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CATALAN TRIANGULATIONS OF THE MÖBIUS BAND

Paul H. Edelman
Victor Reiner

University of Minnesota

Abstract. A Catalan triangulation of the Möbius band is an abstract simplicial complex triangulating the Möbius band which uses no interior vertices, and has vertices labelled $1, 2, \ldots, n$ in order as one traverses the boundary. We prove two results about the structure of this set, analogous to well-known results for Catalan triangulations of the disk. The first is a generating function for Catalan triangulations of M having n vertices, and the second is that any two such triangulations are connected by a sequence of diagonal-flips.

I. Introduction and definitions.

By a *Catalan triangulation* of the disk (resp. Möbius band), we mean an abstract simplicial complex triangulating the disk (resp. Möbius band) which uses no interior vertices, and has vertices labelled $1, 2, \ldots, n$ in order as one traverses the boundary. We will refer to a triangle in such a triangulation by the unordered triple ijk of vertices it contains, and similarly for edges ij and vertices i.

Let $\mathcal{A}_n$ and $\mathcal{M}_n$ denote the set of Catalan triangulations of the disk and Möbius band , respectively. The present paper was motivated by two very well-known facts about $\mathcal{A}_n$ (see, e.g., [STT]):

(1) The cardinality a_n of $\mathcal{A}_n$ is the Catalan number $\frac{1}{n-1}\binom{2n-4}{n-2}$, which has generating function

$$A(x) = \sum_{n \geq 2} a_n x^n = \frac{2x^2}{1 + \sqrt{1-4x}}.$$

(here we are using the convention that $a_2 = 1$).

(2) Any two triangulations in $\mathcal{A}_n$ may be connected by a sequence of operations which we will call *diagonal flips*, in which two triangles ijk and ikl which share the diagonal edge ik within the quadrangle $ijkl$ are replaced by the two other triangles ijl and jkl in the quadrangle.

The two main results of this paper (proven in Sections 2 and 3) are:

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Theorem 1. *The cardinality m_n of $\mathcal{M}_n$ has generating function*

$$M(x) = \sum_{n \geq 5} m_n x^n = \frac{x^2((2-5x-4x^2)+\sqrt{1-4x}(-2+x+2x^2))}{(1-4x)(1-4x+2x^2+\sqrt{1-4x}(1-2x))}$$
$$= x^5 + 14x^6 + 113x^7 + \ldots$$

Theorem 2. *Any two triangulations in $\mathcal{M}_n$ may be connected by a sequence of diagonal flips.*

A few remarks are in order about the connection of these results to previous work on triangulations of surfaces S with boundary. In the case where S is the disk, let $\mathcal{A}_{n,m}$ be the set of triangulations using exactly m interior vertices and n boundary vertices. Brown [Br] generalized the Catalan enumeration by giving a simple closed form for the cardinality of $\mathcal{A}_{n,m}$. A more recent result of Pachner [Pa, Theorem 5.3] shows that any two triangulations of simplicial d-polytopes having the same number of vertices are connected by *bistellar operations* that preserve the number of vertices. In the special case $d = 3$, diagonal flips are the only bistellar operations which preserve the number of vertices, and Pachner's proof along with Steinitz's Theorem (see, e.g., [Gr, p. 235]) actually shows that any two triangulations in $A_{n,m}$ are connected by diagonal flips.

Moving on to surfaces without boundary, the analogous results to Theorems 1 and 2 are known for labelled triangulations of 2-spheres. A result of Tutte [Tu] counting *rooted planar triangulations* may be converted to a simple closed form for the number of labelled triangulations of a 2-sphere having n vertices, and Wagner [Wa] was the first to prove that all such triangulations are connected by diagonal flips. Dewdney [De] proves that **unlabelled** triangulations of the 2-torus are all connected by diagonal flips. However, this is not true for labelled triangulations as there is exactly one unlabelled triangulation of the 2-torus using 7 vertices which has no neighbors under diagonal flips, but can be labelled in more than one way.

For surfaces S of arbitrary genus, both orientable and non-orientable, there has been a great deal of work on enumerating the *triangular maps* embeddable on S (see [Gao] for some references). However these results differ from Theorem 1 in that a triangular map on S is *not* a simplicial complex, but rather a regular cell complex in which all maximal cells (along with their boundaries) are isomorphic to triangles. For example, in a planar map on S, two distinct edges can have the same pair of endpoint vertices, which is not allowed in a simplical complex (see e.g. [Gao], Fig. 4.1).

II. Enumeration.

Before embarking on the enumeration of M_n, we establish a diagrammatic notation which will be useful in this and the next section. We will sometimes draw a picture of the disk having two of its vertices labelled with the same label r (see Fig. 1(b)). When we talk about a *triangulation* of this figure, we mean an abstract simplicial complex which triangulates the quotient space of the disk in which these two vertices are identified. This is equivalent to a triangulation of the disk itself in which these two vertices do not share an edge, nor are they both adjacent by an edge to some third vertex. Sometimes we will draw a picture of the disk in which there is an extra edge going around the outside of the disk connecting two vertices s and t on its boundary (see Fig. 1(c)). By a triangulation of this figure we mean a triangulation of the disk in which there is no edge connecting s and t. Sometimes we will have both of these anomalies in our picture of the disk (see, e.g., Fig. 1(d)), and this means that both of the above restrictions apply to triangulations of the object drawn.

The usual method for proving facts about triangulations and triangular maps on surfaces with boundary is to distinguish a boundary edge e, and then decompose the set of all triangulations according to the third vertex in the unique triangle to which e belongs. The following lemma is the key fact we need about such decompositions for the Möbius band.

Lemma 3. *Let T be a triangulation in $\mathcal{M}_n$ and let $1kn$ be the unique triangle of T containing the boundary edge $1n$. Let M' be obtained from the Möbius band M by removing the interiors of $1kn$ and $1n$. Let T' be the simplicial complex in which $1kn$ and $1n$ have been removed and the vertex k has been "split" into two copies. Then there are two possibilities (see Fig. 2):*

1. *M' is disconnected, in which case T' consists of two disconnected simplicial complexes, one triangulating a disk and the other a Möbius band.*
2. *M' is connected, in which case T' is a triangulation of Fig. 1(d) and we must have $3 \leq k \leq n-2$.*

Proof. As a preliminary step, shrink the triangle $1kn$ into an edge e by a homotopy which keeps k fixed and coalesces 1 and n into a single vertex. The result is a Möbius band $\hat{M}$ with an edge e having both endpoints on the boundary, and $\hat{M} - e = M'$. Note that the topology of the complement M' is unchanged by homotopies of the edge e or by sliding the endpoints of e around the boundary of M. For this reason, it suffices to examine the the quotient space RP^2 of M in which the boundary is identified to a single basepoint. The edge e maps under the quotient to a loop with this basepoint, and gives rise an element e_* of the fundamental group $\pi_1(RP^2) \cong \mathbb{Z}/2$. By our previous comment about homotopies, the topology of M' only depends on whether e_* is trivial or not. In either case, one can easily check (by choosing one's favorite edge e giving rise to the appropriate element e_* in $\pi_1(RP^2)$) that the assertions of Case 1 or Case 2 follow, depending on whether the group element e_* is trivial or non-trivial respectively. Furthermore, in Case 2 one must have $k \geq 3$ since when $k = 2$, the triangle $12n$ shrinks to an edge e contained entirely in the boundary of M so that e_* is the trivial element in $\pi_1(RP^2)$. By symmetry, one must also have $k \leq n-2$ in Case 2. ■

As in the introduction, let $\mathcal{A}_n$ be the set of Catalan triangulations of the disk in Figure 1(a), and $\mathcal{M}_n$ the set of Catalan triangulations of the Möbius band . Also let $\mathcal{B}_{i,j}$ (resp. $\mathcal{D}_{i,j}$) be the set of triangulations of Figure 1(b) (resp. 1(d)) in which there are $i-1$ vertices traversed counterclockwise strictly between the two vertices labelled r, and $j-1$ vertices traversed clockwise strictly between them. Note that in Figure 1(d), one copy of the vertex r is adjacent to s and the other is adjacent to t. Let $m_n, a_n, b_{i,j}$, and $d_{i,j}$ be the respective cardinalities of the sets $\mathcal{M}_n, \mathcal{A}_n, \mathcal{B}_{i,j}, \mathcal{D}_{i,j}$, where we have adopted the convention that $a_2 = 1$.

Proposition 4. *We have the recurrences*

(R1)

$$m_n = \sum_{i+j=n+1} a_i m_j + m_i a_j + d_{i,j}$$

(R2)

$$d_{i,j} = b_{i,j} - (a_{i+1} - a_i)(a_{j+1} - a_j)$$

(R3)

$$b_{i,j} = \sum_{k=3}^{i-1} a_{k-1} b_{i-k+2,j} + \sum_{k=2}^{j-1} a_k (a_{i+j-k+1} - a_i a_{j-k+3} - a_{i+1} a_{j-k+2} + a_i a_{j-k+2})$$

and initial values

$$m_i = 0 \text{ for } i \leq 4$$
$$a_i = 0 \text{ for } i \leq 1$$
$$b_{i,j} = d_{i,j} = 0 \text{ for } i \leq 2 \text{ or } j \leq 2$$

Proof. The first two terms on the right-hand side of (R1) come from Case 1 of Lemma 3, depending upon whether the edge $1k$ or the edge kn forms part of the boundary of the Möbius band component of T' (see Figure 2). The last term on the right-hand side of (R1) comes from Case 2 of the Lemma.

Recurrence (R2) comes from the fact that a triangulation of Figure 1(d) is the same as a triangulation of Figure 1(b) which does *not* have an edge from s to t. It then suffices to show that the number of triangulations of Figure 1(b) which *do* have the edge st is

$$(a_{i+1} - a_i)(a_{j+1} - a_j).$$

This follows from the fact that the edge st will divide the triangulation into triangulations of two smaller disks having $i+1$ and $j+1$ vertices respectively. One of these smaller triangulations must not use an rs edge, and hence is counted by $a_{i+1} - a_i$, while the other must not use an rt edge, and is counted by $a_{j+1} - a_j$.

Recurrence (R3) uses the usual decomposition technique. Assume we have a triangulation T of Figure 1(b). Distinguish the top vertex labelled r, let s be the vertex counter-clockwise adjacent to it, and let rst be the unique triangle of T which contains the edge rs. We then classify the triangulation according to the location of t. The first sum counts triangulations in which the t is on the shortest path between the two vertices labelled r which passes through s, and the summation index k is one more than the number of edges on the path upward from t to the top vertex labelled r. In this case, removing the triangle rst and splitting the vertex t into two copies leaves two disjoint simplicial complexes, one triangulating a smaller disk (counted by a_{k-1}) and the other triangulating a smaller version of Figure 1(b) (counted by $b_{i-k+2,j}$). The second sum counts triangulations where t is on the other side, and the summation index has the same meaning as before. In this case, removing the triangle rst and splitting t into two copies again leaves two disjoint simplicial complexes. One triangulates a disk (counted by a_k), and the other triangulates a disk which must avoid using either of the edges rt, rs, which is counted by

$$a_{i+j-k+1} - a_i a_{j-k+3} - a_{i+1} a_{j-k+2} + a_i a_{j-k+2}$$

using inclusion-exclusion. ■

Once these recurrences are known, it is then straightforward to find the desired generating functions. We summarize one approach in several steps below.

Step 1. Rewrite (R3) using the Catalan recurrence

$$a_n = \sum_{i+j=n+1} a_i a_j$$

on the last three terms in the second sum, to obtain the recurrence (R3')

$$b_{i,j} = \sum_{k=3}^{i-1} a_{k-1} b_{i-k+2,j} - a_i a_{j+2} + 2a_i a_{j+1} - a_{i+1} a_{j+1} + a_{i+1} a_j + e_{i,j}$$

where

$$e_{i,j} = \begin{cases} \sum_{\substack{l+m=i+j+1 \\ m \leq j-1}} a_l a_m & \text{if } i, j \geq 3 \\ 0 & \text{else.} \end{cases}$$

Step 2. Find the generating function for $e_{3,j}$. Since

$$\begin{aligned} e_{3,j} &= \sum_{\substack{l+m=j+4 \\ m \leq j-1}} a_l a_m \\ &= \sum_{\substack{l+m=j+4 \\ l \geq 5}} a_l a_m \\ &= \sum_{l+m=j+4} a_l a_m - a_4 a_j - a_3 a_{j+1} \\ &= a_{j+3} - a_4 a_j - a_3 a_{j+1} \end{aligned}$$

one can sum over j to get an expression for $E_3(y) = \sum_{j \geq 3} e_{3,j} y^j$ in terms of the known generating function $A(x) = \sum_{n \geq 2} a_n x^n$.

Step 3. Once $E_3(x)$ is known, summing the recurrence

$$e_{i,j} - e_{i+1,j-1} = a_{i+2} a_{j-1}$$

over i, j gives an expression for $E(x,y) = \sum_{i,j \geq 3} e_{i,j} x^i y^j$.

Step 4. Summing (R3') over i, j gives an expression for $B(x,y) = \sum_{i,j \geq 3} b_{i,j} x^i y^j$ in terms of $E(x,y), A(x), A(y)$.

Step 5. Summing (R2) over i, j gives an expression for $D(x,y) = \sum_{i,j \geq 3} b_{i,j} x^i y^j$ in terms of $B(x,y), A(x), A(y)$. ¿From this one obtains an expression for

$$D(x,x) = lim_{y \to x} D(x,y)$$

using L'Hôpital's rule.

Step 6. Summing (R1) over n gives an expression for $M(x) = \sum_{n \geq 5} m_n x^n$ in terms of $D(x,x)$ and $A(x)$, which yields Theorem 1.

III. Connectivity by diagonal-flips.

The goal of this section is to prove Theorem 2, but in fact we will prove something slightly more general.

Thereom 2'. *For a fixed number of vertices, and for each of the diagrams in Figures 1(a)-(d) and the Mobius band M, any two triangulations are connected by a sequence of diagonal flips.*

Proof. Assume that we have chosen a particular one of the diagrams in Figures 1(a)-(d) or the Möbius band, and have fixed the number of boundary vertices. As in the previous section, the strategy will be to decompose the set $\mathcal{T}$ of triangulations in the following way: Given a triangulation T in $\mathcal{T}$, we distinguish a boundary edge ab and decompose

$$\mathcal{T} = \bigsqcup_k \mathcal{T}_k$$

according to the third vertex k in the unique triangle abk of T which contains the edge ab. As usual, we let T' be the simplicial complex obtained from T by removing ab, abk and splitting the vertex k into two copies, and check that T' is always a disjoint union of one or two simplicial complexes triangulating diagrams appearing in Figures 1(a)-(d), but with *fewer* boundary vertices than T. By induction this will imply that any two triangulations in $\mathcal{T}_k$ are connected by diagonal flips, and will complete Step 1 of the proof. In Step 2, we linearly order the vertices k which can appear in the decomposition $\mathcal{T} = \bigsqcup_k \mathcal{T}_k$, and check that for any two adjacent values k, k' in this order there exists a triangulation T in $\mathcal{T}_k$ and T' in $\mathcal{T}_{k'}$ which are related by a single diagonal flip. This will complete the proof of the Theorem.

In the cases where T triangulates one of the Figures 1(a)-(d), the argument for Step 1 is relatively simple, and is summarized in Figure 3. For each case, the distinguished boundary edge ab is shown darkened, the alternatives for the location of k are displayed, and the connected components of T' are labelled with the diagram that they triangulate.

The case where T triangulates the Möbius band is slightly more bothersome. In this case, we choose the distinguished boundary edge ab to be $1n$. Note that here the description of the decomposition $\mathcal{T} = \bigsqcup_k \mathcal{T}_k$ is slightly inaccurate, in that we need to not only decompose the triangulations according to the vertex k in the unique triangle $1kn$ containing $1n$, but also according to the alternatives given in the Lemma 3, and furthermore according to whether the edge $1k$ or the edge kn is contained in the boundary of the Möbius strip in Case 1 of Lemma 3 (see Figure 2). Strictly speaking, we should rename this decomposition

$$\mathcal{T} = \bigsqcup_\alpha \mathcal{T}_\alpha$$

in this case. We need to know that in this finer decomposition, one can apply induction to conclude that any two triangulations in $\mathcal{T}_\alpha$ are connected by diagonal flips. However one cannot simply induct on the number of vertices, since in Case 2 of Lemma 3, the simplicial complex T' has the same number of vertices as T. However, in this case T' triangulates the "simpler" object of Figure 1(d), and so induction still applies. This completes Step 1.

To carry out Step 2 for triangulations of Figures 1(a)-(d), in each of these cases we linearly order the vertices k that can appear in the decomposition $\mathcal{T} = \bigsqcup_k \mathcal{T}_k$ in clockwise order, beginning to the left of the distinguished boundary edge ab. Given two adjacent values k, k' in this order, we wish to exhibit triangulations T, T' in $\mathcal{T}, \mathcal{T}'$ that are connected by a diagonal flip. To do this, we have drawn in Figures 4(a)-(d) pictures of the appropriate figures along with the quadrangles $abkk'$ and both of the diagonals of the quadrangle. For each of these pictures, it suffices to exhibit a triangulation of the rest of the figure outside this quadrangle which does not use either of these diagonal edges. In each case (a)-(d), this is equivalent to triangulating the disconnected set of figures shown in Figures 4(a)-4(d), and it is easy to check that at least one such triangulation exists (one really need only worry about cases in which the diagrams have few vertices). Note that some of the diagrams which arise in these disconnected sets are "new" in the sense that we have not encountered them in our previous analysis, but this does not concern us since we only need to show that they have at least one triangulation.

Step 2 for triangulations of the Möbius band is similar, except that the α's which appear in the finer decomposition $\mathcal{T} = \bigsqcup_\alpha \mathcal{T}_\alpha$ discussed above must be linearly ordered differently. We first put in order of increasing k those sets $\mathcal{T}_\alpha$ which are in Case 1 of Lemma 3 and where kn is an edge of the Möbius band component of T'. Next we put in order of decreasing k those sets $\mathcal{T}_\alpha$ which are in Case 1 of Lemma 3 and where $1k$ is an edge of the Möbius band component of T'. Last we put in order of increasing k those sets $\mathcal{T}_\alpha$ which are in Case 2 of Lemma 3. The pictures in Figure 5, like those in Figure 4, show for any adjacent α, α' in this order exactly what sort of triangulations are needed to produce a pair of triangulations T and T' which differ by a diagonal flip. It is then easy to check that in each case such triangulations exist. ∎

Remarks:

(1) Figure 6 shows the unique triangulation in $\mathcal{M}_5$ and the graph of $\mathcal{M}_6$ in which the vertices are triangulations, and two triangulations are connected by a diagonal flip. Notice that the dihedral group D_{2n} of order $2n$ acts as symmetries of the boundary triangulation of the Möbius band, and hence acts on $\mathcal{M}_n$ and on this graph.

There are several questions one might ask about this graph, by analogy to the graph on $\mathcal{A}_n$, which is known to be the 1-skeleton of a *simple* $(n-3)$-polytope called the *associahedron* or *Stasheff polytope* (see [Lee]). The last fact implies that the graph on $\mathcal{A}_n$ has edge-connectivity and vertex-connectivity $n-3$ by Balinski's Theorem (see [Ba]), and the diameter of the graph on $\mathcal{A}_n$ was proven to be $2n-10$ for n sufficiently large in [STT]. What are the edge-connectivity, vertex-connectivity, and diameter of the graph on $\mathcal{M}_n$? Computer calculations show that $\mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7$, and $\mathcal{M}_8$ have diameters $0, 5, 10$, and 16 respectively.

(2) In light of Theorem 2, and also the result of Pachner [Pa, Theorem 6.3] which says that any two triangulations of homeomorphic PL-manifolds (possibly with boundary) are connected by a sequence of *shelling* and *inverse-shelling* operations (possibly altering the number of vertices), one might be tempted to make the following

Tempting conjecture. *For a fixed surface (2-manifold with boundary) with fixed boundary triangulation and fixed number of interior vertices, any two triangulations are connected by a sequence of diagonal flips.*

However, this is false. For example, let S be the Möbius band with three boundary vertices labelled $1, 2, 3$ and three interior vertices labelled $4, 5, 6$. Then there are a total of 6 such triangulations of S, all isomorphic (they come from removing the 123 triangle in any triangulation of RP^2 as the quotient of the boundary of an icosahedron by the antipodal map), but each has no neighbors under diagonal flips.

Lest the reader think this phenomenon is confined to non-orientable surfaces S or to triangulations which use interior vertices, consider the example where S is the punctured 2-torus (i.e. a disk with one handle) and triangulations having *no interior vertices* and 6 boundary vertices labelled $1, 2, 3, 4, 5, 6$. There are a total of two such labelled triangulations

$$\{124, 135, 136, 145, 235, 246, 256, 346\}$$

$$\{125, 134, 135, 146, 236, 245, 246, 356\}$$

which are isomorphic as unlabelled triangulations, and neither has any neighbors under diagonal flips.

Nevertheless, computer data suggest the following conjecture.

Conjecture. *Let $\mathcal{S}$ be the set of all triangulations of a fixed surface S having a fixed boundary triangulation and fixed number of interior vertices. If the total number of vertices (both boundary and interior) is not minimal, i.e. there exists a triangulation of S having fewer total number of vertices, then any two triangulations in $\mathcal{S}$ are connected by a sequence of diagonal flips.*

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail: edelman@math.umn.edu, reiner@math.umn.edu