ON SOME INSTANCES OF THE GENERALIZED BAUES PROBLEM

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ABSTRACT. We present an approach applicable to certain instances of the generalized Baues problem of Billera, Kapranov, and Sturmfels. This approach involves two applications of Alexander/Spanier-Whitehead duality. We use this to show that the weak generalized Baues problem has a positive answer for the surjective map of cyclic polytopes $C(n,d) \to C(n,2)$ if $n < 2d$ and $d \geq 10$.

1. INTRODUCTION

The generalized Baues problem (GBP) of Billera, Kapranov, and Sturmfels [3, §3] asks whether a certain poset associated to an affine surjection $\pi : P \to Q$ of polytopes has the homotopy type of a sphere, when the poset is endowed with a standard topology. Although that it is known that this question has a negative answer in general, there are many interesting special cases for which the answer is known or conjectured to be positive. For motivation and a survey of general results on the GBP, see [15].

The purpose of this paper is to outline an approach to the GBP under certain conditions on the polytope P and the map π. We apply this approach to positively answer the GBP in the case of the natural surjection of a cyclic polytope $C(n,d)$ onto the cyclic polygon $C(n,2)$ if $n < 2d$ and $d \geq 10$. The approach uses Alexander (or more strongly, Spanier-Whitehead) duality twice, in order to work with posets that may be more tractable than the original. This approach was partly inspired by the somewhat special result on the GBP obtained in [1, Theorem 1.2]. It is also extremely similar to a double-usage of Alexander duality occurring in work of Stanley [18, Lemma 2.8] in a somewhat different context.

2. THE APPROACH

We first introduce subdivisions and the Baues poset $\omega(P \xrightarrow{\pi} Q)$. Let $\pi : P \to Q$ be an affine surjection of polytopes P, Q, of dimensions d', d respectively. Denote by V the vertex set of P, and say V has cardinality n. Let $\mathcal{A}$ be the point set $\pi(V)$, and we assume for ease of exposition that $\mathcal{A}$ also has cardinality n, that is, no two vertices of P have the same image under π; this assumption is not essential for our results. Note that Q is the convex hull $\operatorname{conv}(\mathcal{A})$.

A ***subdivision*** of $\mathcal{A}$ is a collection of pairs $\{(A_\alpha, Q_\alpha)\}$ where

- A_α are subsets of $\mathcal{A}$,
- each Q_α is the convex hull of A_α and is d-dimensional,

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- the union of the Q_α covers Q,
- for any α, β, the intersection of Q_α, Q_β is a face F (possibly empty) of each, and $A_\alpha \cap F = A_\beta \cap F$

The set of subdivisions of $\mathcal{A}$ is ordered by *refinement* in the following fashion: $\{(Q_\alpha, A_\alpha)\} \leq \{(Q'_\beta, A'_\beta)\}$ if and only if for every α there exists some β with $A_\alpha \subseteq A'_\beta$ (and hence also $Q_\alpha \subseteq Q'_\beta$). We will denote by $\omega(\mathcal{A})$ the poset of subdivisions of $\mathcal{A}$ ordered by refinement.

Certain subdivisions of $\mathcal{A}$ called *coherent* subdivisions are singled out by a geometric property which we will not explain here; see [9, Chapter 7] or [11]. However, we will use the following fundamental theorem of Gelfand, Kapranov, and Zelevinsky [9, Chapter 7, Theorem 2.4] about the subset of coherent subdivisions, which partly explains their importance:

Theorem 1. *The subposet of coherent subdivisions inside $\omega(\mathcal{A})$ is isomorphic to the poset of faces of an $(n-d-1)$-dimensional convex polytope, called the secondary polytope $\Sigma(\mathcal{A})$.*

In particular, $\omega(\mathcal{A})$ contains a top element $\hat{1}$ corresponding to the very coarse subdivision $\{(Q, \mathcal{A})\}$.

Our approach to the GBP will only apply in situations where all subdivisions of $\mathcal{A}$ are coherent. This is a serious restriction, although there are examples known to have this property- see [11]. In particular, when $\mathcal{A}$ is the set of vertices of a convex polygon in the plane, all of its subdivisions are coherent.

In this paper, when we talk about the topology of a poset, we are implicitly identifying the poset with the geometric realization of its *order complex*, that is the simplicial complex of chains in the poset [5, §9]. With this understanding, if we assume that all subdivisions of $\mathcal{A}$ are coherent, then the poset $\omega(\mathcal{A}) - \hat{1}$ triangulates the sphere $\mathbb{S}^{n-d-2}$. This follows because $\omega(\mathcal{A}) - \hat{1}$ is the face poset of the boundary complex of the $(n-d-1)$-dimensional secondary polytope $\Sigma(\mathcal{A})$, and hence triangulates the barycentric subdivision of this complex.

A subdivision of $\mathcal{A}$ is *π-induced* if each of the sets $\mathcal{A}_\alpha$ has the property that the corresponding subset $V_\alpha \subset V$ is the set of vertices of a boundary face of P. The Baues poset $\omega(P \xrightarrow{\pi} Q)$ is defined to be the subposet of $\omega(\mathcal{A})$ consisting of the π-induced subdivisions of $\mathcal{A}$. With this defined, we can now phrase the GBP:

Question 2. (The generalized Baues problem) [3, §3]
Is the Baues poset $\omega(P \xrightarrow{\pi} Q)$ homotopy equivalent to the sphere $\mathbb{S}^{d'-d-1}$?

This problem has been resolved positively in many special cases, in particular when $d = 1$ [3] or when $d' - d \leq 2$ [14], but has a negative answer in general [14]; see [15] for a survey of these results.

For our approach to the GBP, in addition to the Baues poset $\omega(P \xrightarrow{\pi} Q)$ will consider its complement $X(P \xrightarrow{\pi} Q) := \omega(\mathcal{A}) - \omega(P \xrightarrow{\pi} Q) - \hat{1}$. Identifying a face of P with the set of its vertices, we let Faces(P) denote the poset of proper non-empty faces of P, considered as an induced subposet of the Boolean algebra 2^V. We will also consider its complementary subposet Nonfaces$(P) := 2^V - F(P) - \{V\}$.

For two topological spaces X, Y, we will write $X \equiv Y$, $X \approx Y$, and $X \overset{s}{\approx} Y$ resp. to mean that X is *homeomorphic*, *homotopy equivalent*, or *stably homotopy equivalent* to Y, respectively. Recall that $X \overset{s}{\approx} Y$ means there exists some nonnegative integer p such that their p-fold suspensions are homotopy equivalent, that is, $\text{Susp}^p X \approx \text{Susp}^p Y$. In particular, when $X \overset{s}{\approx} Y$ they share the same integral homology groups in each dimension, due to the homotopy invariance of homology groups and the suspension isomorphism $H_i(X, \mathbb{Z}) \cong H_{i+1}(\text{Susp}X, \mathbb{Z})$.

The following lemma is the crux of our approach to the GBP.

Lemma 3. *Using the notations already established, assume that $\pi : P \to Q$ satisfy the two hypotheses that*

- *all subdivisions of $\mathcal{A}$ are coherent, and*
- $X(P \overset{\pi}{\to} Q) \approx \text{Nonfaces}(P)$.

Then

$$\omega(P \overset{\pi}{\to} Q) \overset{s}{\approx} \mathbb{S}^{d'-d-1}.$$

Proof. Recall [17] Spanier-Whitehead duality asserts that for every subcomplex X of a CW-sphere $\mathbb{S}^m$, there is a another CW-complex $D_m X$ having the same homotopy type as $\mathbb{S}^m - X$, and the stable homotopy type of $D_m X$ is determined by the stable homotopy type of X. In other words, if $A \overset{s}{\approx} B$, then $D_m A \overset{s}{\approx} D_m B$.

Recall also that when a poset Λ triangulates $\mathbb{S}^m$, any subposet $X \subseteq \Lambda$ has the property that the order complexes of $X, \Lambda - X$ are deformation retracts of each others complements within $\mathbb{S}^m$ [6, Lemma 4.7.27]. Hence $D_m X \overset{s}{\approx} \Lambda - X$ in this situation. With this in mind, we have the following chain of stably homotopy equivalences, which are explained below:

$$\begin{aligned}
\omega(P \overset{\pi}{\to} Q) &\overset{s}{\approx} D_{n-d-2} X(P \overset{\pi}{\to} Q) \\
&\overset{s}{\approx} D_{n-d-2}\text{Nonfaces}(P) \\
&\overset{s}{\approx} D_{n-d-2} D_{n-2}\text{Faces}(P) \\
&\overset{s}{\approx} D_{n-d-2} D_{n-2}\mathbb{S}^{d'-1} \\
&\overset{s}{\approx} D_{n-d-2}\mathbb{S}^{n-d'-2} \\
&\overset{s}{\approx} \mathbb{S}^{d'-d-1}
\end{aligned}$$

The first line is an application of Spanier-Whitehead duality to the subspaces

$$\omega(P \overset{\pi}{\to} Q), X(P \overset{\pi}{\to} Q) \quad \hookrightarrow \quad \omega(\mathcal{A}) - \hat{1} \equiv \mathbb{S}^{n-d-2}$$

where we have used the assumption that all subdivisions of $\mathcal{A}$ are coherent to conclude that $\omega(\mathcal{A}) - \hat{1} \equiv \mathbb{S}^{n-d-2}$. The second line comes from our assumption that $X(P \overset{\pi}{\to} Q) \approx \text{Nonfaces}(P)$. The third is another application of Spanier-Whitehead duality, this time to

$$\text{Faces}(P), \text{Nonfaces}(P) \quad \hookrightarrow \quad 2^V - \{\varnothing, V\} \equiv \mathbb{S}^{n-2}.$$

The fourth comes from the fact that Faces(P) triangulates the boundary of P, a d'-dimensional polytope. The last two lines follow from the fact that $D_m \mathbb{S}^k \overset{s}{\approx} \mathbb{S}^{m-k-1}$. □

Remark 4.
The conclusion of Lemma 3 is stronger than the assertion which follows from a double usage of Alexander duality, namely that $\omega(P \overset{\pi}{\to} Q)$ has the same integral homology groups as $\mathbb{S}^{d'-d-1}$, but is weaker than the desired conclusion of the GBP, i.e. that $\omega(P \overset{\pi}{\to} Q) \approx \mathbb{S}^{d'-d-1}$.

However, it is not as weak as it might seem at first glance, as we now explain. First, since the GBP is known to hold whenever $d' - d \leq 2$ [14], we may assume that $d' - d \geq 3$, and hence $\mathbb{S}^{d'-d-1}$ is simply-connected. The following well-known lemma, whose proof we include for completeness, says that when the conclusion of Lemma 3 holds, we only need to check whether the fundamental group of $\omega(P \xrightarrow{\pi} Q)$ is trivial:

Lemma 5. *Let X be a CW-complex with $X \overset{s}{\approx} \mathbb{S}^k$ for some $k \geq 2$, and with X simply connected. Then $X \approx \mathbb{S}^k$.*

Proof. Since $X \overset{s}{\approx} \mathbb{S}^k$, we know that X has the same integral homology groups as $\mathbb{S}^k$. Since X and $\mathbb{S}^k$ are simply-connected, the Hurewicz Theorem [16, p. 397] says that they have the same homotopy groups. In particular, $\pi_k(X) \cong \pi_k(\mathbb{S}^k) \cong \mathbb{Z}$, so there is a map $f : \mathbb{S}^k \to X$ whose homotopy class corresponds under these isomorphisms to $1 \in \mathbb{Z}$. It follows from the definition of the Hurewicz homomorphism that f induces an isomorphism between the k-dimensional homology groups of $\mathbb{S}^k$ and X. Since both $X, \mathbb{S}^k$ have all other homology groups trivial, an application of a Whitehead theorem [16, p. 399] says that f induces an isomorphism between all the homotopy groups of $\mathbb{S}^k$ and X. But then since X is a CW-complex, another Whitehead theorem [16, p. 405] implies that f induces a homotopy equivalence between $\mathbb{S}^k$ and X. □

3. Applying the main lemma

In order to apply Lemma 3, we need tools to compare the homotopy type of the posets Nonfaces(P) and $X(P \xrightarrow{\pi} Q)$. Our approach will be to find good coverings of spaces homotopy equivalent to these posets, and then compare the nerves of these covers. For the remainder of the paper, we will assume that

- P is a simplicial polytope, i.e. that its boundary faces are all simplices, and
- $\mathcal{A}$ has only coherent subdivisions.

Because of our assumption that P is simplicial, the poset Nonfaces(P) forms an order filter in the Boolean algebra 2^V, and hence is dual or oppposite to the face poset of a simplicial complex Δ. We can therefore replace Nonfaces(P) by Δ up to homeomorphism. Every minimal nonface N of P corresponds to a maximal face F_N of Δ, and we let $\mathcal{F} = \{F_N\}$ be the covering of Δ by these maximal faces. This is a good covering in the sense that any intersection $\cap_{i=1}^r F_{N_i}$ is either empty or contractible, and note that the latter happens if and only if $\cup_{i=1}^r N_i \subsetneq V$. Hence by the usual Nerve Lemma [5, (10.6)] one can replace Nonfaces(P) by the nerve($\mathcal{F}$) up to homotopy equivalence. We summarize our conclusions in the following proposition:

Proposition 6. *Assuming P is simplicial,* Nonfaces(P) $\approx$ nerve($\mathcal{F}$). *Here* nerve($\mathcal{F}$) *has vertices indexed by the minimal non-faces of P, and a face for each collection $N_1, \ldots, N_r$ of minimal faces with $\cup_{i=1}^r N_i \subsetneq V$.*

Remark 7.
One might ask whether the assumption that P is simplicial is important in the previous covering/nerve construction. Even without assuming that P is simplicial, one can cover the order complex of the poset Nonfaces(P) by the order complexes of the subposets P_N where P_N is the set of non-faces of P which contain N, and N ranges over the minimal non-faces of P. On the other hand, this turns out not

to always be a good cover. For example, let P be a triangular prism whose two triangular faces have vertices labelled a, b, c and a', b', c' in the obvious way, so that a, a' span an edge, as do b, b' and c, c'. Then the minimal non-faces $N_1 = ab', N_2 = a'b$ have the property that P_{N_1}, P_{N_2} are each contractible, but they intersect in a poset with only two incomparable elements $\{aa'bb'c, aa'bb'c'\}$ which is neither contractible nor empty.

We wish to also replace $X(P \xrightarrow{\pi} Q)$ by something homotopy equivalent, and to do this, we need to review the notion of the secondary fan associated to $\mathcal{A}$. Without loss of generality, assume the points of $\mathcal{A}$ affinely span $\mathbb{R}^d$. Identify $\mathcal{A}$ with a $(d+1) \times n$ matrix whose columns give the coordinates of the points in $\mathcal{A}$ with an extra $(d+1)^{st}$ coordinate equal to 1 appended to each. Any matrix $(n-d-1) \times n$ matrix $\mathcal{A}^*$ whose row space coincides the nullspace of $\mathcal{A}$ is called a *Gale transform* of $\mathcal{A}$, and we regard $\mathcal{A}^*$ as a configuration of n points in $\mathbb{R}^{n-d-1}$ which are its columns. If a is a point given by some column of the matrix $\mathcal{A}$, let a^* be the point given by the corresponding column of $\mathcal{A}^*$. Given a subset $A^* \subset \mathcal{A}^*$, let $\text{cone}(A^*)$ denote the set of nonnegative linear combinations of the element in a^*. The *secondary fan* of $\mathcal{A}$ is the common refinement of all cones $\text{cone}(A^*)$ for $A^* \subset \mathcal{A}^*$. It turns out that the secondary fan is the normal fan to the secondary polytope $\Sigma(\mathcal{A})$.

Theorem 8. [2] *The poset of coherent subdivisions of $\mathcal{A}$ is dual (or opposite) to the the poset of non-zero cones in the secondary fan of $\mathcal{A}$. Specifically, a coherent subdivision uses a pair (A, Q) with $A = \{a_1, \ldots, a_r\} \subset \mathcal{A}$ if and only if the corresponding cone of the secondary fan lies in the relative interior of* $\text{cone}(\mathcal{A}^* - \{a_1^*, \ldots, a_r^*\})$.

As a consequence, in the case of interest for us when all subdivisions of $\mathcal{A}$ are coherent, the poset of subdivisions of $\mathcal{A}$ is the face poset of a regular cell complex homeomorphic to $\mathbb{S}^{n-d-2}$, namely the decomposition of the unit sphere in $\mathbb{R}^{n-d-1}$ by the cones of the secondary fan of $\mathcal{A}$. Call this cell complex K, and let K' be the subspace which is the union of all cells of K indexed by elements in $X(P \xrightarrow{\pi} Q)$. Although K' need not in general be a subcomplex of K, by [1, Lemma ?] it is homotopy equivalent to $X(P \xrightarrow{\pi} Q)$.

We wish to find a good cover of K'. For any minimal non-face N of P, let K_N be the union of all cells of K indexed by subdivisions $\{(A_\alpha, Q_\alpha)\}$ of $\mathcal{A}$ which use some A_α containing N. Then we have a covering $\mathcal{E} = \{K_N\}$ of K' by letting N range over the minimal non-faces of P.

Proposition 9. *Assuming P is simplicial and all subdivisions of $\mathcal{A}$ are coherent, $X(P \xrightarrow{\pi} Q) \approx \text{nerve}(\mathcal{E})$. Here $\text{nerve}(\mathcal{E})$ has vertices indexed by the minimal non-faces of P. A collection $N_1, \ldots, N_r$ of minimal faces spans a face of $\text{nerve}(\mathcal{E})$ if and only if there exists a single proper subdivision $\{(A_\alpha, Q_\alpha\}$ of $\mathcal{A}$ having some A_{α_i} containing N_i for each i.*

Proof. We already have seen that we can replace $X(P \xrightarrow{\pi} Q)$ by the space K' up to homotopy equivalence. So it suffices to check that $\mathcal{E}$ is a good covering of K'. It is easy to check from the correspondence between coherent subdivisions and cones given in Theorem 8 that each K_N corresponds to a convex union of cones in the secondary fan. Therefore, any intersection $\cap_{i=1}^r K_{N_i}$ corresponds to a convex union of cones (possibly the 0 cone), and hence is either empty or contractible as a subspace of the spherical complex K.

The last assertion in the Proposition follows from the last assertion in Theorem 8. □

By Lemmas 6, 9 we can compare the nerves of the two covers $\mathcal{E}, \mathcal{F}$ in place of comparing the homotopy types of Nonfaces(P) and $X(P \xrightarrow{\pi} Q)$.

Proposition 10. *With the above notation, there is an inclusion of simplicial complexes*

$$i : \text{nerve}(\mathcal{F}) \quad \hookrightarrow \quad \text{nerve}(\mathcal{E})$$

Proof. Both nerves have vertex sets indexed by the minimal non-faces of P. Assume $N_1, \ldots, N_r$ are minimal non-faces which span a face of nerve($\mathcal{F}$), that is $N := \cup_{i=1}^r N_i \subsetneq V$. Then there always exist proper subdivisions of $\mathcal{A}$ whose restriction to conv(N) contains only the pair (N, conv(N)), that is leaving conv(N) completely unsubdivided. For example, one can use the *pulling* construction of Lee [11] on the remaining vertices $V - N$. Therefore $N_1, \ldots, N_r$ span a face of nerve($\mathcal{E}$). □

Finally, we apply these results in a concrete situation, relating to cyclic polytopes. The *cyclic polytope* $C(n,d)$ is the convex hull of any n points on the *moment curve* $\{(t, t^2, \ldots, t^d)\}$ in $\mathbb{R}^d$. Although this polytope depends upon the choice of the x_1-coordinates $t_1 < \ldots < t_n$ of the points on the moment curve, much of the combinatorics of these polytopes does not depend upon this choice. In particular, Gale's Evenness Criterion [19, Theorem 0.7] describes the face lattice of $C(n,d)$ independent of this choice.

The map $\pi : \mathbb{R}^{d'} \to \mathbb{R}^d$ which forgets the last $d'-d$ coordinates clearly restricts to a surjection $\pi : C(n,d') \to C(n,d)$, and one can show that the set of subdivisions of $C(n,d)$ and also the subset of π-induced subdivisions are independent of the choice of the t_i. Subdivisions and the GBP for these maps between cyclic polytopes have been studied a great deal in the recent past [1, 7, 12, 8, 13], and in [15, Conjecture 19] we conjectured the that the weak GBP always has a positive answer for these maps $\pi : C(n,d') \to C(n,d)$. This is known to be true in the following cases:

- $d = 1$ [3]
- $d' - d \leq 2$ [14]
- $n = d' + 1$ [13]
- $n = d' + 2$ and $d = 2$ [1].

We will consider the case where $d = 2$ (and rename d' by d for ease of notation), i.e. $\pi : C(n,d) \to C(n,2)$. Note that $P = C(n,d)$ is always a simplicial polytope, and it is well-known [10, 11] that all triangulations of $Q = C(n,2)$ are coherent. Therefore our approach will apply whenever we can say something about the inclusion $i : C(n,d) \hookrightarrow C(n,2)$.

Lemma 11. *Keeping the same notation as above, for the map $\pi : C(n,d) \to C(n,2)$, whenever $n < 2d$, the map $i :$ nerve($\mathcal{F}$) $\hookrightarrow$ nerve($\mathcal{E}$) is an isomorphism.*

Proof. Let $C(n,d)$ have vertex set $[n] := \{1, 2, \ldots, n\}$, where i denotes the vertex $(1, t_i, i_i^2, \ldots, t_i^d)$. We give the proof only when d is even, since the description of the minimal non-faces of $C(n,d)$ is slightly simpler in this case. The case when d is odd is similar. One can check using Gale's Evenness Criterion that if d is even, the subsets of $[n]$ which index minimal non-faces of $C(n,d)$ are exactly the subsets of cardinality $\frac{d}{2} + 1$ which contain no two consecutive residues modulo n.

We already know that i is an injective simplicial map, so we need only show it is surjective. In other words, we must show that if $N_1, \ldots, N_r$ are minimal non-faces of $C(n,d)$ whose projections into $C(n,2)$ each lie inside some polygon of a fixed proper subdivision S, and if $\cup_{i=1}^r N_i = [n]$, then $n \geq 2d$.

Because S is a proper subdivision of the convex n-gon $C(n,2)$, we can find two subpolygons Q_1, Q_2 used in S, each of whose boundaries contain only one interior edge of the polygon. Since $\cup_{i=1}^{r} N_i = [n]$, for each $j = 1, 2$, the polygon Q_j must contain a collection of projections of the N_i which cover almost all of its vertices, missing at most two vertices of Q_j (namely the two vertices on the boundary edge of Q_j which forms an interior edge of $C(n,2)$). By the description of minimal non-faces of $C(n,d)$ given in the first paragraph, this implies each of Q_1, Q_2 must have at least $d+1$ vertices, so Q must have at least $2d$ vertices. In other words, $n \geq 2d$. □

From Lemma 3 we immediately deduce the following.

Corollary 12. $\omega(C(n,d) \xrightarrow{\pi} C(n,2)) \overset{s}{\approx} \mathbb{S}^{d'-3}$ *whenever* $n < 2d$. □

Remark 13.
It is possible to do a finer analysis in Lemma 11 and get the same conclusion in the corollary whenever $n < 2d+2$, but it is not clear that this is worth the effort. One mostly wants to know that the conclusion holds for n less than approximately $2d$.

In light of Corollary 12 and Lemma 5, we are particularly interested in knowing when $\omega(P \xrightarrow{\pi} Q)$ is simply connected.

Lemma 14. $\omega(C(n,d) \xrightarrow{\pi} C(n,2))$ *is simply connected whenever* $d \geq 10$.

Proof. We begin by observing that whenever P is simplicial and $\mathcal{A}$ has only coherent subdivisions, the Baues poset $\omega(P \xrightarrow{\pi} Q)$ is actually the face poset of a regular cell complex. To see this, recall from Theorem 1 that the poset $\omega(\mathcal{A})$ of all subdivisions of $\mathcal{A}$ is the face poset of the secondary polytope $\Sigma(\mathcal{A})$, which is a regular cell complex. One can then check using Theorem 8 that since P is simplicial, the subposet $\omega(P \xrightarrow{\pi} Q)$ is an order ideal in $\omega(\mathcal{A})$, and hence indexes the cells of some regular cell subcomplex L of the secondary polytope $\Sigma(\mathcal{A})$.

Recall also that the fundamental group of a regular cell complex can be computed in terms of its 2-skeleton alone. Therefore it suffices for us to show that when $d \geq 10$, the 2-skeleton of the cell complex L is simply connected.

In fact, we claim that L has the same 2-skeleton as $\Sigma(\mathcal{A})$ when $d \geq 10$, due to the neighborliness of cyclic polytopes. Recall that Gale's Evenness Criterion implies $C(n,d)$ is a $\lfloor d/2 \rfloor$-*neighborly* polytope [19, Corollary 0.8], meaning that every subset of its vertices having cardinality $\lfloor d/2 \rfloor$ or less spans a boundary face. We recall [10] the description of 0-cells, 1-cells, 2-cells in $\Sigma(\mathcal{A})$ (the *associahedron*), and analyze when they are π-induced from the surjection $\pi : C(n,d) \to C(n,2)$:

- 0-cells of $\Sigma(\mathcal{A})$ correspond to triangulations of $C(n,2)$, and since these subdivisions are made up of polygons with at most 3 vertices, they will all be π-induced if $d \geq 6$.
- 1-cells of $\Sigma(\mathcal{A})$ correspond to subdivisions of $C(n,2)$ having mostly triangles and exactly one quadrangle, so they will all be π-induced if $d \geq 8$.
- 2-cells of $\Sigma(\mathcal{A})$ correspond to subdivisions of $C(n,2)$ having mostly triangles, and either two quadrangles or one pentagon, so they will all be π-induced if $d \geq 10$.

We therefore conclude that when $d \geq 10$, the fundamental group of $\omega(C(n,d) \xrightarrow{\pi} C(n,2)$ coincides with that of $\Sigma(\mathcal{A}) \equiv \mathbb{S}^{n-4}$, which is simply connected since $n \geq d \geq 10$. □

From the previous result, Corollary 12, and Lemma 5 we immediately deduce

Corollary 15. *The weak GBP has a positive answer for* $\pi : C(n,d) \to C(n,2)$ *when* $n < 2d$ *and* $d \geq 10$.

4. Questions

1. Can one do a more detailed analysis of the 2-skeleton of $\omega(C(n,d) \xrightarrow{\pi} C(n,2)$ and weaken the hypothesis that $d \geq 10$ in Lemma 14?
2. Can one show that the inclusion of nerves as in Lemma 11 is sometimes a homotopy equivalence when it is not an isomorphsim, thereby weakening the hypothesis that $n < 2d$ in Lemma 11?
3. One might think of applying our method to the projections $\pi : C(n,d') \to C(n,d)$ when $d > 2$. It was shown in [1] that the only other non-trivial special cases where $Q = C(n,d)$ has only coherent subdivisions are $(n,d) = \{(7,3),(8,3),(8,4)\}$. Bearing in mind that the GBP in this situation is already settled positively in the cases $d = 1$ or $d' - d \leq 2$ or $n = d' + 1$, this means that there is only one remaining case with $d \geq 3$ where our method might apply, namely $\pi : C(8,6) \to C(8,3)$. However, it turns out that this case is also covered by [1, Theorem 1.2], hence there seems to be nothing new to be proved for projections of cyclic polytopes by this method when $d > 2$.
4. As a step in the proof of Lemma 3 we observed that
$$\mathrm{Nonfaces}(P) \stackrel{s}{\approx} D_{n-2}\mathbb{S}^{d'-1} \stackrel{s}{\approx} \mathbb{S}^{n-d'-2}.$$
This suggests the following stronger question:

Question 16. *For any d-dimensional polytope P with n vertices, is the poset* Nonfaces(P) *homotopy equivalent to* $\mathbb{S}^{n-d-2}$, *not just stably homotopy equivalent?*

We suspect that the answer is "Yes", and the geometry of *Gale diagrams* [19, Lecture 6] can be used to prove this, but have not been able to carry this out. We also suspect that the answer is "No" if instead we only look at the poset of non-faces in some regular cell complex homeomorphic to $\mathbb{S}^{d-1}$, rather than the non-faces of of a convex polytope.

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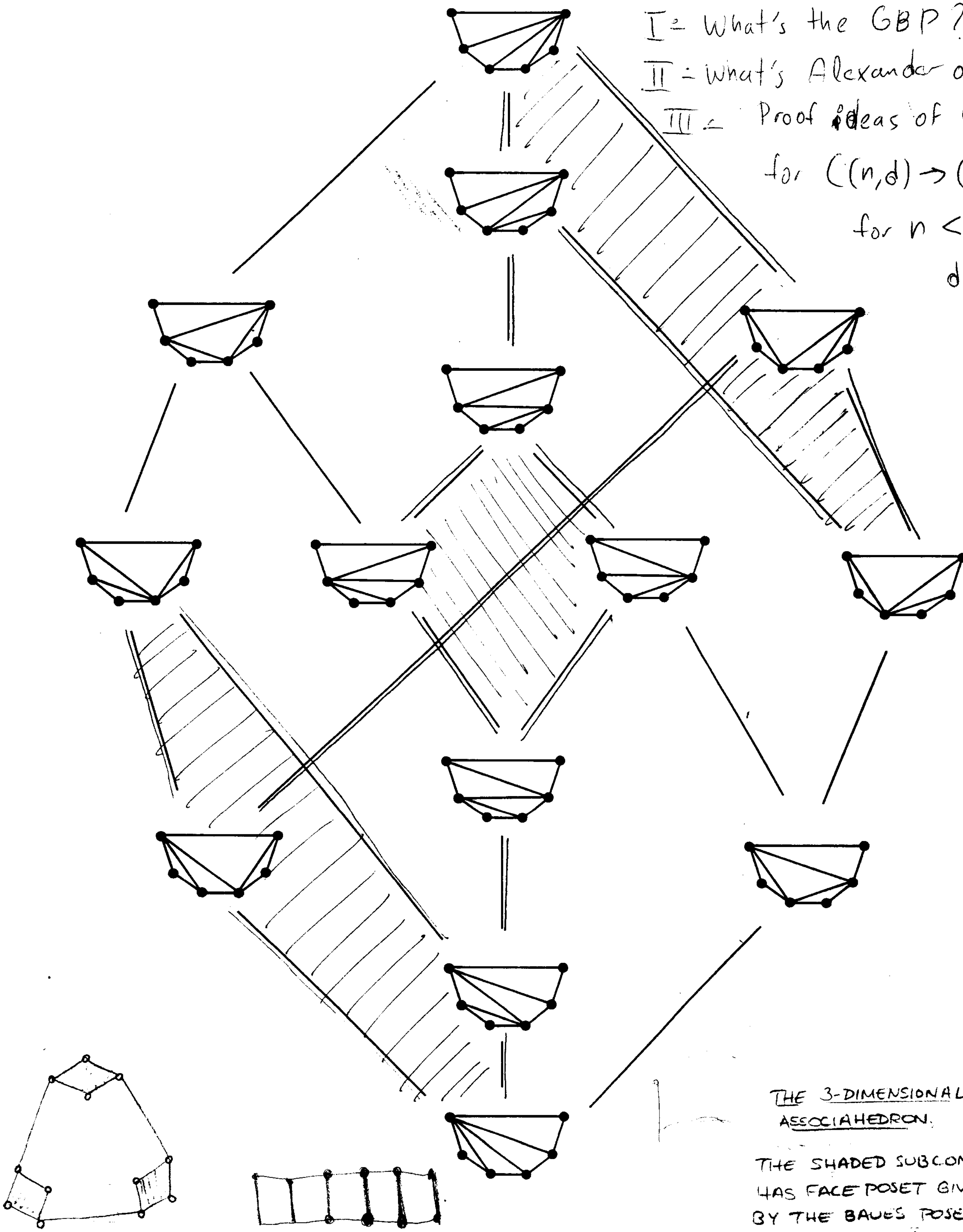
Alexander Duality and the GBP.

Outline:

I: What's the GBP?

II: What's Alexander duality?

III: Proof ideas of GBP for $C(n,d) \to C(n,2)$ for $n < 2d+2$, $d \geq 10$.



THE 3-DIMENSIONAL ASSOCIAHEDRON.

THE SHADED SUBCOMPLEX HAS FACE POSET GIVEN BY THE BAUES POSET $\omega(C(6,4) \to C(6,2))$

II - Alexander Duality $X \subseteq S^d$ a nice subspace.

$\Rightarrow \tilde{H}_i(X, \mathbb{Z}) \cong \tilde{H}^{d-1-i}(S^d - X, \mathbb{Z})$.

<u>Idea:</u> Look at $X(P \xrightarrow{\pi} Q) \simeq$ Subdivisions of Q which are <u>not</u> projections of a family of faces upstairs (use some non-face).

$X(P \xrightarrow{\pi} Q) \subseteq W(\Delta^{n-1} \to Q) \cong S^{n-d}$

And is Alexander dual to $W(P \xrightarrow{\pi} Q) \subseteq W(\Delta^{n-1} \to Q)$.

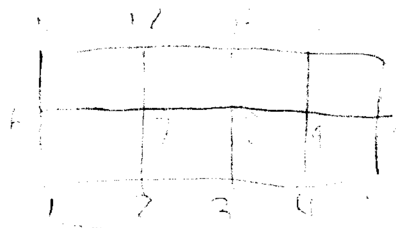
Still what good is it if $H_i(W(P \xrightarrow{\pi} Q), \mathbb{Z})$

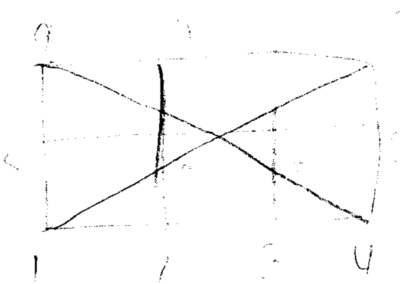
$= H_i(S^{\dim P - \dim Q - 1}, \mathbb{Z})$?

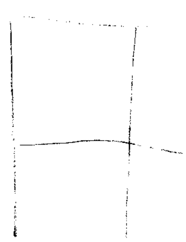
<u>Thm:</u> If $\pi_1(W(P \xrightarrow{\pi} Q)) =$ ~~the~~

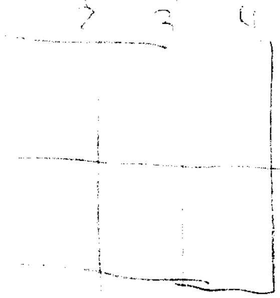
then X is homotopic equivalent to $S^d \simeq X$.

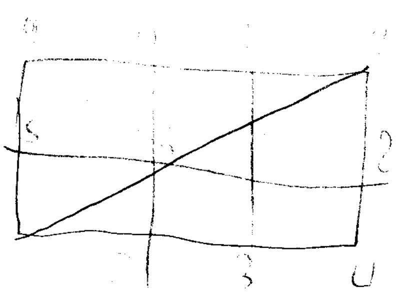
<u>step:</u>











(i)

Notes on Bauer for $C(n,d) \to C(n,2)$

~~PROPOSITION~~ LEMMA: Let $\begin{matrix} P \\ \downarrow \pi \\ Q \end{matrix}$ satisfy these 2 hypotheses:

(i) $\mathcal{A} = \pi(\text{vertices}(P))$ has only coherent subdivisi…

(ii) $\left\{ \begin{matrix} \text{subdivisions of } \mathcal{A} \\ \text{using a non-face of } P \end{matrix} \right\} \underset{\text{homotopy equiv.}}{\approx} \left\{ \begin{matrix} \text{proper} \\ \text{non-faces} \\ \text{of } P \end{matrix} \right\}$

Then $\omega(P \xrightarrow{\pi} Q)$ has the homology of $\mathbb{S}^{\dim P - \dim Q - …}$

proof: Let X, Y be the posets in LHS, RHS of (ii).
Let $n = |\text{vertices}(P)|$.

Then we have

(a) $\omega(P \xrightarrow{\pi} Q)$, $X \hookrightarrow \partial \Sigma(\mathcal{A}) \underset{\text{homeomorphic}}{\cong} \mathbb{S}^{n - \dim Q - …}$

(b) $\{\text{proper faces of } P\}$, $Y \hookrightarrow \partial \Delta^{n-1} \cong \mathbb{S}^{n-2}$

$\{\text{proper faces of } P\} \cong \mathbb{S}^{\dim P - 1}$

Applying Alex. duality to (b) gives $H_\circ(Y) = H_\circ(\mathbb{S}^{n-2} - \mathbb{S}^{\dim…})$

$= H_\circ(\mathbb{S}^{n-2-(\dim P - 1)})$

$= H_\circ(\mathbb{S}^{n - \dim P - 2})$

Using (ii) we get $H_\circ(X) = H_\circ(Y) = H_\circ(\mathbb{S}^{n - \dim P - 2})$

and then applying Alex. duality to (a) gives

$H_\circ(\omega(P \xrightarrow{\pi} Q)) = H_\circ(\mathbb{S}^{n - \dim Q - 2} - \mathbb{S}^{n - di…})$

$= H_\circ(\mathbb{S}^{n - \dim Q - 2 - (n - \dim P - 2)})$

$= H_\circ(\mathbb{S}^{\dim P - \dim Q - 1})$ ∎

(2)

Consider the covering of $X = \left\{ \begin{array}{l} \text{subdivisions of } A \\ \text{using a non-face of } P \end{array} \right\}$

by $\mathcal{C} = \left\{ \underbrace{\begin{array}{l} \text{subdivisions of } A \\ \text{containing some cell} \\ \text{larger than } \pi(N) \end{array}}_{C_N} \right\}_{N \text{ a minimal non-face of } P}$

PROPOSITION: $\mathcal{C}$ is always a good covering in the sense that $\bigcap_{i=1}^{r} C_{N_i}$ is either empty or contractible for any collection $N_1, \ldots, N_r$ (so $\text{nerve}(\mathcal{C}) \approx X$).

proof: Check that each C_N indexes the cells of a convex cone in the secondary fan for P, so $\bigcap_{i=1}^{r} C_{N_i}$ indexes the cells in the intersection. This intersection is also convex, so $\bigcap_{i=1}^{r} C_{N_i}$ is empty or contractible as a poset by the usual lemma ∎

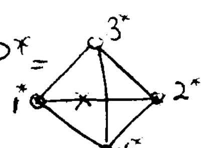
Consider the covering of $Y = \left\{ \begin{array}{l} \text{proper} \\ \text{non-faces of } P \end{array} \right\}$

by $\mathcal{D} = \left\{ \underbrace{\begin{array}{l} \text{proper non-faces of } P \\ \text{containing } N \end{array}}_{D_N} \right\}_{N \text{ a minimal non-face of } P}$

PROPOSITION: If P is a simplicial polytope, then $\mathcal{D}$ is a good covering.

proof: When P is simplicial, non-faces of P form a filter in $2^{[n]}$ and hence the opposite of a simplicial complex. Then $\mathcal{D}$ corresponds to covering this simplicial complex by its facets ∎

N.B. False if we don't assume P simplicial, e.g. $P^* =$



$N_1 = 1^*2^*3^*$ $N_2 = 1^*2^*4^*$

(3)

PROPOSITION: If P is simplicial and $\mathcal{A} = \pi(\text{vertices}(P))$ is in convex position, then $\exists$ an inclusion

$$\text{nerve } \mathcal{D} \hookrightarrow \text{nerve } \mathcal{C}$$

proof: $C_{N_1} \cap \ldots \cap C_{N_r} \neq \phi$ iff $\exists$ a proper subdivision of $\mathcal{A}$ containing a face larger than $\pi(N_i)$ for $i = 1, \ldots, r$.

$D_{N_1} \cap \ldots \cap D_{N_r} \neq \phi$ iff $\exists$ a proper non-face of P containing $\bigcup_{i=1}^{r} N_i$, i.e. iff $\bigcup_{i=1}^{r} N_i \subsetneq [n]$ since P is simplicial.

But whenever $\bigcup_{i=1}^{r} N_i \subsetneq [n]$, pick a vertex v of P in $[n] - \bigcup_{i=1}^{r} N_i$ and ~~then~~ consider the subdivision of $\mathcal{A}$ which leaves $\text{conv}([n] - \pi(v))$ unsubdivided and "cones over $\pi(v)$". ~~This~~ This contain a face larger than each $\pi(N_i)$ for each $i = 1, \ldots, r$ because $\mathcal{A}$ is in convex position

PROPOSITION: The above inclusion is an isomorphism for $\begin{matrix} P = C(n,d) \\ \downarrow \quad \downarrow \\ Q = C(n,2) \end{matrix}$ if $n < 2d$.

sketch proof: Need to show that if $N_1, \ldots, N_r$ are minimal non-faces of $C(n$ that can be "captured" by a proper subdivision S of $C(n,2)$, then $\bigcup_{i=1}^{r} N_i \subsetneq [n]$. Since S is proper, it has some non-trivial diagonal dividing $C(n,2)$ into 2 polygons A & B. (If $\bigcup N_i = [n]$) Roughly speaking, each polygon A & B must contain $\pi(N_i)$'s that cover all of its vertices. Since minimal non-faces of $C(n,d)$ are again roughly speaking, sets of cardinality $\frac{d}{2}$ with no consecu... ...ls of $[n]$ mod n, this requires A & B to each have si...

(4)

COROLLARY: If $n < 2d$ then $\omega(C(n,d) \to C(n,2)) \underset{s}{\approx} S^{\prime}$

stabl
homot
equiva

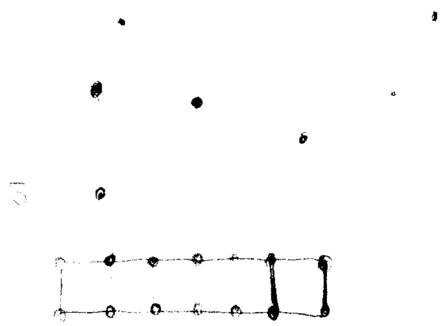
PROPOSITION: If $d \geq 10$ then $\pi_1(\omega(C(n,d) \to C(n,2))) = 1$

proof: If $d \geq 10$ then $C(n,d)$ is 4-neighborly,
i.e. every 5-subset of $[n]$ spans a boundary face
of $C(n,d)$ (as does every 4-subset
3-subset
2-subset
⋮).

One can see that this implies that $\omega(C(n,d) \to C(n,2$
indexes the cells of a regular CW-subcomplex
of the associahedron $\Sigma(C(n,2))$ which contains
the entire 2-skeleton of $\Sigma(C(n,2))$.

Therefore $\pi_1(\omega(C(n,d) \to C(n,2))) =$
$\pi_1(\Sigma(C(n,2))) = 1$ since
$n \geq d \geq 1C$
implies
$\Sigma(C(n,2)) \cong S$
is simply connect

COROLLARY: If $n < 2d$ and $d \geq 10$ then
the weak GBP holds for $C(n,d)$
$\downarrow$
$C(n,2)$



$(6, 4)$

$\downarrow$

$C(12, 2)$?