exist for some strictly convex distances. Thus the class of strictly convex distances is not included in the class of nice distances. However, among the strictly convex distances, those which have a semialgebraic curve as the boundary of the unit ball are nice in the sense of Klein and Wood, as it is proved in the next theorem. With this result we are able to construct many and easy to handle examples of nice distances, by giving a finite number of algebraic conditions.

Theorem 5. *Let d be a strictly convex distance such that the boundary of the unit ball is a semialgebraic curve. Then the bisector of any two points is also a semialgebraic curve. As a consequence the intersection of any two bisectors has a finite number of connected components.*

Proof. Let P and Q be any two points. There is no loss of generality in supposing that P is the origin $O = (0,0)$ and that Q has coordinates $(q,0)$. By symmetry it is enough to study $Bi(P,Q)$ in the half plane $y \geq 0$. Let:

$$f_1(x,y) = 0, ..., f_r(x,y) = 0, g_1(x,y) \geq 0, ..., g_s(x,y) \geq 0, y \geq 0 \qquad (3.1)$$

$$f_1'(x,y) = 0, ..., f_r'(x,y) = 0, g_1'(x,y) \geq 0, ..., g_s'(x,y) \geq 0, y \geq 0 \qquad (3.2)$$

be the equations and inequations which define, on the halfplane $y \geq 0$, the boundaries B_P and B_Q of the unit balls centered at P and Q respectively. A point

$X = (x_0, y_0)$ is in $Bi(P,Q)$ if and only if $d(R,P) = d(R,Q)$, that, according to the definition of d, means:

$$\frac{\sqrt{x_0^2+y_0^2}}{\sqrt{x_1^2+y_1^2}} = \frac{\sqrt{(x_0-q)^2+y_0^2}}{\sqrt{(x_2-q)^2+y_2^2}} \tag{3.3}$$

where (x_1, y_1) is the unique point of intersection of line PX with B_P in $y \geq 0$ and (x_2, y_2) is the unique point of intersection of line QX with B_Q in $y \geq 0$. This means that the bisector is the set of solutions of (3.3), where (x_1, y_1) is the unique solution of the system of equations and inequations (3.1) plus the linear equation $y = (y_0/x_0)x$; and (x_2, y_2) is the unique solution of the system of equations and inequations (3.2) plus the linear equation $(x-x_0)/(y-y_0) = (q-x_0)/(-y_0)$. Being therefore the bisector the projection of a semialgebraic set, it itself is semialgebraic [8]. Being a semialgebraic set with empty interior and with an infinite number of points, it must be a semialgebraic curve.

Last assertion follows from the fact that the intersection of two semialgebraic sets is a semialgebraic set and any semialgebraic set has a finite number of connected components [8]. ⊔

The next example consists on a strictly convex distance d such that any pair of points P and Q, lying on an horizontal line, have a bisector $Bi(P,Q)$ without asymptotic line in the band determined by the lines $r_m(P)$ and $r_m(Q)$. This fact will have, as a consequence, the existence of a convex distance violating second part of (iv) in the definition of nice distances.

Example. We are going to define a strictly convex distance d, with the property of having some pairs of bisectors with infinitely many intersection points. The unit d-circle C will be the union of two arcs c_1 and c_2 which are both isometric copies of the graph G of the function g defined in the next paragraphs, pasted in a convenient way.

First consider function $f : [-1,1] \rightarrow R$ defined as:

$$f(x) = \begin{cases} x, & \text{if } x \leq 0; \\ 2^n, & \text{if } 2^n - 2^{n-2} < x \leq 2^{n+1} - 2^{n-1} \text{ with n an integer.} \end{cases}$$

Note that on the negative X-axis f is the identity function and on the positive X-axis f is a staircase function which is constant and equal to 2^n in each interval $I_n = (2^n - 2^{n-2}, 2^{n+1} - 2^{n-1}]$ with n an integer. Now define $g : [-1,1] \rightarrow R$ as:

$$g(x) = \int_0^x f(x)dx$$

Obviously function g is continuous and equals function $x^2/2$ on the negative X-axis. Function g equals $x^2/2$ on the positive X-axis *only at the points* of the form 2^n with n an integer, because from 0 to 2^n, the area under function f equals the area under the identity function. At the rest of the points, that is on each of the intervals of the form $(2^n, 2^{n+1})$, function g is linear. Think of the graph of function g as the lower half of the boundary of a convex body. As it is not strictly convex, we want to modify function f above. This modification will be made in two steps:

Step 1. Consider any two consecutives steps of function f, say step S_n over the interval I_n and step S_{n+1} over the interval I_{n+1}, and connect them by a line segment of big slope M, through the "midpoint" between the two steps. Function f with this modification becomes a continuous function which looks like a true staircase, where the jonction between two consecutive steps is a line segment of slope M.

Step 2. The horizontal steps of the staircase described in Step 1 will now be slightly sloped, connecting any two consecutive jonctions of the staircase, say J_n and J_{n+1} around step S_n, by a line segment of small slope m through the point $(2^n, 2^n)$ of step S_n (see Fig.6 below).

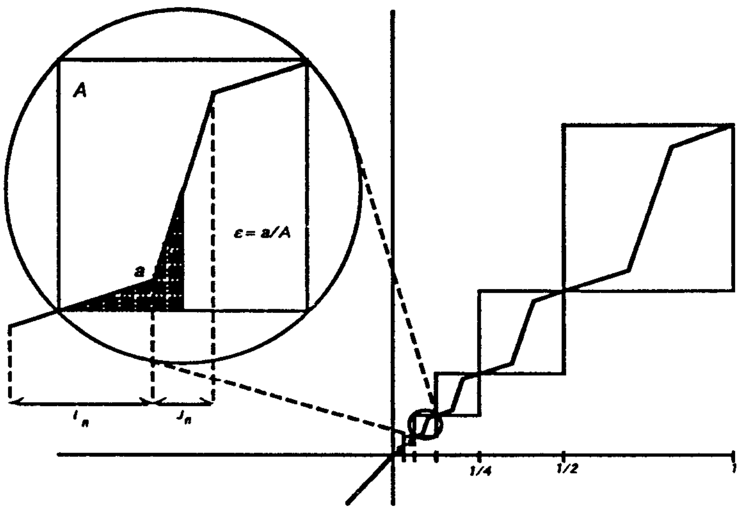

Fig. 6

Note that the modifications made in Steps 1 and 2 on function f affect f only on the positive X-axis, on the negative X-axis remaining unchanged, i. e. the identity function.

Let us call f again, function f modified as indicated in Steps 1 and 2 and let us consider as before $g : [-1, 1] \to R$ defined as:

$$g(x) = \int_0^x f(x)dx$$

Function g is of class C^1 because f is continuous. g equals function $x^2/2$ on the negative X-axis and equals function $x^2/2$ on the positive X-axis *only at the points* of the form 2^n with n integer, because at these points the areas above and below the graph of function $f(x) = x$ are compensated (see Fig.6 again). At the rest of the points, that is on each of the intervals of the form $(2^n, 2^{n+1})$, being the integral of a linear function, graph of function g is an arc of parabola. As $g' = f$ is strictly increasing on the interval $[0,1]$, the graph of g above this interval is a strictly convex curve. On the interval $[-1,0]$, being a single arc of the parabola $x^2/2$, the graph of g is also strictly convex.

Call G the graph of g and let the arc c_1 be $c_1 = G$ and c_2 equal to c_1 rotated around point $(0, 1/2)$ an angle equal to π. The unit d-circle C will be $c_1 \cup c_2$.

The key of this example is to observe that limit l doesn't exist for bisectors of pairs of points in a horizontal line as we are going to prove now.

First note that functions $s_1(h)$ and $s_2(h)$ defined on Theorem 4 are the two branches γ_1 and γ_2 of the "inverse" function of g; because of this we can calculate $s_1(h) = \gamma_1(h) = \sqrt{2h}$, and $s_2(h) = \gamma_2(h)$ must be the value z such that the area under f between the lines $x = 0$ and $x = z$ equals h.

For points of the form $h = 2^{2n-1}$ with n integer is $s_1(h) = s_2(h) = 2^n$ and then its quotient:

$$\frac{s_1(h)}{s_2(h)} = 1$$

Let ϵ be the quotient a/A, where A is the area of a square $[2^n, 2^{n+1}]^2$ and a is the portion of A under the graph of f and between the lines $x = 2^n$ and $x = 2^{n+1} - 2^{n-1}$ (See Fig.6).

For points of the form $h = 2^{2n}(1+\epsilon)$ with n integer, we have $s_1(h) = 2^n\sqrt{2(1+\epsilon)}$ and $s_2(h) = 2^{n+1} - 2^{n-1}$ and so:

$$\frac{s_1(h)}{s_2(h)} = 2/3\sqrt{2(1+\epsilon)}$$

Because $0 < \epsilon < 1/8$ (see Fig.6) we have:

$$2/3\sqrt{2(1+\epsilon)} < 2/3\sqrt{2(1+1/8)} = 1$$

and then we can conclude that :

$$\lim_{h \to 0} \frac{s_1(h)}{s_2(h)}$$

does not exist as:

$$\frac{s_1(h)}{s_2(h)}$$

takes the values 1 and $2/3\sqrt{2(1+\epsilon)}$ infinitely many times when $h \to 0$. Accordingly with Remark 1 following Theorem 4, the bisector $Bi(P,Q)$ of any pair of horizontal points P and Q must have infinitely many inflection points.

Now we can obtain the wanted conclusion: as bisectors are invariant through simultaneous translation of the pair of points P and Q, we can apply some *small* translation to the pair of points P and Q in the direction of the band in which $Bi(P,Q)$ is contained. The bisector for the new pair of points P' and Q', $Bi(P',Q')$, would intersect $Bi(P,Q)$ in an infinite number of points. This strictly convex distance d would not be nice in the sense of Klein and Wood as it has pairs of bisectors whose intersection has an infinite number of connected components. This result is really intriguing because the boundary of the unit d-circle is of class C^1 (from the continuity of f) and has second derivative except at a countable set of points with only one limit point.

4. The Topology of Voronoi Regions

Though strictly convex distances are in general not nice, they produce, as nice distances do[3], Voronoi diagrams with very good properties.

Let d be a strictly convex distance on the plane and A a finite collection of points. Let $H(P,Q) = \{X \in R^2 : d(X,P) - d(X,Q) < 0\}$. Then:

$$R_A(P) = \bigcap_{Q \in A - \{P\}} H(P,Q)$$

is *the Voronoi region of P with respect to A* and:

$$Vor_d(A) = \bigcup_{P \in A} Bd\, R_A(P)$$

is *the Voronoi diagram of A with respect to the distance d.*

Theorem 6. *(Properties of Voronoi regions) Let d be a strictly convex distance on the plane and A a finite collection of points. Then:*

(i) $R_A(P)$ is an open and not empty subset of the plane and $R_A(P) = \{X \in R^2 : d(X,P) < d(X,Q)$, for every $Q \in A - \{P\}\}$.

(ii) $R_A(P)$ is star-shaped as seen from P.

(iii) $Cl\, R_A(P) = \{X \in R^2 : d(X,P) \leq d(X,Q)$, for every $Q \in A - \{P\}\}$, where Cl denotes the topological closure.

(iv) $\bigcup_{P \in A} Cl\, R_A(P) = R^2$.

Proof. (i) Note that function $d_P : R^2 \to R^+$ defined as $d_P(X) = d(P,X)$ is continuous and then function $g : R^2 \to R$ defined as $g(X) = d_P(X) - d_Q(X)$ is also a continuous function. $H(P,Q)$ is the set of points where function g takes strictly negative values and so is open and it is not empty, as P always belongs to $H(P,Q)$. Being $R_A(P)$ a finite intersection of open sets, it is itself an open set containing always point P. Last equality in (i) follows obviously from the definition.

(ii) is derived from the d-star-shapedness of regions, as d is additive along lines. To prove it we follow Klein[3]. Being d-star-shapedness stable through intersections it suffices to prove that $H(P,Q)$ is d-star-shaped as seen from P . Let $X \in H(P,Q)$, and $Y \in R^2$ such that:

$$d(P,X) = d(P,Y) + d(Y,X)$$

If $Y \notin H(P,Q)$ then $d(Q,X) \leq d(Q,Y) + d(Y,X) \leq d(P,Y) + d(Y,X) = d(P,X)$ contradicting the fact that $X \in H(P,Q)$. So $Y \in H(P,Q)$.

(iii) Let us call F the set of points $X \in R^2$ verifying: $d(X,P) \leq d(X,Q)$ for every $Q \in A - \{P\}$. Being F a closed set containing $R_A(P)$, it follows that F contains the closure $Cl\, R_A(P)$ of $R_A(P)$.

Conversely, if $X \in F$ let us prove that X is an accumulation point for $R_A(P)$ by proving that the whole open segment (P,X) is contained in every $H(P,Q)$, for every $Q \in A - \{P\}$ (and then the whole open segment (P,X) will be contained in $R_A(P)$). Let $Q \in A - \{P\}$:

(a) If $d(X,P) < d(X,Q)$, then $X \in H(P,Q)$ and therefore not only the open segment (P,X) but the closed segment $[P,X]$ is contained in $H(P,Q)$.

(b) If $d(X,P) = d(X,Q)$, let $Y \in (P,X)$. If $Y \notin H(P,Q)$ (i. e. $d(Y,P) \geq d(Y,Q)$) then: $d(X,Q) = d(X,P) = d(X,Y) + d(Y,P) \geq d(X,Y) + d(Y,Q)$, that together with the triangular inequality leads to: $d(X,Q) = d(X,Y) + d(Y,Q)$. But this last equality would mean Y belongs to the open segment (X,Q), contrary to the assumption of $Y \in (X,P)$. Then Y must belong to $H(P,Q)$.

(iv) is obvious. ⊔

5. Intersection Properties of Associated Bisectors

In the Euclidean case bisectors are straight lines and so any two bisectors intersect at most once and transversaly. In the general case of an arbitrarly strictly convex distance d, we just know that bisectors are simple curves and therefore we ask how does the intersection of two bisectors look like. We have seen in Section 3 some wild behaviour of the intersection for some pairs of bisectors. In this section we study this problem in the case of ***associated*** bisectors, i. e. when they are any pair among the bisectors determined by three given points.

If the closure of the unit d-ball $Cl\, B_d(0,1)$ has a unique supporting line through each point of its boundary (***smooth boundary***), then given any two points P and Q, the set of d-circles passing through the two points fills the whole plane except the points in the line through P and Q different from P and Q. Because of this, in the case of smooth boundary, given three points P, Q and R, one of the two following facts holds:

(i) Either they lie on some d-circle (***d-cocircular***), and then $Bi(P,Q)$ and $Bi(P,R)$ intersect at the center of the unit d-circle containing P, Q and R.

(ii) Or they lie on a straight line (***collinear***), and then $Bi(P,Q)$ and $Bi(P,R)$ don't intersect.

If the closure of the unit d-ball $Cl\, B_d(0,1)$ has more than one (and then infinitely many) supporting line through some point of its boundary, then sets of three points can exist that lie neither on any straight line nor on any d-circle. As an example of, think of a strictly convex distance d whose unit d-ball has the shape shown in Fig. 7(a) and note that the set of d-circles, passing through any two points P and Q situated on a vertical line, just fills the shaded region of the plane of Fig.7(b).

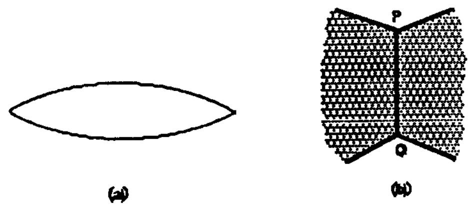

Fig. 7

A third point R out of this region and not on the straight line PQ, together with points P and Q, form a set of three points which are neither d-cocircular nor collinear. Note that, in this case, their associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ cannot intersect and they have the same asymptotic directions.

Proposition 4. *Let P, Q and R be three points in the plane. Let m and m' be the asymptotic directions of the associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ respectively. Then:*

(i) The associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect at most once.

(ii) The associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect exactly once if and only if the points P, Q and R are d-cocircular.

(iii) In the case of smooth boundary, the associated bisectors intersect if and only if its asymptotic directions m and m' are different.

(iv) In the case of non smooth boundary, if the asymptotic directions m and m' are different, the bisectors intersect exactly once but nothing can be said if the directions coincide.

Proof. For (i) note that $Bi(P,Q) \bigcap Bi(P,R)$ is not empty if and only if P, Q and R are d-cocircular. From Theorem 1 of Section 1, at most one d-circle exists containing three given points. (ii) is obvious from (i). For (iii), if $m \neq m'$ then bisectors intersect as a consequence of Theorems 2 and 3 of Section 2. Conversely, if bisectors intersect, we know that P, Q and R are d-cocircular and therefore not collinear. This means that the direction of the line PQ is different from the direction of the line PR. Now, having smooth boundary, the contact points of the supporting lines in this two directions must be different. Thus the asymptotic directions of the bisectors are also different. ⊔

We will say that the bisectors $Bi(P,Q)$ and $Bi(P,R)$ associated to three given points P, Q and R intersect *transversaly* if the following two conditions holds:

(i) They intersect (necessarily in exactly one point).

(ii) It exists an homeomorphism from the plane onto the plane sending each of the coordinate axes onto each bisector.

The two open regions that $Bi(P,Q)$ determines are, each one, characterized as the set of points where function $f(X) = d(X,P) - d(X,Q)$ is less than (respectively greater than) zero ($\{f < 0\}$ and $\{f > 0\}$). We can say then that $Bi(P,Q)$ divides the plane into two regions each of them having an *associated sign* $<$ or $>$. Note that taking $-f$ instead of f leads to the interchange of signs in the regions.

Similarly, the two open regions that $Bi(P,R)$ determines are characterized by the signs of function $g(X) = d(X,P) - d(X,R)$.

If $Bi(P,Q)$ and $Bi(P,R)$ intersect, then around the intersection point appears a set of regions characterized each of them by a pair of signs, first sign being the one of function f, second sign the one of function g. This pair of signs will be called *a combination of signs*. The possible combinations are $<<$, $<>$, $><$ and $>>$.

Transversality in the intersection of bisectors $Bi(P,Q)$ and $Bi(P,R)$ is equivalent to the appearence of four regions, each of them with one of the possible combination of signs[5]. Non appearence of some combination indicates that the intersection is not transversal and in this case one of the remaining signed regions will be not connected as shown in Fig.8.

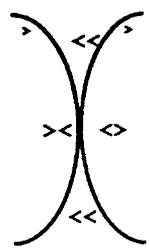

Fig. 8

Theorem 7. *Let P, Q and R be three points in the plane. If their associated bisectors $Bi(P,Q)$ and $Bi(P,R)$ intersect, then they do it transversaly.*

Proof. Suppose the intersection is not transversal and that the combination of signs that doesn't appear is $>>$ as in Fig.8. Other cases can be reduced to this one by changing the signs of functions f or g.

$Bi(Q,R)$ is the set of points where function $h(X) = d(X,Q) - d(X,R)$ equals zero ($Bi(Q,R) = \{h = 0\}$) and depending on the signs of functions f and g, four different cases can occur:

1. Case $++$. $h = -f + g$ (if f and g are both as defined)
2. Case $--$. $h = +f - g$ (if f and g both change signs)
3. Case $-+$. $h = +f + g$ (if only f changes sign)
4. Case $+-$. $h = -f - g$ (if only g changes sign)

Let us study separately two of the four cases, the other two being similar.

Case $++$: $f(X) = d(X,P) - d(X,Q)$ and $g(X) = d(X,P) - d(X,R)$. Note that the region signed $<<$, that is $\{f < 0\} \cap \{g < 0\}$, is by definition the Voronoi region of point P, that would then be not connected, contradicting the connectedness of Voronoi regions.

Case $-+$: $f(X) = d(X,Q) - d(X,P)$ and $g(X) = d(X,P) - d(X,R)$. Note that in this case $h(X) = d(X,Q) - d(X,R) = f(X) + g(X)$ can equal zero only where

f and g have opposite signs, therefore $Bi(Q,R) = \{h = 0\}$ must be a subset of the unions of the region signed >< with the region signed <>.

$Bi(Q,R) = \{h = 0\}$ passes through the intersection point of $Bi(P,Q) = \{f = 0\}$ with $Bi(P,R) = \{g = 0\}$ and then has not any other point of intersection with them.

Suppose first that $\{h = 0\}$ intersects $\{f = 0\}$ and $\{g = 0\}$ transversaly. Then it exists a non empty region of points X satisfying $f(X) < 0$, $g(X) < 0$ and $h(X) > 0$, i. e. it exists points X for which $d(X,Q) < d(X,P) < d(X,R)$ and $d(X,Q) > d(X,R)$ hold simultaneously, which is impossible. Therefore $\{h = 0\}$ must be enterely contained in either >< or <> region.

If $\{h = 0\}$ is a subset of >< region. Then the region $\{h < 0\} \cap \{f < 0\}$, being the Voronoi region of point Q, must be non empty and then the region $\{h < 0\} \cap \{g > 0\}$, which is the Voronoi region of point R, is empty.

If $\{h = 0\}$ is a subset of <> region, then either region $\{h < 0\} \cap \{f < 0\}$, which is the Voronoi region of point Q, or the region $\{h > 0\} \cap \{g > 0\}$, which is the Voronoi region of point R, is not connected.

In all cases we arrive to a contradiction. ⊔

6. Acknowledgements

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Appendix

Proof of Proposition 1. Let x_1 and x_2 be the two unique solutions of the equation $f(x) - f(x_0) = h$ (obviously x_1 and x_2 depends on h). Note that, from the existence of lateral derivatives, both:

$$\lim_{h\to 0}\frac{s_1(h)}{h} \text{ and } \lim_{h\to 0}\frac{h}{s_2(h)}$$

exist. If $m \neq \infty$ then:

$$\lim_{h\to 0}\frac{s_1(h)}{s_2(h)} = \lim_{h\to 0}\frac{s_1(h)}{h}\lim_{h\to 0}\frac{h}{s_2(h)} =$$

$$= \lim_{h\to 0}\frac{(x_0-x_1)-h/m}{f(x_0)-f(x_1)}\lim_{h\to 0}\frac{f(x_0)-f(x_2)}{(x_2-x_0)+h/m} = \frac{1/f'_-(x_0)-1/m}{1/m-1/f'_+(x_0)}.$$

If $m=\infty$ then:

$$\lim_{h\to 0}\frac{s_1(h)}{s_2(h)} = \lim_{h\to 0}\frac{s_1(h)}{h}\lim_{h\to 0}\frac{h}{s_2(h)} =$$

$$= \lim_{h\to 0}\frac{x_0-x_1}{f(x_0)-f(x_1)}\lim_{h\to 0}\frac{f(x_0)-f(x_2)}{x_2-x_0} = \frac{-f'_+(x_0)}{f'_-(x_0)}.$$

□

Proof of Proposition 2. Let $p_1(h)$ and $p_2(h)$ be the lenghts of the segments that the perpendicular through S to PQ determines on chord $c(h)$. All we need to prove is that:

$$\lim_{h\to 0}\frac{s_1(h)}{s_2(h)} = \lim_{h\to 0}\frac{p_1(h)}{p_2(h)}.$$

But $s_i(h)$ for $i=1,2$ differs from $p_i(h)$ for $i=1,2$ in $\pm h/m$, where m is the slope of line OS and thus is a finite constant $-\infty < m < +\infty$. For instance, if m is negative we would have $s_1(h) = p_1(h) + h/m$ and $s_2(h) = p_2(h) - h/m$ and then:

$$\lim_{h\to 0}\frac{s_1(h)}{s_2(h)} = \lim_{h\to 0}\frac{p_1(h)+h/m}{p_2(h)-h/m} =$$

$$= \lim_{h\to 0}\frac{p_1(h)}{p_2(h)}\lim_{h\to 0}\frac{1+h/mp_1(h)}{1-h/mp_2(h)}$$

So it suffices to prove that:

$$\lim_{h\to 0} h/p_i(h) = 0$$

for $i=1,2$. Let $S=(x_0,y_0)$ and suppose as before that $f:(x_0-\delta, x_0+\delta)\to R$ is a function such that $f(x_0)=S$ and whose graph equals C in this neighbourhood of x_0. As S is a local maximum of C, f must have a local maximum in x_0, and then

$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = 0$$

Being f continuous in x_0, when $h\to 0$ is $x_i \to x_0$ for $i=1,2$. As for $i=1,2$:

$$\frac{|h|}{|p_i(h)|} = \frac{|f(x_i)-f(x_0)|}{|x_i-x_0|}$$

it follows that:

$$\lim_{h\to 0}\frac{h}{p_i(h)}=\lim_{x_i\to x_0}\frac{f(x_i)-f(x_0)}{x_i-x_0}=0.$$

⊔

Proof of Proposition 3. Let x_1 and x_2 be the solutions of the equation $f(x)-f(x_0)=h$. By the generalized Cauchy Theorem applied to $[x_1,x_0]$ and to $[x_0,x_2]$ we have:

$$\frac{f(x_1)-f(x_0)}{(x_1-x_0)^{p+1}}=\frac{f^p(\xi)}{(p+1)!(\xi-x_0)}$$

and

$$\frac{f(x_2)-f(x_0)}{(x_2-x_0)^{p+1}}=\frac{f^p(\eta)}{(p+1)!(\eta-x_0)}$$

for some $\xi\in[x_1,x_0]$ and some $\eta\in[x_0,x_2]$. Then, as $s_1(h)=x_1-x_0$ and $s_2(h)=x_2-x_0$, we obtain that:

$$\lim_{h\to 0}\frac{s_1(h)}{s_2(h)}=\lim_{h\to 0}\sqrt[p+1]{\frac{(f^p(\eta)-f^p(x_0))/(\eta-x_0)}{(f^p(\xi)-f^p(x_0))/(\xi-x_0)}}=\sqrt[p+1]{\frac{f_+^{p+1}(x_0)}{f_-^{p+1}(x_0)}}$$

as claimed. ⊔