Homotopy Type of Posets of Subgroups *

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Contents

1 Introduction

Given a group G, there are several interesting and naturally defined simplicial complexes and partially ordered sets on which G acts as simplicial maps or order

*Work submitted for the Preliminary Oral Examination, School of Mathematics, University of Minnesota, under supervision of Peter Webb preserving maps. When this happens, the homology groups acquire a G-module structure and so have importance in representation theory (see [Web87]), but in this work we will mainly be interested in the equivariant topological structure of such posets and simplicial complexes. We survey what is known about equivariant homotopy type in this context.

Section 2 contains the basic definitions and constructions. In Section 3 we include theorems that allow us to recognize, for example, when a G-poset is G-contractible and which conditions are sufficient to ensure that two G-posets have the same G-homotopy type. The main result in this direction is a theorem due to Quillen. Section 4 contains applications of the theorems of section 3 to different G-posets. Finally in section 5 we discuss techniques that have been applied only recently, and possible directions in which this research may continue.

2 G-posets

We refer the reader to Chapter 3 of Stanley's book ([Sta86]) for many of the basic concepts about posets, lattices and simplicial complexes. In order to review notation and make the definitions in an equivariant setting, we recall some of them now.

2.1 Definitions

Let G be a group. A *G-poset* P is a poset in which a G-action is defined such that $x \leq y$ and $g \in G$ imply $gx \leq gy$. A *morphism* of G-posets then is a map of G-sets which is also order-preserving.

A *G-subposet* P' of P is a subposet invariant under the action of G. Given an element $x \in P$ one can define the subposet $P_{\leq x} = \{ y \in P \mid y \leq x \}$, analogously one defines $P_{\geq x}$, $P_{<x}$, $P_{>x}$, etc. If x is fixed by G, then clearly $P_{\leq x}, P_{\geq x}$, etc. are G-subposets. More generally we see that $P_{\leq x}$, $P_{\geq x}$, etc. are G_x-subposets.

If P and Q are G-posets, we can define a G-action on the product poset $P \times Q$ by $g(p, q) = (gp, gq)$. This product together with the natural projections have the usual universal properties. If P and Q are posets on disjoint sets, then the disjoint union $P + Q$ has also a natural G-action. The disjoint union together with the inclusions form the coproduct in the category of G-posets.

The ordinal sum $P \oplus Q$ of Stanley will be called here the *join* of two posets P, Q, and denoted by $P * Q$. It has the disjoint union of P and Q as underlying set and the same order relations as in $P + Q$ together with the additional relations that every element in P will be less than every element in Q. The join also becomes a G-poset if we define the natural action on it. Two special instances of this construction are the *cone* of the poset P, defined as $CP = \{\hat{0}\} * P$ and the *suspension* ΣP of P, defined as $\Sigma P = \{\hat{0}, \hat{0}'\} * P$, in which $\{\hat{0}, \hat{0}'\}$ is a poset of two noncomparable elements with trivial G-action. We also define, given P, the poset $\hat{P} = \{\hat{0}\} * P * \{\hat{1}\}$.

If A is a subset of the poset P we say that $x \in P$ is an *upper bound* of A if $a \leq x$ for every $a \in A$. We say that x is the *join* of A, denoted $\vee A$, if x is the least upper bound of A. We can observe that the join of A is unique when it exists. We similarly define lower bound and we call the *meet* the greatest lower bound. If every finite subset of P has a meet and a join, then P is a *lattice.* If P is a lattice that has a maximum element $\hat{1}$ and a minimum element $\hat{0}$, we call P a *bounded lattice.* In this case $P^0 = P - \{\hat{0}, \hat{1}\}$ is called the *proper part* of the lattice.

If A is a poset such that no two elements of A are comparable, we say that A is an *antichain.*

We say that a subposet $A \subset P$ is *bounded* if it has either a lower bound or an upper bound in P. (We will use this concept in a different context than that of a bounded lattice, so hopefully, no confusion will arise).

If $A \subset P$, we denote by $P(A)$ the subposet of all elements comparable to all elements of A. The subset A is *astral* if $P(A) \neq \emptyset$. Given $C \subset P$, we denote by $\Gamma(P, C)$ the simplicial complex of finite nonempty astral subsets of C. We observe that if P is a G-poset and $C \subset P$ is G-invariant, we can define a natural G-action on $\Gamma(P, C)$, because if $A \subset C$ is astral, then $gA \subset C$ is astral as well.

A subset $C \subset P$ is called a *cutset* if every maximal chain in P has an element in C. A cutset C is *coherent* if every A in $\Gamma(P, C)$ which is bounded has either a meet or a join in P. (For example, in a lattice every cutset is coherent).

2.2 Implications of the G-action

If P is a G-poset, the fact that every element of G defines a poset automorphism $P \to P$ has several implications about the poset structure on P.

Let P be a G-poset and $A = \{x_1, x_2, \ldots, x_n\} \subset P$ be a subset that has a join. We then denote $\vee A$ by $x_1 \vee x_2 \vee \cdots \vee x_n$. In this case, given that multiplication by g is a poset automorphism, we have

$$g\,(\vee A) = g(x_1 \vee x_2 \vee \cdots \vee x_n) = gx_1 \vee gx_2 \vee \cdots \vee gx_n. \tag{1}$$

If in addition, A is invariant under G, then clearly

$$g\,(\vee A) = gx_1 \vee gx_2 \vee \cdots \vee gx_n = x_1 \vee x_2 \vee \cdots \vee x_n = \vee A \tag{2}$$

that is, $\vee A$ is a fixed point. In particular, if P is a G-poset with a maximum element $\hat{1}$, note that $\hat{1} = \vee P$, and so $\hat{1}$ has to be fixed by G.

Remember that if x and a are elements in a bounded lattice P such that $x \wedge a = \hat{0}$ and $x \vee a = \hat{1}$, we say that x and a are *complements.* We denote the set of complements of a by $a^{\perp}$. If every element in the lattice P has a complement, we say that P is a *complemented* lattice. By calculations similar to equations (1) and (2) above, we can see that if a is a point fixed by G in a G-lattice, then $a^{\perp}$ is G-invariant. Other classes of subsets which are invariant by G is the set of minimal elements, the set of maximal elements, and the set of elements with a given fixed rank.

Note that if we have $gx \leq x$, with $g \in G$ of finite order, then

$$x = g^{o(g)}x \leq \cdots \leq g^2x \leq gx \leq x \tag{3}$$

that is, $gx = x$. Hence every G-orbit is an antichain. Clearly, we obtain the same conclusion if P is finite.

2.3 The order complex and other constructions

A *G-simplicial complex* Δ is a simplicial complex such that there is a G-action defined on its set of vertices in such a way that every $g \in G$ acts simplicially. We say that a G-simplicial complex is *admissible* if G_σ acts trivially on σ for every simplex σ in Δ.

We can associate to any poset P a simplicial complex $\Delta(P)$, called the *order complex* of P, that has finite chains $x_0 < x_1 < \cdots < x_n$ of elements of P as simplices. We can see that a map $\phi: P \to Q$ of posets induces a simplicial map $\Delta(\phi): \Delta(P) \to \Delta(Q)$, also if P is a G-poset we can define a natural action of G on $\Delta(P)$ such that $\Delta(P)$ becomes a G-simplicial complex, which can easily be seen to be admissible. On the other hand, the G-simplicial complex $\Gamma(P, C)$ defined above is not admissible in general.

Now, starting with Δ a G-simplicial complex, one can define a G-poset $P(\Delta)$, called the *face poset* of Δ, with points the faces of Δ and order relation given by face inclusion.

Given a simplicial complex Δ we denote its geometric realization by $|\Delta|$. We will use $|P|$ to denote $|\Delta(P)|$ in case P is a poset. Again, if Δ is a G-simplicial complex, we can define an action on $|\Delta|$ that makes it a G-topological space. It is by means of $|P|$ that one can associate topological concepts to a poset P.

If X and Y are G-spaces, a *G-homotopy* from X to Y is a continuous map $H: X \times [0,1] \to Y$ such that $H(gx, t) = gH(x,t)$ for all $g \in G$, $x \in X$ and $t \in [0,1]$. We say that two G-posets are G-homotopic if $|P|$, $|Q|$ are G-homotopic. Two G-maps between G-posets $\phi, \psi: P \to Q$ are G-homotopic if there is a G-homotopy from $|P|$ to $|Q|$ such that $H(x, 0) = |\phi|(x)$ and $H(x, 1) = |\psi|(x)$. Using this definition, we can speak about G-homotopy equivalence of posets, for example, we say that P is G-contractible if P is G-homotopy equivalent to a point. The relation of being G-homotopic is denoted by $\simeq_G$.

Given a G-poset P, its *barycentric subdivision* $\mathrm{sd}(P)$ is the poset $P(\Delta(P))$, this is the poset of all chains of P ordered by inclusion. It is known that $|P|$ and $|\mathrm{sd}(P)|$ are G-homeomorphic ([CR87], p.589). Similarly, the barycentric subdivision of a simplicial complex can be defined as $\Delta(P(\Delta))$, and we have $\Delta \cong_G \Delta(P(\Delta))$ (G-homeomorphic). Hence

$$P_1 \simeq_G P_2 \iff \Delta(P_1) \simeq_G \Delta(P_2)$$

by definition, but also

$$\Delta_1 \simeq_G \Delta_2 \iff P(\Delta_1) \simeq_G P(\Delta_2).$$

Remember that if $x \leq y$ in P, the interval $[x,y]$ is $\{ z \in P \mid x \leq z \leq y \}$. Given P, we then define the *poset of intervals* of P: $\mathrm{Int}(P) = \{ [x,y] \subset P \mid x \leq y \}$ ordered by inclusion. The poset $\mathrm{Int}(P)$ can also be given a G-action that makes it a G-poset. We have that the map $\mathrm{Int}(P) \to |P|$ given by $[x,y] \mapsto \frac{1}{2}x + \frac{1}{2}y$ defines a G-homeomorphism between $|\mathrm{Int}(P)|$ and $|P|$ (see [Wal88]).

We define the *join* of two G-simplicial complexes Δ, Δ' as:

$$\Delta * \Delta' = \Delta \cup \Delta' \cup \{ \sigma \cup \tau \mid \sigma \in \Delta,\ \tau \in \Delta' \}$$

2.4 Canonical homeomorphisms

We present some well-known homeomorphisms relating various of the constructions we have considered. Where these have not appeared before, we give proofs which relate to the equivariant situation.

Theorem 2.4.1 *Let P and Q be G-posets. We have the following G-homeomorphism:*

$$|P * Q| \cong_G |P| * |Q|$$

in which $|P| * |Q|$ has the associated compactly generated topology.

Proof: It is really just a matter of looking at the definitions to see that $\Delta(P * Q)$ and $\Delta(P) * \Delta(Q)$ may be canonically identified (in particular, in a G-equivariant fashion). Thus it is enough to prove that for two G-simplicial complexes Δ, Δ' we have

$$|\Delta * \Delta'| \cong_G |\Delta| * |\Delta'|$$

Define the map $f\colon |\Delta| * |\Delta'| \to |\Delta * \Delta'|$ by $[x,y,t] \mapsto (1-t)x + ty$. It is proven in [Wal88] that this is a homeomorphism. It is immediate that it is equivariant. ∎

Of course, this also means that $|\Sigma P| \cong_G \Sigma|P|$.

Theorem 2.4.2 *Let P and Q be G-posets. We have the following G-homeomorphism:*

$$|P \times Q| \cong_G |P| \times |Q|$$

in which $|P| \times |Q|$ has the associated compactly generated topology.

Proof: See [CR87], p. 592, also [Wal88]. ■

Theorem 2.4.3 *Let P and Q be G-posets, and $x \in P$, $y \in Q$ fixed by G. Then we have the following G-homeomorphism:*

$$|(P \times Q)_{>(x,y)}| \cong_G |P_{>x}| * |Q_{>y}|$$

in which $|P_{>x}| * |Q_{>y}|$ has the associated compactly generated topology.

Proof: In [Wal88], the author defines the map $(P \times Q)_{>(x,y)} \to |P_{>x} * Q_{>y}|$ that sends $(x, y') \mapsto y'$, $(x', y) \mapsto x'$ and $(x', y') \mapsto \frac{1}{2}x' + \frac{1}{2}y'$ for $x' > x$ and $y' > y$ and proves that it is a homeomorphism. It is immediate to check that is is also equivariant. ■

As a particular case of this theorem, we have the following G-homeomorphism ([Qui78], Prop. 1.9):

$$|CP \times CQ - \{\hat{0}, \hat{0}\}| \cong_G |(CP \times CQ)_{>\{\hat{0},\hat{0}\}}| \cong_G |CP_{>\hat{0}}| * |CQ_{>\hat{0}}| = |P| * |Q| \cong_G |P * Q|.$$

Theorem 2.4.4 *Let P be a G-poset, and $x, y \in P$ fixed by G, with $x < y$ Then we have the following G-homeomorphism:*

$$|\mathrm{Int}\,(P)_{<[x,y]}| \cong_G \Sigma|(x,y)|.$$

Proof: See [Wal88]. We have that

$$\mathrm{Int}\,(P)_{<[x,y]} = \mathrm{Int}\,([x,y)) \cup \mathrm{Int}\,((x,y])$$

and that $|\mathrm{Int}\,([x,y))| \cong_G |[x,y)| \cong_G x * |(x,y)|$, $|\mathrm{Int}\,((x,y])| \cong_G |(x,y]| \cong_G |(x,y)| * y$ and these G-homeomorphisms restrict to the canonical homeomorphism $|\mathrm{Int}\,((x,y))| \cong_G |(x,y)|$. So $\mathrm{Int}\,(P)_{<[x,y]}$ is the union of two cones over $|(x,y)|$, which is $\Sigma|(x,y)|$. ■

As a particular case, we have the following:

$$|\mathrm{Int}\,(\hat{P}) - \{[\hat{0}, \hat{1}]\}| = |\mathrm{Int}\,(\hat{P})_{<[\hat{0},\hat{1}]}| \cong_G \Sigma|(\hat{0}, \hat{1})| = \Sigma|P|.$$

2.5 Orbit posets

If P is a G-poset, we would like to define the *orbit poset* P/G with points the orbits of G on P and order relation

$$[x] \leq [y] \text{ if there is a } g \in G \text{ with } gx \leq y. \tag{4}$$

It turns out that this relation is reflexive and transitive, but not always antisymmetric. However, a sufficient condition for condition (4) to define a poset is that every G-orbit is an antichain, which we have seen to be satisfied if either the poset or the group is finite. One can also prove easily that for any G-poset P, the G-poset $\mathrm{sd}\,(P)$ satisfies this property.

We can observe that if $\phi: X \to Y$ is a G-homotopy equivalence of G-spaces, then ϕ induces an ordinary homotopy equivalence $X/G \to Y/G$. But on the other hand, for a G-poset P such that P/G is defined, it turns out that $|P/G|$ is not always related to $|P|/G$. A sufficient condition for them to be homeomorphic can be given (see [Bre72]):

> For every chain $x_1 \leq \cdots \leq x_n$ in P and every sequence of elements $g_1, \ldots, g_n$ in G for which we get a chain $g_1x_1 \leq \cdots \leq g_nx_n$, there is a $g \in G$ such that $g_ix_i = gx_i$.

We can call a G-poset satisfying such condition a *regular* G-poset. It can be proven that if P satisfies the condition that the orbits are antichains, then $\mathrm{sd}\,(P)$ is regular and so in this case

$$|P|/G \cong |\mathrm{sd}\,(P)|/G \cong |\mathrm{sd}\,(P)/G|$$

and so if we want to consider the orbit space $|P|/G$ as the geometric realization of a poset, we can take $\mathrm{sd}\,(P)/G$ as such.

3 Homotopy Equivalences

In this section we include some theorems which are basic tools when dealing with homotopy type of G-posets. Most of the results presented here appear in a nonequivariant form in [Bjö95].

3.1 The Order Homotopy Theorem and its Consequences

This is 1.3 in Quillen's paper [Qui78], and (10.11) in [Bjö95]. The equivariant case appeared in [TW91].

Theorem 3.1.1 *Let P, Q be G-posets and $\phi, \psi: P \to Q$ two G-maps such that $\phi(x) \leq \psi(x)$ for all $x \in P$. Then ϕ and ψ are G-homotopic.*

Sketch of Proof: Let $\{0 < 1\}$ be a G-poset with trivial G-action. Then the G-map $H: P \times \{0 < 1\} \to Q$ given by $H(x,0) = \phi(x)$, $H(x,1) = \psi(x)$ induces a G-homotopy $|P| \times [0,1] = |P| \times |\{0 < 1\}| \cong_G |P \times \{0 < 1\}| \overset{|H|}{\to} |Q|$ between $|\phi|$ and $|\psi|$. ■

Corollary 3.1.2 *Let P be a G-poset and $\phi: P \to P$ be a G-map such that $\phi(x) \geq x$ for all $x \in P$. Then $\phi: P \to \mathrm{Im}\,(\phi)$ is a G-homotopy equivalence, with the inclusion $\imath: \mathrm{Im}\,(\phi) \hookrightarrow P$ as G-homotopy inverse. (Dually, the conclusion also holds if $\phi(x) \leq x$ for all $x \in P$.)*

Proof: Apply the previous theorem to show that both $\phi \circ \imath$ and $\imath \circ \phi$ are both homotopic to the identity map on P. ■

Now we have the basic tools to prove that a poset is contractible

Corollary 3.1.3 *Let P be a G-poset that has a maximum (or a minimum) element. Then P is G-contractible.*

Proof: The constant map that sends P to the distinguished element is a G-map that satisfies the hypothesis of the previous corollary. ■

Corollary 3.1.4 *Let P be a G-poset and a and element fixed by G such that $x \vee a$ is defined for every $x \in P$ (respectively, if $x \wedge a$ is defined for every $x \in P$). Then P is G-contractible.*

Proof: The map $\phi: P \to P$ given by $x \mapsto x \vee a$ is a G-map by equation (1) in section 2.2 and satisfies $\phi(x) \geq x$ for all $x \in P$. By corollary 3.1.2, we have that P and $\mathrm{Im}\,(\phi)$ are G-homotopy equivalent. But then $\mathrm{Im}\,(\phi)$ has a as a minimum element and so is G-contractible. ■

A poset P as in the previous corollary is said to be *join-contractible* via a (Respectively, we define a *meet-contractible* poset). The inequalities used in the proof are stated frequently in the literature as

$$x \leq \phi(x) = x \vee a \geq a.$$

3.2 Contractible Carriers

These were used by Walker in [Wal81] to give an elementary proof of Quillen's theorem. But we can also see some more immediate applications. They were made equivariant in [TW91].

Definition 3.2.1 *Let Δ be an admissible G-simplicial complex and X a G-topological space. A* contractible G-carrier *from Δ to X is a map C that sends simplices of Δ to subspaces of X such that:*

1. *$C(\sigma)$ is contractible for every simplex σ in Δ,*
2. *If $\tau \subset \sigma$ then $C(\tau) \subset C(\sigma)$,*
3. *$C(g\sigma) = gC(\sigma)$ for all $g \in G$ and all simplices σ,*
4. *G_σ acts trivially on $C(\sigma)$ for all simplices σ.*

We say that a G-map $f\colon |\Delta| \to X$ is carried *by C if $f(|\sigma|) \subset C(\sigma)$ for all simplices σ of Δ*

The following lemma is basic to prove some important facts.

Lemma 3.2.2 *If C is a contractible G-carrier from Δ to X, then:*

1. *There is a continuous G-map $f\colon |\Delta| \to X$ carried by C,*
2. *Any two continuous G-maps carried by C are G-homotopic.*

Proof: See [TW91] p. 176. ■

A first consequence of this is the following generalization of Theorem 3.1.1.

Corollary 3.2.3 *Let P, Q be G-posets and $\phi, \psi\colon P \to Q$ maps of G-posets. If $\phi(x)$ and $\psi(x)$ are comparable for every $x \in P$, then ϕ, ψ are G-homotopic.*

Proof: Define $C(\sigma) = |\phi(\sigma) \cup \psi(\sigma)|$ for a chain σ in P. We have that $C(\sigma)$ is contractible because $\phi(\sigma) \cup \psi(\sigma) \subset Q$ always has a minimum element. Then C is a contractible G-carrier from $\Delta(P)$ to $|Q|$ that carries both ϕ and ψ, hence they are G-homotopic by the previous lemma. ■

(See [BW83] p. 14 for the non-equivariant case). Also we have the following generalization of the join-contractibility of section 3.1.1.

Theorem 3.2.4 *Let P be a G-poset and $a \in P$ an element fixed by G such that:*

1. *For all $x \in P$, either $a \wedge x$ or $a \vee x$ exists,*
2. *Whenever $x < y$, $a \wedge x$ does not exist but $a \wedge y$ exists, we have that $(a \wedge y) \vee x$ exists.*

Then P is G-contractible.

Sketch of Proof: Let $M = \{ y \in P \mid a \wedge y \text{ exists} \}$ and $M^c = P - M$. Then define, for σ a chain in P.

$$C(\sigma) = |\sigma \cup \{a\} \cup \{ a \vee x \mid x \in \sigma \cap M^c \} \cup \{ a \wedge y \mid y \in \sigma \cap M \} \\ \cup \{ (a \wedge y) \vee x \mid x < y, x \in \sigma \cap M^c, y \in \sigma \cap M \}|.$$

Let z be the minimum element in σ. Then if $z \in M$ we have that $C(\sigma)$ is meet-contractible via z and if $z \in M^c$, $C(\sigma)$ is join-contractible via z. So C is a contractible G-carrier from $\Delta(P)$ to $|P|$ that carries both the identity map and the constant map a. ■

Remember that in section 2 we defined the proper part of a bounded G-lattice P as $P^0 = P - \{\hat{0}, \hat{1}\}$.

Corollary 3.2.5 *Let P a bounded G-lattice, and $a \in P^0$ an element fixed by G. If $a^{\perp} = \{ x \in P \mid x \wedge a = \hat{0}, x \vee a = \hat{1} \}$ is the set of complements of a, then $P^0 - a^{\perp}$ is a G-contractible G-subposet of P.*

Sketch of Proof: Apply the previous theorem to the G-poset $P^0 - a^{\perp}$ and the element $a \in P^0 - a^{\perp}$. ■

Corollary 3.2.6 *In this situation, if a has no complements, then P^0 is G-contractible.* ■

The last theorem and its corollaries appeared first in [BW83] and the equivariant form appeared in [Wel95].

3.3 Quillen's Theorem

Then we have the most powerful tool to prove G-homotopy equivalences.

Theorem 3.3.1 (Quillen's Theorem) *Let P and Q be G-posets and $\phi : P \to Q$ a map of G-posets. If for all $y \in Q$ we have that $\phi^{-1}(Q_{\leq y})$ is G_y-contractible, then ϕ is a G-homotopy equivalence. (Dually, the conclusion also holds if $\phi^{-1}(Q_{\geq y})$ is G_y-contractible for all $y \in Q$).*

This was proved in the nonequivariant case by Quillen [Qui78] and extended to an equivariant result in [TW91].

It is an easy consequence of this theorem that if $x \in P$ is a point such that $P_{>x}$ is contractible, then the inclusion $P - \{x\} \hookrightarrow P$ is a homotopy equivalence. But Bouc observed ([Bou84]) that in a poset of finite length we can remove all such x simultaneously preserving the homotopy type:

Corollary 3.3.2 *Let P be a G-poset of finite length and define*

$$P^* = \{\, x \in P \mid P_{>x} \text{ is not } G_x\text{-contractible} \,\}$$

(Define P_* dually using $P_{<x}$). Then for any G-subposet $P' \subset P$ such that $P^* \subset P' \subset P$ we have that the inclusions $P^* \hookrightarrow P' \hookrightarrow P$ are G-homotopy equivalences. (Similarly for a G-subposet $P' \subset P$ such that $P_* \subset P' \subset P$).

Proof: See [TW91] p. 177. ■

Remember that an *ideal* of the poset P is a subposet I such that $i \in I$ and $p \leq i$ imply $p \in I$.

Corollary 3.3.3 *Let P and Q be G-posets and $R \subset P \times Q$ a G-invariant ideal in the product poset. If $R_x = \{\, y \in Q \mid (x, y) \in R \,\}$ is G_x-contractible for all $x \in P$ and $R_y = \{\, x \in P \mid (x, y) \in R \,\}$ is G_y-contractible for all $y \in Q$, then P and Q are G-homotopy equivalent.*

Proof: Consider the projection $\pi: R \to P$, which is a G-map. Define $F_x = \pi^{-1}(P_{\geq x}) = \{\, (z, y) \in R \mid z \geq x \,\}$ and G_x-maps $\phi: F_x \to R_x$ given by $(z, y) \mapsto y$ and $\psi: R_x \to F_x$ given by $y \mapsto (x, y)$. The map ϕ is well-defined because R is an ideal. We have then $(\psi \circ \phi)(z, y) \leq (z, y)$ and $(\phi \circ \psi)(y) = y$, hence F_x is G_x-homotopy equivalent to R_x by section 3.1.1, hence G_x-contractible. By Quillen's Theorem, R and P are G-homotopy equivalent. Similarly, R and Q are homotopy equivalent, hence so are P and Q. ■

The nonequivariant case appeared first in [Qui78].

Lemma 3.3.4 *Let P be a G-poset, C a coherent cutset and $A \subset C$ a G-invariant astral subset. Then $P(A)$ is a G-contractible G-subposet of P.*

Proof: Let $x \in P(A)$ and $A' = \{\, a \in A \mid a \geq x \,\}$. Assume first that $A' \neq \varnothing$. Then A' is G-invariant, because if $a \in A'$ and $ga \notin A'$, then $ga < x \leq a$, contradicting equation 3 of section 2.2. Then A' is a bounded subset of the coherent cutset C so it has a meet or a join, which will be fixed by G. If A' has a join $\vee A'$, then $\vee A' \in P(A)$ (because $\vee A' \geq x > a$ for all $a \in A - A'$), and any element in $P(A)$ is comparable with $\vee A'$ (Let $y \in P(A)$. If $y \leq a'$ for some $a' \in A'$ then $y \leq a' \leq \vee A'$. Otherwise $y \geq a'$ for all $a' \in A'$, hence $y \geq \vee A'$). In case A' has a meet $\wedge A'$, then $a < x \leq \wedge A'$ for all $a \in A - A'$ so again $\wedge A' \in P(A)$. Again, let $y \in P(A)$. If $y \geq a'$ for some $a' \in A$, then $y \geq a' \geq \wedge A'$. Otherwise $y \leq a'$ for all $a' \in A$, hence $y \leq \wedge A'$. In any case, by corollary 3.1.4, $P(A)$ is G-contractible. Now, in case $A' = \varnothing$, then A is bounded above, so it has a meet or a join. We can prove then as before that every element in $P(A)$ is comparable to such meet or join. ∎

Theorem 3.3.5 (Cutset Theorem) *If C is a G-invariant coherent cutset in P, then P and $\Gamma(P,C)$ are G-homotopy equivalent.*

Proof: Consider the G-invariant ideal in the product of G-posets $\mathrm{sd}\,(P) \times \Gamma(P,C)$.

$$R = \{\,(x, A) \in \mathrm{sd}\,(P) \times \Gamma(P,C) \mid x \subset P(A)\,\}.$$

If $x \in \mathrm{sd}\,(P)$ we have

$$R_x = \{\,A \in \Gamma(P,C) \mid x \subset P(A)\,\} = \{\,A \in \Gamma(P,C) \mid A \subset P(x)\,\},$$

and this is the set of all non-empty subsets of $P(x) \cap C$, which has a maximal element $P(x) \cap C$ and so is G_x-contractible. Now

$$R_A = \{\,x \in \mathrm{sd}\,(P) \mid x \subset P(A)\,\} = \mathrm{sd}\,(P(A)).$$

Given that $P(A)$ is G_A-contractible by the previous lemma, we can apply 3.3.3. ∎

(See [Wal81] p. 380 for the non-equivariant case).

3.4 Wedge decomposition.

Now we will state another kind of result. The following lemma is used in the proof of the theorem of this section.

Lemma 3.4.1 *Let P be a G-poset and $P' \subset P$ a G-contractible G-subposet of P. Then P and the quotient $|P|/|P'|$ are G-homotopy equivalent.* ∎

Assuming P and P' as in the lemma, suppose further that $A = P - P'$ is an antichain. Given that P' is invariant, A will also be invariant. Consider the space

$$Q = \bigvee_{x \in A} (\{x, p\} * (|P_{<x}| * |P_{>x}|))$$

(i.e. a wedge of the suspensions of the spaces $|P_{<x}| * |P_{>x}|$), where p is a new point disjoint from P which is taken to be the wedge point. Define a G-action on it by fixing the point p, permuting the elements $x \in A \subset P$ in the same fashion as G did already and sending $y \in P_{<x}$ to $gy \in P_{<gx}$, etc.

Theorem 3.4.2 *In this situation, P and Q are G-homotopy equivalent.* ■

This is Proposition 2.5 in [Wel95]. The theorem is particularly useful in the case when P is a G-lattice, because as we saw before in Corollary 3.2.5, $P^0 - a^\perp$ is always contractible and in some cases $a^\perp$ is an antichain. This will be used in section 4.4.1 to describe the structure of the lattice of subgroups of a solvable group.

4 Applications

In this section we apply the previous theorems to posets defined in terms of subgroups of a group.

4.1 The p-subgroup complex

Let G be a finite group and p a prime such that $p \mid o(G)$. We define the G-poset

$$\mathcal{S}_p(G) = \{ P \leq G \mid P \text{ is a non-identity } p\text{-subgroup} \}$$

on which G acts by conjugation. This poset was considered by K. Brown in [Bro75] where he proved the formula

$$\chi(\mathcal{S}_p(G)) \equiv 1 \pmod{o(G)_p}$$

where $o(G)_p$ is the order of a p-Sylow subgroup of G.

We may ask when is $\mathcal{S}_p(G)$ contractible. Let $O_p(G)$ be the largest normal p-subgroup of G.

Theorem 4.1.1 *$O_p(G) > 1$ if and only if $\mathcal{S}_p(G)$ is G-contractible.*

Proof: Let $O_p(G) > 1$. We have that $O = O_p(G) \in \mathcal{S}_p(G)$ is such that $H \vee O = \langle H, O \rangle$ is defined for any $H \in \mathcal{S}_p(G)$ and that O is fixed by G (because $O \lhd G$). Hence $\mathcal{S}_p(G)$ is contractible by Corollary 3.1.4. On the other hand, if $\mathcal{S}_p(G)$ is G-contractible to a point p, then inclusion of p in $|\mathcal{S}_p(G)|$ must be a G-map, hence the image of such point is a fixed point in $|\mathcal{S}_p(G)|$. Given that $\mathcal{S}_p(G)$ is admissible, this means that a chain is fixed by G, and so that chain consists of normal subgroups. ■

Of course G-contractibility implies contractibility, so $O_p(G) > 1$ implies $\mathcal{S}_p(G)$ is contractible. Quillen conjectured in [Qui78] the converse of this very last statement. The conjecture is still open, but has been proved for special classes of groups.

Conjecture 4.1.2 (Quillen) *Let G be a finite group. If $\mathcal{S}_p(G)$ is contractible, then $O_p(G) > 1$.*

Theorem 4.1.3 *Let*

$$\mathcal{A}_p(G) = \{\, P \in \mathcal{S}_p(G) \mid P \text{ is elementary abelian} \,\}$$

and

$$\mathcal{B}_p(G) = \{\, P \in \mathcal{S}_p(G) \mid P = O_p(N_G(P)) \,\}.$$

Then, in the notation of 3.3.2, we have $\mathcal{A}_p(G) = \mathcal{S}_p(G)_*$ and $\mathcal{B}_p(G) = \mathcal{S}_p(G)^*$.

Proof: See [TW91], p. 178. ∎

Corollary 4.1.4 *Let Q be a subposet of $\mathcal{S}_p(G)$ containing either $\mathcal{B}_p(G)$ or $\mathcal{A}_p(G)$. Then the inclusion of Q in $\mathcal{S}_p(G)$ is a G-homotopy equivalence.* ∎

The homotopy equivalence of $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ was discovered by Quillen [Qui78], and the one between $\mathcal{B}_p(G)$ and $\mathcal{S}_p(G)$ by Bouc [Bou84]. The fact that the homotopy equivalence can be taken to be G-equivariant was first observed in [TW91].

We may also observe that $H \in \mathcal{A}_p(G)$ being equivalent to $\mathcal{S}_p(G)_{<H}$ not being $N_G(H)$-contractible, is also equivalent to $\mathcal{S}_p(G)_{<H}$ not being contractible. Because if $H \in \mathcal{A}_p(G)$, then $\mathcal{S}_p(G)_{<H}$ is the lattice of proper subspaces of a vector space, which is well known to be homotopy equivalent to a wedge of spheres or empty. See Theorem 4.4.1 below. On the other hand, the assertion that $H \in \mathcal{B}_p(G)$ implies $\mathcal{S}_p(G)_{>H}$ is not contractible is in fact equivalent to Quillen's conjecture: By Proposition 6.1 in [Qui78] we have $\mathcal{S}_p(G)_{>H} \simeq \mathcal{S}_p(N_G(H)/H)$. Assuming Quillen's conjecture, then

$$\begin{aligned} H \in \mathcal{B}_p(G) \Rightarrow O_p(N_G(H)/H) &= O_p(H_G(H))/H = 1 \\ &\Rightarrow \mathcal{S}_p(N_G(H)/H) \simeq \mathcal{S}_p(G)_{>H} \text{ is not contractible} \end{aligned}$$

On the other hand, if G is a group with $O_p(G) = 1$, then $1 = O_p(N_G(1))$, so the trivial subgroup satisfies the property that defines $\mathcal{B}_p(G)$, hence $\mathcal{S}_p(G)_{>1} = \mathcal{S}_p(G)$ would be not contractible, this is Quillen's conjecture.

The importance of the subgroups in $\mathcal{B}_p(G)$ is that in the case of a group which has a building they are the unipotent radicals of the (proper) parabolic subgroups. The inclusion relation on these subgroups is the opposite of the inclusion of the parabolics, and also $\mathcal{B}_p(G)$ is the opposite of the poset which defines the building. Thus $\Delta(\mathcal{B}_p(G))$ may be identified with the (barycentric subdivision of the) building. This means that $\mathcal{B}_p(G)$ and hence $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ can be regarded as the generalization to all finite groups, at every prime, of the notion of a building.

Regarding orbit posets, we have that if G is finite, then $\mathcal{S}_p(G)$ is finite and so $\mathcal{S}_p(G)/G$ is defined. Now, by Sylow's theorem, $\mathcal{S}_p(G)/G$ has a maximum element and so is contractible. Webb conjectured in [Web87] that $|\mathcal{S}_p(G)|/G$ was contractible, and this was recently proved by P. Symonds.

4.2 Complexes mentioned by Alperin.

The complexes mentioned here appeared in [Alp90], where it is stated without explicit proof that they are homotopy equivalent to $\mathcal{S}_p(G)$. We show here as an application of the crosscut theorem that they are in fact G-homotopy equivalent to $\mathcal{S}_p(G)$.

Theorem 4.2.1 *Let Δ_1 be the G-simplicial complex having as vertex set the set of Sylow p-subgroups of G and simplices the sets of vertices having nontrivial intersection. Then Δ_1 and $\mathcal{S}_p(G)$ are G-homotopy equivalent.*

Proof: Let $P = \mathcal{S}_p(G)$ and $C = \{\text{Sylow } p\text{-subgroups of } G\}$. Then C is an invariant cutset in P, we check it is coherent. Since every element in C is maximal, no subset T of C is bounded above. So it reduces to prove that if $T \subset C$ is bounded below, it has a meet in P. Let $T = \{P_1, \ldots, P_r\}$ such that there is $H \in P$ which is a lower bound of T. But then $H \subset \cap_{i=1}^r P_i = P_0$ and $P_0 = \wedge T \neq 1$ so $P_0 \in \mathcal{S}_p(G)$ is the meet of T in P. Hence C is a coherent cutset, and by the cutset theorem (3.3.5), we have $\mathcal{S}_p(G) \simeq \Gamma(P, C)$. But it is immediate that $A = \{P_1, \ldots, P_r\}$ is astral iff $\cap_{i=1}^r P_i \neq 1$. Hence $\Gamma(P, C)$ is the complex described in the statement of the theorem. ∎

Theorem 4.2.2 *Let Δ_2 be the G-simplicial complex having as vertex set the set of subgroups of G of order exactly p and simplices the sets of vertices that commute pairwise. Then Δ_2 and $\mathcal{S}_p(G)$ are G-homotopy equivalent.*

Proof: We show that Δ_2 and $\mathcal{A}_p(G)$ are G-homotopy equivalent from which the result follows by theorem 4.1.3. Let $P = \mathcal{A}_p(G)$ and $C \subset P$ the set of subgroups of order p. We have that C is an invariant cutset in P. To check that it is coherent, let $\{T_1, T_2, \ldots, T_r\} \subset C$ bounded above by T. This means that the T_i belong to the lattice of subgroups of the group T, and so they have a join in T, hence in P. So C is a coherent cutset. We can then see that a subset A of C is astral iff the subgroups in A commute pairwise. Hence $\Gamma(P, C)$, which is G-homotopy equivalent to $P = \mathcal{A}_p(G)$ by the cutset theorem, is precisely the poset described in the statement of the theorem. ∎

4.3 The poset of subgroups of p-power index

In [WW93], the authors consider the G-poset

$$\mathcal{S}^p(G) = \{\, H \leq G \mid [G : H] = p^i, i \neq 0 \,\}$$

and prove that for all finite groups G, this poset is homotopy equivalent to a join of antichains, hence homotopy equivalent to a wedge of spheres of the same dimension. Also they prove the equivalence of the following statements:

1. $\mathcal{S}^p(G)$ is G-contractible,
2. $\mathcal{S}^p(G)$ is contractible,
3. $\chi(\mathcal{S}^p(G)) = 1$,
4. $O^p(G) \neq G$,

where $O^p(G)$ is the minimal normal subgroup of index a power of p.

4.4 The lattice of subgroups

We have the following theorem, which applies the wedge decomposition from 3.4 to the proper part $L(G)^0 = L(G) - \{1, G\}$ of the lattice of subgroups of G.

Theorem 4.4.1 *Let G be a finite solvable group. If $L(G)$ is a complemented lattice, then $L(G)^0$ is G-homotopy equivalent to a wedge of spheres of dimension $k-2$, where k is the length of a chief series. Otherwise, if $L(G)$ is noncomplemented, then $L(G)^0$ is G-contractible.* ■

This theorem was proved in a nonequivariant way in [Thé85], the equivariant version is in [Wel95]. The proof is by induction on the length of a chief series of G and is an application of a remark at the end of section 3.4. We give a sketch of it: Let N be a minimal normal subgroup of G which appears in a fixed chief series. Then we can see that the set of complements of N in $L(G)$ is an antichain, and each $C \in N^{\perp}$ is a solvable group with a chief series of length one less than that of G. If there is any subgroup that has no complements, then $L(G)^0$ is G-contractible by 3.2.6.

This also gives a proof of the Solomon-Tits theorem in the case of the building of $GL(n,p)$, because in this case the building is $L(C_p^n)^0$, where C_p is the cyclic group of order p.

For nonsolvable groups, Welker mentions in [Wel95]: $L(A_5)^0$ is homotopy equivalent to a wedge of 60 spheres $\mathbb{S}^1$, $L(A_6)^0$ is homotopy equivalent to a wedge of 720 spheres $\mathbb{S}^2$, $L(A_7)^0$ is homotopy equivalent to a wedge of 2520 spheres $\mathbb{S}^3$ and conjectures that $L(A_n)^0$ is homotopy equivalent to a wedge of $n!/2$ spheres $\mathbb{S}^{n-4}$ for $n > 6$. On the other hand, $L(GL(3,2))^0$ is homotopy equivalent to a wedge of 2-spheres and 1-spheres.

We consider now orbit posets. Under the hypothesis of last theorem, it is also proven in [Wel95] that both $L(G)^0/G$ and $|L(G)^0|/G$ have the homotopy type of a wedge of spheres $\mathbb{S}^{k-2}$. For non solvable groups, we have that $L(A_n)^0/A_n \simeq |L(A_n)^0|/A_n$ for $n = 5,6,7$, also $L(M_{11})^0/M_{11} \simeq |L(M_{11})^0|/M_{11}$ for the Mathieu group M_{11}. But $|L(M_{12})^0|/M_{12}$ is contractible while $L(M_{12})^0/M_{12}$ is not.

5 Open problems and other techniques

There are still unanswered questions about the structure of these posets.

1. Quillen's conjecture (4.1.2). This has been proven, for example, in the case that G is p-solvable and for G of Lie type in the same characteristic p, and some other cases. See [AS93].

2. Homotopy type of $\mathcal{S}_p(G)$ and $L(G)^0$. Are the connected components of these posets always a wedge of spheres, of possibly different dimensions?

3. The A_7 geometry. There is a 2-dimensional geometry Δ_1 for A_7 discovered by Neumaier ([Neu84]) and another 2-dimensional geometry Δ_2 which arises because $A_7 \leq A_8 \cong GL(4,2)$ acts on the building of $GL(4,2)$. It is known that
$$\widetilde{H}_*(\Sigma\mathcal{A}_2(A_7)) \cong_{\mathbb{Z}G} \widetilde{H}_*(\Delta_1) \oplus \widetilde{H}_*(\Delta_1^\sigma) \oplus \widetilde{H}_*(\Delta_2),$$
where σ is the outer automorphism of order 2 of A_7. We wonder if $\Sigma\mathcal{A}_2(A_7)$ is of the same A_7-homotopy type as $\Delta_1 \vee \Delta_1^\sigma \vee \Delta_2$.

4. What happens if G is not finite? There are various questions one may ask to do with extending the results stated earlier to suitable classes of infinite groups. For example, one may ask if the result of theorem 4.4.1 is true when G is a polycyclic group. (A *polycyclic group* is a group G with a finite chain of subgroups $1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_r = G$ so that G_i/G_{i-1} is cyclic for all $1 \leq i \leq r$).

Some techniques that might be applied are:

1. The non-pure shellability of Björner and Wachs (see [BW96]). A shellable poset is homotopy equivalent to a wedge of spheres, of possible different dimensions. Even though, as Welker points out in [Wel94], we cannot expect this concept to be conclusive in the study of subgroup complexes, it can still be useful.

2. Diagrams of spaces. They have already been used in [PW95] to give a wedge decomposition of $\mathcal{A}_p(G)$ in some particular cases using the notion of homotopy limits (i.e. homotopy colimits), but without the group action. In order extend this to the equivariant situation, we propose the following definition: (compare with [WZŽ95], [PW95]). A *diagram of spaces* $\mathcal{D}$ over the G-poset P is an assignment of spaces D_x to the elements $x \in P$, of maps $d_{xy}\colon D_x \to D_y$ to the order relations $x \geq y$ in such a way that d_{xx} is the identity and $d_{xy} \circ d_{yz} = d_{xz}$ if $x \geq y \geq z$, and of maps $\eta_{g,x}\colon D_x \to D_{gx}$ for

$x \in P$, $g \in G$ such that $\eta_{1,x}$ is the identity, and the following diagrams are commutative:

$$\begin{array}{ccc} D_x & \xrightarrow{\eta_{g_2,x}} & D_{g_2x} \\ & {\scriptstyle \eta_{g_1g_2,x}} \searrow & \downarrow {\scriptstyle \eta_{g_1,g_2x}} \\ & & D_{g_1g_2x} \end{array}$$

for $x \in P$, $g_1, g_2 \in G$,

$$\begin{array}{ccc} D_x & \xrightarrow{d_{xy}} & D_y \\ {\scriptstyle \eta_{g,x}} \downarrow & & \downarrow {\scriptstyle \eta_{g,y}} \\ D_{gx} & \xrightarrow{d_{gx,gy}} & D_{gy} \end{array}$$

for $g \in G$ and $x \geq y$ in P. Remember that the *homotopy limit* $\operatorname{holim} \mathcal{D}$ of the P-diagram $\mathcal{D}$ is defined as the quotient of the disjoint union of topological spaces

$$\sqcup_{x \in P}[\Delta(P_{\leq x}) \times D_x]$$

by the equivalence relation generated by the identifications

$$(z, a) \sim (z, d_{xy}(a))$$

where $z \leq y \leq x$, $a \in D_x$ and we consider $(z, a) \in \Delta(P_{\leq x}) \times D_x$ and $(z, d_{xy}(a)) \in \Delta(P_{\leq y}) \times D_y$. We then define first a G-action on the disjoint union above by the following rule: If $(z, a) \in \Delta(P_{\leq x}) \times D_x$, then $g(z, a) = (gz, \eta_{g,x}(a)) \in \Delta(P_{\leq gx}) \times D_{gx}$. We then can check that this is a G-action that can be defined in the quotient $\operatorname{holim} \mathcal{D}$. In order to use this notion we need to establish a series of lemmas identifying the equivariant homotopy type of this construction, but this remains to be done.

3. Traditional methods of algebraic topology. For example, Symonds proved Webb's conjecture by establishing that the space $|\mathcal{S}_p(G)|/G$ has trivial homology in all dimensions and trivial fundamental group.

4. Computer assisted methods. They have already been used in [Kut93] to prove that $\mathcal{S}_p(G)$ is shellable for a certain group of size 5832, and in [Wel95] for some of the results presented in section 4.4. We plan to use the package GAP and its feature GRAPE (designed to handle graphs) to adapt it to the analysis of G-simplicial complexes and G-posets.

5. Representation theory, as in [AS93].

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