where $\eta(k)$ is given in (8.6). A simple check verified that $\delta > 0$ whence the latter error term in (10.3) is subsumed by the former—and, *mirabile dictu*, Theorem 2 is proved.

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ITERATED FIBER POLYTOPES

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Dedicated to the memory of Paul Filliman

Abstract. The construction of the fiber polytope $\Sigma(P, Q)$ of a projection $\pi: P \to Q$ of polytopes is extended to flags of projections. While the faces of the fiber polytope are related to subdivisions of Q induced by the faces of P, those of an iterated fiber polytope are related to discrete homotopies between polyhedral subdivisions. In particular, in the case of projections

$$\mathbf{R}^{n+2} \to \mathbf{R}^3 \to \mathbf{R}^2 \to \mathbf{R}^1,$$

starting with an $(n+1)$-simplex, vertices of the successive iterates correspond to, respectively, subsets, permutations and sequences of permutations of an n-set. The first iterate will always be combinatorially an n-cube, and, under certain conditions, the second will have the structure of the $(n-1)$-dimensional permutohedron.

§1. *Introduction.* We recall the definition of the fiber polytope given in [4]. Let $P \subset \mathbf{R}^n$ be any polytope, and let $\pi: \mathbf{R}^n \to \mathbf{R}^d$ be a linear map. Then the image $Q = \pi(P)$ is a polytope in $\mathbf{R}^d$, and each fiber $\pi^{-1}(x)$, $x \in Q$, is a polytope in $\mathbf{R}^n$. We define a subset of $\mathbf{R}^n$ by averaging all fibers of the projection π

$$\Sigma(P, Q) = \frac{1}{\text{vol}\,(Q)} \int_{x \in Q} \pi^{-1}(x) dx. \qquad (1.1)$$

The set-valued integral (1.1) can be understood in three different but equivalent ways:

(1) as the Riemann-type limit of all Minkowski sums $1/N \sum_{i=1}^{N} \pi^{-1}(x_i)$, where $x_i \in Q$;

(2) as the set of points $\int_{x \in Q} \gamma(x) dx \in \mathbf{R}^n$ where $\gamma: Q \to P$ runs over all sections of π; or

(3) as the convex set whose support function is the pointwise integral of the support functions of the fibers.

In [4] it is proved that the set $\Sigma(P, Q) \subset \mathbf{R}^n$ is a polytope of dimension $\dim P - \dim Q$, called the *fiber polytope* of the map $\pi: P \to Q$. When $P = \Delta_n$, the n-simplex, $\Sigma(P, Q) = \Sigma(Q)$, the *secondary polytope* of [8], whose vertices correspond to regular triangulations of Q. In general, the face poset of $\Sigma(P, Q)$ is isomorphic to the poset of P-coherent polyhedral subdivisions of Q. In particular, if $\dim Q = 1$ then $\Sigma(P, Q)$ is a polytope of dimension $\dim P - 1$, whose vertices are in bijection with certain monotone edge paths on P. This construction was applied in [3] to resolve a topological problem posed in [1].

The thrust of the present paper is to investigate functorial properties of the fiber polytope operator $\Sigma(\cdot,\cdot)$. We extend this operator to flags of projections. This results in a polyhedral model for homotopies between subdivisions of a given polytope. The case where that polytope is 1-dimensional is the subject of [3].

The point of departure in [3] was the observation that the fiber polytope of a generic map of an $(n+1)$-simplex Δ_{n+1} onto a line is combinatorially isomorphic to the n-cube. As an indication of this, note that a triangulation of a set of $n+2$ points on a line is determined by a subset of the n non-extreme points, giving the association with the vertices of an n-cube. Similarly, the fiber polytope of a generic map of a regular n-cube onto a line has the structure of an $(n-1)$-dimensional *permutohedron* (defined as the convex hull of the orbit of a point with distinct coordinates in $\mathbf{R}^n$ under the action of the symmetric group S_n). Assuming the minimum and maximum points in the cube with respect to this projection to be $(0,\dots,0)$ and $(1,\dots,1)$, then vertices of this fiber polytope correspond to monotone paths in the cube between $(0,\dots,0)$ and $(1,\dots,1)$; each such is determined by the order in which the coordinates are turned from 0 to 1, and thus are in bijection with the vertices of the permutohedron. The face posets in both these cases are examined in detail in Section 4.

One can now take the permutohedron and project it onto a line in such a way that the vertex corresponding to the identity permutation $e=123\dots n$ is first while its reverse $w_0=n\dots 321$ is last. What is the fiber polytope of this map? Its vertices correspond to certain (but not all [1, Section III.7, p. 121]) maximal chains in the weak Bruhat order on S_n. An explicit example for $n=4$ is worked out in Section 5. In that example the iterated fiber polytope is a planar polygon with 12 vertices, corresponding to 12 of the 16 maximal chains in the weak Bruhat order on S_4.

The paper is organized as follows. In Section 2 we examine the iterated fiber polytope of two consecutive projections. A basic construction is that of a fan obtained *via* projection of a higher dimensional fan. We also study coherent homotopies of polyhedral subdivisions arising from this iterated fiber polytope. The section ends with a brief discussion of how results in this paper can be interpreted in terms of quotients of toric varieties [9].

In Section 3 we introduce the *flag polytope* of a given polytope with respect to a general flag of projections. For instance, the usual fiber polytope arises from a flag of length 1, and the iterates discussed above arise from certain flags of length 2 or 3. Our main result here concerns Minkowski sum relations among flag polytopes.

In Section 4 we concentrate on the case of a flag of projections $\Delta_{n+1}\to Q_3\to Q_2\to Q_1$. We show that the flag polytope $\Sigma(\Sigma(\Delta_{n+1}, Q_1), \Sigma(Q_2, Q_2))$ is combinatorially a permutohedron whenever Q_2 is an $(n+2)$-gon in convex position. A complete example for $n=4$ is worked out in Section 5. The next iterate, the polytope $\Sigma(\mathbf{F}_{123})=\Sigma(\Delta_{n+1}, \mathbf{F}_{123})$ of (4.2) and (5.2), corresponds to projecting this permutohedron onto a line. For general n, it remains an open problem to give a combinatorial description of this flag polytope.

§2. *Iterated projection and homotopies of polyhedral subdivisions.* Let $P \xrightarrow{\pi} Q \xrightarrow{\theta} R$ be surjective linear maps of polytopes. By Lemma 2.3 of [**4**], the fiber polytope $\Sigma(Q, R)$ equals the image of the fiber polytope $\Sigma(P, R)$ under the linear map π. We will study the iterated fiber polytope $\Sigma(\Sigma(P, R), \Sigma(Q, R))$.

Note that if $\dim P = n$ and $\dim Q = d$, then both $\Sigma(P, Q)$ and $\Sigma(\Sigma(P, R), \Sigma(Q, R))$ are $(n-d)$-polytopes in $V = \mathbf{R}^n$. Our first theorem establishes a natural surjection from the fact lattice of the latter polytope to the face lattice to the former polytope.

THEOREM 2.1. *$\Sigma(P, Q)$ is a Minkowski summand of $\Sigma(\Sigma(P, R), \Sigma(Q, R))$.*

Before starting the proof, we describe some basics about normal fans in the context of fiber polytopes. Let $W \subset V$ are vector spaces and $\mathrm{proj}_{V/W}$ the canonical projection from V to V/W. If $C \subset V$ is a cone, then the projection of C onto V/W is

$$\mathrm{proj}_{V/W} C = \{x + W \mid (x + W) \cap C \neq \emptyset\};$$

$\mathrm{proj}_{V/W} C$ is again a cone. If $\mathscr{F}$ is a fan in V, then define the fan $\mathrm{proj}_{V/W} \mathscr{F}$ as follows: the relatively open cells of $\mathrm{proj}_{V/W} \mathscr{F}$ are the equivalence classes under the relation

$$x + W \sim x' + W \Leftrightarrow \{C \in \mathscr{F} \mid (x + W) \cap C \neq \emptyset\} = \{C \in \mathscr{F} \mid (x' + W) \cap C \neq \emptyset\}, \quad (2.1)$$

where elements of $\mathscr{F}$ are considered to be relatively open cones. Alternatively, $\mathrm{proj}_{V/W} \mathscr{F}$ is the common refinement of the set of cones $\{\mathrm{proj}_{V/W} C \mid C \in \mathscr{F}\}$.

If V is the ambient vector space of the polytope P, we consider its normal fan $\mathscr{N}(P)$ as the collection of relatively open normal cones $N(P, F)$ to the faces F of P; these cones lie in the dual vector space V^*. Considering $\Sigma(P, Q) \subset \ker \pi$, it follows that $\mathscr{N}(\Sigma(P, Q))$ is a fan in the space $(\ker \pi)^*$, which we identify with $V^*/\mathrm{im}\, \pi^*$, where π^* is the adjoint to π. The same is true of each $\mathscr{N}(\pi^{-1}(x))$. In fact,

$$\mathscr{N}(\pi^{-1}(x)) = \{\mathrm{proj}_{(\ker \pi)^*} N(P, F) \mid \pi^{-1}(x) \cap \mathrm{rel\,int}\, F \neq \emptyset\}, \quad (2.2)$$

and $\mathscr{N}(\Sigma(P, Q))$ is the common refinement of these fans as x runs over $Q = \pi(P)$. Here $\mathrm{proj}_{(\ker \pi)^*}$ means $\mathrm{proj}_{V^*/\mathrm{im}\, \pi^*}$. These observations prove the following result (*cf.* [**9**, Prop. 2.3]).

PROPOSITION 2.2. *The normal fan of the fiber polytope $\Sigma(P, Q)$ is the projection of the normal fan of P, i.e.,*

$$\mathscr{N}(\Sigma(P, Q)) = \mathrm{proj}_{(\ker \pi)^*} \mathscr{N}(P). \quad (2.3)$$

Proof of Theorem 2.1. Applying Proposition 2.2 twice, we find that the normal fan of the polytope $\Sigma(\Sigma(P, R), \Sigma(Q, R))$ is the projection, under the map

$$\wp : V^*/\mathrm{im}\, (\theta \circ \pi)^* \to (V^*/\mathrm{im}\, (\theta \circ \pi)^*)/(\mathrm{im}\, \pi^*/\mathrm{im}\, (\theta \circ \pi)^*) \cong V^*/\mathrm{im}\, \pi^*$$

of the fan $\mathcal{N}(\Sigma(P, R)) = \mathrm{proj}_{(\ker \theta \circ \pi)^*}(\mathcal{N}(P))$. This fan refines the fan $\mathcal{N}(\Sigma(P, Q)) = \mathrm{proj}_{(\ker \pi)^*}\mathcal{N}(P)$ because

$$\mathrm{proj}_{(\ker \pi)^*} = \wp \circ \mathrm{proj}_{(\ker \theta \circ \pi)^*}.$$

This completes the proof, because the relation on polytopes "is a Minkowski summand" is equivalent to the relation "has a coarser normal fan".

A key element in the theory of fiber polytopes is the notion of coherent polyhedral subdivision. Let $\pi: V \to W$, P a polytope in V, $Q = \pi(P)$ and $\psi \in V^*$. The *P-coherent subdivision* $\Pi(\psi)$ of Q consists of all $\pi(F)$, where F is a face of P such that $N(P, F) \cap (\psi + \mathrm{im}\, \pi^*) \neq \emptyset$. If $\pi(F) \in \Pi(\psi)$, we will say that ψ *picks F under* π.

We will see now that the faces of the polytope $\Sigma(\Sigma(P, R), \Sigma(Q, R))$ correspond to what might be called *coherent homotopies of polyhedral subdivisions.* In the special case where $d = \dim Q = \dim R + 1$, $\Sigma(Q, R)$ is a line segment contained in Q, and $\Sigma(P, R)$ is an $(n-d+1)$-polytope in P, whose vertices correspond to P-coherent subdivisions of R. The polytope $\Sigma(\Sigma(P, R), \Sigma(Q, R))$ has vertices corresponding to certain paths (*homotopies*) between the subdivisions of R induced by the top and bottom of Q with respect to θ. Theorem 2.1 relates these homotopies to P-coherent subdivisions of Q. Such homotopies, in the case $\dim R = 2$ and $\dim Q = 3$, played an important role in the construction in [**11**, Section 2.6]. For an example see the last table of Section 5, which gives all seven coherent homotopies between the top and the bottom of a cyclic 3-polytope with six vertices.

For simplicity, we assume for the rest of this section that $P = \Delta_n$, the n-simplex in $V = \mathbf{R}^{n+1}$, $d = \dim Q = \dim R + 1$, and the matrices defining the maps π and θ have all their maximal minors non-zero. This genericity assumption implies that the polytopes Q and R are simplicial, and that their corresponding oriented matroids are uniform [**5**].

Let $h = (h_0, \ldots, h_n) \in V^*$ be a height vector of the vertices of Q over their projections in R, i.e., h is any element in $\mathrm{im}\, \pi^* \backslash \mathrm{im}\, (\theta \circ \pi)^*$. The faces $\Sigma(P, R)^{-h}$ and $\Sigma(P, R)^h$ are thus defined and independent of the choice of h. By our genericity assumption, both are vertices of $\Sigma(P, R)$; they correspond, respectively, to the triangulations Π_{top} of R induced by the "top" of Q, and the triangulation Π_{bottom} of R induced by the "bottom" of Q. We define a *(Q, R)-homotopy* to be a sequence $H = (\Pi_0, \Pi_1, \Pi_2, \ldots, \Pi_k)$ of triangulations of R such that $\Pi_0 = \Pi_{\mathrm{top}}$, $\Pi_k = \Pi_{\mathrm{bottom}}$, and all adjacent pairs Π_i, Π_{i+1} differ by exactly one *bistellar flip* [**2**; (3.8)]. For regular triangulations, such flips correspond to edges of $\Sigma(P, R)$, the secondary polytope of R (see [**8**; Theorem 3A.8]).

A generic $\psi \in V^*$ defines a (Q, R)-homotopy $H(\psi)$ as follows. Consider the sequence of vertices $\Sigma(P, R)^{\psi + \alpha h}$, as α ranges from $-\infty$ to $+\infty$. This gives a sequence of regular triangulations of R, which is a (Q, R)-homotopy. (Q, R)-homotopies of the form $H(\psi)$ are called *coherent*.

It follows from [**4**; Lemma 2.3] that the polytope $\Sigma(\Sigma(P, R), \Sigma(Q, R))$ equals the *monotone path polytope* $\Sigma_h(\Sigma(P, R))$, which parametrizes the monotone vertex paths of the form $\{\Sigma(P, R)^{\psi + \alpha h} \mid \alpha \in (-\infty, +\infty)\}$, between $\Sigma(P, R)^{-h}$ and $\Sigma(P, R)^h$. This leads to the following interpretation.

THEOREM 2.3. *The vertices of* $\Sigma(\Sigma(P, R), \Sigma(Q, R))$ *are in one-to-one correspondence with the coherent* (Q, R)*-homotopies connecting* Π_{top} *with* Π_{bottom}.

Theorem 2.1 implies that there is a surjection from the set of vertices of the polytope $\Sigma(\Sigma(P, R), \Sigma(Q, R))$ to those of $\Sigma(P, Q)$. We will describe this surjection more explicitly under the same genericity assumption as above.

Let ψ be a linear functional on Δ_n, which supports a vertex of $\Sigma(\Sigma(\Delta_n, R), \Sigma(Q, R))$. By Theorem 2.1, ψ also supports a vertex of $\Sigma(\Delta_n, Q) = \Sigma(Q)$. Let $\Pi(\psi)$ be the corresponding regular triangulation of Q. For any two consecutive triangulations Π_i and Π_{i+1} in the coherent (Q, R)-homotopy $H(\omega)$, there exists a unique d-simplex $\tau_i = \tau(\Pi_i, \Pi_{i+1}) = \{x_0, x_1, \ldots, x_d\}$ which is involved in the bistellar flip between Π_i and Π_{i+1}. More precisely, there exists a partition $A_i \cup B_i$ of τ_i such that

$$\Pi_i \backslash \Pi_{i+1} = \{\{x_0, x_1, \ldots, x_d\} \backslash \{a\} : a \in A_i\},$$

and (2.4)

$$\Pi_{i+1} \backslash \Pi_i = \{\{x_0, x_1, \ldots, x_d\} \backslash \{b\} : b \in B_i\}.$$

COROLLARY 2.4. *Each coherent* (Q, R)*-homotopy* $H = (\Pi_0, \Pi_1, \Pi_2, \ldots, \Pi_k)$ *defines a regular triangulation*

$$T(H) = \{\tau(\Pi_i, \Pi_{i+1}) \mid i = 0, 1, \ldots, k-1\}$$

of Q. *More precisely*, $T(H(\psi)) = \Pi(\psi)$.

Proof. We will show that $T(H(\psi))$ equals the regular triangulation $\Pi(\psi)$ of Q. The above lifting vector h is a linear functional in $\operatorname{im} \pi^* \backslash \operatorname{im} (\theta \circ \pi)^*$; in fact, we have

$$\operatorname{im} \pi^* = \operatorname{span}(h) + \operatorname{im}(\theta \circ \pi)^*. \tag{2.5}$$

Let $\tau_i = \tau(\Pi_i, \Pi_{i+1})$ be any d-simplex in $T(H(\psi))$. There exists a real number α_i such that, for all $\varepsilon > 0$ sufficiently small:

(i) $\psi(\alpha_i - \varepsilon)h$ picks the $(d-1)$-simplices in $\Pi_i \backslash \Pi_{i+1}$ under $\theta \circ \pi$;

(ii) $\psi + \alpha_i h$ picks the d-simplex τ_i under $\theta \circ \pi$;

(iii) $\psi + (\alpha_i + \varepsilon)h$ picks the $(d-1)$-simplices in $\Pi_{i+1} \backslash \Pi_i$ under $\theta \circ \pi$.

Statement (ii) is equivalent to

$$N(\Delta_n, \tau_i) \cap (\psi + \alpha_i h + \operatorname{im}(\theta \circ \pi)^*) \neq \emptyset.$$

This implies

$$N(\Delta_n, \tau_i) \cap (\psi + \operatorname{im} \pi^*) \neq \emptyset,$$

which means that τ_i is picked by ψ under π.

Conversely, let τ be any d-simplex which is picked by ψ under π. By the definition of "picking", there exists a real number α such that (ii) holds, and there exists a partition $A \cup B$ of τ such that (i) and (iii) hold. Therefore $\tau = \tau_i$ and $\alpha = \alpha_i$ for some $i \in \{0, \ldots, k-1\}$.

The map $H \mapsto T(H)$ is usually not injective, as can be seen in the example of Section 5.

All results in this paper can be translated into the language of algebraic geometry. It was shown in [9] that the construction of the fiber polytope corresponds to taking the *Chow quotient* of a projective toric variety modulo a subtorus of its dense torus. Hence every statement about fiber polytopes is also a statement about Chow quotients of toric varieties. For instance, Theorem 2.1 translates as follows.

COROLLARY 2.5. *Let X be a projective toric variety with dense torus $(\mathbf{C}^*)^n$, and let $T_1 \subset T_2 \subset (\mathbf{C}^*)^n$ be subtori. Then there exists an equivariant morphism from the iterated Chow quotient $(X//T_1)//(T_1/T_2)$ onto the Chow quotient $X//T_2$.*

All results to follow are given only in terms of combinatorial convexity. In each case, there is a relatively straightforward toric translation.

§3. *Flags of polytope projections.* The construction in Theorem 2.1 can be iterated as follows. Let $\mathscr{F}_{d_1,\ldots,d_k}(\mathbf{R}^n)$ denote the *flag variety* consisting of all flags of linear subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V_k \subset V_{k+1} = \mathbf{R}^n$, where $\dim V_i = d_i$. This is a real algebraic manifold of dimension $\sum_{i=1}^{k} (d_{i+1} - d_i)d_i$. Elements in $\mathscr{F}_{d_1,\ldots,d_k}(\mathbf{R}^n)$ will be denoted by $\mathbf{F}_k$, or by $\mathbf{F}_{d_1,\ldots,d_k}$. Each flag $\mathbf{F}_k \in \mathscr{F}_{d_1,\ldots,d_k}(\mathbf{R}^n)$ gives rise to a chain of orthogonal projections

$$\mathbf{F}_k : V_k \xrightarrow{\pi_{k-1}} V_{k-1} \xrightarrow{\pi_{k-2}} \ldots \xrightarrow{\pi_1} V_1 \xrightarrow{\pi_0} V_0 = \{0\}, \tag{3.1}$$

which is preceded by one additional map $\mathbf{R}^n \xrightarrow{\pi_k} V_k$. We write $\mathbf{F}_{k-1}$ for the natural image of $\mathbf{F}_k \in \mathscr{F}_{d_1,d_2,\ldots,d_k}$ in $\mathscr{F}_{d_2,\ldots,d_k}$.

Fix a polytope $P \subset \mathbf{R}^n$ (*e.g.*, a simplex), and consider its projections into V_k and its further projections under (3.1). This situation is abbreviated by writing $P \xrightarrow{\pi_k} \mathbf{F}_k$.

We now define the iterated fiber polytope $\Sigma(P, \mathbf{F}_k)$. We proceed by induction on k. For $k=0$ we set $\Sigma(R, \mathbf{F}_0) = R$, for any R. For $k>0$, let $Q_k = \pi_k(P)$ and notice that both $P \xrightarrow{\pi_{k-1} \circ \pi_k} \mathbf{F}_{k-1}$ and $Q_k \xrightarrow{\pi_{k-1}} \mathbf{F}_{k-1}$. By induction, both $\Sigma(P, \mathbf{F}_{k-1})$ and $\Sigma(Q_k, \mathbf{F}_{k-1})$ are defined. By repeated application of [4; Lemma 2.3], they are seen to be related by the projection

$$\Sigma(P, \mathbf{F}_{k-1}) \xrightarrow{\pi_k} \Sigma(Q_k, \mathbf{F}_{k-1}). \tag{3.2}$$

We define $\Sigma(P, \mathbf{F}_k)$ to be the fiber polytope

$$\Sigma(P, \mathbf{F}_k) = \Sigma(\Sigma(P, \mathbf{F}_{k-1}), \Sigma(Q_k, \mathbf{F}_{k-1})). \tag{3.3}$$

This polytope is an invariant of the polytope P and the flag $\mathbf{F}_k$. We call $\Sigma(P, \mathbf{F}_k)$ the *flag polytope.* Note that for $k=1$ the flag polytope $\Sigma(P, \mathbf{F}_1) = \Sigma(P, \pi_1(P))$ is just the usual fiber polytope. For $k=2$ we get the polytope in Theorem 2.1. In order to generalize Theorem 2.1 to arbitrary flag polytopes, we need the next lemma.

LEMMA 3.1. *Let π be a linear map such that $\pi: P_1+P_2 \to Q_1+Q_2$ and $\pi: P_1 \to Q_1$. Then $\Sigma(P_1, Q_1)$ is a Minkowski summand of $\Sigma(P_1+P_2, Q_1+Q_2)$.*

Proof. Refinement of fans is preserved by projection. Since $\mathcal{N}(P_1+P_2)$ refines $\mathcal{N}(P_1)$, the fan $\text{proj}_{(\ker \pi)^*}\mathcal{N}(P_1+P_2)=\mathcal{N}(\Sigma(P_1+P_2, Q_1+Q_2))$ refines the fan $\text{proj}_{(\ker \pi)^*}\mathcal{N}(P_1)=\mathcal{N}(\Sigma(P_1, Q_1))$.

THEOREM 3.2. *For any $k>0$, the flag polytope $\Sigma(P, \mathbf{F}_k)$ has dimension $\dim P-\dim Q_k$, and it has the fiber polytope $\Sigma(P, Q_k)$ as a Minkowski summand.*

Proof. By induction on k. For $k=1$, the first conclusion is known and the second is vacuous, and for $k=2$ this is Theorem 2.1. Let $\prec$ denote the relation of being a Minkowski summand.

Assume the result is true for chains of length $k-1$. Then we have both $\Sigma(P, Q_{k-1}) \prec \Sigma(P, \mathbf{F}_{k-1})$ and $\Sigma(Q_k, Q_{k-1}) \prec \Sigma(Q_k, \mathbf{F}_{k-1})$. Further, $\Sigma(P, \mathbf{F}_{k-1})$ has dimension $\dim P-\dim Q_{k-1}$ and $\Sigma(Q_k, \mathbf{F}_{k-1})$ has dimension $\dim Q_k-\dim Q_{k-1}$, and so by (3.3) $\Sigma(P, \mathbf{F}_k)$ has dimension $\dim P-\dim Q_k$. By [4, Lemma 2.3],

$$\Sigma(P, Q_{k-1}) \xrightarrow{\pi_k} \Sigma(Q_k, Q_{k-1}), \tag{3.4}$$

and so by Theorem 2.1, (3.2) and Lemma 3.1,

$$\Sigma(P, Q_k) \prec \Sigma(\Sigma(P, Q_{k-1}), \Sigma(Q_k, Q_{k-1})) \prec \Sigma(P, \mathbf{F}_k).$$

Here $P_1=\Sigma(P, Q_{k-1})$, $P_1+P_2=\Sigma(P, \mathbf{F}_{k-1})$ and $\pi=\pi_k$ in (3.2) and (3.4).

It is well known that the coordinates of any secondary polytope $\Sigma(\Delta_n, \mathcal{A})$ are given by linear functions in the volumes of the spanning simplices in $\mathcal{A}$ (see *e.g.*, [7] or [**2**, Section 3]). When $\mathcal{A}$ lies on an affine hyperplane, these volumes are the absolute values of the Plücker coordinates of the corresponding linear subspace $\mathbf{F}_1$. Hence for fixed P the flag polytope $\Sigma(P, \mathbf{F}_1)$ is a piecewise linear function on the Grassmannian in its Plücker embedding. By this we mean that the support function of $\Sigma(P, \mathbf{F}_1)$ depends piecewise-linearly on the Plücker coordinates of the non-trivial subspace V_1 in the flag $\mathbf{F}_1$. Moreover, if $P=\Delta_{n-1}$, the standard $(n-1)$-simplex, then $\Sigma(P, \mathbf{F}_1)$ is a linear function on each oriented matroid stratum of the Grassmannian (*cf.* [**5**], [**6**]).

The Plücker coordinates on the flag variety $\mathcal{F}_{d_1, \ldots, d_k}(\mathbf{R}^n)$ are indexed by chains of subsets $I_1 \subset I_2 \subset \ldots \subset I_k \subset \{1, 2, \ldots, n\}$ with $\text{card}(I_j)=d_j$. The Plücker coordinate of $\mathbf{F}_k$ with index $(I_1, \ldots, I_k)$ is the product over $j=1, \ldots, k$ of the Plücker coordinates indexed I_j of the subspaces V_j. The *flag oriented matroid strata* are defined to be the largest regions in $\mathcal{F}_{d_1, \ldots, d_k}(\mathbf{R}^n)$ on which the sign of each Plücker coordinate is constant. The following proposition can be proved by induction on k.

PROPOSITION 3.3. *For each fixed polytope P lying on an affine hyperplane in $\mathbf{R}^n$, the flag polytope $\Sigma(P, \mathbf{F}_k)$ is a piecewise linear function on the flag variety $\mathcal{F}_{d_1, \ldots, d_k}(\mathbf{R}^n)$ (in its Plücker embedding). In particular, if $P=\Delta_{n-1}$, the standard $(n-1)$-simplex, then $\Sigma(P, \mathbf{F}_k)$ is a linear function on each flag oriented matroid stratum.*

It would be of interest to have an explicit formula for the vertex coordinates of flag polytopes.

§4. *Simplex, cube, permutohedron and beyond.* The sequence of constructions in [**1**, Ch. III] can be interpreted as computing the iterated fiber polytopes given by a flag

$$\mathbf{F}_{123}: V_3 \xrightarrow{\pi_2} V_2 \xrightarrow{\pi_1} V_1 \xrightarrow{\pi_0} V_0 = \{0\},$$

where $\dim V_i = i$. If $\mathbf{R}^{n+2} \xrightarrow{\pi_3} V_3$, and $\Delta_{n+1} \subset \mathbf{R}^{n+2}$ is the standard simplex, then $\Sigma(\Delta_{n+1}, \mathbf{F}_1)$ is combinatorially an n-cube. Under certain conditions, $\Sigma(\Delta_{n+1}, \mathbf{F}_{12})$ has vertices corresponding to permutations in S_n and $\Sigma(\Delta_{n+1}, \mathbf{F}_{123})$ has vertices corresponding to certain *sequences* of these permutations. It is the purpose of this section to study this sequence of flag polytopes.

We first describe certain facets of monotone path polytopes. These correspond to "coarsest" possible coherent cellular strings in a polytope $P \subset \mathbf{R}^n$ with respect to a linear objective function $\gamma \in (\mathbf{R}^n)^*$. For any polytope $F \subset \mathbf{R}^n$, denote by F^γ the face of F on which γ achieves its maximum, that is,

$$F^\gamma = \{x \in F \mid \langle x, \gamma \rangle \geqslant \langle y, \gamma \rangle, \text{ for all } y \in F\}.$$

A *cellular string* on P with respect to γ is an ordered list $F_1, F_2, \ldots, F_k$ of faces of P, $0 < \dim F_i < \dim P$, having the property that $F_1^{-\gamma} = P^{-\gamma}$, $F_k^\gamma = P^\gamma$, and, for $i < k$, $F_i^\gamma = F_{i+1}^{-\gamma}$. A string is said to be *coherent* if there is a $\theta \in (\mathbf{R}^n)^*$ satisfying $\tilde{\theta} \in \tilde{N}(F_1) \cap \tilde{N}(F_2) \cap \ldots \cap \tilde{N}(F_k)$, where

$$\tilde{\theta} = \mathrm{proj}_{(\mathbf{R}^n)^*/\mathrm{span}\,\gamma}\theta \qquad \text{and} \qquad \tilde{N}(F) = \mathrm{proj}_{(\mathbf{R}^n)^*/\mathrm{span}\,\gamma} N(P, F).$$

The monotone path polytope $\Sigma_\gamma(P)$ is defined to be the fiber polytope of P with respect to the map $\gamma: \mathbf{R}^n \to \mathbf{R}$. Its faces correspond to coherent cellular strings of P (see [**4**], [**3**]).

PROPOSITION 4.1. *If F_1, F_2 is a cellular string in P with $F_1 \cap F_2$ a simple vertex and* $\dim F_1 + \dim F_2 = \dim P$, *then it is coherent and corresponds to a facet of $\Sigma_\gamma(P)$.*

Proof. Let $N(P, F_1 \cap F_2) = \mathrm{cone}\{v_1, \ldots, v_n\}$, where cone denotes the *strictly* positive span of $\{v_1, \ldots, v_n\}$. After possible reordering, we have $N(P, F_1) = \mathrm{cone}\{v_1, \ldots, v_m\}$, $N(P, F_2) = \mathrm{cone}\{v_{m+1}, \ldots, v_n\}$,

$$N(F_1, F_1 \cap F_2) = \mathrm{span}\{v_1, \ldots, v_m\} + \mathrm{cone}\{v_{m+1}, \ldots, v_n\}, \qquad \text{and}$$

$$N(F_2, F_1 \cap F_2) = \mathrm{cone}\{v_1, \ldots, v_m\} + \mathrm{span}\{v_{m+1}, \ldots, v_n\},$$

with $1 \leqslant m \leqslant n$. Now

$$\gamma \in N(F_1, F_1 \cap F_2) \cap -N(F_2, F_1 \cap F_2) = \mathrm{cone}\{-v_1, \ldots, -v_m, v_{m+1}, \ldots, v_n\},$$

so $\gamma = \theta_2 - \theta_1$, where $\theta_1 \in N(P, F_1)$ and $\theta_2 \in N(P, F_2)$. Projected into $(\mathbf{R}^n)^*/\mathrm{span}\,\gamma$, we get $\tilde{\theta}_1 = \tilde{\theta}_2 \in \tilde{N}(F_1) \cap \tilde{N}(F_2)$. Hence the cellular string F_1, F_2 is coherent. It corresponds to a facet of $\Sigma_\gamma(P)$ because $F_1 \cap F_2$ is simple and $\dim F_1 + \dim F_2 = \dim P$, so it cannot be coarsened.

Suppose we specify the linear map $\pi_3 : \mathbf{R}^{n+2} \to V_3$ by means of a $3 \times (n+2)$ matrix

$$A = \begin{pmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{n+1} \\ \omega_0 & \omega_1 & \dots & \omega_{n+1} \\ \nu_0 & \nu_1 & \dots & \nu_{n+1} \end{pmatrix}$$

and the maps π_i, $i=2, 1$ as projections on the first i coordinates, so that the composition $\pi_1 \circ \pi_2 \circ \pi_3 : \mathbf{R}^{n+2} \to V_1$ is given by the inner product with $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n+1})$. For the rest of the discussion we take $\gamma = (0, 1, \dots, n+1)$, although any values $\gamma_0 < \gamma_1 < \dots < \gamma_{n+1}$ would yield the same results.

Suppose that Q_3, defined as the convex hull of the columns of A, is a 3-polytope, and let Q_2 be the polygon $\mathrm{conv}\{(0, \omega_0), (1, \omega_1), \dots, (n+1, \omega_{n+1})\}$. By the results in Section 3, we have the following flag polytopes and Minkowski sum relations among them:

$$\Sigma(\Delta_{n+1}, \mathbf{F}_1) = \Sigma_\gamma(\Delta_{n+1});$$

$$\Sigma(Q_2) \prec \Sigma(\Delta_{n+1}, \mathbf{F}_{12}) = \Sigma_\omega(\Sigma_\gamma(\Delta_{n+1})); \tag{4.1}$$

and

$$\Sigma(Q_3) \prec \Sigma(\Delta_{n+1}, \mathbf{F}_{23}) \prec \Sigma(\Delta_{n+1}, \mathbf{F}_{123}) = \Sigma_\nu(\Sigma_\omega(\Sigma_\gamma(\Delta_{n+1}))).$$

Here the polytope in the first row has dimension n, the two polytopes in the second row have dimension $n-1$, and the three polytopes in the third row have dimension $n-2$. To see $\Sigma(\Delta_{n+1}, \mathbf{F}_{12})$ as an iterated path polytope, note that, by definition, $\Sigma(\Delta_{n+1}, \mathbf{F}_{12}) = \Sigma(\Sigma_\gamma(\Delta_{n+1}), \Sigma(Q_2, \mathbf{F}_1))$. Here $\Sigma(Q_2, \mathbf{F}_1)$ is the monotone path polytope given by the projection of the polygon $Q_2 \subset V_2$ onto the first coordinate, and so it lies in a complementary one-dimensional subspace. Thus $\Sigma(\Delta_{n+1}, \mathbf{F}_{12}) = \Sigma_\omega(\Sigma_\gamma(\Delta_{n+1}))$, where the linear functional ω acts on a point of $\Sigma_\gamma(\Delta_{n+1})$ by integrating ω over any γ-monotone path in Δ_{n+1} representing that point.

The following result is known, *e.g.*, [**8**; §3E.1] or [7; Cor. 23]. We include a proof for completeness.

Proposition 4.2. $\Sigma(\Delta_{n+1}, \mathbf{F}_1)$ *is combinatorially equivalent to an n-cube.*

Proof. A cellular string $F_1, \dots, F_k$ on Δ_{n+1} with respect to the objective function γ is given by a sequence of integers $0 = i_0 < i_1 < \dots < i_k = n+1$ and sets $F_j \subset \{i_{j-1}, i_{j-1}+1, \dots, i_j\}$, with $\{i_{j-1}, i_j\} \subset F_j$, $j=1, \dots, k$. The correspondence to faces of the n-cube $C^n = [0, 1]^n$ is given by sending the string $F_1, \dots, F_k$ to the face

$$\{(x_1, \dots, x_n) \in C^n \mid x_{i_1} = x_{i_2} = \dots = x_{i_{k-1}} = 1, \quad \text{and } x_j = 0, \quad j \notin F_1 \cup \dots \cup F_k\}. \tag{4.2}$$

To see that each such string $F_1, \ldots, F_k$ is coherent, note first that for any face F of Δ_{n+1} (considered as a subset of $\{0, 1, \ldots, n+1\}$), the inner normal cone is given by $N(\Delta_{n+1}, F) = \text{cone}\{e_i^* | i \notin F\} + \text{span}\{e\}$, where $e = e_0^* + e_1^* + \ldots + e_{n+1}^*$ is the linear functional defining the hyperplane $\mathit{aff}\,\Delta_{n+1}$. To show $F_1, \ldots, F_k$ to be coherent, we must show

$$\bigcap_{j=1}^{k} (N(\Delta_{n+1}, F_j) + \text{span}\{\gamma\}) \neq \emptyset.$$

For any $\theta = (\theta_0, \theta_1, \ldots, \theta_{n+1}) \in (\mathbf{R}^n)^*$, let, f_θ be the *convexification* of θ, that is, the largest convex function on the real interval $[0, n+1]$ with $f_\theta(i) \leqslant \theta_i$ for $i = 0, 1, \ldots, n+1$. Given the cellular string $F_1, \ldots, F_k$ as above, we can find θ satisfying

$$f_\theta(i) = \theta_i \text{ for } i \in F_1 \cup \ldots \cup F_k, f_\theta(i) < \theta_i \text{ for } i \in \{1, \ldots, n\} \backslash F_1 \cup \ldots \cup F_k \tag{4.3}$$

and

$$f_\theta \text{ is given by a distinct affine linear function } f_j \text{ on each interval } [i_{j-1}, i_j]. \tag{4.4}$$

Then by the convexity of f_θ, the function $f_\theta - f_j$ is zero on $i \in F_j$ and positive on $i \in \{0, \ldots, n+1\} \backslash F_j$. Writing

$$\sum_{i=0}^{n+1} f_j(i) e_i^* = \alpha\gamma + \beta e$$

for suitable $\alpha, \beta \in \mathbf{R}$, this says $\theta - \alpha\gamma - \beta e \in \text{cone}\{e_i^* \mid i \notin F_j\}$ and so $\theta \in N(\Delta_{n+1}, F_j) + \text{span}\{\gamma\}$.

The properties (4.3) and (4.4) of the function f_θ define the normal cone $N(F_1, \ldots, F_k)$ to the cube $\Sigma(\Delta_{n+1}, \mathbf{F}_1)$ at the face corresponding to $F_1, \ldots, F_k$. Facets of $\Sigma(\Delta_{n+1}, \mathbf{F}_1)$ correspond either to cellular strings F_1, F_2, where $F_1 = \{0, 1, \ldots, i\}$ and $F_2 = \{i, i+1, \ldots, n+1\}$ (corresponding to the facets $x_i = 1$ in C^n), or to single-element strings $F = \{0, 1, \ldots, n+1\} \backslash \{i\}$ (the facets $x_i = 0$), for $1 \leqslant i \leqslant n$.

To compute $\Sigma(\Delta_{n+1}, \mathbf{F}_2) = \Sigma_\omega(\Sigma_\gamma(\Delta_{n+1}))$, it is enough to specify the action of ω on vertices of $\Sigma_\gamma(\Delta_{n+1})$, *i.e.*, on cellular strings $F_1, \ldots, F_k$ consisting entirely of edges of Δ_{n+1} (each $|F_j| = 2$). If $0 = i_0 < i_1 < \ldots < i_k = n+1$ are the vertices of this string then

$$\omega(F_1, \ldots, F_k) = \frac{1}{n+1} \sum_{i=0}^{k} (i_j - i_{j-1}) \frac{\omega_{i_{j-1}} + \omega_{i_j}}{2}$$

(see [**4**; Theorem 5.3]).

In the following theorem, we need the additional hypothesis that the points $(0, \omega_0), (1, \omega_1), \ldots, (n+1, \omega_{n+1})$ are in strict convex position.

THEOREM 4.3. *If Q_2 is an $(n+2)$-gon then the flag polytope $\Sigma(\Delta_{n+1}, \mathbf{F}_{12})$ is combinatorially isomorphic to the $(n-1)$-dimensional permutohedron P_n.*

Proof. The boundary of the polygon Q_2 is divided into two paths with respect to the projection onto the first coordinate, one on "top" and the other on the "bottom", having vertices indexed by sets T and B, respectively, with $T \cup B = \{0, 1, \ldots, n+1\}$ and $T \cap B = \{0, n+1\}$. We consider *ω-decreasing* monotone paths on the cube $\Sigma_\gamma(\Delta_{n+1})$. Faces of $\Sigma_\omega(\Sigma_\gamma(\Delta_{n+1}))$ correspond to cellular strings in $\Sigma_\gamma(\Delta_{n+1})$ joining the initial vertex, corresponding to the top path, to the final vertex, corresponding to the bottom path. Since Q_2 is an $(n+2)$-gon, these paths correspond to complementary vertices of the n-cube by (4.2) and so there are no proper faces containing both vertices. We show now that all cellular strings on $\Sigma_\gamma(\Delta_{n+1})$ are coherent and that $\Sigma_\omega(\Sigma_\gamma(\Delta_{n+1}))$ is combinatorially equivalent to P_n.

The face lattice of the permutohedron P_n is isomorphic to the lattice of ordered partitions of the set $\{1, 2, \ldots, n\}$, ordered by refinement (see, *e.g.*, [**12**; Chap. 5, §3]). We give an isomorphism between this poset and the poset of cellular strings on the regular n-cube C^n with respect to the linear functional $\Sigma_{i \in T} e_i^* - \Sigma_{i \in B} e_i^*$. An ordered partition $A_1, A_2, \ldots, A_k$ corresponds to a cellular string $G_1, G_2, \ldots, G_k$ on C^n given by

$$\begin{aligned} G_i = \{(x_1, \ldots, x_n) \in C^n \mid x_j = 1,\\ &j \in (B \cap (A_1 \cup \ldots \cup A_{i-1})) \cup (T \cap (A_{i+1} \cup \ldots \cup A_k));\\ &x_j = 0, \;\; j \in (T \cap (A_1 \cup \ldots \cup A_{i-1})) \cup (B \cap (A_{i+1} \cup \ldots \cup A_k))\}. \end{aligned} \tag{4.5}$$

This correspondence determines an isomorphism of posets because G_i determines A_i.

To prove the theorem, we need to construct a similar isomorphism to the poset of ω-monotone cellular strings on the non-regular n-cube $\Sigma_\gamma(\Delta_{n+1})$.

Given an ordered partition $A_1, A_2, \ldots, A_k$ of $\{1, \ldots, n\}$, define a sequence of paths $T_0, T_1, \ldots, T_k$ between vertices of Q_2 as follows: $T_0 = T$ and for $i \geqslant 1$, $T_i = (T_{i-1} \setminus A_i) \cup (B \cap A_i)$. Note that $T_k = B$ and that the path T_i together with the region between T_{i-1} and T_i consists of a string $F_{i,1}, \ldots, F_{i,l_i}$ of edges and one or more polygons. This is the cellular string on Δ_{n+1} corresponding to the face G_i of $\Sigma_\gamma(\Delta_{n+1}) \cong C^n$, as can be verified by comparing (4.2) and (4.5). $G_1, G_2, \ldots, G_k$ is a cellular string on $\Sigma_\gamma(\Delta_{n+1})$ because the vertex corresponding to the path T_i is ω-minimum on G_i and ω-maximum on G_{i+1}.

Conversely, every ω-monotone cellular string $G_1, G_2, \ldots, G_k$ on $\Sigma_\gamma(\Delta_{n+1})$ arises this way. Indeed, if T_i is the vertex path on Δ_{n+1} corresponding to the vertex $G_i \cap G_{i+1}$, $T_0 = T$ corresponding to the maximum vertex of G_1 and $T_k = B$ to the minimum of G_k, then define $A_i = (T_i \setminus T_{i-1}) \cup (T_{i-1} \setminus T_i)$. Since T_i lies below T_{i-1} in their projection into Q_2, we have $T_i \setminus T_{i-1} \subset B$ and $T_{i-1} \setminus T_i \subset T$. Consequently, $A_1, A_2, \ldots, A_k$ is an ordered partition of $\{1, \ldots, n\}$, inverting the correspondence above. In each direction, the correspondence is order preserving.

We have shown that the poset of all cellular strings in $\Sigma_\gamma(\Delta_{n+1})$ is isomorphic to the face lattice of the $(n-1)$-dimensional permutohedron P_n. The theorem follows by an applciation of

LEMMA 4.4. *Suppose $\pi: P \to Q$ is a linear map and $\mathcal{P}$ is the poset of all P-induced polyhedral subdivisions of Q. If $\mathcal{P}$ is isomorphic to the face poset of a polytope of dimension* $\dim P - \dim Q$, *then all induced subdivisions are coherent and $\mathcal{P}$ is the face poset of the fiber polytope $\Sigma(P, Q)$.*

Proof. This follows since a full-dimensional spherical subcomplex of a sphere must be the entire complex (see, *e.g.*, [**10**; Thm. 6.6 and Exer. 6.9, pp. 67–68]).

Figure 1 illustrates the situation with respect to a face F of the 14-dimensional permutohedron $\Sigma_\omega(\Sigma_\gamma(\Delta_{16}))$. The corresponding ordered partition $A_1, A_2, \ldots, A_k$ satisfies $A_1 \cup \ldots \cup A_{i-1} = \{1, 2, 11, 12, 14, 15\}$, $A_i = \{3, 4, 8, 13\}$ and $A_{i+1} \cup \ldots \cup A_k = \{5, 6, 7, 9, 10\}$. T_{i-1} is the path 0–2–3–5–7–9–12–13–14–16, T_i is 0–2–4–5–7–8–9–12–14–16, and so $F_{i,1} \ldots F_{i,7}$ is the string

$$\{0, 2\} - \{2, 3, 4, 5\} - \{5, 7\} - \{7, 8, 9\} - \{9, 12\} - \{12, 13, 14\} - \{14, 16\}$$

and G_i is the 4-dimensional face of the 15-cube $\Sigma_\gamma(\Delta_{16})$ corresponding to the face

$$\{x \in \mathbf{R}^{15} \mid x_2 = x_5 = x_7 = x_9 = x_{12} = x_{14} = 1,\ x_1 = x_6 = x_{10} = x_{11} = x_{15} = 0,\ 0 \leqslant x_j \leqslant 1\}$$

of the regular 15-cube.

We describe now the normal cone $N(\mathscr{P})$ to the polytope $\Sigma(\Delta_{n+1}, \mathbf{F}_{12}) \cong P_n$ at the face F corresponding to the ordered partition $\mathscr{P} = (A_1, A_2, \ldots, A_k)$ of $\{1, \ldots, n\}$. Again, let the sequence of paths $T_0, T_1, \ldots, T_k$ between vertices of Q_2 be given by $T_0 = T$, $T_k = B$ and for $0 < i < k$, $T_i = (T_{i-1} \backslash A_i) \cup (B \cap A_i)$, and let the cellular string $F_{i,1}, \ldots, F_{i,l_i}$ on Δ_{n+1} corresponding to the face G_i of $\Sigma_\gamma(\Delta_{n+1})$ be as above.

A linear functional $\theta = (\theta_0, \theta_1, \ldots, \theta_{n+1})$ supports $\Sigma(\Delta_{n+1}, \mathbf{F}_{12})$ at F, if, and only if, it defines the cellular string $G_1, G_2, \ldots, G_k$ in the cube $\Sigma_\gamma(\Delta_{n+1})$. Thus

$$N(\mathscr{P}) = \bigcap_{i=1}^{k} \{N(F_{i,1}, \ldots, F_{i,l_i}) + \operatorname{span}\{\omega\}\}.$$

So $\theta \in N(\mathscr{P})$, if, and only if, there are real numbers $t_1, t_2, \ldots, t_k$ such that $\theta + t_i\omega$ satisfy conditions (4.3) and (4.4) with respect to $N(F_{i,1}, \ldots, F_{i,l_i})$. Consider a lifting of the polygon Q_2 to a 3-polytope Q with vertices given by the

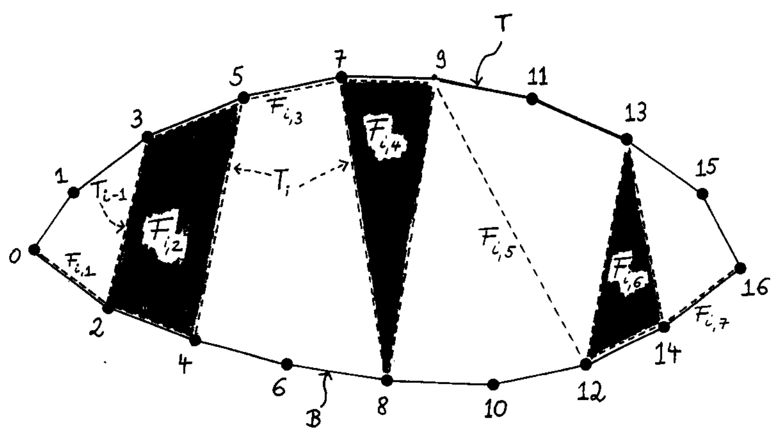

Figure 1. A face of the 14-dimensional permutohedron $\Sigma_\omega(\Sigma_\gamma(\Delta_{16}))$.

columns of

$$\begin{pmatrix} 0 & 1 & \dots & n+1 \\ \omega_0 & \omega_1 & \dots & \omega_{n+1} \\ \theta_0 & \theta_1 & \dots & \theta_{n+1} \end{pmatrix}.$$

To specify a $\theta \in N(\mathscr{P})$ it is necessary and sufficient that the image of Q under the shear $s_t : (x, y, z) \mapsto (x, y, z+ty)$ followed by the projection $p_{1,3} : (x, y, z) \mapsto (x, z)$ is a polygon whose "bottom" is the convexification of a linear functional in $(\mathbf{R}^{n+2})^*$ which supports the chain $F_{i,1}, \dots, F_{i,l_i}$ as in (4.3) and (4.4) for $t = t_i$.

Note that if (a, b, c) is normal to a face F of Q then $(a, b-tc, c)$ is normal to the face $s_t(F)$ of $s_t(Q)$. The bottom of the polygon $(p_{1,3} \circ s_t)(Q)$ consists of the images of faces of Q having normals (a, b, c) with $b-tc=0$ and $c<0$. If we normalize so that $c=-1$, then we conclude that $t=-b$. Thus we want to arrange Q so that the desired faces are sorted by the y-coordinates of their normal vectors, that is, so that the faces $F_{i,j}$ allow normals $(a_{ij}, -t_i, -1)$ for $t_1 < t_2 < \dots < t_k$. This can be achieved by folding Q_2 along the chords appearing in the paths $T_1, \dots, T_{k-1}$.

Remark 4.5. Recall that the 1-skeleton of the permutohedron P_n is the Hasse diagram of the weak Bruhat order on the symmetric group S_n. Thus, under the hypothesis of Theorem 4.3, vertices of $\Sigma(\Delta_{n+1}, \mathbf{F}_{123}) = \Sigma_v(\Sigma_\omega(\Sigma_\gamma(\Delta_{n+1})))$ correspond to paths in this graph. It would be of interest to determine conditions under which these paths are in fact maximal chains in the weak Bruhat order. An example of this phenomenon is given in the next section.

§5. *An example.* We illustrate the results of the previous section for $n=4$ and the example

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5.1}$$

Let Q_i denote the convex hull of the column vectors from the first i rows of A. Thus Q_1 is a segment, Q_2 is a hexagon, and Q_3 is a cyclic 3-polytope with six vertices. The sequence of projections $\Delta_5 \to Q_3 \to Q_2 \to Q_1$ gives rise to the following iterated fiber polytopes

$$\left.\begin{aligned} &\Sigma(\mathbf{F}_1) = \Sigma(\Delta_5, Q_1), \\ &\Sigma(\mathbf{F}_{12}) = \Sigma(\Sigma(\Delta_5, Q_1), \Sigma(Q_2, Q_1)) \succ \Sigma(\mathbf{F}_2) = \Sigma(\Delta_5, Q_2), \\ &\Sigma(\mathbf{F}_{123}) = \Sigma(\Sigma(\Sigma(\Delta_5, Q_2), \Sigma(Q_2, Q_1)), \Sigma(\Sigma(Q_3, Q_2), \Sigma(Q_2, Q_1))) \\ &\qquad \succ \Sigma(\mathbf{F}_{23}) = \Sigma(\Sigma(\Delta_5, Q_2), \Sigma(\Delta_3, Q_2)) \succ \Sigma(\mathbf{F}_3) = \Sigma(\Delta_5, Q_3). \end{aligned}\right\} \tag{5.2}$$

All six polytopes lie in $\mathbf{R}^6$. We now describe them explicitly. When listing the vertices, we always clear denominators to get integer coordinates.

The polytope $\Sigma(\mathbf{F}_1)$ is the secondary polytope of six equidistant points on the line. It is combinatorially isomorphic to the 4-dimensional cube. We list its 16 vertices in decreasing order with respect to the value of the linear functional $\gamma_2 = (1, 2, 0, 0, 2, 1)$:

$$v_{\{1,4\}} = (140041), \quad v_{\{1\}} = (150004), \quad v_{\{4\}} = (400051), \quad v_{\{1,2,4\}} = (123031),$$
$$v_{\{1,3,4\}} = (130321), \quad v_{\{\}} = (500005), \quad v_{\{1,2,3,4\}} = (122221), \quad v_{\{1,3\}} = (130402),$$
$$v_{\{2,4\}} = (204031), \quad v_{\{1,2\}} = (124003), \quad v_{\{3,4\}} = (300421), \quad v_{\{1,2,3\}} = (122302),$$
$$v_{\{2,3,4\}} = (203221), \quad v_{\{2\}} = (205003), \quad v_{\{3\}} = (300502), \quad v_{\{2,3\}} = (203302).$$

The flag polytope $\Sigma(\mathbf{F}_{12})$ parametrizes edge paths on the cube $\Sigma(\mathbf{F}_1)$ which are monotone with respect to γ_2. It is combinatorially isomorphic to the 3-dimensional permutohedron (Theorem 4.3). We list its 24 vertices in decreasing order with respect to value of the linear functional $\gamma_3 = (0, -1, 0, 0, 1, 0)$.

Permutation	Monotone edge path on $\Sigma(\mathbf{F}_1)$	Coordinates
[1234]	$[v_{\{1,4\}}, v_{\{4\}}, v_{\{2,4\}}, v_{\{2,3,4\}}, v_{\{2,3\}}]$	(71, 12, 56, 19, 91, 31)
[1243]	$[v_{\{1,4\}}, v_{\{4\}}, v_{\{2,4\}}, v_{\{2\}}, v_{\{2,3\}}]$	(71, 12, 68, 3, 87, 39)
[1324]	$[v_{\{1,4\}}, v_{\{4\}}, v_{\{3,4\}}, v_{\{2,3,4\}}, v_{\{2,3\}}]$	(81, 12, 21, 49, 86, 31)
[1342]	$[v_{\{1,4\}}, v_{\{4\}}, v_{\{3,4\}}, v_{\{3\}}, v_{\{2,3\}}]$	(87, 12, 3, 63, 82, 33)
[1423]	$[v_{\{1,4\}}, v_{\{4\}}, v_{\{\}}, v_{\{2\}}, v_{\{2,3\}}]$	(99, 12, 33, 3, 52, 81)
[1432]	$[v_{\{1,4\}}, v_{\{4\}}, v_{\{\}}, v_{\{3\}}, v_{\{2,3\}}]$	(105, 12, 3, 33, 52, 75)
[2134]	$[v_{\{1,4\}}, v_{\{1,2,4\}}, v_{\{2,4\}}, v_{\{2,3,4\}}, v_{\{2,3\}}]$	(41, 42, 71, 19, 76, 31)
[2143]	$[v_{\{1,4\}}, v_{\{1,2,4\}}, v_{\{2,4\}}, v_{\{2\}}, v_{\{2,3\}}]$	(41, 42, 83, 3, 72, 39)
[2314]	$[v_{\{1,4\}}, v_{\{1,2,4\}}, v_{\{1,2,3,4\}}, v_{\{2,3,4\}}, v_{\{2,3\}}]$	(37, 50, 61, 31, 70, 31)
[3124]	$[v_{\{1,4\}}, v_{\{1,3,4\}}, v_{\{3,4\}}, v_{\{2,3,4\}}, v_{\{2,3\}}]$	(45, 54, 21, 67, 62, 31)
[3142]	$[v_{\{1,4\}}, v_{\{1,3,4\}}, v_{\{3,4\}}, v_{\{3\}}, v_{\{2,3\}}]$	(51, 54, 3, 81, 58, 33)
[3214]	$[v_{\{1,4\}}, v_{\{1,3,4\}}, v_{\{1,2,3,4\}}, v_{\{2,3,4\}}, v_{\{2,3\}}]$	(37, 58, 37, 55, 62, 31)
[2341]	$[v_{\{1,4\}}, v_{\{1,2,4\}}, v_{\{1,2,3,4\}}, v_{\{1,2,3\}}, v_{\{2,3\}}]$	(31, 62, 55, 37, 58, 37)
[2413]	$[v_{\{1,4\}}, v_{\{1,2,4\}}, v_{\{1,2\}}, v_{\{2\}}, v_{\{2,3\}}]$	(33, 58, 81, 3, 54, 51)
[2431]	$[v_{\{1,4\}}, v_{\{1,2,4\}}, v_{\{1,2\}}, v_{\{1,2,3\}}, v_{\{2,3\}}]$	(31, 62, 67, 21, 54, 45)
[3241]	$[v_{\{1,4\}}, v_{\{1,3,4\}}, v_{\{1,2,3,4\}}, v_{\{1,2,3\}}, v_{\{2,3\}}]$	(31, 70, 31, 61, 50, 37)
[3412]	$[v_{\{1,4\}}, v_{\{1,3,4\}}, v_{\{1,3\}}, v_{\{3\}}, v_{\{2,3\}}]$	(39, 72, 3, 83, 42, 41)
[3421]	$[v_{\{1,4\}}, v_{\{1,3,4\}}, v_{\{1,3\}}, v_{\{1,2,3\}}, v_{\{2,3\}}]$	(31, 76, 19, 71, 42, 41)
[4123]	$[v_{\{1,4\}}, v_{\{1\}}, v_{\{\}}, v_{\{2\}}; v_{\{2,3\}}]$	(75, 52, 33, 3, 12, 105)
[4132]	$[v_{\{1,4\}}, v_{\{1\}}, v_{\{\}}, v_{\{3\}}, v_{\{2,3\}}]$	(81, 52, 3, 33, 12, 99)
[4213]	$[v_{\{1,4\}}, v_{\{1\}}, v_{\{1,2\}}, v_{\{2\}}, v_{\{2,3\}}]$	(33, 82, 63, 3, 12, 87)
[4231]	$[v_{\{1,4\}}, v_{\{1\}}, v_{\{1,2\}}, v_{\{1,2,3\}}, v_{\{2,3\}}]$	(31, 86, 49, 21, 12, 81)
[4312]	$[v_{\{1,4\}}, v_{\{1\}}, v_{\{1,3\}}, v_{\{3\}}, v_{\{2,3\}}]$	(39, 87, 3, 68, 12, 71)
[4321]	$[v_{\{1,4\}}, v_{\{1\}}, v_{\{1,3\}}, v_{\{1,2,3\}}, v_{\{2,3\}}]$	(31, 91, 19, 56, 12, 71)

The polytope $\Sigma(\mathbf{F}_2)$ is the *associahedron*. Its 14 vertices are labeled by the triangulations of the hexagon Q_2. We list them in decreasing order with respect to γ_3. Since $\Sigma(\mathbf{F}_2)$ is a summand of $\Sigma(\mathbf{F}_{12})$, each vertex corresponds to one or more of the 24 permutations.

#	Triangulation of Q_2	Permutations	Coordinates
(1)	{014, 024, 234, 345}	[1234]	(9, 3, 8, 5, 14, 3)
(2)	{014, 023, 034, 345}	[1324], [1342]	(11, 3, 1, 11, 13, 3)
(3)	{014, 024, 235, 245}	[1243]	(9, 3, 11, 1, 13, 5)
(4)	{014, 025, 045, 235}	[1423]	(13, 3, 6, 1, 8, 11)
(5)	{014, 023, 035, 045}	[1432]	(14, 3, 1, 6, 8, 10)

(6)	{012, 124, 234, 345}	[2134], [2314], [2341]	(3, 9, 11, 5, 11, 3)
(7)	{012, 124, 235, 245}	[2143], [2413], [2431]	(3, 9, 14, 1, 10, 5)
(8)	{013, 023, 134, 345}	[3124], [3142], [3412]	(5, 10, 1, 14, 9, 3)
(9)	{012, 123, 134, 345}	[3214], [3241], [3421]	(3, 11, 5, 11, 9, 3)
(10)	{015, 025, 145, 235}	[4123]	(10, 8, 6, 1, 3, 14)
(11)	{015, 023, 035, 145}	[4132]	(11, 8, 1, 6, 3, 13)
(12)	{013, 023, 135, 145}	[4312]	(5, 13, 1, 11, 3, 9)
(13)	{012, 125, 145, 235}	[4213], [4231]	(3, 13, 11, 1, 3, 11)
(14)	{012, 123, 135, 145}	[4321]	(3, 14, 5, 8, 3, 9)

The flag polytope $\Sigma(\mathbf{F}_{123})$ is a 12-gon. Each vertex is labeled by a maximal chain in the weak Bruhat order on S_4. We list the 12 vertices in their order on the polygon.

#	Chain in the weak Bruhat order	Coordinates
(*a*)	[4321, 3421, 3241, 3214, 2314, 2134, 1234]	(12606, 18077, 13173, 14133, 18077, 12414)
(*b*)	[4321, 3421, 3241, 2341, 2314, 2134, 1234]	(12414, 18077, 14133, 13173, 18077, 12606)
(*c*)	[4321, 4231, 2431, 2341, 2314, 2134, 1234]	(12414, 17555, 18831, 7431, 17555, 14694)
(*d*)	[4321, 4231, 2431, 2413, 2143, 2134, 1234]	(12542, 17299, 20495, 5255, 17299, 15590)
(*e*)	[4321, 4231, 4213, 2413, 2143, 2134, 1234]	(12806, 16963, 22199, 2879, 16963, 16670)
(*f*)	[4321, 4231, 4213, 2413, 2143, 1243, 1234]	(13046, 16723, 23159, 1439, 16723, 17390)
(*g*)	[4321, 4231, 4213, 4123, 1423, 1243, 1234]	(24244, 11633, 12979, 1439, 11633, 26552)
(*h*)	[4321, 4312, 4132, 1432, 1342, 1324, 1234]	(26552, 11633, 1439, 12979, 11633, 24244)
(*i*)	[4321, 4312, 3412, 3142, 1342, 1324, 1234]	(17390, 16723, 1439, 23159, 16723, 13046)
(*j*)	[4321, 3421, 3412, 3142, 1342, 1324, 1234]	(16670, 16963, 2879, 22199, 16963, 12806)
(*k*)	[4321, 3421, 3412, 3142, 3124, 1324, 1234]	(15590, 17299, 5255, 20495, 17299, 12542)
(*l*)	[4321, 3421, 3241, 3214, 3124, 1324, 1234]	(14694, 17555, 7431, 18831, 17554, 12414)

The weak Bruhat order on S_4 has 16 maximal chains. The four chains which are missing in our list are not coherent, *i.e.*, the corresponding monotone edge paths on the permutohedron $\Sigma(\mathbf{F}_{12})$ are not coherent with respect to γ_3. Two of the four "missing vertices" lie in the interior of the 12-gon, while the other two lie on the edge (g, h).

The polytope $\Sigma(\mathbf{F}_{23})$ is a 7-gon. Its vertices correspond to the seven coherent (Q_3, Q_2)-homotopies, each running from the top surface to the bottom surface of the cyclic polytope Q_3. Such a homotopy sweeps out a regular triangulation of the cyclic polytope Q_3, and thus defines a vertex of $\Sigma(\mathbf{F}_3)$, the secondary polytope of Q_3. The latter polytope is a pentagon. We list the vertices of $\Sigma(\mathbf{F}_{23})$ in their order on the polygon.

(Q_3, Q_2)-homotopy	Triangulation of Q_3	Chains	Coordinates
(1, 6, 9, 14)	[0124, 1234, 1345]	(*a*, *b*)	(186, 413, 325, 325, 413, 186)
(1, 6, 7, 13, 14)	[0124, 2345, 1245, 1235]	(*c*, *d*, *e*)	(186, 395, 487, 127, 395, 258)
(1, 3, 7, 13, 14)	[2345, 0124, 1245, 1235]	(*f*)	(198, 383, 535, 55, 383, 294)
(1, 3, 4, 10, 13, 14)	[2345, 0245, 0145, 0125, 1235]	(*g*)	(429, 278, 325, 55, 278, 483)
(1, 2, 5, 11, 12, 14)	[0234, 0345, 0145, 0135, 0123]	(*h*)	(483, 278, 55, 325, 278, 429)
(1, 2, 8, 12, 14)	[0234, 0134, 1345, 0123]	(*i*)	(294, 383, 55, 535, 383, 198)
(1, 2, 8, 9, 14)	[0234, 0134, 0123, 1345]	(*j*, *k*, *l*)	(258, 395, 127, 487, 395, 186)

The column labeled "Chains" gives the vertices of $\Sigma(\mathbf{F}_{123})$ corresponding to the given vertex of $\Sigma(\mathbf{F}_{23})$. Geometrically, the 7-gon is a Minkowski summand of the 12-gon. The column labeled "Triangulation of Q_3" describes the surjection onto the vertices of $\Sigma(\mathbf{F}_3)$, the secondary polytope of Q_3. Geometrically, the pentagon $\Sigma(F_3)$ is a Minkowski summand of the given 7-gon. Note that the coordinate vectors in the previous two tables lie in 2-dimensional affine subspaces of $\mathbf{R}^6$ parallel to the kernel of A and $(1, 1, 1, 1, 1, 1)$.

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COXETER-ASSOCIAHEDRA

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Abstract. Recently M. M. Kapranov [Kap] defined a poset KPA_{n-1}, called the *permuto-associahedron*, which is a hybrid between the face poset of the *permutohedron* and the *associahedron*. Its faces are the partially parenthesized, ordered, partitions of the set $\{1, 2, \ldots, n\}$, with a natural partial order.

Kapranov showed that KPA_{n-1} is the face poset of a regular CW-ball, and explored its connection with a category-theoretic result of MacLane, Drinfeld's work on the Knizhnik-Zamolodchikov equations, and a certain moduli space of curves. He also asked the question of whether this CW-ball can be realized as a convex polytope.

We show that indeed, the permuto-associahedron corresponds to the type A_{n-1} in a family of convex polytopes KPW associated to the classical Coxeter groups, $W = A_{n-1}$, B_n, D_n. The embedding of these polytopes relies on the *secondary polytope* construction of the associahedron due to Gel'fand, Kapranov, and Zelevinsky. Our proofs yield integral coordinates, with all vertices on a sphere, and include a complete description of the facet-defining inequalities.

Also we show that for each W, the dual polytope KPW^* is a refinement (as a CW-complex) of the *Coxeter complex* associated to W, and a coarsening of the barycentric subdivision of the Coxeter complex. In the case $W = A_{n-1}$, this gives a combinatorial proof of Kapranov's original sphericity result.

§0. *Introduction.* This paper is concerned with the construction of polytopes with prescribed combinatorial structure. In fact, there is a three-part problem associated with combinatorial objects like permutohedra, associahedra, . . . :

1. the first part is the combinatorial description of a finite poset (*definition*);
2. the second part asks for a proof that the poset under consideration is the face poset of a regular CW-ball (*sphericity*); and
3. the third part is the construction of a convex polytope whose face lattice is isomorphic to the poset (*realization*).

Note that *realization* gives a proof of *sphericity*, since every convex polytope is a regular CW-ball (*cf.* [Bj2], [BLSWZ, Sect. 4.7]).

For the permutohedron, the *definition* and *realization* are classical. For the associahedron, the *definition* is due to Stasheff [Stas] (and later independently to Perles [Per]). *Sphericity* was proved by Stasheff, *realization* was achieved by Milnor (unrecorded), Haiman [Hai] and Lee [Lee]. A "systematic" construction method for the associahedron was achieved by Gel'fand, Zelevinsky and Kapranov [GZK1, Remark 7c] with their construction of *secondary polytopes*,

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