the pentagon $\Sigma(F_3)$ is a Minkowski summand of the given 7-gon. Note that the coordinate vectors in the previous two tables lie in 2-dimensional affine subspaces of $\mathbf{R}^6$ parallel to the kernel of A and $(1, 1, 1, 1, 1, 1)$.

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References

1. H. J. Baues. Geometry of loop spaces and the cobar construction. *Mem. Amer. Math. Soc.*, 25 (1980), No. 230.
2. L. J. Billera, P. Filliman and B. Sturmfels. Constructions and complexity of secondary polytopes. *Advances in Mathematics*, 83 (1990), 155–179.
3. L. J. Billera, M. M. Kapranov and B. Sturmfels. Cellular strings on polytopes. *Proc. Amer. Math. Soc.*, 122 (1994), to appear.
4. L. J. Billera and B. Sturmfels. Fiber polytopes. *Annals of Math.*, 135 (1992), 527–549.
5. A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler. *Oriented Matroids* (Cambridge University Press, 1993).
6. P. Filliman. Exterior algebra and projections of polytopes. *Discrete Comput. Geometry*, 5 (1990), 305–322.
7. I. M. Gel'fand, M. M. Kapranov and A. V. Zelevinsky. Newton polytopes of the classical resultant and discriminant. *Advances in Mathematics*, 84 (1990), 237–254.
8. I. M. Gel'fand, A. V. Zelevinsky and M. M. Kapranov. Discriminants of polynomials in several variables and triangulations of Newton polyhedra. *Algebra i Analiz*, 2 (1990), 1–62. (English translation in *Leningrad Math. J.*, 2 (1991), 449–505.)
9. M. M. Kapranov, B. Sturmfels and A. Zelevinsky. Quotients of toric varieties. *Mathem. Annalen*, 290 (1991), 643–655.
10. W. S. Massey. *Singular homology theory* (Springer-Verlag, New York, 1980).
11. D. D. Sleator, R. E. Tarjan and W. P. Thurston. Rotation distance, triangulations, and hyperbolic geometry. *J. American Math. Soc.*, 1 (1988), 647–681.
12. V. A. Yemelichev, M. M. Kovalev and M. K. Kravtsov. *Polytopes, Graphs and Optimisation* (Cambridge University Press, 1984).
13. I. M. Gel'fand, M. M. Kapranov and A. V. Zelevinsky. *Discriminants, Resultants and Multi-dimensional Determinants* (Birkhaüser, Boston, 1994).

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COXETER-ASSOCIAHEDRA

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Abstract. Recently M. M. Kapranov [Kap] defined a poset KPA_{n-1}, called the *permuto-associahedron*, which is a hybrid between the face poset of the *permutohedron* and the *associahedron*. Its faces are the partially parenthesized, ordered, partitions of the set $\{1, 2, \ldots, n\}$, with a natural partial order.

Kapranov showed that KPA_{n-1} is the face poset of a regular CW-ball, and explored its connection with a category-theoretic result of MacLane, Drinfeld's work on the Knizhnik-Zamolodchikov equations, and a certain moduli space of curves. He also asked the question of whether this CW-ball can be realized as a convex polytope.

We show that indeed, the permuto-associahedron corresponds to the type A_{n-1} in a family of convex polytopes KPW associated to the classical Coxeter groups, $W = A_{n-1}$, B_n, D_n. The embedding of these polytopes relies on the *secondary polytope* construction of the associahedron due to Gel'fand, Kapranov, and Zelevinsky. Our proofs yield integral coordinates, with all vertices on a sphere, and include a complete description of the facet-defining inequalities.

Also we show that for each W, the dual polytope KPW^* is a refinement (as a CW-complex) of the *Coxeter complex* associated to W, and a coarsening of the barycentric subdivision of the Coxeter complex. In the case $W = A_{n-1}$, this gives a combinatorial proof of Kapranov's original sphericity result.

§0. *Introduction.* This paper is concerned with the construction of polytopes with prescribed combinatorial structure. In fact, there is a three-part problem associated with combinatorial objects like permutohedra, associahedra, . . . :

1. the first part is the combinatorial description of a finite poset (*definition*);
2. the second part asks for a proof that the poset under consideration is the face poset of a regular CW-ball (*sphericity*); and
3. the third part is the construction of a convex polytope whose face lattice is isomorphic to the poset (*realization*).

Note that *realization* gives a proof of *sphericity*, since every convex polytope is a regular CW-ball (*cf.* [Bj2], [BLSWZ, Sect. 4.7]).

For the permutohedron, the *definition* and *realization* are classical. For the associahedron, the *definition* is due to Stasheff [Stas] (and later independently to Perles [Per]). *Sphericity* was proved by Stasheff, *realization* was achieved by Milnor (unrecorded), Haiman [Hai] and Lee [Lee]. A "systematic" construction method for the associahedron was achieved by Gel'fand, Zelevinsky and Kapranov [GZK1, Remark 7c] with their construction of *secondary polytopes*,

[MATHEMATIKA, **41** (1994), 364–393]

and then generalized and explained by the construction of *fiber polytopes* by Billera and Sturmfels [BS1].

For the *permuto-associahedron* KPA_{n-1}, we owe *definition* and *sphericity* to Kapranov [Kap], who denotes the object by "KP_n". Here we contribute a combinatorial proof for *sphericity* (Section 2) which gives some extra information about the relation between the permutohedron and permuto-associahedron, and a construction that solves the *realization* problem (Sections 3 and 4).

Furthermore, we also *define* and *realize Coxeter-associahedra* KPW for Coxeter groups of types B and D.

The two main theorems of the paper are:

THEOREM 1 (*Sphericity*). *For* $W = A_{n-1}, B_n, D_n$ *the dual* KPW^* *of the Coxeter-associahedron poset* KPW *is the face poset of a regular* CW-*ball whose boundary refines the Coxeter complex* ∂PW^* *and is refined by its barycentric subdivision* $\mathrm{sd}(\partial PW^*)$, *i.e.*,

$$\mathrm{sd}(\partial PW^*) \prec \partial KPW^* \prec \partial PW^*.$$

THEOREM 2 (*Realization*). *There exists a realization of the associahedron* K_{n-2} *in* $\mathbb{R}^n$ *inside the fundamental chambers of the Weyl groups* $W = A_{n-1}, B_n, D_n$ *such that the polytope given by the convex hull of the orbit* WK_{n-2} *of* K_{n-2} *under the canonical action of* W *on* $\mathbb{R}^n$ *has face lattice isomorphic to the Coxeter-associahedron poset* KPW.

We note that the two proofs can be followed independently: the proof of Theorem 1 is completed in Section 2. The proof of Theorem 2 in Sections 3 and 4 does not rely on this, and proceeds directly from the definitions of Section 1.

§1. *Combinatorics*. In this section, we review the combinatorial description of the face posets for the Coxeterhedra PA_{n-1}, PB_n and PD_n, and define analogously the face posets for the Coxeter-associahedra KPA_{n-1}, KPB_n and KPD_n. Our convention is (as in [Zie]) that the subscript on the name of a ranked poset indicates its length minus one, and thus the dimension of the corresponding polytope.

The classical Coxeter groups A_{n-1}, B_n, D_n. The Coxeter groups A_{n-1} and D_n are both subgroups of the *signed permutation group* B_n, which consists of all permutations and sign changes of the coordinates in $\mathbb{R}^n$. A_{n-1} is the subgroup of permutations with no sign changes (*i.e.*, the *symmetric group* on n letters), and D_n is the subgroup of signed permutations with an *even* number of sign changes. We use the following one-line notation for signed permutations w:

$$w = w_1 w_2 \dots w_n,$$

where

$$w_i = \begin{cases} j & \text{if } w(\mathbf{e}_i) = +\mathbf{e}_j \\ \bar{j} & \text{if } w(\mathbf{e}_i) = -\mathbf{e}_j \end{cases}$$

and $\mathbf{e}_i$ denotes the i-th standard basis vector in $\mathbb{R}^n$. For example, $w = 2\bar{4}51\bar{3}$ is the element sending

$$\begin{aligned} \mathbf{e}_1 &\mapsto +\mathbf{e}_2 \\ \mathbf{e}_2 &\mapsto -\mathbf{e}_4 \\ \mathbf{e}_3 &\mapsto +\mathbf{e}_5 \\ \mathbf{e}_4 &\mapsto +\mathbf{e}_1 \\ \mathbf{e}_5 &\mapsto -\mathbf{e}_3 \end{aligned}$$

We will think of the three groups $\mathrm{W} = \mathrm{A}_{n-1}$, B_n, D_n as *Coxeter systems* (W, S), *i.e.*, each may be given a distinguished set of generators S having certain properties (see [Bro] or [Hum] for definitions). For the symmetric group A_{n-1} the set S consists of the *adjacent transpositions* $\{s_i\}_{1 \leq i \leq n-1}$, where s_i interchanges the i-th and the $(i+1)$-st coordinate. The set S for B_n contains an extra generator s_n which changes the sign of the last coordinate, while the set S for D_n contains an extra generator s_n which swaps *and* changes the signs of the last two coordinates.

Face lattices of the Coxeterhedra.

Definition 3. For any Coxeter system (W, S), the subgroups W_J generated by subsets $J \subseteq S$ are called *parabolic subgroups* of W. The *Coxeterhedron* PW is the finite poset of all cosets

$$\{w\mathrm{W}_J\}_{w \in W, J \subseteq S}$$

of all parabolic subgroups of W, ordered by *inclusion*.

Remark 4 (*Realization*). For any Coxeter system (W, S), there is a simple polytope that has PW as its face lattice. See Fig. 1 for examples. This polytope may be constructed in at least two ways,

(i) as the convex hull of the orbit of a generic point in $\mathbb{R}^n$ under the action of W as a reflection group on $\mathbb{R}^n$,

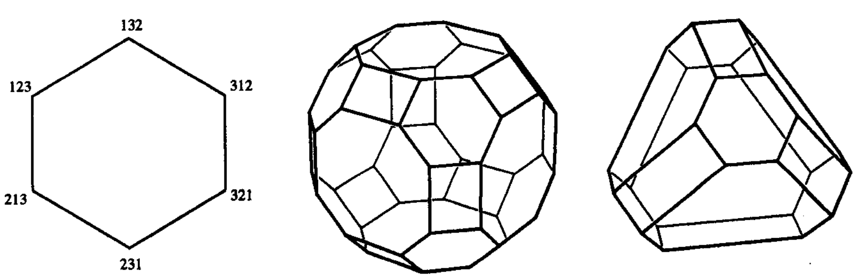


Figure 1. The Coxeterhedra PA_2, PB_3, PD_3.

(ii) as the *zonotope* generated by the *root system* for (W, S).

We will abuse notation and refer to both the poset and to its geometric realization as the Coxeterhedron PW.

The polytopes PW are simple, so their polar duals are simplicial polytopes. Thus the boundary complex of the *dual* polytope PW* is a simplicial complex, called the *Coxeter complex*. We refer to [BLSWZ, Sect. 2.3], [Bro] or [Hum] for further discussions.

For the classical groups $\mathrm{W} = \mathrm{A}_{n-1}, \mathrm{B}_n, \mathrm{D}_n$, it will be useful to have a unified terminology for the cosets of parabolic subgroups, which we now describe. Given a coset $w\mathrm{W}_J$, we will

(1) place dots between certain of the letters of w, thereby breaking $w = w_1 w_2 \ldots w_n$ into blocks,
(2) possibly introduce a single box that surrounds some of the blocks,
(3) alter w within the coset $w\mathrm{W}_J$ to obtain a coset representative in a canonical form (described below).

Step (1) proceeds by placing a dot between w_i and w_{i+1} if the adjacent transposition s_i is *not* in J. Step (2) proceeds by

(2a) circling *no* blocks, if $\mathrm{W} = \mathrm{A}_{n-1}$ or s_n is not in J,
(2b) circling the *entire* last block, if $\mathrm{W} = \mathrm{B}_n$ and s_n is in J,
(2c) circling those blocks which contain the *last two* letters if $\mathrm{W} = \mathrm{D}_n$ and s_n is in J. Thus the box, if present, encloses either the last block (if that block has at least two elements), or the last two blocks (if the last block is a singleton).

Step (3) proceeds by using the subgroup W_J to alter the coset representative w until it satisfies the following conditions.

(3a) Within each block and within the box (if present), the numbers are in increasing order.
(3b) If $\mathrm{W} = \mathrm{B}_n$, there can be no bars inside the box.
(3c) If $\mathrm{W} = \mathrm{D}_n$ and the last block is the only one boxed, then only the last letter can have a bar.

It is easy to check that exactly one coset representative satisfies these conditions in each case. We will call this dotted, boxed, canonical coset representative the *string* corresponding to $w\mathrm{W}_J$. Here are some examples:

$$
\begin{array}{rcll}
2.146.35 & \leftrightarrow & 261453\mathrm{W}_{\{s_2,s_3,s_5\}} & \text{in } \mathrm{PA}_5, \\
2.1\overline{46}.\bar{3}5 & \leftrightarrow & 2\overline{64}15\bar{3}\mathrm{W}_{\{s_2,s_3,s_5\}} & \text{in } \mathrm{PB}_6, \\
2.1\overline{46}.\boxed{35} & \leftrightarrow & 2\overline{64}15\bar{3}\mathrm{W}_{\{s_2,s_3,s_5,s_6\}} & \text{in } \mathrm{PB}_6, \\
\bar{2}.1\overline{46}.\bar{3}5 & \leftrightarrow & \overline{264}15\bar{3}\mathrm{W}_{\{s_2,s_3,s_5\}} & \text{in } \mathrm{PD}_6, \\
2.1\bar{4}.\boxed{35\bar{6}} & \leftrightarrow & 2\bar{4}1\overline{563}\mathrm{W}_{\{s_2,s_4,s_5,s_6\}} & \text{in } \mathrm{PD}_6, \\
2.1\bar{4}.\boxed{3\bar{5}.6} & \leftrightarrow & 2\bar{4}1\overline{563}\mathrm{W}_{\{s_2,s_4,s_6\}} & \text{in } \mathrm{PD}_6.
\end{array}
$$

The inclusion relation on cosets $w\mathrm{W}_J$ corresponds to the following order relation on strings: $\alpha \leqslant \beta$, if, and only if, the string β is obtained from α by any combination of the following two operations.

(1) Combining a consecutive sequence of blocks into one block. For example,

$$6.\bar{2}.14\bar{7}.\bar{3}5.\boxed{8} < 6.1\overline{23}45\bar{7}.\boxed{8} \quad \text{in } \mathrm{PB}_8,$$

$$6.\bar{2}.14\bar{7}.\bar{3}5.\boxed{8} < 6.\bar{2}.14\bar{7}.\boxed{358} \quad \text{in } \mathrm{PB}_8,$$

$$6.\bar{2}.14\bar{7}.\boxed{35.8} < 6.\bar{2}.\boxed{1345\bar{7}.8} \quad \text{in } \mathrm{PD}_8.$$

(2) Adding in the box.

$$6.\bar{2}.14\bar{7}.\bar{3}5 < 6.\bar{2}.14\bar{7}.\boxed{35} \quad \text{in } \mathrm{PB}_7,$$

$$6.\bar{2}.14.\overline{37}.5 < 6.\bar{2}.14.\boxed{\overline{35}.7} \quad \text{in } \mathrm{PD}_7.$$

Face posets of the Coxeter-associahedra. We now define the face poset of the Coxeter-associahedron KPW for $W=A_{n-1}$, B_n, D_n. In Sections 4 and 5 we will prove that KPW is the face lattice of a convex polytope, which (by abuse of notation) we will also call KPW.

Definition 5. For $W=A_{n-1}$, B_n or D_n, the *Coxeter-associahedron* KPW is a partially ordered set, defined as follows. The elements of KPW are the strings (canonical coset representatives) in PW, *partially parenthesized*: this means that the blocks are treated as if they were being multiplied together and some of them are grouped together by parentheses to indicate order of multiplication. In particular, every pair of parentheses encloses at least two blocks. In the cases $W=B_n$, D_n, but not in case $W=A_{n-1}$, there is always an extra *virtual* parenthesis pair around the entire string, if, and only if, there is *more than one block* and *no box* is present.

The order relation on these parenthesized strings is defined as follows: $A \leqslant B$, if, and only if, B is obtained from A by any combination of the following three operations:

(1) removing a parenthesis pair (possibly the virtual one), and combining all the blocks within it into one block;
(2) adding in the box (and hence removing the virtual parenthesis pair);
(3) removing a non-virtual parenthesis pair.

Finally, an extra minimum element $\hat{0}$ and an extra maximum element $\hat{1}$ are included in the posets KPW.

For examples, Fig. 2 shows the posets KPA_2, KPB_2, KPD_2: they are the face lattices of a 12-gon, an octagon and a square, respectively.

Here are some larger examples of the order relation:

$$((9(4.2.6.8)(3.5))1) < (9.2468(3.5)1) \quad \text{in } \mathrm{KPA}_9$$

$$(7(\bar{9}.8)(2(\bar{6}(1\bar{5}.\bar{3}4)))) < 7.\bar{9}.8.2(\bar{6}(1\bar{5}.\boxed{\bar{3}4})) \quad \text{in } \mathrm{KPB}_9$$

$$(((\bar{9}.8)2(\bar{6}(1\bar{5}.\bar{3}7)))4) < 8\bar{9}.2(\bar{6}(1\bar{5}.\boxed{34))\bar{7}} \quad \text{in } \mathrm{KPD}_9$$

Fig. 3 shows the polytopes KPA_2, KPB_3, KPD_3. Note that KPA_{n-1} is embedded as the principal order ideal below the face $123\ldots n-1n$ in both

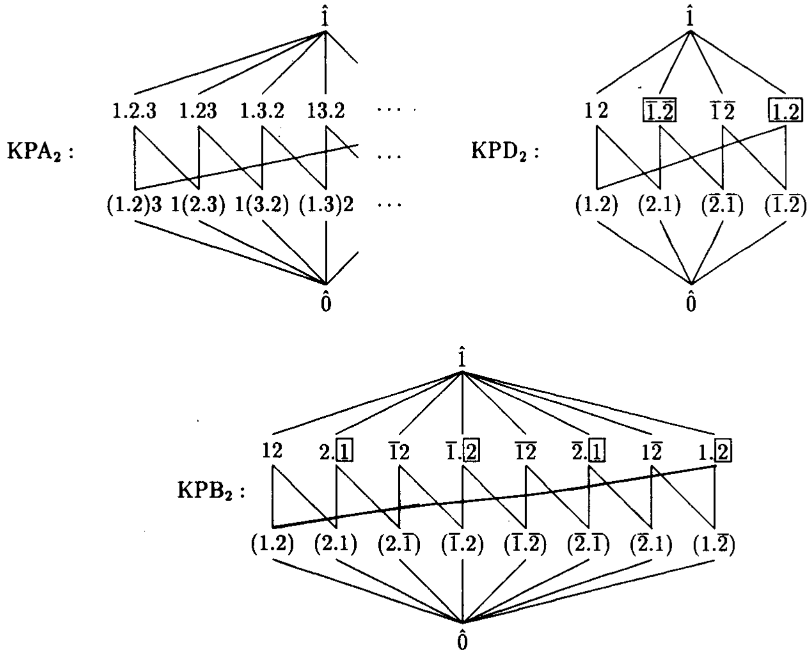


Figure 2. The posets KPA$_2$, KPB$_2$, and KPD.

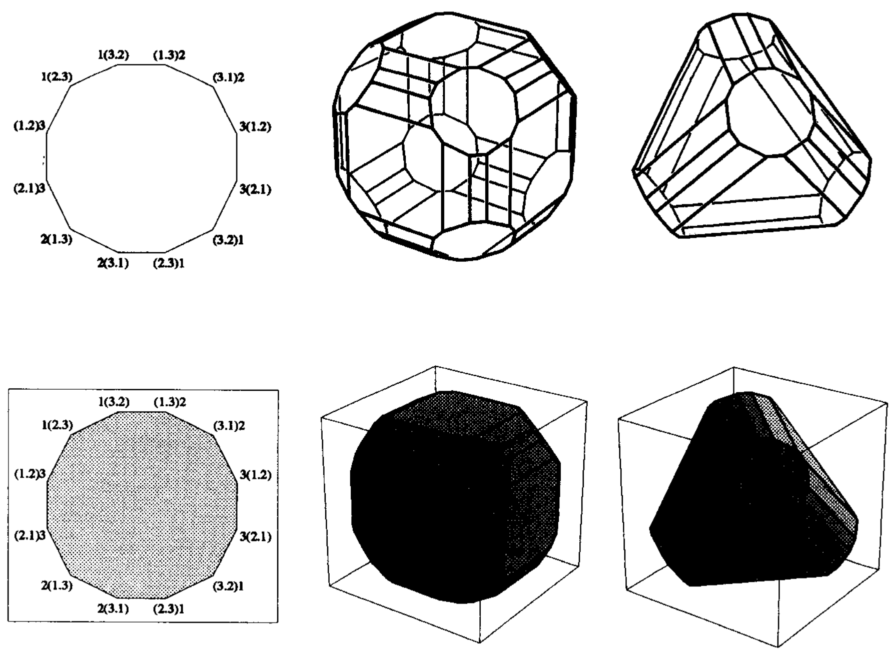


Figure 3. KPA$_2$, KPB$_3$, KPD$_3$.

KPB_n and KPD_n. For example, in Fig. 3, the 12-gon KPA_2 is isomorphic to the face labelled 123 in KPB_3 or KPD_3.

The next two lemmas are needed for the proof of *realization* (Theorem 2) in Sections 4 and 5.

LEMMA 6. *The posets* KPW *are ranked lattices.*

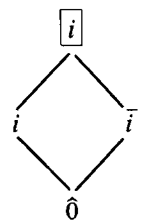
Proof. Since every covering relation $x \lessdot y$ in KPW involves removing exactly one parenthesis pair from x to obtain y, it follows that

$$\text{rank}\,(x) = n - \#\{\text{parenthesis pairs in } x\}$$

defines a rank function on KPW for $W = A_{n-1}, B_n, D_n$.

Since KPW is a finite poset with $\hat{0}$ and $\hat{1}$, to show it is a lattice it will suffice to show that every two elements x, y have a greatest lower bound $x \wedge y$ in KPW (see [Stan, Prop. 3.3.1]). Given x, y in KPB_n, we describe $x \wedge y$ in stages, and leave it to the reader to check that this actually defines their greatest lower bound. This also defines $x \wedge y$ in KPA_{n-1} as a special case, and the description of $x \wedge y$ in KPD_n requires only minor modifications which we omit.

Stage 1. For each $i \in \{1, 2, \ldots, n\}$ determine whether i appears *without* a bar, *with* a bar, or *boxed* in $x \wedge y$, by taking the greatest lower bound in the lattice shown below of i's appearances in x and y:



This means, for example, that if i appears without a bar in x and boxed in y, then it will appear without a bar in $x \wedge y$. If it appears with a bar in x and without a bar in y, then $x \wedge y$ will be $\hat{0}$ and the description process is done.

Stage 2. Determine the (unordered) block structure of $x \wedge y$ by intersecting the blocks of x and y (*i.e.*, the usual greatest lower bound for set partitions.)

Stage 3. Determine the order on the blocks of $x \wedge y$ by placing block B before block B', if, and only if, all numbers in B appear in earlier blocks or in the same block as all the numbers in B' in both x and y. If two numbers i, j lie in different blocks in both x and y, and appear in different order in x than they do in y, then $x \wedge y = \hat{0}$, and the description process is done. Up to this stage, we have determined the underlying (unparenthesized) string of $x \wedge y$. For example,

$$\bar{1}(23) \;\wedge\; 1(2.3) = \hat{0}$$

$$1.23 \;\wedge\; 2.13 = \hat{0}$$

while the underlying string of

$$1\bar{2}34.\overline{56}.7\bar{8} \;\wedge\; 1\,.\,\bar{2}34.\overline{567}\bar{8}$$

will be

$$1.\bar{2}34.\overline{56}.7\bar{8}$$

Stage 4. Identify parenthesis pairs in x, y with the subset of numbers they enclose. For each parenthesis pair in x or in y, include the same parenthesis pair in $x \wedge y$. Whenever a sequence $B_1 . B_2 . \cdots . B_k$ of consecutive blocks in $x \wedge y$ is combined into a single block of x or of y, include a parenthesis pair around $B_1 . B_2 . \cdots . B_k$ in $x \wedge y$. If two parenthesis pairs in $x \wedge y$ conflict, *i.e.*, they are not disjoint but neither one encloses the other, then $x \wedge y = \hat{0}$. For example,

$$(1.2.3)\boxed{45} \;\wedge\; 12.3.\bar{4}.5 = ((1.2)3)(\bar{4}.5)$$

$$1.23 \;\wedge\; 12.3 = \hat{0}$$

This completes the description of $x \wedge y$ in KPB_n, and the proof that KPW is a lattice.

LEMMA 7. *The lattices* KPW *are atomic and coatomic.*

Proof. The description of the rank function in the previous lemma implies that atoms of KPW are the completely parenthesized (signed) permutations in W, while coatoms are the completely unparenthesized strings in PW.

The proofs are straightforward combinatorial arguments, and we include here only the argument for atomicity.

Given x in KPW, and y an element of KPW which lies above all the atoms below x, we must show that $y \geqslant x$. We will explain why $y \geqslant x$ in stages, using an example in KPB_{11}:

$$x = ((7.\bar{9})8\ 10\ 11)2(\bar{6}(1\bar{5}\,\boxed{34})).$$

First of all, if i appears boxed in x, then it must appear boxed in y, since y lies above atoms containing i and atoms containing $\bar{i}$. *E.g.*, in our example, y must have 3 boxed since y lies above atoms of the form

$$\cdots (\bar{5}(\bar{3}.4)) \cdots,$$

$$\cdots (\bar{5}(3.4)) \cdots.$$

If i or $\bar{i}$ appears unboxed in y, then it must appear the same way in x: otherwise x would lie above an atom that contains the same letter with the opposite sign, $\bar{i}$ resp. i. *E.g.*, if $\bar{7}$ appears in y but not in x, then we get a contradiction since x lies above an atom that contains 7,

$$\cdots ((7.\bar{9})8) \cdots.$$

If i, j appear in the same block of x, they must appear in the same block of y, since y lies above atoms having i, j in either order. *E.g.*, y must have 8, 10, 11 in the same block since it lies above atoms of the form

$$\cdots (8(10.\overline{11})) \cdots,$$

$$\cdots (\overline{11}(10.8)) \cdots.$$

If i appears to the left of j in x, then i's block must appear weakly to the left of j's block in y, since y lies above atoms having i to the left of j. *E.g.*, y must have 7's block weakly left of 11's block, since y lies above atoms of the form

$$((((7.\bar{9})8)10)\overline{11})\cdots.$$

So far, we have shown that the underlying (unparenthesized) string of y lies above the string of x in PW. Now we discuss the parenthesization of these strings. Identify a parenthesis pair with the set of numbers it encloses. We claim that every parenthesis pair in y is also in x. If not, then without loss of generality, y has the form $\cdots A_1(A_2 . A_3)\cdots$ while x has the form $\cdots A_1 . A_2 . A_3\cdots$ and there would be atoms of the form $\cdots A_1 . A_2 . A_3\cdots$ below x but not below y. We further claim that whenever $A_1 . \cdots . A_k$ is a consecutive sequence of blocks in x which is combined into a single block of y, there must be a parenthesis pair $(A_1 . \cdots . A_k)$ around them in x. If not, then there would be atoms of the form $(A_0 . A_1)A_2 . \cdots . A_k$ below x, but not below y.

Finally, the last two claims imply that $y \geqslant x$, completing the proof.

Remarks 8 (*Vertices, edges and facets of the Coxeter-associahedra*). The vertices of the polytope KPA_{n-1} correspond to complete parenthesizations of permutations of the letters $1, 2, \ldots, n$. The edges are of two types: they correspond to either a single re-parenthesization, or to a transposition of two adjacent letters that are grouped together. The facets correspond to the ordered partitions of $\{1, 2, \ldots, n\}$ into at least two blocks.

The vertices of the polytope KPB_n correspond to complete parenthesizations of signed permutations of the letters $1, 2, \ldots, n$. The edges are of three types: they correspond to either a single re-parenthesization, to a transposition of two

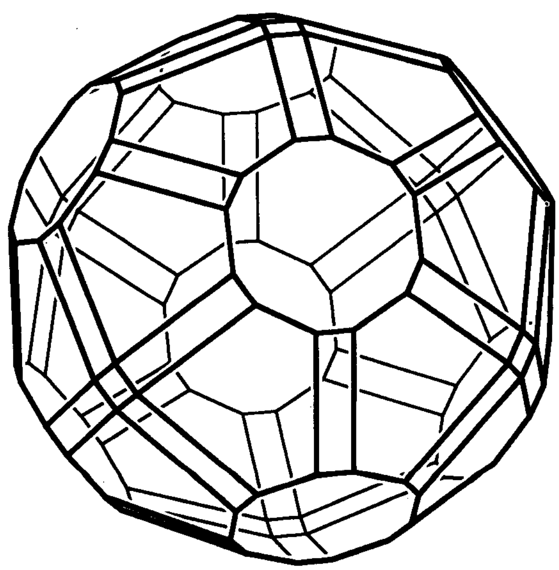


Figure 4. KPA_3.

adjacent letters that are grouped together, or to inverting the sign of the last letter in the permutation. The facets correspond to ordered partitions that either have only one block, or have a box around the last block, but not both.

The vertices of the polytope KPD_n correspond to complete parenthesizations of signed permutations of the letters $1, 2, \ldots, n$, having an even number of minus-signs. The edges are again of three types: they correspond to either a single re-parenthesization, to a transposition of two adjacent letters that are grouped together, or to exchanging the last two letters in the permutation *and* inverting their signs. (The last operation is allowed even if the last two letters are not grouped together.) There are three types of facets: partitions with exactly one block (and an even number of minus signs), ordered partitions with more than one block, where the last block is boxed (so this last block contains more than one element) and partitions where the last two blocks are boxed (and the last block is a singleton).

Observe that KPA_3 and KPD_3 are not equivalent, although the associated Coxeter systems A_3 and D_3 are isomorphic.

§2. *Sphericity.* In this section we prove Theorem 1: the dual poset KPW^* for $\mathrm{W}=\mathrm{A}_{n-1}$, B_n, D_n describes the inclusion of faces in a regular CW-ball. Our strategy is as follows: for any Coxeter system (W, S), the *polar dual* PW^* to the Coxeterhedron PW is a simplicial polytope, whose boundary complex $\partial\mathrm{PW}^*$ is called the *Coxeter complex* (see [Bro], [Hum]). For $\mathrm{W}=\mathrm{A}_{n-1}$, B_n, D_n, we will define a surjective set map

$$\mathbf{\Phi}\colon \mathrm{sd}\,(\partial\mathrm{PW}^*)\to\mathrm{KPW}\backslash\{\hat{0}\}$$

from the *barycentric subdivision* of the Coxeter complex to the Coxeter-associahedron poset with its bottom element $\hat{0}$ removed, (*i.e.*, the dual of the face poset of $\partial\mathrm{KPW}^*$). This map will have the following properties.

($\mathbf{\Phi}$1) For all faces A in $\mathrm{KPW}\backslash\{\hat{0}\}$, the pair

$$\left(\bigcup_{B\geqslant A}\mathbf{\Phi}^{-1}(B),\ \bigcup_{B>A}\mathbf{\Phi}^{-1}(B)\right)$$

is a pair of subcomplexes of $\mathrm{sd}\,(\partial\mathrm{PW}^*)$, and homeomorphic as a pair to $(\mathbf{B}^d, \partial\mathbf{B}^d)$ for some d. Here $\mathbf{B}^d$ denotes a topological d-ball and $\partial\mathbf{B}^d$ its boundary.

($\mathbf{\Phi}$2) If we let $\alpha(A)$ denote the underlying string of a parenthesized string A in KPW^*, then the usual barycentric subdivision homeomorphism (see [Mun])

$$h\colon \|\mathrm{sd}\,(\partial\mathrm{PW}^*)\|\to\|\partial\mathrm{PW}^*\|$$

maps $\bigcup_{B\geqslant A}\mathbf{\Phi}^{-1}(B)$ into the face of $\partial\mathrm{PW}^*$ corresponding to $\alpha(A)$.

This will then complete the proof of Theorem 1, namely that

$$\bigcup_{A\in\mathrm{KPW}\backslash\{\hat{0}\}}\left(\bigcup_{B\geqslant A}\mathbf{\Phi}^{-1}(B)\right)$$

is a regular CW-decomposition of a sphere which refines $\partial\mathrm{PW}^*$, and which is refined by $\mathrm{sd}(\partial \mathrm{PW}^*)$, *i.e.*,

$$\mathrm{sd}(\partial \mathrm{PW}^*) \prec \partial \mathrm{KPW}^* \prec \partial \mathrm{PW}^*.$$

Figure 5 shows these sequences of refinements for $\mathrm{W} = \mathrm{A}_3, \mathrm{B}_3, \mathrm{D}_3$.

The map Φ is easy to define once we have identified what faces in the barycentric subdivisions look like. By the definition of barycentric subdivision, a face in $\mathrm{sd}(\mathrm{PW}^*)$ is a chain C of strings

$$\alpha_1 < \alpha_2 < \ldots < \alpha_k$$

where $<$ is the order relation on strings previously defined. This means that for each i, α_{i+1} is obtained from α_i by combining consecutive blocks and/or adding in the box. To define $\Phi(C) = A$ as a parenthesized string, let the underlying string $\alpha(A)$ be α_1. Then at each step $\alpha_i < \alpha_{i+1}$, if some consecutive blocks of α_i are combined together, put a parenthesis pair around the corresponding blocks of α_1. This defines $\Phi(C) = A$. For example, if C is the chain

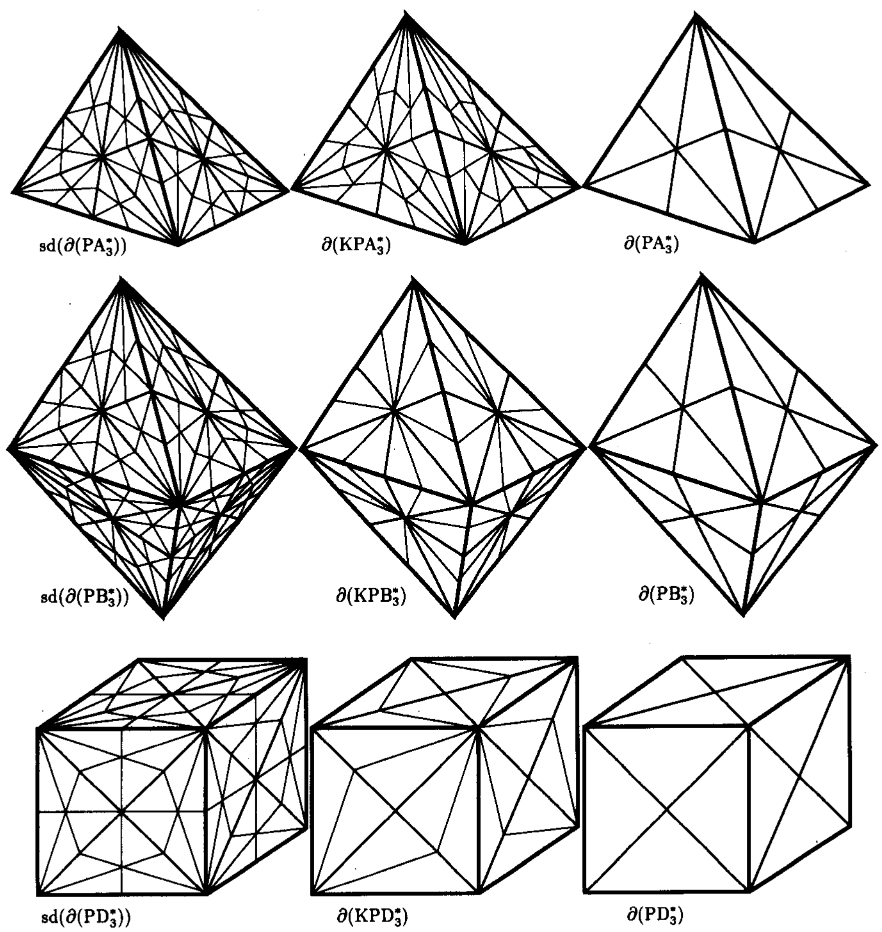


Figure 5. $\mathrm{sd}(\partial \mathrm{PW}^*) \prec \partial \mathrm{KPW}^* \prec \partial \mathrm{PW}^*$ for $\mathrm{W} = \mathrm{A}_3, \mathrm{B}_3, \mathrm{D}_3$.

in PD_8 given by

$$7.1\bar{4}.8.\bar{5}.26 < 1\bar{4}7.8.\bar{5}.\boxed{26} < 1\bar{4}78.\boxed{25\bar{6}},$$

then $\Phi(C)$ is

$$((7.1\bar{4})8)(\bar{5}.26).$$

Property (Φ2) of the map Φ will follow from the following lemma, whose proof is straightforward.

LEMMA 9. *If* $A \leqslant B$ *in* KPW *then* $\alpha(A) \leqslant \alpha(B)$ *in* PW.

To show property (Φ2), assume $C \in \bigcup_{B \geqslant A} \Phi^{-1}(B)$ *i.e.*, $\Phi(C) = B \geqslant A$. Under the homeomorphism h we know that the chain C of strings

$$\alpha_1 < \ldots < \alpha_k$$

will be mapped inside the face α_1 of PW*. Since $\alpha_1 = \alpha(B) \geqslant \alpha(A)$ by the lemma, we know that α_1 is a subface of $\alpha(A)$, and so C is mapped inside of $\alpha(A)$ by h as desired.

Property (Φ1) is not quite as obvious. The fact that $\bigcup_{B \geqslant A} \Phi^{-1}(B)$ and $\bigcup_{B > A} \Phi^{-1}(B)$ are both subcomplexes of sd(PW*) follows immediately from the next lemma, whose proof is again straightforward.

LEMMA 10. *If* $C \subseteq D$ *in* sd(PW*) *then* $\Phi(C) \geqslant \Phi(D)$ *in* KPW.

To show that $\bigcup_{B \geqslant A} \Phi^{-1}(B)$ is a ball with boundary $\bigcup_{B > A} \Phi^{-1}(B)$, we need to review a bit of the theory of *signed posets* and their associated B_m-*distributive* lattices $J(P)$ from [Rei]. A *signed poset* P on m elements is a subset P of the root system B_m

$$B_m = \{\pm \mathbf{e}_i \pm \mathbf{e}_j\}_{1 \leqslant i < j \leqslant m} \cup \{\pm \mathbf{e}_i\}_{1 \leqslant i \leqslant m}$$

satisfying two axioms related to irreflexivity and transitivity for posets.

(SP1) If $\mathbf{u}$ is in P, then $-\mathbf{u}$ is *not* in P.
(SP2) If $\mathbf{u}$, $\mathbf{v}$ are in P, and $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$ is in B_m for some c_1, $c_2 > 0$, then $\mathbf{w}$ is in P.

An *order ideal* I of P is a vector I in $\{0, +1, -1\}^m$ whose (usual) inner product with any vector in P is non-negative, *i.e.*,

$$\langle I, \mathbf{u} \rangle \geqslant 0, \qquad \forall \mathbf{u} \in P.$$

These order ideals are analogous to the usual notion of order ideals in a poset on m elements, if we think of the order relations in the poset as a subset of the root system A_{m-1}, and identify an order ideal with its characteristic vector in $\{0, 1\}^m$. The order ideals of P are then ordered component-wise using the

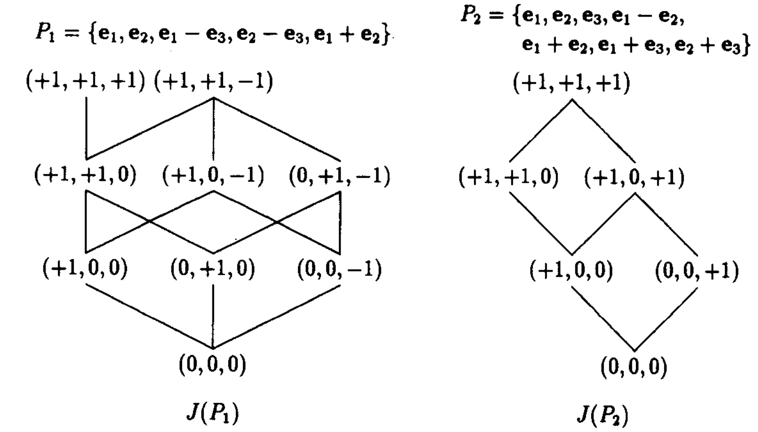


Figure 6. Two examples of the B_m-distributive lattices $J(P)$.

order $0 < +1, -1$ in each component to form a poset $J(P)$. In [Rei], $J(P)$ is called a B_m*-distributive lattice*. Two examples are shown in Fig. 6.

Why are these $J(P)$ relevant? Let $\Delta J(P)$ denote the *order complex* of $J(P)$, that is the simplicial complex of *chains* in $J(P)$. A theorem of [Rei] shows that $\Delta J(P)$ is EL-shellable in the sense of Björner [Bj1], and it then follows from a theorem of Danaraj and Klee [DK] [Bj2] [BLSWZ, Sect. 4.7] that $\Delta J(P)$ is homeomorphic to a ball. Our next goal then is to show that $\bigcup_{B \geqslant A} \Phi^{-1}(B)$ is isomorphic as a simplicial complex to $\Delta J(P_A)$ for a certain signed poset P_A.

To do this, label the parenthesis pairs in A by $p_1, p_2, \ldots, p_m$ in such a way that the virtual pair (if present) is labelled p_m. Then define the signed poset P_A by

$$P_A = \{\mathbf{e}_i - \mathbf{e}_j\colon \text{pair } p_i \text{ encloses } p_j\} \cup \{\mathbf{e}_i\colon p_i \text{ is } \textit{not} \text{ the virtual pair}\}$$
$$\cup \{\mathbf{e}_i + \mathbf{e}_j\colon \text{neither } p_i \text{ nor } p_j \text{ is virtual}\}.$$

It is easy to see that P_A always satisfies axioms (SP1), (SP2) of signed posets. For example, let

$$A_1 = ((\bar{3}.14)(2\bar{6}.5\bar{7}8)) \quad \text{in } \mathrm{KPB}_8$$
$$A_2 = (\bar{1}2(6.\bar{3}))(\bar{8}.\boxed{45)\bar{\bar{7}}} \quad \text{in } \mathrm{KPD}_8$$

and number the parenthesis pairs p_1, p_2, p_3 in such a way that p_3 encloses p_1, p_2 in A_1 and so that p_2 encloses p_1 in A_2. In this case, P_{A_1}, P_{A_2} coincide with the examples P_1, P_2 from Fig. 6.

We must now produce a simplicial isomorphism

$$f_A\colon \bigcup_{B \geqslant A} \Phi^{-1}(B) \to \Delta J(P_A).$$

First we define f_A on vertices. A vertex of $\bigcup_{B \geqslant A} \Phi^{-1}(B)$ is a single string α satisfying $\Phi(\alpha) = \alpha \geqslant A$. Here the string α on the right-hand side of the equality is thought of as a parenthesized string in KPW with the empty set of parentheses. Let $f_A(\alpha)$ be the vector I in $\{0, +1, -1\}^m$ specified by

$$I_i = \begin{cases} +1, & \text{if the numbers enclosed by pair } p_i \text{ in } A \\ & \quad \text{are all inside a single block of } \alpha, \\ -1, & \text{if } i=m \text{ and } A \text{ contains the virtual pair } p_m \\ & \quad \text{and } \alpha \text{ has any boxed blocks,} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that I is an ideal of P_A, and furthermore that f_A gives a bijection

$$\begin{array}{ccc} \text{vertices of } \bigcup_{B \geqslant A} \Phi^{-1}(B) & \leftrightarrow & \text{vertices of } \Delta J(P_A) \\ (=\text{strings } \alpha \geqslant A) & & (=\text{ideals of } P_A) \end{array}$$

Having defined f_A on the vertices, we need to know that vertices lying in a common face of $\bigcup_{B \geqslant A} \Phi^{-1}(B)$ map into a face of $\Delta J(P_A)$ in order for f_A to induce a simplicial map. This fact and the stronger fact that f_A induces a simplicial isomorphism are immediate from the following lemma (whose proof is again straightforward).

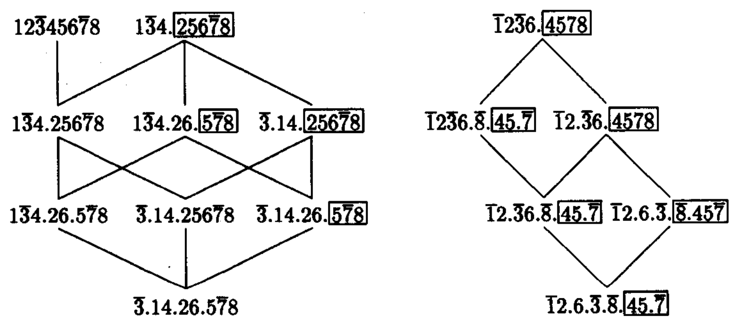
LEMMA 11. *Let α, β be two strings, with $\alpha, \beta \geqslant A$ (again we are thinking of α, β as elements of* KPW *with empty set of parentheses). Then $\alpha < \beta$ in* PW, *if, and only if, $f_A(\alpha) < f_A(\beta)$ as ideals in $J(P_A)$.*

Figure 7 shows the chains C in sd (PW*) that lie in $\bigcup_{B \geqslant A} \Phi^{-1}(B)$ for $A = A_1, A_2$ as in the previous example. Compare this with Fig. 6.

It only remains to verify that

$$f_A\left(\bigcup_{B > A} \Phi^{-1}(B)\right) = \partial \Delta J(P_A).$$

This is a routine exercise in the definitions, which we will not go through in detail. However, it does help to point out that the boundary $\partial\Delta J(P_A)$ is described completely once we know its maximal faces. These maximal faces are the chains C of ideals in P that miss exactly *one* rank of $J(P)$, and have a



$A_1 = ((\bar{3}.14)(2\bar{6}.5\bar{7}8))$ in KPB_8 $\qquad$ $A_2 = (\bar{1}2(6.\bar{3}))(\bar{8}.\boxed{45)\bar{7}}$

Figure 7. Two examples of the chains in $\bigcup_{B \geqslant A} \Phi^{-1}(B)$.

unique extension to the missing rank. One may then classify such a chain C according to whether the rank it misses is the top, bottom, or among the middle ranks, and this classification helps to show that $f_A^{-1}(C)$ lies in $\bigcup_{B>A} \Phi^{-1}(B)$, *i.e.*, $\Phi(f_A^{-1}(C)) > A$.

This completes the proof of

THEOREM 1. *For* $W = A_{n-1}, B_n, D_n$, *the order dual* KPW^* *is the face poset of a regular CW-ball, and there is a sequence of subdivisions*

$$\text{sd}\,(\partial PW^*) \prec \partial KPW^* \prec \partial PW^*.$$

COROLLARY 12. *For* $W = A_{n-1}, B_n, D_n$, *the topological space* $\|\Delta(KPW \backslash \{\hat{0}, \hat{1}\})\|$ *associated to the order complex of* KPW *is a sphere.*

§3. *Associahedra.* In this section, we start with a brief review of the construction of *fiber polytopes* due to Billera and Sturmfels [BS1] (see also [BS2], [Stu]), which generalizes and re-interprets the construction of *secondary polytopes* of Gel'fand, Zelevinsky & Kapranov [GZK1, GZK2]. The intuition of this construction motivates our construction of the Coxeter-associahedra, and provides the principal "building blocks" for it. Our sketch is supposed to provide geometric intuition for our construction of the permuto-associahedra, and (especially nice) coordinates for the associahedra $K_{n-2} \subseteq \mathbb{R}^n$.

Let $P \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^q$ be polytopes. Consider a projection

$$\pi: P \to Q,$$

of these polytopes, *i.e.*, an affine map $\pi: \mathbb{R}^p \to \mathbb{R}^q$ such that $Q = \pi(P)$. A *section* of π is a continuous map $\gamma: Q \to P$ which satisfies $\pi \circ \gamma = \text{id}_Q$, that is, $\pi(\gamma(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in Q$.

Definition 13. The *fiber polytope* $\Sigma(P, Q) \subseteq \mathbb{R}^p$ of a polytope projection $\pi: P \to Q$ is the set of all average values of the sections of π, that is,

$$\Sigma(P, Q) = \left\{ \frac{1}{\text{vol}\,(Q)} \int_Q \gamma(\mathbf{x}) d\mathbf{x} : \gamma \text{ is a section of } \pi \right\}.$$

Without loss of generality, we need only consider sections that are piecewise linear over a finite polyhedral subdivision of Q. Thus we can integrate the sections (component-wise) using classical Riemann integrals.

It is quite trivial to see that the fiber polytope is a convex set that is contained in the fiber of the barycentre,

$$\Sigma(P, Q) \subseteq \pi^{-1}(\mathbf{q}_0),$$

where the barycentre of Q is given by

$$\mathbf{q}_0 = \frac{1}{\text{vol}\,(Q)} \int_Q \mathbf{x} d\mathbf{x}.$$

Here we use that for a linear function f on a polytope R, one has the formula

$$\int_R f(\mathbf{x})d\mathbf{x} = \mathrm{vol}\,(R)\,f(\mathbf{r}_0),$$

where $\mathbf{r}_0$ is the barycentre of R. In the following we will mostly ignore the scaling factor vol (Q), which is needed for the above inclusion but irrelevant for our discussion. The following is the key result from Billera and Sturmfels [BS1].

THEOREM 14. [BS1] *$\Sigma(P, Q)$ is a polytope of dimension* $\dim(P)-\dim(Q)$, *whose faces correspond to the coherent subdivisions of Q by faces of P. Here the vertices of $\Sigma(P, Q)$ correspond to the finest subdivisions, while the facets correspond to the coarsest proper subdivisions.*

For every polytope Q with p vertices, there is a canonical map $\pi: \Delta_{p-1} \to Q$ from the simplex with p vertices to Q. In this case the vertices of $\Sigma(Q) = \Sigma(\Delta_{p-1}, Q)$ correspond to the regular triangulations of Q—this $\Sigma(Q)$ is the *secondary polytope* of Gel'fand, Zelevinsky and Kapranov [GZK1, GZK2].

The following construction from [Zie] explains the construction of *coherent subdivisions*, and thus of vertices and facets of $\Sigma(P, Q)$. For any linear functional $\mathbf{x} \mapsto \mathbf{c}\mathbf{x}$ on $\mathbb{R}^p$, we can consider the projection $\hat{\pi}: \mathbf{x} \mapsto (\pi(\mathbf{x}), -\mathbf{c}\mathbf{x})$, which maps P to $\hat{Q} := \hat{\pi}(P) \subseteq \mathbb{R}^{q+1}$. Thus the projection π factors into $P \xrightarrow{\hat{\pi}} \hat{Q} \to Q$, where the second map just forgets the last coordinate. Interpreting the last coordinate as a "height function" on $\hat{Q}$, we get a subdivision of Q from the "bottom faces" of $\hat{Q}$. Thus every linear functional $\mathbf{c}\mathbf{x}$ on P defines a subdivision of Q. Also, a generic linear function will induce a finest coherent subdivision of Q, which describes a unique section $\gamma: Q \to P$; in the case where Q is a simplex, the finest coherent subdivisions are triangulations.

Conversely, suppose we are given any coherent subdivision of Q and a convex function $f: Q \to \mathbb{R}$ which induces it, that is, such that with $\hat{Q} = \mathrm{conv}\,\{(\mathbf{x}, f(\mathbf{x})): \mathbf{x} \in Q\}$ we get the original subdivision as the projection of the bottom faces of $\hat{Q}$ to Q. From this we can define a linear function $\mathbf{c}^f$ on P by setting $\mathbf{c}^f\mathbf{v} := f(\pi(\mathbf{v}))$ for the vertices of $P = \Delta_{p-1}$, and extending linearly over Δ_{p-1}. If the original subdivision was a coarsest non-trivial one, then the linear functional $\mathbf{c}^f$ obtained from it will induce the corresponding facet of $\Sigma(\Delta_{p-1}, Q)$.

Instead of a detailed discussion and proofs we refer to [GZK2], [BS1] and [Zie, Lect. 9]. Here we will only discuss the two main examples that are relevant for the permuto-associahedra.

Example 15 (*Permutohedron*). [BS1, Ex. 5.4] Let $P = [0, 1]^n \subseteq \mathbb{R}^n$ be the unit cube in $\mathbb{R}^n$, and let $Q = [0, n] \subseteq \mathbb{R}^1$, then the map $\pi: \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto \Sigma_{i=1}^n x_i$ defines a projection $\pi: [0, 1]^n \to [0, n]$.

Here the "extreme sections" map $[0, n]$ to paths in the 1-skeleton of $[0, 1]^n$ that are increasing with respect to the height function $\sum_{i=1}^n x_i$. These correspond to permutations: the permutation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ corresponds to the path

$$\gamma^\sigma: \mathbf{0} \to \mathbf{e}_{\sigma(1)} \to \mathbf{e}_{\sigma(1)} + \mathbf{e}_{\sigma(2)} \to \ldots \to \mathbf{e}_{\sigma(1)} + \mathbf{e}_{\sigma(2)} + \ldots + \mathbf{e}_{\sigma(n)} = \mathbf{1},$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the zero vector and the all-ones vector in $\mathbb{R}^n$, and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denotes the standard basis in $\mathbb{R}^n$. The integral of γ^σ is given by the sum

$$\int_0^n \gamma^s(\mathbf{x})d\mathbf{x} = \tfrac{1}{2}((\gamma^\sigma(0) + \gamma^\sigma(1)) + (\gamma^\sigma(1) + \gamma^\sigma(2)) + \ldots + (\gamma^\sigma(n-1) + \gamma^\sigma(n)))$$

$$= \tfrac{1}{2}((2n-1)\mathbf{e}_{\sigma(1)} + (2n-3)\mathbf{e}_{\sigma(2)} + \ldots + (1)\mathbf{e}_{\sigma(n)})$$

$$= \tfrac{1}{2} \sum_{i=1}^{n} (2n+1-2i)\mathbf{e}_{\sigma(i)}$$

$$= \frac{2n+1}{2}\mathbf{1} - (\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(n)).$$

Thus the fiber polytope of the projection π turns out to be an affine image of the "usual" representation of the permutohedron, which represents the permutation σ by the vector whose entries are given by σ^{-1}:

$$\Sigma([0, 1]^n, [0, n]) \cong \mathrm{PA}_{n-1}.$$

Remark 16. There seems to be no similarly straightforward way to obtain the other Coxeterhedra as fiber polytopes, without admitting an extra group action.

We now turn to the $(n-2)$-dimensional associahedron, which was constructed as the secondary polytope of an $(n+1)$-gon by Gel'fand, Zelevinsky and Kapranov [GZK1, GZK2]. Viewed in terms of the fibre polytope construction, for any projection of an n-simplex to an $(n+1)$-gon the resulting fiber polytope is an associahedron.

For our purpose, however, we need a very special choice both of the n-simplex and of the $(n+1)$-gon, as follows.

Example 17 (*Associahedron*). Define $\mathbf{f}_i = \mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_i$ for $0 \leqslant i \leqslant n$, with $\mathbf{f}_0 = \mathbf{0}$. We use

$$\Delta_n = \mathrm{conv}\,\{\mathbf{0}, \mathbf{f}_1, \ldots, \mathbf{f}_n\} = \{\mathbf{x} \in \mathbb{R}^n : 1 \geqslant x_1 \geqslant \ldots \geqslant x_n \geqslant 0\}$$

as our *standard simplex*.

Consider the (linear) projection map $\pi: \Delta_n \to \mathbb{R}^2$ that maps $\mathbf{0}$ to $(0, 0)$ and

$$\pi: \mathbf{f}_i \mapsto (i, i^2),$$

$$\mathbf{f}_i - \mathbf{f}_{i-1} = \mathbf{e}_i \mapsto (1, 2i-1) = (i, i^2) - (i-1, (i-1)^2),$$

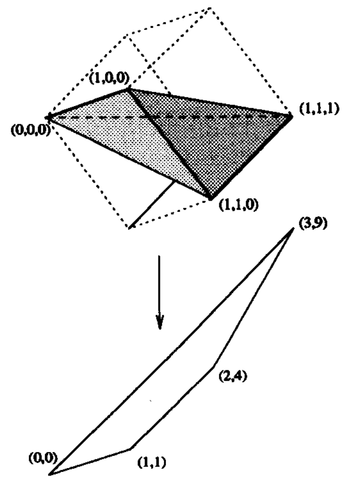


Figure 8. The projection $\pi: \Delta_n \to C_2(n+1)$ for $n=3$.

for $1 \leqslant i \leqslant n$. The image $\pi(\Delta_n)$ is the "cyclic" convex $(n+1)$-gon

$$C_2(n+1) = \text{conv}\,\{(i, i^2): 0 \leqslant i \leqslant n\}.$$

One can calculate the volume and the barycentre $\mathbf{q}_0$ of this $(n+1)$-gon as

$$\text{vol}\,(C_2(n+1)) = \binom{n+1}{3}, \qquad \mathbf{q}_0(C_2(n+1)) = \left(\frac{n}{2}, \frac{6n^2+1}{15}\right).$$

There is a well-known correspondence between the complete parenthesizations of a string of n letters and the triangulations T of the $(n+1)$-gon $C_2(n+1)$, as follows. For a word of length n, label with the numbers $0, 1, 2, \ldots, n$ the parenthesis positions before the word, between the letters and after the word. Then a parenthesis pair placed at positions i and j corresponds to a diagonal (i, j), and if the parenthesis pair groups two blocks together, one from positions i to j, the second from positions j to k, then this corresponds to the triangle

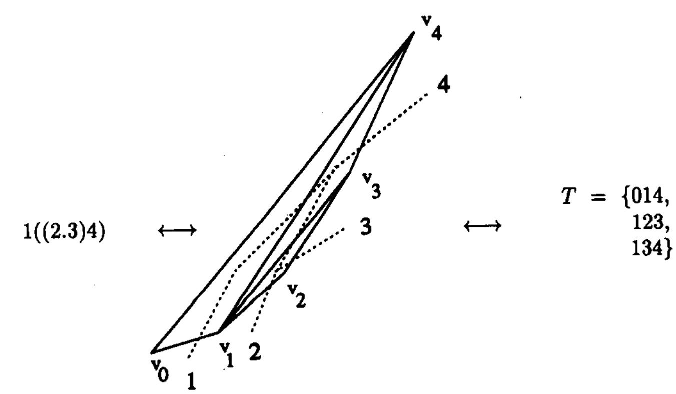


Figure 9. The correspondence between parenthesizations and triangulations.

$[ijk]$ having vertices

$$(i, i^2), \qquad (j, j^2), \qquad (k, k^2).$$

Note that if a parenthesization is complete (with $n-2$ parenthesis pairs), then every pair groups only two blocks together.

We denote by $\mathscr{T}_n$ the set of all these triangulations, viewed as sets of triangles. So, for example, we get (omitting set brackets for the triples),

$$\mathscr{T}_3 = \{\{013, 123\}, \{012, 023\}\},$$

$$\mathscr{T}_4 = \{\{014, 124, 234\}, \{014, 123, 134\}, \{024, 012, 234\}, \{013, 034, 123\}, \{012, 023, 034\}\}.$$

With this the fiber polytope [BS1] of the projection

$$\pi : \Delta_n \to C_2(n+1)$$

is given by

$$3\binom{n+1}{3} \Sigma(\Delta_n, C_2(n+1)) = \operatorname{conv} \{\mathbf{v}^T : T \in \mathscr{T}_{n+1}\},$$

where

$$\mathbf{v}^T = \sum_{(i,j,k) \in T} \tfrac{1}{2}(j-i)(k-i)(k-j)(\mathbf{f}_i + \mathbf{f}_j + \mathbf{f}_k) \in \mathbb{Z}^n,$$

for all triangulations T of $C_2(n+1)$ without new vertices. Here the sum is over all triples $i<j<k$ such that $(\pi(\mathbf{f}_i), \pi(\mathbf{f}_j), \pi(\mathbf{f}_k))$ is a triangle in the triangulation T, of area $\frac{1}{2}(j-i)(k-i)(k-j)$.

This yields a specific embedding of the associahedron K_{n-2}. We use the scaled fiber polytope

$$\mathrm{K}_{n-2} = \operatorname{conv} \{\mathbf{v}^T : T \in \mathscr{T}_n\} \subseteq \mathbb{R}^n$$

as our *standard associahedron*. It is realized in an $(n-2)$-dimensional affine subspace, which can be derived from the condition that $\pi(\Sigma(\Delta_n, C_2(n+1))) = \{\mathbf{q}_0(C_2(n+1))\}$. Thus we derive the equations

$$\sum_{i=1}^{n} x_i = 3\binom{n+1}{3}\frac{n}{2} = \frac{n^2(n^2-1)}{4},$$

$$\sum_{i=1}^{n} (2i-1)x_i = 3\binom{n+1}{3}\frac{6n^2+1}{15} = \frac{6n^5-5n^3-n}{30}.$$

Now we derive defining inequalities for the facets of the standard associahedron, using the method described above. The diagonals of $C_2(n+1)$ (which describe the coarsest possible subdivisions of $C_2(n+1)$) correspond to the pairs (i,j) with $0 \leqslant i < j \leqslant n$ and $2 \leqslant j-i \leqslant n-1$. With every such (i,j) we associate the convex function

$$f^{ij}(x,y)=\max\{0,\,-y+(i+j)x-ij\}$$

which defines the subdivision of $C_2(n+1)$ by the diagonal (i,j), because it is linear except for a break at the line through (i,i^2) and (j,j^2). We calculate $\mathbf{c}^{ij}\in(\mathbb{R}^n)^*$ from

$$\begin{aligned}\mathbf{c}^{ij}\mathbf{f}_k&=f^{ij}(\pi(\mathbf{f}_k))=f^{ij}(k,k^2)=\max\{0,\,-k^2-(i+j)k-ij\}\\&=\max\{0,\,(k-i)(j-k)\},\end{aligned}$$

and thus

$$\begin{aligned}\mathbf{c}^{ij}\mathbf{x}&=\mathbf{c}^{ij}\sum_{k=1}^{n}x_k\mathbf{e}_k=\mathbf{c}^{ij}\sum_{k=1}^{n}x_k(\mathbf{f}_k-\mathbf{f}_{k-1})\\&=\sum_{k=1}^{n}x_k(\max\{0,\,(k-i)(j-k)\}-\max\{0,\,(k-1-i)(j-k+1)\})\\&=\sum_{k=i+1}^{j}((k-i)(j-k)-(k-1-i)(j-k+1))x_k\end{aligned}$$

—so $c_k^{ij}=(-2k+1)+i+j$ for $i<k\leqslant j$, and $c_k^{ij}=0$ otherwise. Knowing $\mathbf{c}^{ij}$, the facets of K_{n-2} are given by

$$\mathbf{c}^{ij}\mathbf{x}\geqslant\binom{j-i+1}{3}\frac{3(j-i)^2-2}{10}\qquad\text{for}\qquad 0\leqslant i<j\leqslant n,\quad 2\leqslant j-i\leqslant n-1.$$

Here the right-hand side of the inequalities is $\min\{\mathbf{c}^{ij}\mathbf{v}^T\colon T\in\mathcal{T}_{n+1}\}$, where the minimum is achieved exactly by those triangulations that use the diagonal (i,j). The formula for the minimum was computed by integrating f^{ij} over $C_2(n+1)$.

The coordinates for this standard associahedron have further special properties. For example, the points $\mathbf{x}\in\mathrm{K}_{n-2}$ of this associahedron satisfy

$$3\binom{n+1}{3}>x_1>x_2>\ .\ .\ .>x_n>0.$$

In fact, this holds for the vertices by construction, and thus also for the convex hull.

A more "miraculous" effect is that for this special coordinatization of K_{n-2} the vertices lie on a sphere around the origin: for all $T\in\mathcal{T}_n$, we have

$$\sum_{i=1}^{n}(\mathbf{v}_i^T)^2=\binom{n+1}{3}\frac{30n^4-33n^2+2}{70}.$$

We have an algebraic proof for this, by analyzing the situation along an edge, corresponding to a single reparenthesization/change-of-diagonal, but no really good explanation.

§4. *Realization.* The Coxeter-associahedra will be constructed below, using the special realization of the associahedra obtained in the last section. The verification that the constructed objects have the desired face lattices KPW is a problem of polyhedral combinatorics. We recommend Schrijver's book [Sch] as a reference for terminology, results and techniques of this field; see also Grötschel and Padberg [GrP] for a valuable introduction.

In producing irredundant descriptions, it is of great advantage to deal with full-dimensional polytopes, since for these the facet defining inequalities are unique (up to a positive scalar). Therefore, we treat here the case of $\mathrm{KPB}_n \subseteq \mathbb{R}^n$ in detail (Theorem 20). From this we get the case of $\mathrm{KPA}_{n-1} \subseteq \mathbb{R}^n$, which is a facet of KPB_n (Corollary 21). The case of KPD_n is handled analogously, where we omit some details (Theorem 22).

For the construction of polytopes with specified combinatorics (and this is the principal object of this paper) it suffices to establish that the constructed polytope has the correct vertex-facet incidences. Here is the precise criterion we use to establish the combinatorial structure of the Coxeter-associahedra.

LEMMA 18. *Let L be a finite lattice that is atomic and coatomic. Let there be a map that associates a point $\mathbf{v}^\alpha \in \mathbb{R}^n$ with every atom $\alpha \in \mathrm{atom}\,(L)$, and let*

$$P = \mathrm{conv}\,\{\mathbf{v}^\alpha : \alpha \in \mathrm{atom}\,(L)\} \subseteq \mathbb{R}^n$$

be the convex hull of these points. Now assume that the following two conditions hold.

(i) *There is a linear functional $\mathbf{c}^\varphi \in (\mathbb{R}^n)^*$ for every coatom φ in L, such that the atoms below φ maximize $\mathbf{c}^\varphi$ among the points $\mathbf{v}^\alpha$, that is,*

$$\mathbf{c}^\varphi \mathbf{v}^\alpha = \max\,\{\mathbf{c}^\varphi \mathbf{v}^\beta : \beta \in \mathrm{atom}\,(L)\}, \quad \textit{if, and only if,} \quad \alpha \leqslant \varphi.$$

(ii) *Every $\mathbf{c} \in (\mathbb{R}^n)^*$ can be written as a non-negative sum of the functionals in a set S^α of the form $S^\alpha = \{\mathbf{c}^\varphi : \varphi \in \mathrm{coatom}\,(L), \varphi \geqslant \alpha\}$, for some $\alpha \in \mathrm{atom}\,(L)$.*

Then L is the face lattice of P, and we have an equality

$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}^\varphi \mathbf{x} \leqslant \max\,\{\mathbf{c}^\varphi \mathbf{v}^\alpha : \alpha \in \mathrm{atom}\,(L)\}, \textit{for all } \varphi \in \mathrm{coatom}\,(L)\}.$$

Proof. Let Q denote the subset of $\mathbb{R}^n$ defined by the right-hand side of the last equation. We have $P \subseteq Q$ by construction. Now by condition (ii), every linear function $\mathbf{c} \in (\mathbb{R}^n)^*$ be written as a positive sum of functions $\mathbf{c}^\varphi$ that are compatible with some $\alpha \in \mathrm{atom}\,(L)$. From condition (i) we derive that $\mathbf{v}^\alpha \in P$ maximizes $\mathbf{c}$ over Q. Thus every linear function on Q is maximized by some vertex in P, and this proves $P = Q$.

With this, condition (i) shows that the inequalities associated with $\varphi \in \mathrm{coatom}\,(L)$ exactly define the facets of P. This is enough to determine the complete combinatorics, since vertex sets of faces of a polytope are all the intersections of vertex sets of facets. (Abstractly, this follows since for a finite lattice that is atomic and coatomic, the subposet of atoms and coatoms completely determines the lattice [Stan, Ex. 3.12].)

Let us recall (and rephrase) Theorem 2 from the introduction:

THEOREM 2. *The polytopes* KPA_{n-1}, KPB_n *and* KPD_n *can be realized as the convex hulls*

$$|\mathrm{KPA}_{n-1}| = \mathrm{conv}\,\{w\mathbf{v}^T\colon T\in\mathcal{T}_n,\ w\in \mathrm{A}_{n-1}\} = \mathrm{conv}\,(\mathrm{A}_{n-1}\cdot \mathrm{K}_n),$$
$$|\mathrm{KPB}_n| = \mathrm{conv}\,\{w\mathbf{v}^T\colon T\in\mathcal{T}_n,\ w\in \mathrm{B}_n\} = \mathrm{conv}\,(\mathrm{B}_n\cdot \mathrm{K}_n),$$
$$|\mathrm{KPD}_n| = \mathrm{conv}\,\{w\mathbf{v}^T\colon T\in\mathcal{T}_n,\ w\in \mathrm{D}_n\} = \mathrm{conv}\,(\mathrm{D}_n\cdot \mathrm{K}_n),$$

where A_{n-1}, B_n *and* D_n *act on* $\mathbb{R}^n$ *by permutation and sign change of the coordinates (as usual).*

Here we have an inclusion $|\mathrm{KPD}_n| \subseteq |\mathrm{KPB}_n|$, *and* $|\mathrm{KPA}_{n-1}|$ *is the common facet of* $|\mathrm{KPB}_n|$ *and of* $|\mathrm{KPD}_n|$ *given by*

$$|\mathrm{KPA}_{n-1}| \subseteq |\mathrm{KPB}_n| \cap \left\{x\in\mathbb{R}^n\colon \sum_{i=1}^{n} x_i = \tfrac{1}{4}(n^4-n^2)\right\}.$$

Proof. The statements for KPA_{n-1} follow from those for KPB_n. Those in turn we derive below, in Theorem 20, after we have constructed an explicit description of the vertices and the facet-defining inequalities for KPB_n. Similarly, the proof for KPD_n is given in Theorem 22.

Remark 19. Let us indicate some of the geometric motivation for the construction in Theorem 2. The realization of the associahedron that we obtained in the last section already has many special properties. It can be viewed as the scaled fiber polytope of the projection $\Delta_n \to [0, n]$, where $\Delta_n \subseteq [0, 1]^n$ is a simplex whose images under the action of A_{n-1} (resp. B_n) cover the cubes $[0, 1]^n$ (resp. $[-1, 1]^n$). Each of these simplices has a canonical map to $C_2(n+1)$. Those maps fit together to give a non-linear "folding map" from $[0, 1]^n$ resp. $[-1, 1]^n$ to $C_2(n+1)$. Thus our construction can be viewed as a generalized fiber polytope associated with this non-linear projection map, or as an "equivariant fiber polytope", where we have a combination of compatible projection and group action.

We start now with the explicit description of the vertices and the facet-defining inequalities for our realization of KPB_n. Consider α, a completely parenthesized, signed permutation of length n, corresponding to an atom of the lattice KPB_n. Let $\sigma^\alpha = \sigma_1\sigma_2 \ldots \sigma_n$ be the permutation given by the letters of α, let $\kappa^\alpha \in \{+1, -1\}^n$ be the vector of signs, where $\kappa_i^\alpha = -1$ if the letter 'i' has a bar in α, and $\kappa_i^\alpha = +1$ otherwise, and let $T = T^\alpha$ be the triangulation associated with the parenthesization of α. Here $w^\alpha = [\sigma^\alpha, \kappa^\alpha]$ represents an element of B_n. The string α will be represented by the point $\mathbf{v}^\alpha = w^\alpha \mathbf{v}^{T^\alpha} \in \mathbb{R}^n$, whose important property is that

$$\kappa_{\sigma_1}\mathbf{v}^\alpha_{\sigma_1} > \kappa_{\sigma_2}\mathbf{v}^\alpha_{\sigma_2} > \ldots > \kappa_{\sigma_n}\mathbf{v}^\alpha_{\sigma_n}.$$

For example, for $\alpha = \bar{2}((1\bar{5})((46)3))$ we get the permutation $\sigma^\alpha = 215463$, the sign vector $\kappa^\alpha = (+1, -1, +1, +1, -1, +1)$, and the triangulation

$$T^\alpha = \{016, 136, 123, 356, 345\},$$

with areas

$$\text{vol}\,[016]=15, \quad \text{vol}\,[136]=15, \quad \text{vol}\,[123]=1,$$
$$\text{vol}\,[356]=3, \quad \text{vol}\,[345]=1.$$

Thus we compute

$$\begin{aligned}
\mathbf{v}^\alpha &= 15\{2(-\mathbf{e}_2)+1(+\mathbf{e}_1)+1(-\mathbf{e}_5)+1(+\mathbf{e}_4)+1(+\mathbf{e}_6)+1(+\mathbf{e}_3)\}\\
&\quad + 15\{3(-\mathbf{e}_2)+2(+\mathbf{e}_1)+2(-\mathbf{e}_5)+1(+\mathbf{e}_4)+1(+\mathbf{e}_6)+1(+\mathbf{e}_3)\}\\
&\quad + \ 1\{3(-\mathbf{e}_2)+2(+\mathbf{e}_1)+1(-\mathbf{e}_5)\}\\
&\quad + \ 3\{3(-\mathbf{e}_2)+3(+\mathbf{e}_1)+3(-\mathbf{e}_5)+2(+\mathbf{e}_4)+2(+\mathbf{e}_6)+1(+\mathbf{e}_3)\}\\
&\quad + \ 1\{3(-\mathbf{e}_2)+3(+\mathbf{e}_1)+3(-\mathbf{e}_5)+2(+\mathbf{e}_4)+1(+\mathbf{e}_6)\}\\
&= \quad -90\mathbf{e}_2 \quad +59\mathbf{e}_1 \quad -58\mathbf{e}_5 \quad +38\mathbf{e} \quad 4+37\mathbf{e}_6 \quad +33\mathbf{e}_3\\
&= (59, -90, 33, 38, -58, 37) \in \text{vert}\,(\text{KPB}_6).
\end{aligned}$$

which satisfies $-v_2^\alpha > v_1^\alpha > -v_5^\alpha > v_4^\alpha > v_6^\alpha > v_3^\alpha > 0$.

The facets of KPB_n correspond to strings without parentheses that either have only one block, or have a box, but not both. With each such string φ we associate a vector $\mathbf{c}^\varphi$, as follows.

Assume that the string φ has $p \geqslant 1$ blocks, where the first letter of the r-th block is the i_r-th letter of the string, and the last letter of the r-th block is the j_r-th letter of the string. Thus the string φ has a 'block structure' given by

$$i_1 \cdots j_1 . i_2 \cdots j_2 . \cdots \cdots . i_p \cdots j_p$$

with

$$1 = i_1 \leqslant j_1, \quad j_1 + 1 = i_2 \leqslant j_2, \ldots, j_{p-1} + 1 = i_p \leqslant j_p = n.$$

Again, we get a sign vector $\lambda^\varphi \in \{-1, +1\}^n$ to indicate which letters in φ have a bar:

$\lambda^i = -1$ if the letter 'i' has a bar in φ, and

$\lambda^i = +1$ if the letter 'i' has no bar.

Also, we read off a permutation $\tau^\varphi = \tau_1 \tau_2 \ldots \tau_n$ from φ. With these conventions, we define $\mathbf{c}^\varphi \in (\mathbb{R}^n)^*$ as

$$c^\varphi = \lambda^\varphi, \qquad \text{if} \quad p = 1, \quad \text{and}$$

$$c_k^\varphi = \lambda_k^\varphi (i_p + j_p - i_r - j_r)$$

if $p > 1$, and the letter 'k' lies in the r-th block of φ.

The first important property we need of this construction is that (in both cases) we have

$$\lambda_{\tau_1} c_{\tau_1}^\varphi \geqslant \lambda_{\tau_2} c_{\tau_2}^\varphi \geqslant \ldots \geqslant \lambda_{\tau_n} c_{\tau_n}^\varphi \geqslant 0,$$

with strict inequality $\lambda_{\tau_k} c_{\tau_k}^\varphi > \lambda_{\tau_{k+1}} c_{\tau_{k+1}}^\varphi$, if, and only if, the k-th and the $(k+1)$-st letter of τ^φ lie in different blocks, that is, if $k=i_r$ and $k+1=j_{r+1}$ for some $r<p$, and with strict inequality $\lambda_{\tau_n}c^\varphi_{\tau_n}>0$, if, and only if, $p=1$.

Again, here are examples, for $\varphi=1\bar{2}3\bar{4}\bar{5}6$ we get $p=1$, $i_1=1, j_1=n=6$, $\tau^\varphi=12\ldots6$, and thus

$$\lambda^\varphi=\mathbf{c}^\varphi=(+1,-1,+1,-1,-1,+1).$$

For $\varphi=\bar{2}.1\bar{5}.\boxed{346}$ we have $p=3$, $1=i_1=j_1$, $2=i_2<j_2=3$, $4=i_3<j_3=6$. We derive $\tau^\varphi=215346$, $\lambda^\varphi=(+1,-1,+1,+1,-1,+1)$, and thus we compute

$$\begin{aligned}\mathbf{c}^\varphi&=(+(4+6-2-3),-(4+6-1-1),0,0,-(4+6-2-3),0)\\&=(+5,-8,0,0,-5,0),\end{aligned}$$

with $-c^\varphi_2>c^\varphi_1=c^\varphi_5>c^\varphi_3=c^\varphi_4=c^\varphi_6=0$.

THEOREM 20. *With $\mathbf{v}^\alpha\in\mathbb{R}^n$ and $\mathbf{c}^\varphi\in(\mathbb{R}^n)^*$ as just constructed, the polytope $|\mathrm{KPB}_n|=\mathrm{conv}\,\{\mathbf{v}^\alpha\colon\alpha\in\mathrm{atom}\,(\mathrm{KPB}_n)\}\subseteq\mathbb{R}^n$ has face lattice KPB_n. A complete linear description is given by*

$$|\mathrm{KPB}_n|=\{\mathbf{x}\in\mathbb{R}^n\colon \mathbf{c}^\varphi\mathbf{x}\leqslant b^\varphi \textit{ for all } \varphi\in\mathrm{coatom}\,(\mathrm{KPB}_n)\},$$

where the right-hand sides are given by $b^\varphi=\frac{1}{4}(n^4-n^2)$ if $p=1$ (i.e., φ consists of one single block), and

$$\begin{aligned}b^\varphi=(i_p+j_p-1)\,&\frac{n^4-n^2}{4}-\frac{6n^5-5n^3-n}{30}\\&-\sum_{r=1}^{p}\binom{j_r-i_r+2}{3}\frac{3(j_r-i_r+1)^2-2}{10}.\end{aligned}$$

otherwise.

Proof. We apply the criterion of Lemma 18. Here $\mathbf{v}^\alpha$ and $\mathbf{c}^\varphi$ have already been constructed. Now every permutation and sign change of a vector $\mathbf{v}^\alpha$ is again a vector of this form. Thus from

$$\kappa_{\sigma_1}\mathbf{v}^\alpha_{\sigma_1}>\kappa_{\sigma_2}\mathbf{v}^\alpha_{\sigma_2}>\ldots>\kappa_{\sigma_n}\mathbf{v}^\alpha_{\sigma_n}$$

we see that for fixed φ, the sum

$$\mathbf{c}^\varphi\mathbf{v}^\alpha=\sum_k c^\varphi_k v^\alpha_k$$

can be maximized over $\{\mathbf{v}^\alpha\colon\alpha\in\mathrm{atom}\,(\mathrm{KPB}_n)\}$ only if all the summands $c^\varphi_k v^\alpha_k$ are positive, *i.e.*, $\kappa_k=\lambda_k$, and the components are ordered compatibly in size, with

$$\kappa_{\sigma_1}c^\varphi_{\sigma_1}\geqslant\kappa_{\sigma_2}c^\varphi_{\sigma_2}\geqslant\ldots\geqslant\kappa_{\sigma_n}c^\varphi_{\sigma_n}\geqslant0.$$

With this we may assume that $\tau=\sigma$. In fact, using the symmetry of the situation we may as well assume $\tau=\sigma=123\ldots n$ and $\kappa_k=\lambda_k=+1$ for all k. This reduces our situation to considering the linear function $\mathbf{c}^\varphi$, optimizing over the vertices of the associahedron in the coordinatization of Section 3, $\mathrm{conv}\,\{\mathbf{v}^T\colon T\in\mathscr{T}_n\}$.

Now we decompose $\mathbf{c}^{\varphi}\mathbf{x}$, as follows:

$$\begin{aligned}-\mathbf{c}^{\varphi}\mathbf{x} &= \sum_{\substack{k=1\\ i_r\leqslant k\leqslant j_r}}^{n} (i_r+j_r-i_p-j_p)x_k\\ &= \sum_{\substack{k=1\\ i_r\leqslant k\leqslant j_r}}^{n} (-2k+i_r+j_r)x_k+\sum_{k=1}^{n} (2k-i_p-j_p)x_k\\ &= \sum_{r=1}^{p}\sum_{k=i_r}^{j_r} ((-2k+1)+(i_r-1)+j_r)x_k+\sum_{k=1}^{n} (2k-1)x_k-(i_p+j_p-1)\sum_{k=1}^{n} x_k.\end{aligned}$$

This last expression shows the following. The last two sums are constant over the associahedron $K_{n-2}\subseteq\mathbb{R}^n$. For the first sum, if $i_r=j_r$, then the coefficient of x_k is zero for $k=i_r=j_r$. If $p=1$ and $i_1=1$, $j_1=n$, then the whole sum is constant over the associahedron. In all other cases we get that the sum $\sum_{k=i_r}^{j_r}((-2k+1)+(i_r-1)+j_r)$ is maximized by $\mathbf{v}^{\alpha}$, if, and only if, T^{α} has a diagonal from $i_r-1=j_{r-1}$ to j_r, that is, if α has a parenthesis pair at the positions i_r-1 and j_r.

In other words, $\mathbf{v}^{\alpha}$ minimizes $-\mathbf{c}^{\varphi}\mathbf{x}$, if, and only if, the string α has a parenthesis pair around every non-trivial block of φ (of length between 2 and $n-1$). The explicit minimal value can now be derived from the data in Section 3.

This completes the argument that $\mathbf{v}^{\alpha}$ minimizes $-\mathbf{c}^{\varphi}\mathbf{x}$, if, and only if, $\alpha\leqslant\varphi$. It only remains to show hypothesis (ii) of Lemma 9, *i.e.*, that every $\mathbf{c}\in(\mathbb{R}^n)^*$ can be written as a positive linear combination of linear functionals that define the facets meeting at a particular vertex. The proof of this gives an algorithm for maximizing the functional $\mathbf{cx}$ over the polytope, which we now describe.

Let $\mathbf{c}\in(\mathbb{R}^n)^*$ be arbitrary. Maximizing $\mathbf{cx}$ over Q, we may use the symmetry of P and Q, to assume $c_1\geqslant c_2\geqslant\ldots\geqslant c_n\geqslant 0$. Now we algorithmically expand $\mathbf{c}$ into a positive combination of vectors $\mathbf{c}^{\varphi}$, where φ is a partition of $\varphi_0=12\ldots n$ into blocks, where the last one is boxed if there is more than one block. Here we have $c^{\varphi_0}=(1,1,\ldots,1)$. First write $\mathbf{c}=c_n\mathbf{c}^{\varphi_0}+\mathbf{c}'$, where the vector $\mathbf{c}'=\mathbf{c}-c_n\mathbf{c}^{\varphi_0}$ satisfies $c_1'\geqslant c_2'\geqslant\ldots\geqslant c_n'=0$. For $\mathbf{c}'=(0,0,\ldots,0)$ we are done. Otherwise $\mathbf{c}'$ has $p>0$ different components, and we can determine i_r, j_r such that

$$c_{i_1}'=\ldots=c_{j_1}'>c_{i_2}'=\ldots=c_{j_2}'>\ldots>c_{i_p}'=\ldots=c_{j_p}'=0.$$

Now set $\varphi_1=i_1\cdots j_1.i_2\cdots j_2.\cdots.i_p\cdots j_p$ and subtract a suitable multiple of $\mathbf{c}^{\varphi_1}$ from $\mathbf{c}'$. In fact, we can rewrite $\mathbf{c}'=t_1\mathbf{c}^{\varphi_1}+\mathbf{c}''$ for

$$t_1=\min_{1\leqslant r<p}\frac{\mathbf{c}_{j_p}'-\mathbf{c}_{i_{p+1}}'}{\mathbf{c}_{j_p}^{\varphi_1}-\mathbf{c}_{i_{p+1}}^{\varphi_1}}.$$

This t_1 is the largest t_1 such that $\mathbf{c}''$ turns out to be decreasing. Then $\mathbf{c}''$ is again decreasing, with the last component 0, and the blocks of components where $\mathbf{c}''$ is constant are unions of such blocks for $\mathbf{c}'$. Furthermore, $\mathbf{c}''$ has fewer different components than $\mathbf{c}'$. Thus if we iterate this procedure, after $k\leqslant n$ steps we have written $\mathbf{c}$ in the form

$$\mathbf{c}=c_n\mathbf{c}^{\varphi_0}+t_1\mathbf{c}^{\varphi_1}+\ldots+t_k\mathbf{c}^{\varphi_k}$$

with $t_i\geqslant 0$, and such that the blocks of φ_{i+1} are unions of blocks of φ_i. Thus there exist complete parenthesizations α of $12\ldots n$ such that $\alpha \leqslant \varphi_i$ for all i. In other words, the ordered partitions φ_i determine a certain set of diagonals in $C_2(n+1)$, one for every block of size $2 \leqslant |B| < n$ occurring in some φ_i. Thus $\mathbf{v}^T$ maximizes $\mathbf{cx}$ over Q, if, and only if, the triangulation T contains this set of diagonals.

We illustrate this algorithm for optimization over Q by an example. Let $n=6$, and $\mathbf{c}_0=(-2, 5, 7, -4, -5, 9)$. We first optimize $\mathbf{c}=(9, 7, 5, 5, 4, 2)$ over Q. This $\mathbf{c}$ is rewritten as follows.

$$(9, 7, 5, 5, 4, 2)=2(1, 1, 1, 1, 1, 1)+(7, 5, 3, 3, 2, 0)$$

$$(7, 5, 3, 3, 2, 0)=\tfrac{1}{3}(10, 8, 5, 5, 2, 0)+(\tfrac{11}{3}, \tfrac{7}{3}, \tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}, 0)$$

$$(\tfrac{11}{3}, \tfrac{7}{3}, \tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}, 0)=\tfrac{1}{4}(10, 8, 4, 4, 4, 0)+(\tfrac{7}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0)$$

$$(\tfrac{7}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0)=\tfrac{1}{15}(10, 5, 5, 5, 5, 0)+(\tfrac{1}{2}, 0, 0, 0, 0, 0)$$

$$(\tfrac{1}{2}, 0, 0, 0, 0, 0)=\tfrac{1}{12}(6, 0, 0, 0, 0, 0)$$

and thus we rewrite

$$\mathbf{c}=2\mathbf{c}^{123456}+\tfrac{1}{3}\mathbf{c}^{1.2.34.5.\boxed{6}}+\tfrac{1}{4}\mathbf{c}^{1.2.345.\boxed{6}}+\tfrac{1}{15}\mathbf{c}^{1.2345.\boxed{6}}+\tfrac{1}{12}\mathbf{c}^{1.\boxed{23456}}.$$

From this we read off that $\mathbf{cx}$ is maximized (over Q) by

$$\begin{aligned}\mathbf{v}^{1((2((3\cdot 4)5))6)} &= 15(\mathbf{f}_0+\mathbf{f}_1+\mathbf{f}_6)+10(\mathbf{f}_1+\mathbf{f}_5+\mathbf{f}_6)\\ &\quad +6(\mathbf{f}_1+\mathbf{f}_2+\mathbf{f}_5)+3(\mathbf{f}_2+\mathbf{f}_4+\mathbf{f}_5)+1(\mathbf{f}_2+\mathbf{f}_3+\mathbf{f}_4)\\ &=(90, 59, 49, 48, 44, 25),\end{aligned}$$

so $\mathbf{c}_0$ is maximized by $\mathbf{v}^{\bar{6}((3((2\cdot\bar{5})\bar{4}))1)}=(-25, 49, 59, -44, -48, 90)$ and by $\mathbf{v}^{\bar{6}((4((2\cdot\bar{5})\bar{3}))1)}=(-25, 48, 59, -44, -49, 90)$.

As a corollary, we get a complete description of the polytope KPA_{n-1}. The facets of KPA_{n-1} correspond to the ordered partitions ψ of $\{1, 2, \ldots, n\}$ into at least 2 blocks. Let (ψ) denote the string ψ surrounded by a pair of parentheses, then (ψ) corresponds to a face of codimension 2 in KPB_n. This face lies below two facets of KPB_n: namely the facet corresponding to $\varphi^0=12\ldots n$, which we identify with KPA_{n-1}, and the facet corresponding to $\varphi(\psi)$, where $\varphi(\psi)$ is obtained from ψ by boxing the last block. (See Fig. 2 for the case $n=2$.)

COROLLARY 21. *With* $\mathbf{v}^\alpha \in \mathbb{R}^n$, $\mathbf{c}^\varphi \in (\mathbb{R}^n)^*$, *and* b^φ *as used in Theorem* 20, *the polytope* $|\mathrm{KPA}_{n-1}|=\mathrm{conv}\,\{\mathbf{v}^\alpha : \alpha \in \mathrm{atom}\,(\mathrm{KPA}_{n-1})\} \subseteq \mathbb{R}^n$ *has face lattice* KPA_{n-1}. *A complete linear description is given by*

$$|\mathrm{KPB}_n| = \left\{\mathbf{x} \in \mathbb{R}^n : \sum_i \mathrm{x}_i = \tfrac{1}{4}(n^4-n^2), \right.$$

$$\left. \mathbf{c}^{\varphi(\psi)}\mathbf{x} \leqslant b^{\varphi(\psi)} \quad \textit{for all } \psi \in \mathrm{coatom}\,(\mathrm{KPA}_{n-1}) \right\},$$

where $\varphi(\psi)$ *is obtained from* ψ *by boxing the last block.*

Analogously, we have a theorem for the case of KPD_n. Here the vertex set is given as

$$\text{vert}\,|KPD_n| = \{\mathbf{v}^\alpha : \alpha \in \text{atom}\,(KPD_n)\} \subseteq \{\mathbf{v}^\alpha : \alpha \in \text{atom}\,(KPB_n)\} = \text{vert}\,|KPB_n|.$$

The lattices KPD_n have three types of coatoms φ: strings with exactly one block (and an even number of minus signs), strings with more than one block such that the last block is boxed (so this last block contains more than one element) and strings where the last two blocks are boxed (and the last block is a singleton). In first two cases, we have already constructed c^φ and b^φ for the linear description of the polytopes KPB_n in Theorem 19. Reusing this, we get a complete description of the polyhedral realization of KPD_n, as follows.

THEOREM 22. *With $\mathbf{v}^\alpha \in \mathbb{R}^n$, $\mathbf{c}^\varphi \in (\mathbb{R}^n)^*$ and $b^\varphi \in \mathbb{R}$ as constructed before, the polytope $|KPD_n| = \text{conv}\,\{\mathbf{v}^\alpha : \alpha \in \text{atom}\,(KPD_n)\} \subseteq \mathbb{R}^n$ has face lattice KPD_n. A complete linear description is given by*

$$|KPD_n| = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}^\varphi \mathbf{x} \leqslant b^\varphi \text{ for all } \varphi \in \text{coatom}\,(KPD_n)\},$$

where b^φ has the same values as in Theorem 19 if at most one block is boxed. If the last two blocks of φ are boxed, $i_p = j_p = n$, then we define $\lambda^\varphi \in \{+1, -1\}^n$ as before, and

$$c_k^\varphi = \lambda_k^\varphi (3n - 1 + i_{p-1} - 2i_r - 2j_r)$$

if $p > 1$, and the letter 'k' lies in the r-th block of φ, and

$$b^\varphi = (3n - 3 + i_{p-1}) \frac{n^4 - n^2}{4} - \frac{6n^5 - 5n^3 - n}{15} - \sum_{r=1}^{p} \binom{j_r - i_r + 2}{3} \frac{3(j_r - i_r + 1)^2 - 2}{5}.$$

Proof. We apply the criterion of Lemma 18. The proof is analogous to that of Theorem 20, so we only remark about two new points.

First, we need a lemma for maximizing a linear function over an orbit of D_n (the same is quite trivial for B_n). For this, let $\mathbf{v} \in \mathbb{R}^n$ with $v_1 > v_2 > \ldots > v_n > 0$ and $\mathbf{c} \in (\mathbb{R}^n)^*$ with $c_1 \geqslant c_2 \geqslant \ldots > c_{i_{p-1}} = \ldots = c_{n-1} = -c_n > 0$. Then

$$\max\,\{\mathbf{c}(w\mathbf{v}) : w \in D_n\} = \mathbf{c}\mathbf{v},$$

and the maximum is achieved by $w = \text{id}$, and by those signed permutations with exactly two minus signs such that $w\mathbf{c}$ is weakly decreasing, except that one component $(w\mathbf{c})_i$ with $i_{p-i} \leqslant i < n$ has a minus sign.

Secondly, it helps to observe that both in KPB_n and in KPD_n, every facet is either isomorphic to KPA_{n-1} or adjacent to such a facet. This implies a strong relationship between the facet defining inequalities of KPB_n and of KPD_n. In fact, assume that the last block of φ is a singleton, and the last two blocks of φ are boxed, so φ defines a facet of KPD_n. Let ψ be the string obtained by boxing only the last singleton block (instead of the box in φ), so ψ defines a facet of KPB_n. Also, let ψ' be the string with only one block, and bars over the same letters as in φ. Then one can see from the combinatorics that the inequality $\mathbf{c}^{\psi} \leqslant b^{\psi}$ has to be a positive combination of the inequalities $\mathbf{c}^{\varphi} \leqslant b^{\varphi}$ and $\mathbf{c}^{\psi'} \leqslant b^{\psi'}$. Indeed, we have

$$\mathbf{c}^{\psi} = 2\mathbf{c}^{\psi} - (i_p + j_p - i_{p-1} - j_{p-1})\mathbf{c}^{\psi'},$$

and

$$b^{\psi} = 2b^{\psi} - (i_p + j_p - i_{p-1} - j_{p-1})b^{\psi'},$$

as is easily checked.

§5. *Remarks.*

1. The three families of Coxeter-associahedra are realized, by construction, with the symmetry of the associated Coxeter group. Additionally, the vertices in our description are integral, and they lie on a sphere around the origin. This last fact follows by construction from the same phenomenon that we observed for the associahedron at the end of Section 3. As stated there, we have a proof, but lack a good explanation, for this phenomenon.

2. It is extremely desirable to have a more conceptual description of the Coxeter-associahedra in terms of fiber polytopes. This suggests an extension of the fiber polytope construction either for piecewise-linear maps, or to an equivariant setting (compare Remark 16).

3. The geometric intuition in Kapranov's paper [Kap] was that one should construct the permuto-associahedron by placing a "small" associahedron at every vertex of a permutohedron, in a suitable way. This suitable way was found and described in Section 3, except that the associahedra were not small (*cf.* Figure 4). However, there is a construction that matches Kapranov's intuition, producing arbitrarily small associahedra at the vertices of a Coxeterhedron. For this we observe that the normal fan for our realizations of KPW refines that of the Coxeterhedron PW, realized as usual as the convex hull of an orbit of W. Thus we get

$$\mathrm{KPW} \cong t\mathrm{KPW} + (1-t)\mathrm{PW} \xrightarrow{t \to 0} \mathrm{PW}$$

from the fact that if the normal fan of P refines that of Q, then the Minkowski sum $P+Q$ is combinatorially isomorphic (and normally equivalent) to P (see [GrZ] [Zie, Prop. 7.12]).

4. The face poset of the associahedron K_n was shown to be EL-*shellable* by Björner (personal communication), and it follows from the realization of KPW as a polytope that its face poset KPW is *CL-shellable* (see [BW]). Are there nice EL- or CL-shellings of KPW?

5. With the proof of Theorem 20, we have a combinatorial, polynomial algorithm for *optimization* over the Coxeter-associahedra. Is there a similarly simple routine for *separation*, *i.e.*, to decide whether a given point lies in KPA_{n-1}?

6. Comments on tools: We have used the program "PORTA" [Chr, CJR] for Fourier–Motzkin computations, yielding complete and irredundant sets of defining equations and inequalities from lists of vertices (and vice versa). This

program is powerful enough to do complete computations for all the 4-dimensional Coxeter-associahedra. The figures 1, 3 and 4, displaying the Coxeterhedra and Coxeter-associahedra for $n=3$ as spatial polytopes, were generated from PORTA output by Jürgen Richter-Gebert (using Mathematica graphics in Fig. 4).

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References

BS1. L. J. Billera and B. Sturmfels. Fiber polytopes. *Annals of Mathematics*, 135 (1992), 527–549.

BS2. L. J. Billera and B. Sturmfels. Iterated fiber polytopes. *Mathematika*, 41 (1994), 348–363.

Bj1. A. Björner. Shellable and Cohen-Macaulay partially ordered sets. *Transactions Amer. Math. Soc.*, 260 (1980), 159–183.

Bj2. A. Björner. Posets, regular CW complexes and Bruhat order. *European J. Combinatorics*, 5 (1984), 7–16.

BLSWZ. A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler. Oriented Matroids. *Encyclopedia of Mathematics* (Cambridge University Press, 1993).

BW. A. Björner and M. Wachs. On lexicographically shellable posets. *Transactions Amer. Math. Soc.*, 277 (1983), 323–341.

Bro. K. S. Brown. *Buildings* (Springer-Verlag, New York, 1989).

Chr. Th. Christof. Ein Verfahren zur Transformation zwischen Polyederdarstellungen. *Diplomarbeit*, *Universität Augsburg*, 1991, 79 pages, plus manual and listing.

CJR. Th. Christof, M. Jünger and G. Reinelt. A complete description of the traveling salesman polytope on 8 nodes. *Operations Research Letters*, 10 (1990), 497–500.

DK. G. Danaraj and V. Klee. Shellings of spheres and polytopes. *Duke Math. J.*, 41 (1974), 443–451.

GZK1. I. M. Gel'fand, A. V. Zelevinsky and M. M. Kapranov. Newton polytopes of principal A-determinants, *Soviet Math. Doklady*, 40 (1990), 278–281.

GZK2. I. M. Gel'fand, A. V. Zelevinsky and M. M. Kapranov. Discriminants of polynomials in several variables and triangulations of Newton polyhedra, *Leningrad Math. Journal*, 2 (1991), 449–505.

GrZ. P. Gritzmann and B. Sturmfels. Minkowski addition of polytopes: Computational complexity and applications to Gröbner bases, *SIAM J. Discrete Math.*, 6 (1993), 246–269.

GrP. M. Grötschel and M. Padberg. Polyhedral Theory. Chapter 8 in: E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, D. B. Schmoys (eds.), *The Traveling Salesman Problem* (Wiley, 1985), 251–360.

Hai. M. Haiman. Constructing the associahedron. Unpublished manuscript, (*MIT*, 1984), 11 pages.

Hum. J. E. Humphreys. *Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Mathematics*, 29 (Cambridge University Press, 1990).

Kap. M. M. Kapranov. Permuto-associahedron, MacLane coherence theorem and asymptotic zones for the KZ equation. *Journal of Pure and Applied Algebra*, 85 (1993), 119–142.

Lee. C. W. Lee. The associahedron and triangulations of the *n*-gon. *European J. Combinatorics*, 10 (1989), 551–560.

Mun. J. R. Munkres. *Elements of Algebraic Topology*, (Addison-Wesley, Menlo Park, 1984).

Per. M. A. Perles. List of problems. *Conference on Convexity*, *Oberwolfach*, *July 1984*.

Rei. V. Reiner. Signed posets. *J. Combinatorial Theory*, *Ser. A*, 62 (1993), 324–360.

Sch. A. Schrijver. *Theory of Linear and Integer Programming*, *Wiley-Interscience Series in Discrete Mathematics and Optimization* (John Wiley & Sons, Chichester, 1986).

Stan. R. P. Stanley. *Enumerative Combinatorics*, Vol. I (Wadsworth & Brooks/Cole, Monterey, 1986).

Stas. J. D. Stasheff. Homotopy associativity of H-spaces I. *Transactions Amer. Math. Soc.*, 108 (1963), 275–292.

Stu. B. Sturmfels. Fiber polytopes: a brief overview. In: "*Special Differential Equations*" (ed. M. Yoshida) (Kyushu University, Fukuoka, 1991), pp. 117–124.

Zie. G. M. Ziegler. *Lectures on Polytopes. Graduate Texts in Math.*, 152 (Springer, 1994).

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REPRESENTATION EXTENSIONS AND AMALGAMATION BASES IN RINGS

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Abstract. The main purposes of this paper are to investigate $\mathbb{Z}$-injective rings with the representation extension property and its dual, to give a necessary and sufficient condition for a $\mathbb{Z}$-injective ring to be an amalgamation base in the class of all rings and to determine structure of $\mathbb{Z}$-injective Noetherian rings which are amalgamation bases. Further, in the class of all commutative rings, it is shown that a commutative ring has the representation extension property, if, and only if, it is an amalgamation base.

§1. *Introduction and preliminaries.* In [**2**], P. M. Cohn investigated initially free products of rings amalgamated with a ring fixed. Consequently, he showed that a regular ring R is embedded in any free product of rings with R amalgamated, thus R is an amalgamation base in the class of all rings. Recently, Renshaw [**9**] gave a criterion for amalgamation bases for rings. The main purposes of this paper are to improve the Renshaw's criterion and to determine the structure of $\mathbb{Z}$-injective Noetherian rings which are amalgamation bases in the class of all rings.

Throughout all the paper all rings are associative rings with an identity, and all modules are unital. Let R be a ring with identity 1. Then the subring of R generated by 1 can be identified with the ring of all integers $\mathbb{Z}$ or its residue ring $\mathbb{Z}/m\mathbb{Z}$, where m is a positive integer >1. If R is $\mathbb{Z}$-injective, that is, injective as an additive group, then either R contains $\mathbb{Q}$, so it is free as $\mathbb{Q}$-module or it is a free $\mathbb{Z}/m\mathbb{Z}$-module, respectively. In the latter R is a direct product of rings whose characteristics are power of primes. For any subset X of R, define $l-\text{ann}\,(X)=\{r\in R | rX=0\}$. It is called a *left annihilator* of X in R. Similarly a *right annihilator* of R is defined.

LEMMA 1. *Let R be a $\mathbb{Z}$-injective ring of characteristic p^s, where p is a prime number, s is a positive integer. Then*

(1) $l-\text{ann}\,(\{p^i\})\ (=r-\text{ann}\,(\{p^i\}))=Rp^{s-i}$,

(2) $Rp^ix\subset Rp^iy$ $(x, y\in R, 0\leqslant i<s)$ *implies* $Rx\subset Ry+Rp^{s-i}$.

Proof. (1) This is easy. (For example, see [**5**, III, 1.7, Exercise 7].)

(2) Let x, $y\in R$ with $Rp^ix\subset Rp^iy$. Then $p^ix=p^ity$ for some $t\in R$. Thus $p^i(x-ty)=0$. By (1), $x-ty\in Rp^{s-i}$. Hence $x\in Ry+Rp^{s-i}$.

LEMMA 2. *Let R be the same as above, and I a left ideal of R with $I\supset Rp$. Then there exist submodules A_1, A_2 of the $\mathbb{Z}$-module R such that $I=A_1+Rp$, $R=A_1\oplus A_2$.*