AN INTERPRETATION FOR THE TUTTE POLYNOMIAL

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ABSTRACT. For any matroid M realizable over $\mathbb{Q}$, we give a combinatorial interpretation of the Tutte polynomial $T_M(x, y)$ which generalizes many of its known interpretations and specializations, including

- Tutte's coloring and flow interpretations of $T_M(1-t, 0), T_M(0, 1-t)$,
- Crapo and Rota's finite field interpretation of $T_M(1-q^k, 0)$,
- the interpretation in terms of the Whitney *corank-nullity* polynomial,
- Greene's interpretation as the weight enumerator of a linear code and its recent generalization to higher weight enumerators by Barg,
- Jaeger's interpretation in terms of linear code words and dual code words with disjoint support,
- Brylawksi and Oxley's two-variable coloring formula.

1. INTRODUCTION

In his 1947 paper [11] Tutte defined a polynomial in two variables x, y associated to every finite graph G which turns out to be a powerful invariant of the graph up to isomorphism. In fact, this polynomial depends only on the *matroid* associate to the graph, and Crapo [5] observed that one can just as easily define the Tutte polynomial $T_M(x, y)$ for an arbitrary matroid. In subsequent years, many interesting interpretations for specializations of $T_M(x, y)$ were found; see [4].

The main result of this paper is a new interpretation for $T_M(x, y)$ when M is a matroid representable over $\mathbb{Q}$, that is when M is the matroid represented by the n column vectors of some $d \times n$ matrix with $\mathbb{Z}$ entries. We will often abuse notion and refer to this $d \times n$ matrix also as M. Note that since M has integer entries, it makes sense to think of it as a matrix over any field $\mathbb{F}$. For a field $\mathbb{F}$, let $Mat_{\mathbb{F}}(M)$ denote the matroid on the ground set $E := \{1, 2, \dots, n\}$ defined by interpreting the columns of M as vectors in $\mathbb{F}^d$. We say that M *reduces correctly* over the field $\mathbb{F}$ if $Mat_{\mathbb{Q}}(M) = Mat_{\mathbb{F}}(M)$, i.e. a subset of columns of M are linearly independent over $\mathbb{Q}$ if and only if they are linearly independent over $\mathbb{F}$. Note that for a fixed integer matrix M, there is a lower bound depending upon M such that any field whose characteristic is greater than this bound has the property that M reduces correctly over $\mathbb{F}$. For example, one can take this bound to be the maximum absolute value of all square subdeterminants of M. Given a vector in $x \in \mathbb{F}^n$, its *support set* is defined to be

$$\text{supp}(\mathbf{x}) := \{i : x_i \neq 0\}.$$

For a matroid M, let $r(M)$ denote the *rank*, that is the cardinality of all bases of M, and let M^* denote its *dual* or *orthogonal* matroid.

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Theorem 1. *Let M be a integer matrix and assume that p, q are prime powers such that M reduces correctly over $\mathbb{F}_p$ and $\mathbb{F}_q$. Letting a, b be indeterminates with $a + b = 1$, we have*

$$T_M\left(\frac{1+(p-1)a}{b}, \frac{1+(q-1)b}{a}\right) = \frac{1}{a^{r(M^*)}b^{r(M)}} \sum_{\substack{(\mathbf{x},\mathbf{y}) \in \mathrm{row}(M) \times \ker M \subseteq \mathbb{F}_p^n \times \mathbb{F}_q^n \\ \mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y}) = \varnothing}} a^{|\mathrm{supp}(\mathbf{x})|} b^{|\mathrm{supp}(\mathbf{y})|}$$

where here $\mathrm{row}(M)$ *is the row-space of M considered as a subspace of $\mathbb{F}_p^n$, and* $\ker M$ *is the kernel of the matrix M considered as a subspace of $\mathbb{F}_q^n$.*

A word or two is in order about the motivation for this result. Conversations with J. Goldman about the result in [9] had led the author to suspect that there might be an interpretation of $T_M(1-p, 1-q)$ for graphic matroids M which generalized Tutte's interpretations of $T_M(1-p, 0)$ and $T_M(0, 1-q)$ in terms of proper colorings and nowhere-zero flows, respectively. This led to Equation (2) in Section 3, which we state here as a separate corollary in the special case of graphic matroids, for the sake of readers interested primarily in graphs:

Corollary 2. *Let G be a graph with $v(G)$ vertices and $c(G)$ connected components. Then for any positive integers p, q, its Tutte polynomial $T_G(x, y)$ satisfies*

$$T_G(1-p, 1-q) = (-p)^{-c(G)}(-1)^{v(G)} \sum_{(\mathbf{x},\mathbf{y})} (-1)^{|\mathrm{supp}(\mathbf{y})|}$$

where the sum runs over pairs $(\mathbf{x}, \mathbf{y})$ in which

- $\mathbf{x}$ *is a vertex coloring of G with p colors,*
- $\mathbf{y}$ *is flow on the edges of G with values in any abelian group of cardinality q, and*
- *each edge contains non-zero flow if and ony if it is colored improperly.*

Here $|\mathrm{supp}(\mathbf{y})|$ *is the number of edges containing non-zero flow in* $\mathbf{y}$*, or equivalently, the number of improperly colored edges in* $\mathbf{x}$*.*

Subsequently, a literature search uncovered Jaeger's paper containing a result [8, Proposition 4] essentially equivalent to the $p = q$ case of Theorem 1, which then begged the question of a generalization in common with Corollary 2. This generalization is Theorem 1. What makes this result more flexible than Jaeger's is the "decoupling" of p and q, which allows them to be specialized independently. As a consequence, we recover (among other things) almost every known interpretation of the Tutte polynomial in terms of colorings, flows, finite fields, and codes.

The paper is structured as follows. Section 2 deduces the proof of Theorem 1 from a Tutte polynomial identity (Theorem 3) valid for all matroids. Section 3 explains how Theorem 1 implies other interpretations of the Tutte polynomial. Section 4 is devoted to remarks and open problems.

2. The main result

In this section we prove Theorem 1. It is possible to deduce it by a deletion-contraction argument exactly as in Jaeger's proof of the $p = q$ case [8, Proposition 4]. However, since Theorem 1 seems at first glance to be a statement only about matroids representable over $\mathbb{Q}$, we prefer to generalize it and deduce it from the following Tutte polynomial identity valid for all matroids. For a matroid M, let $r(M)$ denote the *rank* of M and let $|M|$ denote the cardinality of its ground set.

Theorem 3. *Let a, b, u, v be indeterminates with $a+b=1$. Then for any matroid M with ground set E we have*

$$T_M\left(\frac{1-ua}{b}, \frac{1-vb}{a}\right)$$
$$= \frac{1}{a^{r(M^*)}b^{r(M)}} \sum_{\varnothing \subseteq B \subseteq C \subseteq E} (-1)^{r(M|_B^*)} b^{|B|} T_{M|_B}(0,v) \cdot (-1)^{r(M/C)} a^{|M/C|} T_{M/C}(u,0)$$

Proof. The Tutte polynomial is the unique polynomial in x, y which is an isomorphism invariant of matroids satisfying the following three conditions (see [4]):

(i) $T_{(1)}(x,y) = x, T_{(0)}(x,y) = y$,
(ii) $T_{M_1 \oplus M_2}(x,y) = T_{M_1}(x,y) \cdot T_{M_2}(x,y)$,
(iii) If $e \in E$ is is neither a loop nor an isthmus, then

$$T_M(x,y) = T_{M-e}(x,y) + T_{M/e}(x,y).$$

Letting $f(M)$ denote the right-hand side of the equation in the statement of the theorem, it therefore suffices to check that it satisfies properties (i),(ii),(iii) with $x = \frac{1-ua}{b}, y = \frac{1-vb}{a}$. For properties (i) and (ii) this follows in a completely straightforward fashion from the same properties for $T_M(u,0), T_M(0,v)$. We leave the details to the reader.

To check property (iii), let $e \in E$ be neither a loop nor an isthmus, and let $\overline{f}(M)$ be the summation appearing in the right-hand side of the theorem, that is

$$(1) \quad \overline{f}(M) = \sum_{\varnothing \subseteq B \subseteq C \subseteq E} (-1)^{r(M|_B^*)} b^{|B|} T_{M|_B}(0,v) \cdot (-1)^{r(M/C)} a^{|M/C|} T_{M/C}(u,0)$$

Since e is neither a loop nor an isthmus, we have

$$r(M) = r(M-e) = r(M/e) + 1$$

and therefore (iii) is equivalent to the following:

$$\overline{f}(M) = a \cdot \overline{f}(M-e) + b \cdot \overline{f}(M/e).$$

To check this, we start with the summation (1) defining $\overline{f}(M)$ and decompose it into three sums according to whether $e \in E - C, e \in B, e \in C - E$. One can re-write the first sum using property (iii) applied to $T_{M/C}(u,0)$, and re-write the second sum using (iii) applied to $T_{M|_B}(0,v)$. However, we must first observe that if $e \in E - C$ then it is not an isthmus of M/C (else it would be an isthmus in M) and we can also assume that it is not a loop of M/C (else $T_{M/C}(u,0)$ would vanish). Similarly e is not a loop of $M|_B$ (else it would be a loop in M) and we can also assume that it is not an isthmus of $M|_B$ (else $T_{M|_B}(0,v)$ would vanish). Therefore we get

$$
\begin{aligned}
&\overline{f}(M) \\
&= \sum_{\substack{\varnothing \subseteq B \subseteq C \subseteq E \\ e \in E - C}} (-1)^{r(M|_B^*)} b^{|B|} T_{M|_B}(0,v) \cdot (-1)^{r((M/C)-e)} a^{|(M/C)-e|+1} T_{(M/C)-e}(u,0) \\
&+ \sum_{\substack{\varnothing \subseteq B \subseteq C \subseteq E \\ e \in E - C}} (-1)^{r(M|_B^*)} b^{|B|} T_{M|_B}(0,v) \cdot (-1)^{r((M/C)/e)+1} a^{|(M/C)/e|+1} T_{(M/C)/e}(u,0) \\
&+ \sum_{\substack{\varnothing \subseteq B \subseteq C \subseteq E \\ e \in B}} (-1)^{r((M|_B/e)^*)} b^{|M|_B/e|+1} T_{M|_B/e}(0,v) \cdot (-1)^{r(M/C)} a^{|M/C|} T_{M/C}(u,0) \\
&+ \sum_{\substack{\varnothing \subseteq B \subseteq C \subseteq E \\ e \in B}} (-1)^{r((M|_B-e)^*)+1} b^{|B-e|+1} T_{M|_B-e}(0,v) \cdot (-1)^{r(M/C)} a^{|M/C|} T_{M/C}(u,0) \\
&+ \sum_{\substack{\varnothing \subseteq B \subseteq C \subseteq E \\ e \in C - B}} (-1)^{r(M|_B^*)} b^{|B|} T_{M|_B}(0,v) \cdot (-1)^{r(M/C)} a^{|M/C|} T_{M/C}(u,0)
\end{aligned}
$$

Using the facts that

$$(M/C) - e \cong (M-e)/C \text{ and } M|_B \cong (M-e)|_B$$

when $e \in E - C$, the first sum above is exactly $a \cdot \overline{f}(M-e)$. Using the facts that

$$M|_B/e \cong (M/e)|_B \text{ and } M/C \cong (M/e)/C$$

when $e \in B$. the third sum above is exactly $b \cdot \overline{f}(M/e)$. We can rewrite the second, fourth and fifth sums as

$$(-a-b+1) \sum_{\substack{\varnothing \subseteq B \subseteq C \subseteq E \\ e \in C - B}} (-1)^{r(M|_B^*)} b^{|B|} T_{M|_B}(0,v) \cdot (-1)^{r(M/C)} a^{|M/C|} T_{M/C}(u,0).$$

However $-a-b+1=0$, so we have verified Equation (1).

□

Before using the previous result to prove Theorem 1, we remark that it generalizes the main result of [9]:

Corollary 4 ([9], Theorem 1).

$$T_M(u,v) = \sum_{A \subseteq E} T_{M|_A}(0,v) T_{M/A}(u,0)$$

Proof. In Theorem 3, take the limit as $a \to \infty$, so that $b \to -\infty$ and $\frac{a}{b} \to -1$. □

Proof of Theorem 1. Making the substitution $u = 1 - p, v = 1 - q$ in Theorem 3 gives

$$\begin{aligned} & T_M\left(\frac{1+(p-1)a}{b}, \frac{1+(q-1)b}{a}\right) \\ &= \frac{1}{a^{r(M^*)}b^{r(M)}} \sum_{\varnothing \subseteq B \subseteq C \subseteq E} (-1)^{r(M|^*_B)} b^{|B|} T_{M|_B}(0, 1-q) \times \\ &\qquad\qquad (-1)^{r(M/C)} a^{|M/C|} T_{M/C}(1-p, 0) \\ &= \frac{1}{a^{n-r(M)}b^{r(M)}} \sum_{A,B \subseteq E, A \cap B = \varnothing} a^{|A|} b^{|B|} (-1)^{r(M|^*_B)} T_{M|_B}(0, 1-q) \times \\ &\qquad\qquad (-1)^{r(M/(E-A))} T_{M/(E-A)}(1-p, 0) \end{aligned}$$

where the last equation comes from the substitution $C = E - A$.

When M is an integer matrix that reduces correctly over $\mathbb{F}_p$, the quantities

$$(-1)^{r(M)} T_M(1-p, 0) \quad \text{and} \quad (-1)^{n-r(M)} T_M(0, 1-q)$$

have well-known combinatorial interpretations (see e.g. [3, Theorem 12.4]). The first quantity is the number of vectors $\mathbf{x} \in \text{row}(M) \subseteq \mathbb{F}_p^n$ having no zero coordinates, that is, having $\text{supp}(\mathbf{x}) = E$. The second quantity is the number of vectors $\mathbf{y} \in \ker(M) \subseteq \mathbb{F}_q^n$ with $\text{supp}(\mathbf{y}) = E$. Furthermore, if M reduces correctly over $\mathbb{F}_p, \mathbb{F}_q$, then for any subset $B \subseteq E$ the matrix $M|_B$ obtained by restricting M to the columns indexed by B reduces correctly over $\mathbb{F}_p$. Likewise, for any subset $A \subseteq E$, one can perform row operations on M to obtain the following block triangular form (where here we have assumed for convenience that C is an initial segment of columns):

$$\begin{bmatrix} I_{r(C)} & * & * \\ 0 & 0 & * \\ 0 & 0 & M/C \end{bmatrix}$$

Here $I_{r(C)}$ is an identity matrix of size $r(C)$, and M/C is an integer matrix which represents the quotient matroid $Mat_{\mathbb{Q}}(M)/C$ and reduces correctly over $\mathbb{F}_q$.

We conclude that

$$\begin{aligned} & T_M\left(\frac{1+(p-1)a}{b}, \frac{1+(q-1)b}{a}\right) \\ &= \frac{1}{a^{n-r(M)}b^{r(M)}} \sum_{A,B \subseteq E, A \cap B = \varnothing} a^{|A|} b^{|B|} (-1)^{r(M|^*_B)} T_{M|_B}(0, 1-q) \times \\ &\qquad\qquad (-1)^{r(M/(E-A))} T_{M/(E-A)}(1-p, 0) \\ &= \sum_{A,B \subseteq E, A \cap B = \varnothing} a^{|A|} b^{|B|} \left| \left\{ \begin{array}{c} (\mathbf{x}, \mathbf{y}) \in \text{row}(M/(E-A)) \times \ker(M|_B) : \\ \text{supp}(\mathbf{x}) = A, \text{supp}(\mathbf{y}) = B \end{array} \right\} \right| \\ &= \sum_{A,B \subseteq E, A \cap B = \varnothing} a^{|A|} b^{|B|} |\{(\mathbf{x}, \mathbf{y}) \in \text{row}(M) \times \ker(M) : \text{supp}(\mathbf{x}) = A, \text{supp}(\mathbf{y}) = B\}| \\ &= \sum_{\substack{(\mathbf{x}, \mathbf{y}) \in \text{row}(M) \times \ker(M) \\ \text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y}) = \varnothing}} a^{|\text{supp}(\mathbf{x})|} b^{|\text{supp}(\mathbf{y})|} \end{aligned}$$

where the only tricky equality is the third. This uses the fact that if we suppress the zero coordinates from $\mathbf{x}, \mathbf{y}$ we obtain a bijection between

$$\{(\mathbf{x}, \mathbf{y}) \in \mathrm{row}(M) \times \ker(M) : \mathrm{supp}(\mathbf{x}) = A, \mathrm{supp}(\mathbf{y}) = B\}. \quad \square$$

and

$$\{(\mathbf{x}, \mathbf{y}) \in \mathrm{row}(M/(E-A)) \times \ker(M|_B) : \mathrm{supp}(\mathbf{x}) = A, \mathrm{supp}(\mathbf{y}) = B\}$$

We conclude this section with a series of remarks about Theorem 1.

Matroids representable over other fields.

If p, q are both powers of the same prime, let $\mathbb{F}$ denote the common prime field inside $\mathbb{F}_p, \mathbb{F}_q$., We can then replace our assumption in Theorem 1 that M is an integer matrix which reduces correctly in $\mathbb{F}_p, \mathbb{F}_q$, by the assumption that M is a matrix with entries in $\mathbb{F}$.

If furthermore $p = q$, we can replace this assumption by the assumption that M is a matrix with entries in $\mathbb{F}_p (= \mathbb{F}_q)$. This allows the useful interpretation (as in the references [7, 8]) of $\mathrm{row}(M)$ as an $\mathbb{F}_p$-linear *code* $\mathcal{C}$ and $\ker(M)$ as its *dual code* $\mathcal{C}^\perp$.

Graphic matroids.

Let G be a finite graph G, with some fixed but arbitrarily chosen orientation of its edges. Then the *node-edge incidence matrix* M which represents the graphic matroid corresponding to G is well-known to reduce correctly over any finite field $\mathbb{F}_p$. In this case, Tutte's original interpretations for $T_M(1-p, 0), T_M(0, 1-q)$ in terms of *proper vertex p-colorings* and *nowhere zero q-flows* (see next section) show that it is not important that $\mathbb{F}_p, \mathbb{F}_q$ are fields. One only needs abelian groups of cardinality p, q such as $\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}$. One may also omit the assumption that p, q are prime powers, and all the results still hold for graphic matroids.

The Crapo-Rota finite field trick.

In their seminal work on matroids, Crapo and Rota proved a result [6, Theorem 1, §16.4] which interprets the specialization $T_M(1-p^k, 0)$ of the Tutte polynomial when p is a prime power, and M is a matroid representable over $\mathbb{F}_p$ (see Equation 4 below). It turns out that the full generality of their result can actually be deduced from the special case with $k = 1$, using the fact that $\mathbb{F}_{p^k}$ is a k-dimensional vector space over $\mathbb{F}_p$ whenever p is a prime power. This is not how they proved their result, but we will nevertheless call this process of deducing a result for p^k from the $k = 1$ case the *Crapo-Rota finite field trick.* We now use this same trick to deduce a generalization of Theorem 1 which is in some sense no stronger, but is useful for some of the applications (see e.g. Corollary 11 below).

Theorem 5. *Let M, p, q be as in Theorem 1, and k, k' two positive integers. Then*

$$T_M\left(\frac{1+(p^k-1)a}{b}, \frac{1+(q^{k'}-1)b}{a}\right) = \frac{1}{a^{r(M^*)}b^{r(M)}} \times \sum_{\substack{((\mathbf{x}_1,\dots,\mathbf{x}_k),(\mathbf{y}_1,\dots,\mathbf{y}_{k'})) \in (\mathrm{row}(M))^k \times (\ker M)^{k'} \\ \bigcup_{i=1}^k \mathrm{supp}(\mathbf{x}_i) \cap \bigcup_{j=1}^{k'} \mathrm{supp}(\mathbf{y}_j) = \varnothing}} a^{|\bigcup_{i=1}^k \mathrm{supp}(\mathbf{x}_i)|} b^{|\bigcup_{j=1}^{k'} \mathrm{supp}(\mathbf{y}_j)|}$$

Proof. There is a field embedding $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^k}$ which makes the field $\mathbb{F}_{p^k}$ a k-dimensional vector space over $\mathbb{F}_p$. In other words, $\mathbb{F}_{p^k} \cong (\mathbb{F}_p)^k$ as $\mathbb{F}_p$-vector spaces. If M is a matrix with entries in $\mathbb{F}_p$, one can check that this identifies $\mathrm{row}(M) \subseteq \mathbb{F}_{p^k}^n$ with $\mathrm{row}(M)^k \subseteq (\mathbb{F}_p^n)^k$. Under this identification, an n-vector $\mathbf{x}$ in $\mathbb{F}_{p^k}^n$ is identified with a k-tuple of vectors $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ in $(\mathbb{F}_p^n)^k$ having the property that

$$\mathrm{supp}(\mathbf{x}) = \bigcup_{i=1}^k \mathrm{supp}(\mathbf{x}_i).$$

A similar discussion applies to $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^k}$ and $\ker(M)$, so the result follows from Theorem 1. □

Duality.
Note that both Theorems 1 and 3 agree with the well-known fact that

$$T_{M^*}(x,y) = T_M(y,x).$$

In Theorem 1 this follows from the fact that any matrix M^* having $\mathrm{row}(M^*) = \ker(M)$ represents the matroid dual to the matroid represented by M.

In Theorem 3 this follows from the fact that

$$M/A \cong M^*|_{(E-A)} \text{ and } M|_A \cong M^*/(E-A)$$

for any $A \subseteq E$.

3. Corollaries

In this section we give some of the special cases and corollaries of Theorems 1 and 5 which motivated our study.

Finite fields, colorings, and flows.

Taking the limit as $a \to \infty$ (so $b \to -\infty$) in Theorem 5 gives the following result.

Corollary 6. *Let M, p, q be as in Theorem 1, and k, k' two positive integers. Then*

$$T_M\left(1-p^k, 1-q^{k'}\right) = (-1)^{r(M)} \sum_{\substack{((\mathbf{x}_1,\dots,\mathbf{x}_k),(\mathbf{y}_1,\dots,\mathbf{y}_{k'})) \in (\mathrm{row}(M))^k \times (\ker M)^{k'} \\ \left(\bigcup_{i=1}^k \mathrm{supp}(\mathbf{x}_i)\right) \uplus \left(\bigcup_{j=1}^{k'} \mathrm{supp}(\mathbf{y}_j)\right) = E}} (-1)^{|\bigcup_{j=1}^{k'} \mathrm{supp}(\mathbf{y}_j)|}.$$

When $k = k' = 1$ this gives

$$T_M(1-p, 1-q) = (-1)^{r(M)} \sum_{\substack{(\mathbf{x},\mathbf{y}) \in \text{row}(M) \times \ker M \\ \text{supp}(\mathbf{x}) \uplus \text{supp}(\mathbf{y}) = E}} (-1)^{|\text{supp}(\mathbf{y})|}. \quad \square \tag{2}$$

Setting $q = 1$ in Corollary 6 gives the following result

$$T_M\left(1-p^k, 0\right) = (-1)^{r(M)} \left| \left\{ (\mathbf{x}_1, \dots, \mathbf{x}_k) \in (\text{row}(M))^k : \bigcup_{i=1}^{k} \text{supp}(\mathbf{x}_i) = E \right\} \right| \tag{3}$$

$$= (-1)^{r(M)} p^{k(r(M)-d)} \left| \left\{ \begin{array}{c} (\mathbf{v}_1, \dots, \mathbf{v}_k) \in (\mathbb{F}^d)^k : \\ \text{for all } e \in E \text{ there exists } i \text{ with } \mathbf{v}_i \notin e^{\perp} \end{array} \right\} \right| \tag{4}$$

where here M is a $d \times n$ integer matrix which reduces correctly over $\mathbb{F}_p$, and $e \in E$ denotes a column of the matrix M. Equation (4) follows from equation (3) using the exact sequence

$$0 \to \ker(M)^T \to F^d \overset{M^T}{\to} \text{row}(M) \to 0 \tag{5}$$

and the observation that $|\ker(M)^T| = p^{d-r(M)}$.

Equation (4) is equivalent to the earlier mentioned theorem of Crapo and Rota [6, Theorem 1. §16.4], via the relation between the Tutte polynomial and the *characteristic polynomial* $p_M(\lambda)$ of its associated *geometric lattice* (see [4, (6.20)]):

$$T_M(1-\lambda, 0) = (-1)^{r(M)} p_M(\lambda)$$

Specializing further to $k = 1$ in the equation (4) gives the well-known "finite field" interpretation of $p_M(\lambda)$.

Corollary 7. *Let M, p be as in Theorem 1, and let $\mathcal{A}$ be the arrangement of hyperplanes in $\mathbb{F}_p^d$ perpendicular to the columns of M. Then*

$$p_M(p) = p^{r(M)-d} \left| \mathbb{F}_p^d - \mathcal{A} \right|. \quad \square$$

Athanasiadis [1] used this result very effectively to compute characteristic polynomials for various classes of hyperplane arrangements.

We mention also that for the matroid M coming from a graph G, specializing $k = 1$ in equation (4) gives Tutte's original interpretation (see [4, Proposition 6.3.1]) of $p_M(\lambda)$ in terms of the *chromatic polynomial*

$$\chi_G(\lambda) = \lambda^{d-r(M)} p_M(\lambda).$$

counting the proper vertex-colorings of the graph G. This is because we can interpret $\mathbb{F}^d$ as the set of vertex p-colorings, and $\ker(M)^T$ as the space of colorings which are constant on each connected component of G. In this interpretation, the space $\text{row}(M)$ is sometimes designated the space of *$\mathbb{F}_p$-coboundaries of G* or *$\mathbb{F}_p$-voltage drops* in G. With this point of view in mind, Corollary 2 is the special case of Equation (2) for graphic matroids.

We lastly mention the dual version to Corollary 7 which is the specialization $p = k = k' = 1$ in Corollary 6:

$$T_M(0, 1-q) = (-1)^{n-r(M)} \left|\{\mathbf{y} \in \ker M : \text{supp}(\mathbf{y}) = E\}\right|$$

This is a well-known generalization of the case when M comes from a graph G, where Tutte [11] originally phrased this interpretation of $(-1)^{n-r(M)}T_M(0, 1-q)$ as the number of *nowhere zero* $\mathbb{F}_q$*-flows* on G.

Jaeger's specializations.

In [8, Proposition 4], Jaeger essentially proves the special case of Theorem 1 in which $p = q$. There he adopts the coding point of view, where M is a matrix with $\mathbb{F}_p$ entries whose rows are a spanning set for an $\mathbb{F}_q$-linear code $\mathcal{C} = \text{row}(M)$. He then also takes a limit as $a \to \infty$ to deduce a specialization [8, Proposition 6] equivalent to the $p = q$ case of equation (2). He then further specializes to $q = 2$ to obtain the following result of Rosenstiehl and Read [10, Theorem 9.1]:

Corollary 8. *Let M represent a matroid over $\mathbb{F}_2$, and let $\mathcal{C} := \text{row}(M)$ and $\mathcal{C}^\perp := \ker(M)$. Then we have*

$$\begin{aligned} T_M(-1, -1) &= (-1)^{r(M)} \sum_{\substack{(\mathbf{x}, \mathbf{y}) \in \mathcal{C} \times \mathcal{C}^\perp \\ \text{supp}(\mathbf{x}) \uplus \text{supp}(\mathbf{y}) = E}} (-1)^{|\text{supp}(\mathbf{y})|} \\ &= (-1)^{n - \dim \mathcal{C} \cap \mathcal{C}^\perp} |\mathcal{C} \cap \mathcal{C}^\perp|. \quad \square \end{aligned}$$

The space $\mathcal{C} \cap \mathcal{C}^\perp$ is called the space of *bicycles* of M, and we explain here how the second equality in the corollary follows from the first. First, note that the condition $\text{supp}(\mathbf{x}) \uplus \text{supp}(\mathbf{y}) = E$ implies that $\mathbf{x} = (1, 1, \ldots, 1) - \mathbf{y}$. It is then easy to check that the set Y consisting of those $\mathbf{y}$ which occur in the above sum forms an a coset inside $\mathbb{F}_2^n$ for the bicycle space $\mathcal{C} \cap \mathcal{C}^\perp$. Since every vector in $\mathcal{C} \cap \mathcal{C}^\perp$ is perpendicular to itself, all such vectors have even support, and therefore all vectors $\mathbf{y} \in Y$ contribute the same sign $(-1)^{|\text{supp}(\mathbf{y})|}$ to the sum. To prove the sign is correct in the second equality, one needs to know that for any $\mathbf{y} \in Y$,

$$(-1)^{|\text{supp}(\mathbf{y})|} = (-1)^{n - r(M) - \dim \mathcal{C} \cap \mathcal{C}^\perp}.$$

This is not obvious, however it is a result of De Fraysseix (see [8, p. 253]).

Jaeger also makes the interesting specialization $a = b = 1/2$ in his main result [8, Proposition 8], in order to interpret $T_M(1+q, 1+q)$ and in particular to recover a conjecture of Las Vergnas about $T_M(3, 3)$. If we similarly set $a = b = 1/2$ in Theorem 1, we obtain the following interpretation for $T_M(1+p, 1+q)$.

Corollary 9. *Let M be as in Theorem 1. Then we have*

$$T_M(1+p, 1+q)) = \sum_{\substack{(\mathbf{x}, \mathbf{y}) \in \text{row}(M) \times \ker M \subseteq \mathbb{F}_p^n \times \mathbb{F}_q^n \\ \text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y}) = \varnothing}} 2^{n - (|\text{supp}(\mathbf{x})| + |\text{supp}(\mathbf{y})|)} \quad \square$$

We claim that this result is actually a disguised form of the well-known formula [4, (6.13)] for $T_M(x, y)$ involving the *Whitney corank-nullity polynomial* of M:

$$T_M(1+p, 1+q) = \sum_{A \subseteq E} p^{r(M/A)} q^{r(M|_A^*)}. \tag{6}$$

To see this, we start with (6) and re-interpret:

$$\begin{aligned}
&T_M(1+p,1+q)\\
&=\sum_{A\subseteq E} p^{r(M/A)} q^{r(M|^*_A)}\\
&=\sum_{A\subseteq E} |\mathrm{row}(M/A)|\cdot|\ker(M)|_A|\\
&=\sum_{A\subseteq E} |\{\mathbf{x}\in\mathrm{row}(M):\mathrm{supp}(\mathbf{x})\subseteq E-A)|\cdot|\{\mathbf{y}\in\ker(M):\mathrm{supp}(\mathbf{y})\subseteq A|\\
&=\sum_{\substack{(\mathbf{x},\mathbf{y})\in\mathrm{row}(M)\times\ker M\\ \mathrm{supp}(\mathbf{x})\cap\mathrm{supp}(\mathbf{y})=\varnothing}} |\{A\subseteq E:\mathrm{supp}(\mathbf{x})\subseteq E-A,\mathrm{supp}(\mathbf{y})\subseteq A|\}|\\
&=\sum_{\substack{(\mathbf{x},\mathbf{y})\in\mathrm{row}(M)\times\ker M\\ \mathrm{supp}(\mathbf{x})\cap\mathrm{supp}(\mathbf{y})=\varnothing}} 2^{n-(|\mathrm{supp}(\mathbf{x})|+|\mathrm{supp}(\mathbf{y})|)}.
\end{aligned}$$

Weight enumerators of codes and two-variable coloring.

The specialization $q=1$ in Theorem 5 says the following.

Corollary 10. *Let M,p,k,a,b be as in Theorem 5. Then*

$$T_M\left(\frac{1+(p^k-1)a}{b},\frac{1}{a}\right)=\frac{1}{a^{r(M^*)}b^{r(M)}}\sum_{(\mathbf{x}_1,\dots,\mathbf{x}_k)\in(\mathrm{row}(M))^k} a^{|\bigcup_{i=1}^k \mathrm{supp}(\mathbf{x}_i)|} \tag{7}$$

If $k=1$ this gives

$$T_M\left(\frac{1+(p-1)a}{b},\frac{1}{a}\right)=\frac{1}{a^{r(M^*)}b^{r(M)}}\sum_{\mathbf{x}\in\mathrm{row}(M)} a^{|\mathrm{supp}(\mathbf{x})|} \tag{8}$$

Equation (8) is essentially equivalent to two results in the literature. The first is the result of Greene [7] that the Tutte polynomial $T_M(x,y)$ can be specialized to give the weight enumerator of the $\mathbb{F}_p$-linear code $\mathcal{C}:=\mathrm{row}(M)$. In fact, Barg [2] recently generalized this to *higher weight enumerators*, giving a result equivalent to equation (7), which we now discuss.

Given a subspace $W\subseteq\mathcal{C}$, define its *support*

$$\mathrm{supp}(W):=\bigcup_{w\in W}\mathrm{supp}(w)=\bigcup_{i=1}^m\mathrm{supp}(w_i)$$

where $w_1,\dots,w_m$ is any spanning subset of W. Following Barg [2], we define the m^{th} higher weight enumerator for the code $\mathcal{C}$ to be

$$\mathcal{D}^m(a):=\sum_{\text{subspaces } W\subseteq\mathcal{C}} a^{|\mathrm{supp}(W)|}\cdot[m]_{\dim W}$$

where

$$\begin{aligned}[m]_d &:= \prod_{i=0}^{d-1}(p^m - p^i) \\ &= |\{(v_1, \dots, v_d) \in (\mathbb{F}_p^m)^d : \{v_i\} \text{ are linearly independent in } \mathbb{F}_p^m\}| \\ &= |\{(w_1, \dots, w_m) \in (\mathbb{F}_p^d)^m : \{w_j\} \text{ are a spanning subset of } \mathbb{F}_p^d\}|.\end{aligned}$$

The last equality above comes from identifying the $\{v_i\}$ as the rows of a full rank $d \times m$ matrix over $\mathbb{F}_p$, and then letting $\{w_j\}$ be the columns of the same matrix.

Corollary 11 ([2]). *Let C be an $\mathbb{F}_p$-linear code with $C = \mathrm{row}(M)$. Then*

$$\mathcal{D}^m(a) = a^{r(M^*)}(1-a)^{r(M)} T_M\left(\frac{1+(p^m-1)a}{1-a}, \frac{1}{a}\right).$$

Proof.

$$\begin{aligned}\mathcal{D}^m(a) &= \sum_{\text{subspaces } W \subseteq C} a^{|\mathrm{supp}(W)|}[m]_{\dim W} \\ &= \sum_{\text{subspaces } W \subseteq C} a^{|\mathrm{supp}(W)|}|\{(w_1, \dots, w_m) \in (\mathbb{F}_p^d)^m : \{w_j\} \text{ span } W\}| \\ &= \sum_{(w_1, \dots, w_m) \in C^m} a^{|\bigcup_{j=1}^m \mathrm{supp}(w_i)|} \\ &= a^{r(M^*)}(1-a)^M T_M\left(\frac{1+(p^m-1)a}{1-a}, \frac{1}{a}\right)\end{aligned}$$

where the last equality is equation (7). □

The second known result which comes from equation (8) is a *two-variable coloring formula* for graphs (equivalent to [4, Proposition 6.3.26]). Let G be a graph with d vertices, n edges, and for any vertex-coloring c of G, let mono(c) be the number of *monochromatic edges*, that is edges whose endpoints receive the same color.

Corollary 12. *Let M be the graphic matroid associated to G. Then*

$$\sum_{\text{colorings } c \text{ of } G \text{ with } \lambda \text{ colors}} \nu^{\mathrm{mono}(c)} = \lambda^{d-r(M)}(\nu-1)^{r(M)} T_M\left(\frac{\nu+\lambda-1}{\nu-1}, \nu\right)$$

Proof. Because of the coloring interpretation of the exact sequence (5) (see the discussion following Corollary 7), we have

$$\begin{aligned}&\sum_{\text{colorings } c \text{ of } G \text{ with } \lambda \text{ colors}} \nu^{\mathrm{mono}(c)} \\ &= \lambda^{d-r(M)} \sum_{\mathbf{x} \in \mathrm{row}(M) \subseteq (\mathbb{Z}/\lambda\mathbb{Z})^n} \nu^{n-|\mathrm{supp}(\mathbf{x})|} \\ &= \lambda^{d-r(M)}\nu^n \left[\sum_{\mathbf{x} \in \mathrm{row}(M) \subseteq (\mathbb{Z}/\lambda\mathbb{Z})^n} \nu^{|\mathrm{supp}(\mathbf{x})|}\right]_{\nu \mapsto \nu^{-1}} \\ &= \lambda^{d-r(M)}\nu^n \left[\nu^{r(M^*)}(1-\nu)^{r(M)} T_M\left(\frac{1+(\lambda-1)\nu}{1-\nu}, \frac{1}{\nu}\right)\right]_{\nu \mapsto \nu^{-1}} \\ &= \lambda^{d-r(M)}(\nu-1)^{r(M)} T_M\left(\frac{\nu+\lambda-1}{\nu-1}, \nu\right)\end{aligned}$$

where the third equality above is equation (8). □

We should also mention that in a recent work, Wagner [13] considers a rescaled version of the Tutte polynomial specialization $T_M(\frac{1+(t-1)a}{1-a}, \frac{1}{a})$, which is very similar to the specializations in Corollaries 9,10,11. He furthermore gives a combinatorial interpretation for the coefficients in this rescaled polynomial.

4. Questions and open problems

1. Theorem 1 recovers many of the interpretations of $T_M(x,y)$ involving finite fields, codes, colorings and flows. However, there are some evaluations which it misses, such as Stanley's interpretation of $T_M(1+n,0)$ in terms of *acyclic orientations*, or the dual interpretation of $T_M(0,2)$ in terms of *totally cyclic orientations* (see [4, Examples 6.3.29 and 6.3.32]). Is there any way to relate Theorem 1 to these results?

   Recently Wagner [12] gave an interpretation of $T_M(t^{-1}, 1+t)$ for matroids M coming from a graph G in terms of certain kinds of flows on G. Does Theorem 1 relate to this?
2. Athanasiadis proved a result [1, Theorem 2.2] which is somewhat stronger than Corollary 7. His result counts points in the complements of arrangements of linear subspaces in $\mathbb{F}_p^d$, rather than just arrangements of hyperplanes. Is there some generalization of the Tutte polynomial to subspace arrangements and an accompanying generalization of Theorem 1 which specializes to his result?

   Athanasiadis also gave numerous examples of families of hyperplane arrangements where one can write down $T_M(1-p,0)$ explicitly using the finite field interpretation (Theorem 7) in a strong way. Can one similarly use Corollary 2 to compute $T_M(1-p,1-q)$ for any non-trivial families of matroids?
3. The condition that $\mathrm{supp}(x) \in \mathrm{row}(M)$ and $\mathrm{supp}(y) \in \ker(M)$ have disjoint support in Theorem 1 is very reminiscent of the notion of *complementary slackness* for optimal solutions of the *primal* and *dual* programs in the theory of linear programming. Is there any deeper connection here?

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