

THE GENERALIZED BAUES PROBLEM

VICTOR REINER

1. INTRODUCTION

The generalized Baues problem, or GBP for short, first appears in a paper of Billera, Kapranov, and Sturmfels [13, §3], and relates to the theory of *fiber polytopes* introduced by Billera and Sturmfels [14]. The GBP is a question in combinatorial topology; it asks whether a certain partially ordered set, with its usual topology, has the homotopy type of a sphere of a certain dimension.

usual?

The goal of this survey is to review the motivation for fiber polytopes and the GBP, and discuss recent progress on the GBP and the open questions remaining. Some recommended summary sources on this subject are the introductory chapters in the doctoral theses of Rambau and Richter-Gebert [60, 63], Lecture 9 in Ziegler's book [83], and Sturmfels' paper [77]. We have also included in the bibliography some references which we will not discuss in the text but are still relevant to the GBP.

Before diving into the general setting of fiber polytopes and the GBP, it is worthwhile to ponder three motivating classes of examples.

Triangulations.

Let \mathcal{A} denote a finite set of points in \mathbb{R}^d . A *triangulation* of \mathcal{A} is, roughly speaking, a polyhedral subdivision of the convex hull of \mathcal{A} into simplices, each having the property that their vertices lie in \mathcal{A} . Note that not every point of \mathcal{A} need appear as a vertex of one of the simplices in the triangulation. The set of all triangulations of \mathcal{A} is in general a difficult object to compute, but one that arises in many applications (see [44]). One approach to the study and computation of triangulations is to consider an extra structure on them, namely the connections between them by certain moves called *bistellar operations* (or *perestroikas* or *modifications*). For triangulations of \mathcal{A} in \mathbb{R}^2 , typical bistellar operations are shown in Figure 1, where points of \mathcal{A} that are not being used as a vertex in the triangulation are shown dotted. Figure 2 (borrowed from [44]) depicts the set of all triangulations of a particular configuration of six points in \mathbb{R}^2 , and the bistellar moves which connect them. We remark that the precise coordinates of the points of \mathcal{A} are important in determining which triangulations and bistellar operations are possible, since we are talking about triangulations using straight geometric simplices. This is different from the point of view in the theory of triangulated *planar maps* (see e.g. [32, §2.9]) and also different from the bistellar equivalences of triangulations of PL-manifolds as considered by Pachner [53].

Nabutovsky
view?

The most well-studied example of triangulations occurs when \mathcal{A} is the vertex set of a convex n -gon in \mathbb{R}^2 . It is well-known that the number of triangulations is the

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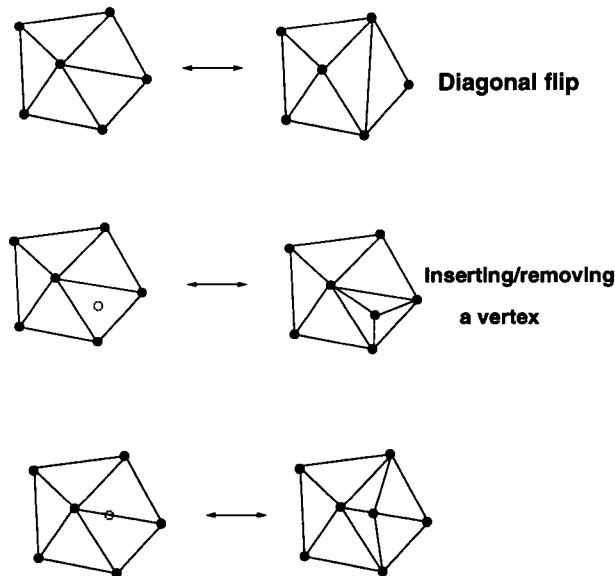


FIGURE 1. Typical bistellar operations/ perestroikas/ modifications in \mathbb{R}^2 .

Catalan number $\frac{1}{n-1} \binom{2n-4}{n-2}$ (see e.g. [73, §3.1] for this and for bijections between triangulations and other standard objects counted by the Catalan number). This is essentially the only non-trivial example of an infinite family of point configurations whose number of triangulations is known (although see Conjecture 23). The only possible bistellar operations in this case are the *diagonal flips* from Figure 1, and it is easy to see that any two triangulations can be connected by a sequence of such flips. There is a well-known bijection between triangulations of an n -gon and associative bracketings of a product $a_1 a_2 \cdots a_{n-1}$, and under this identification, bistellar operations correspond to “rebracketings”. From this point of view, the graph of triangulations of an n -gon and diagonal flips was perhaps first studied in the 1940’s by Tamari [79] and later in collaboration with others [29, 35, 36, 80]. These authors distinguished a direction on each rebracketing and defined a poset on the triangulations having these directed edges as its cover relations. They were able to show that this *Tamari poset* is a lattice [29, 35]. Its Hasse diagram is depicted in Figure 3 for $n = 6$, for a choice of a particular convex 6-gon whose vertices lie on a parabola.

These authors seem also to have been aware (without proof) that this graph appears to be the 1-skeleton of a cellular $(n - 4)$ -sphere, and proved results about how its “facial” structure interacts with the Tamari lattice structure. Meanwhile, similar issues of associativity appeared in the early 1960’s in Stasheff’s work [74] on homotopy associativity. Stasheff vindicated this apparent sphericity by showing that the set of all *polygonal subdivisions* of an n -gon indexes the cells in a *regular cell complex* [17, (12.3)] homeomorphic to the $(n - 4)$ -sphere. Note that in this way of thinking, a diagonal flip bistellar operation corresponds to a polygonal subdivision whose maximal cells are all triangles except for one quadrangle (containing the flipping diagonal), and less refined subdivisions of the n -gon correspond to higher dimensional cells in the sphere. In an unpublished work (see [37, p. 120]), Milnor

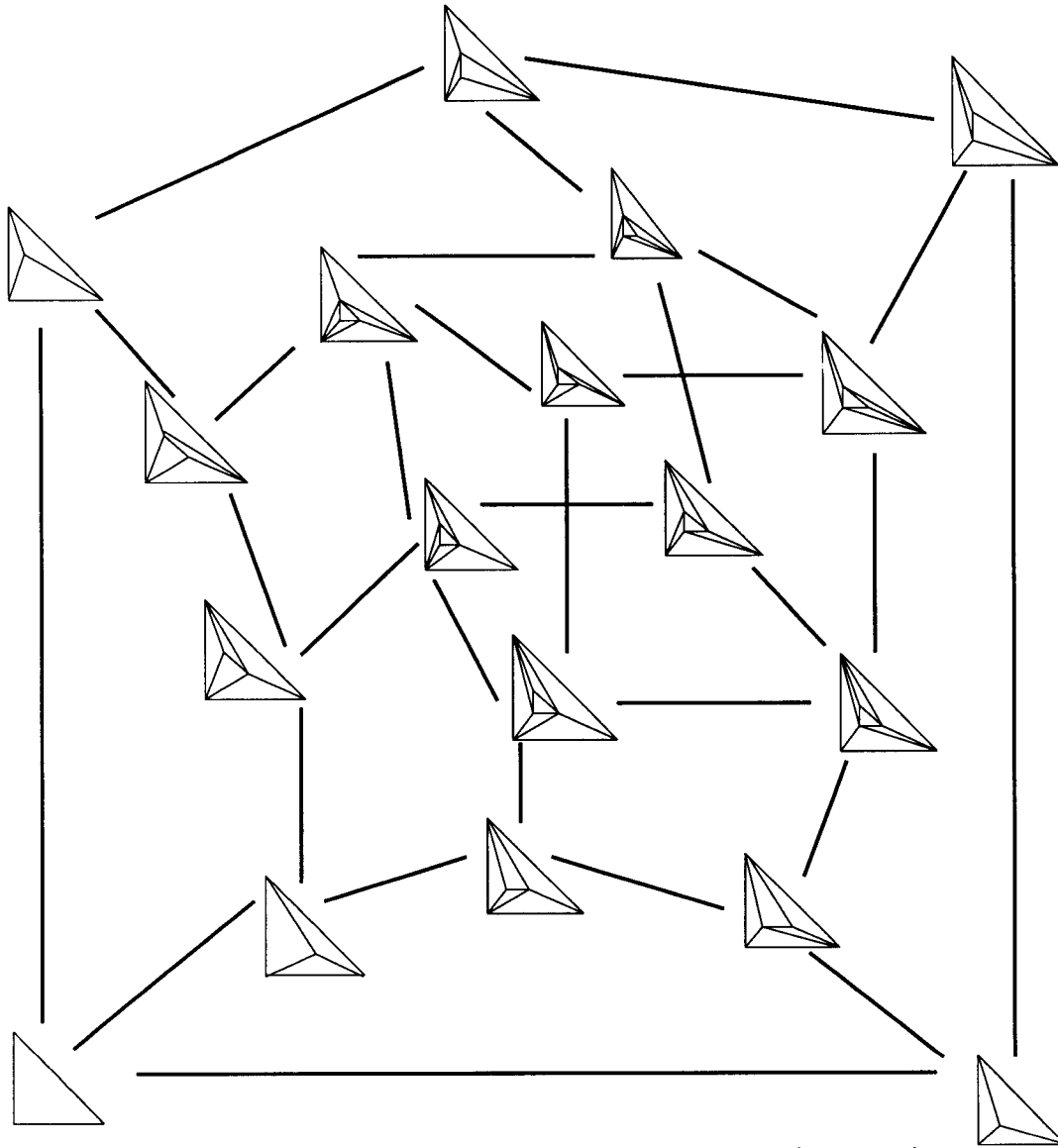


FIGURE 2. All triangulations and bistellar operations for a set A of 6 points in \mathbb{R}^2 . *parallel sides on triangles...*

produced a set of vertex coordinates for the vertices of this $(n-4)$ -sphere which embed it as the boundary complex of an $(n-3)$ -dimensional polytope. Unfortunately, the existence of this polytopal embedding seems to have been unknown in the combinatorial geometry community, and was rediscovered in the mid 1980's after Perles posed the problem of whether this complex was polytopal (see [41]). Independently, Haiman [33] and Lee [41] and constructed this polytope, which Haiman dubbed the *associahedron*. In the work of Gelfand, Kapranov, Zelevinsky et al [39, 31] it is sometimes called the *Stasheff polytope*. A recent preprint of Kapranov and Saito [38] documents its occurrence in other surprising geometric contexts.

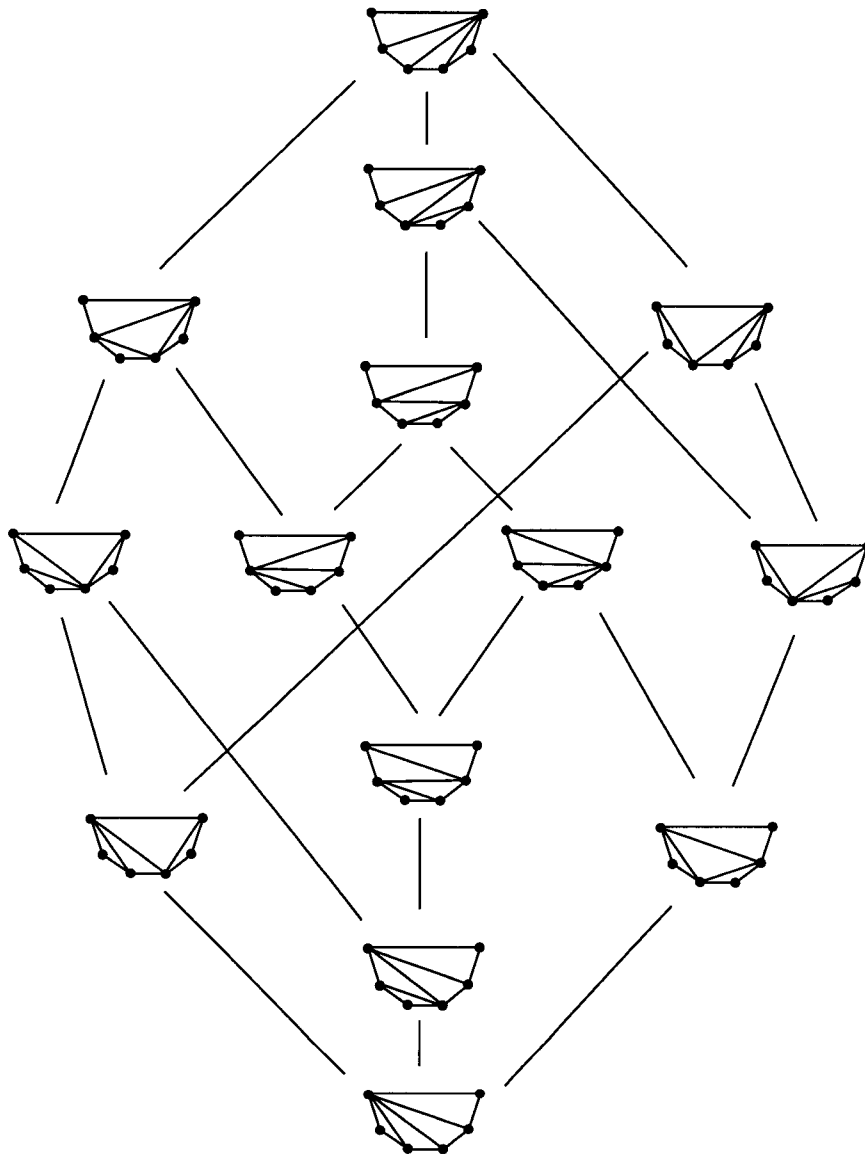


FIGURE 3. Triangulations of a convex 6-gon: the associahedron.

The associahedron also makes its appearance in computer science, where triangulations of an n -gon show up in the equivalent guise of *binary trees*, and bistellar operations correspond to an operation on binary trees called *rotation*. Here Sleator, Tarjan and Thurston [70] were able to determine the diameter of the 1-skeleton of the associahedron (it is at most $2n - 10$ for $n \geq 13$ and is exactly $2n - 10$ for infinitely many values of n). In a series of papers, Pallo [54, 55, 56, 57] studied computational aspects of this 1-skeleton and in particular computed the *Möbius function* (see [71, §3.7]) of the Tamari lattice.

In one of Lee's constructions of the associahedron, he employs the method of *Gale diagrams* [41, §4], and around the same time, Gelfand, Kapranov and Zelevinsky were using these methods for studying triangulations as part of their theory of *A-discriminants*, *A-resultants*, and *A-determinants* (see [31] and the references therein). Briefly, the principal *A-determinant* is a polynomial E_A in a variable set $\{c_a\}_{a \in A}$ indexed by A , which vanishes whenever the sparse d -variate polynomial in x_1, \dots, x_d

$$f := \sum_{a \in A} c_a \mathbf{x}^a$$

has a root (x_1, \dots, x_n) in common with all of the derived polynomials $x_1 \frac{\partial f}{\partial x_1}, \dots, x_d \frac{\partial f}{\partial x_d}$. Their work showed that the *Newton polytope* of E_A , that is, the set of exponent vectors in \mathbb{Z}^A of the non-zero monomials occurring in E_A , is an $(n-d-1)$ -dimensional polytope whose extreme vertices correspond to a subset of the triangulations of A called the *regular* (and later called *coherent*) triangulations. A triangulation T of A is *coherent* if there exists a choice of *heights* α_a in \mathbb{R} for each $a \in A$ which induces T in the following fashion: after "lifting" the points a in \mathbb{R}^d to the points $(a, \alpha_a) \in \mathbb{R}^{d+1}$ and taking the convex hull to form a polytopes P_α , the "lower" facets of P_α (i.e. those facets whose normal vector has negative $(d+1)$ -coordinate) project to the maximal simplices of T under the projection $\mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$. Figure 4 (borrowed from [60]) illustrates a coherent triangulation of a set A in \mathbb{R}^2 along with a choice of heights α which induces it. After seeing the definition, it is perhaps not obvious that one can have *incoherent* triangulations! However, the standard examples already occur in Figure 2, namely the two triangulations shown below. It is a non-trivial exercise to check the impossibility of assigning six heights to these points in such a way as to induce either of these triangulations. In general, checking whether a triangulation is coherent involves checking whether there exists a solution to a certain system of linear inequalities in the heights α_a , where the coefficients in the inequalities depend upon the coordinates of the points in A [44, §1.3].

Gelfand, Kapranov and Zelevinsky called the Newton polytope of E_A the *secondary polytope* $\Sigma(A)$. Knowing that the vertices of $\Sigma(A)$ correspond to the coherent triangulations of A , it is perhaps not surprising that the higher dimensional faces of $\Sigma(A)$ correspond to *coherent subdivisions*, that is, subdivisions into polytopes which are not necessarily simplices, but induced in a similar fashion by a choice of heights α_a for a in A .

Theorem 1. [31, Chapter 7, Theorem 2.4] *The faces of the secondary polytope $\Sigma(A)$ are indexed by the coherent subdivisions of A , and inclusion of faces of $\Sigma(A)$ corresponds to refinement of subdivisions.*

In particular, they showed that every bistellar operation between coherent triangulations corresponds to a coherent subdivision and hence forms an edge in the secondary polytope $\Sigma(A)$. This has a strong consequence: it implies that the subgraph of coherent triangulations and bistellar operations is connected (and even $(n-d-1)$ -vertex-connected in the graph-theoretic sense by Balinski's Theorem [83, 3.5]). Polytopality of $\Sigma(A)$ also has nice implications for computing the particular coherent triangulation induced by a choice of heights α_a , such as the *Delaunay triangulation* of A arising in computational geometry applications (see [27]).

Schönhardt
1928

E. Schönhardt

"Über die
Zerlegung von

Dreiecks polyedern
in Tetraeder"

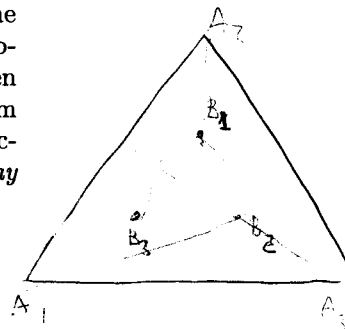
Mathematische
Annalen, 98,
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Connolly-
Henderson

"A convex 3-
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Math. Proc.
Camb. Phil.
Soc.
(1980), 88

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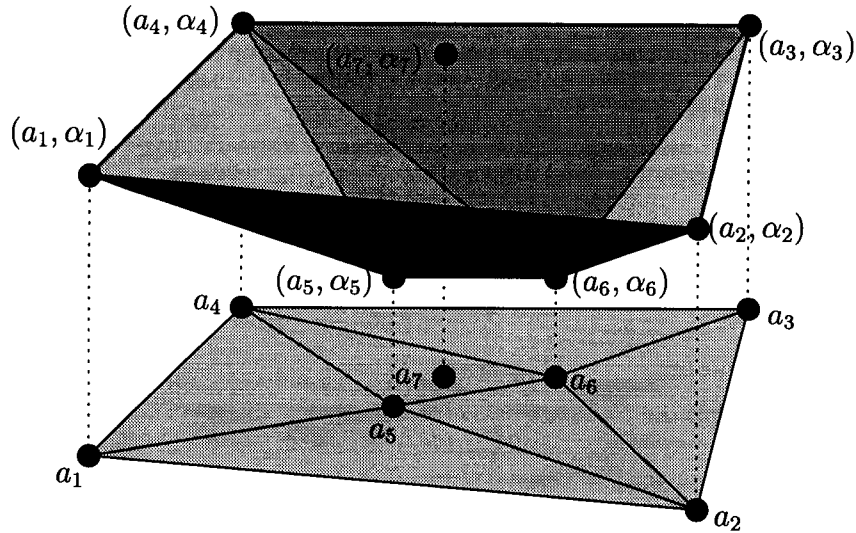


FIGURE 4. A coherent triangulation induced by a choice of heights.

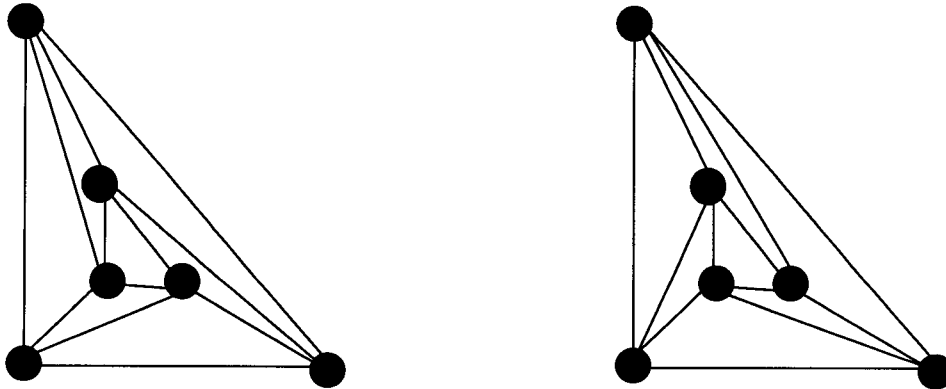


FIGURE 5. The two incoherent triangulations from Figure 2.

We remark that in the case where \mathcal{A} is the set of vertices of a convex n -gon, every subdivision is coherent, and hence the secondary polytope $\Sigma(\mathcal{A})$ is the associahedron encountered earlier.

The fact that the subgraph of coherent triangulations and bistellar operations is highly connected and forms the 1-skeleton of a cellular (even polytopal) sphere raises the following basic question:

Question 2.

Is the graph of all triangulations of \mathcal{A} and their bistellar operations connected?

A glance at Figure 2 illustrates that even in small cases where there are incoherent triangulations, the graph still appears to be connected. We can provide some motivation for the Generalized Baues Problem by performing the following mental exercise while staring at Figure 2. First picture the planar subgraph of coherent triangulations, by ignoring the two vertices corresponding to the incoherent triangulations in Figure 2 (call them T_1 and T_2). When one imagines this planar

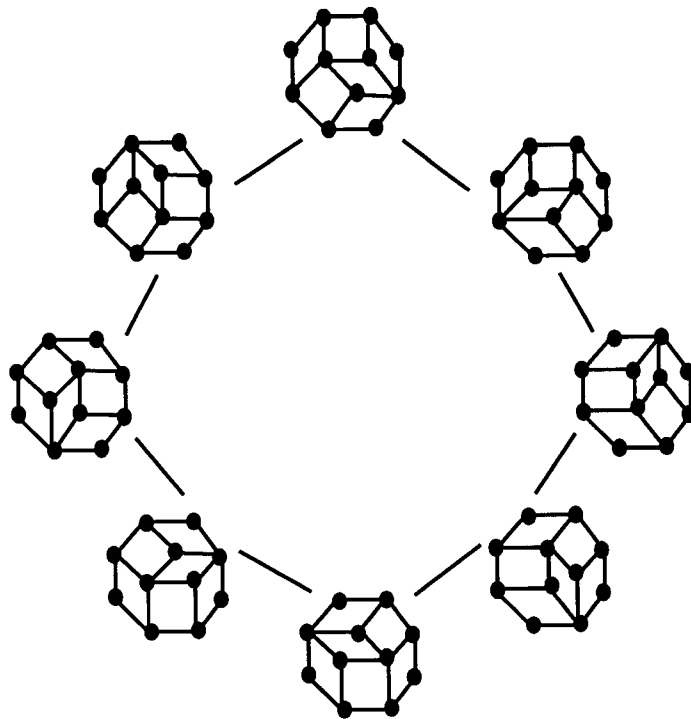


FIGURE 6. The rhombic tilings of an octagon.

subgraph as a 2-dimensional spherical cell complex, that is the boundary of the 3-dim secondary polytope $\Sigma(\mathcal{A})$, the union of the neighbors of T_1 and T_2 form the vertices of a hexagonal cell. Now “inflate” this hexagonal cell on the 2-sphere into a cubical 3-dimensional cell with the extra two vertices corresponding to T_1, T_2 . This gives a 3-dimensional cell complex which is still homotopy equivalent (but not homeomorphic) to a 2-sphere.

Roughly speaking, The Baues question in this context asks whether this behavior is general- Do the *incoherent* triangulations and subdivisions of \mathcal{A} attach themselves to the spherical boundary of $\Sigma(\mathcal{A})$ in such a way as to not change its homotopy type?

Zonotopal tilings.

Consider Figure 6, similar to [14, Figure 1], depicting the tilings of a centrally symmetric octagon having unit side lengths by unit rhombi. As in the case of triangulations of a point set, we have drawn in edges between the tilings corresponding to certain natural operations connecting them, illustrated in Figure 7. Similarly, the graph whose vertices are the tilings of a 10-gon and whose edges are these operations is depicted in Figure 8.

These operations have been given various names in the literature, depending upon the context in which the tilings arise. In the crystalline physics literature [21, 50], where the set of tilings is a model for the possible states of a crystalline solid, these moves are called *elementary flips* or *localized phasons*. Rather than

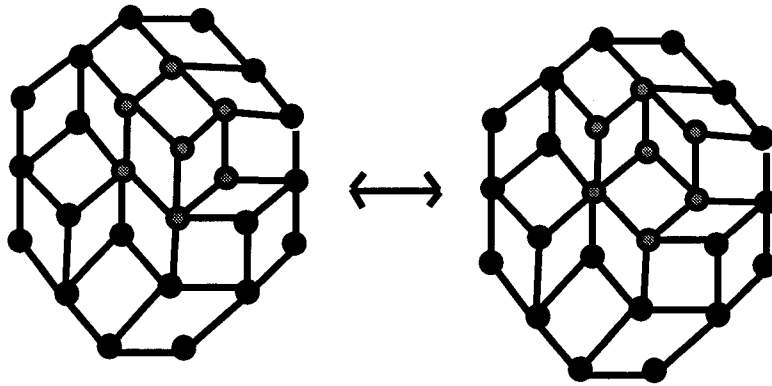


FIGURE 7. A typical cube-flip/ mutation/ triangle-switch/ 1-move/ braid-relation/ Yang-Baxter-relation/ elementary-flip/ localized-phason.

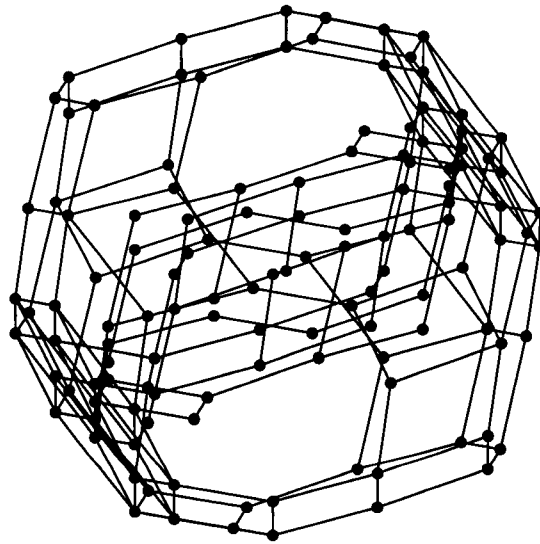


FIGURE 8. The graph of tilings of decagon (cf. [84, Figure 3]).

considering tilings of an $2n$ -gon, an equivalent (and useful) viewpoint comes from consideration of *arrangements of pseudolines* (see [18, Chapter 6] for definition, background and references). An arrangement of n affine pseudolines in the plane labelled $1, 2, \dots, n$ counterclockwise gives rise to a rhombic tiling of a centrally symmetric $2n$ -gon which is “dual” to the line arrangement in the sense of planar maps; see Figure 9.

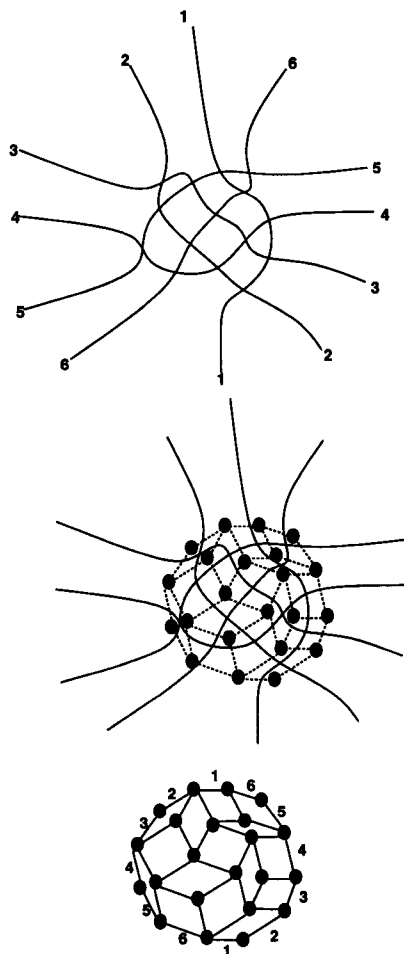


FIGURE 9. A configuration of affine pseudolines and its associated tiling

In the pseudoline picture, the move depicted in Figure 7 corresponds to moving one pseudoline locally across the nearby crossing point of two other pseudolines; such moves are often called *mutations* or *triangle-switches* or *1-moves*. When one thinks of such a pseudoline arrangement as a degenerate *braid diagram* recording a *reduced decomposition* of a permutation (see [18, §6.4]), such moves are sometimes called *braid relations* or *Yang-Baxter relations*.

Rather than restricting our attention to tilings of centrally symmetric polygons, we can more generally consider the set of *zonotopal subdivisions* of a *zonotope*. A *zonotope* Z in \mathbb{R}^d is the Minkowski sum of a set V of line segments in \mathbb{R}^d , and a *zonotopal subdivision* of Z is, roughly speaking, a subdivision of Z into smaller zonotopes, each a translate of a zonotope generated by a subset of the V , and which intersect pairwise along common faces (possibly empty). The subdivision is *cubical* if it is as refined as possible, that is each smaller zonotope in the subdivision is a translate of a cube generated by a linearly independent subset of V . In the case where Z is a centrally symmetric $2n$ -gon, V is a set of n line segments whose slopes match the slopes of the polygon edges. Cubical tilings in this case coincide with

the rhombic tilings depicted earlier, and the “cube-flip” moves which formed the edges in the graphs of Figures 6 and 8 correspond to zonotopal subdivisions of Z in which all of the smaller zonotopes are cubes except for one which is hexagonal.

Note that the graph of tilings in Figure 6 is circular, and the graph of tilings in Figure 8 appears to be planar and possibly even polytopal. This reflects the fact that for centrally symmetric octagons and decagons, all zonotopal subdivisions are *coherent* in a sense which will be described below. A special case of Billera and Sturmfels fiber polytope construction [14, §5] states that the subset of coherent zonotopal subdivisions of a d -dimensional zonotope having n generators index the faces an $(n-d)$ -dimensional polytope (which happens to itself be a zonotope). Thus the graphs in Figures 6 and 8 are the 1-skeleta of these *fiber zonotopes*.

Coherence of a zonotopal subdivision is defined similarly to coherence of a triangulation. A zonotopal subdivision T of a zonotope Z in \mathbb{R}^d having generating line segments V is *coherent* if there exists a choice of segments \hat{V} in \mathbb{R}^{d+1} which project down to V under the forgetful projection $\mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ and induce T in the following fashion: the “upper facets” of the zonotope \hat{Z} generated by \hat{V} project to the maximal simplices of T under the map $\mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$. An example is shown in Figure 10. Again, it is not obvious that incoherent zonotopal subdivisions can exist, but it can be shown that, for example, that the tiling of a 12-gon depicted in Figure 1 is incoherent for certain choices of the slopes of edges in the 12-gon. As with coherence of triangulations, checking coherence of a particular tiling is a problem of existence of a solution to a system of linear inequalities, and the system of inequalities in this case strongly depends upon the slopes of the segments V (although not upon the length of these segments). Again as in the case of triangulations, the fact that the graph of coherent tilings and cube-flips is the 1-skeleton of a polytope has strong consequences for its connectivity. This raises the analogous question to Question 2

Question 3.

Is the graph of all cubical tilings of a zonotope and their cube-flips connected?

One can also view coherence of 2-dimensional tilings in terms of pseudolines and straight lines. A coherent tiling is one whose pseudoline arrangement is isomorphic to a (straight) line arrangement in which each line has slope perpendicular to the slope of the edge in the polygon to which it corresponds (that is, to the edge of the polygon labelled with the same number in Figure 9). Some of this viewpoint is explained in the instructions for the delightful puzzle *Hexa-Grid* [34], which supplies foam rubber versions of the rhombic tiles occurring in Figure 9, and asks the consumer to assemble them into a tiling of a zonotopal 12-gon!

In studying tilings and zonotopal subdivisions of higher dimensional zonotopes, the oriented matroid point of view has become indispensable (see [18, §2.2]). The Bohné-Dress Theorem [20, 64] states that zonotopal subdivisions of Z biject with the *single-element liftings* of the *realized* oriented matroid \mathcal{M} associated with the generating segments V , or using oriented matroid duality, to the *single-element extensions* of the dual oriented matroid \mathcal{M}^* (see [18, §7.1]). From this point of view, the subset of coherent zonotopal subdivisions of Z corresponds to the *coherent liftings* of V [14, §5]. If one views realized oriented matroids and their liftings in terms of sphere and pseudosphere arrangements, then the notion of a coherent lifting was explored in the work of Bayer and Brandt [8] on *discriminantal arrangements*,

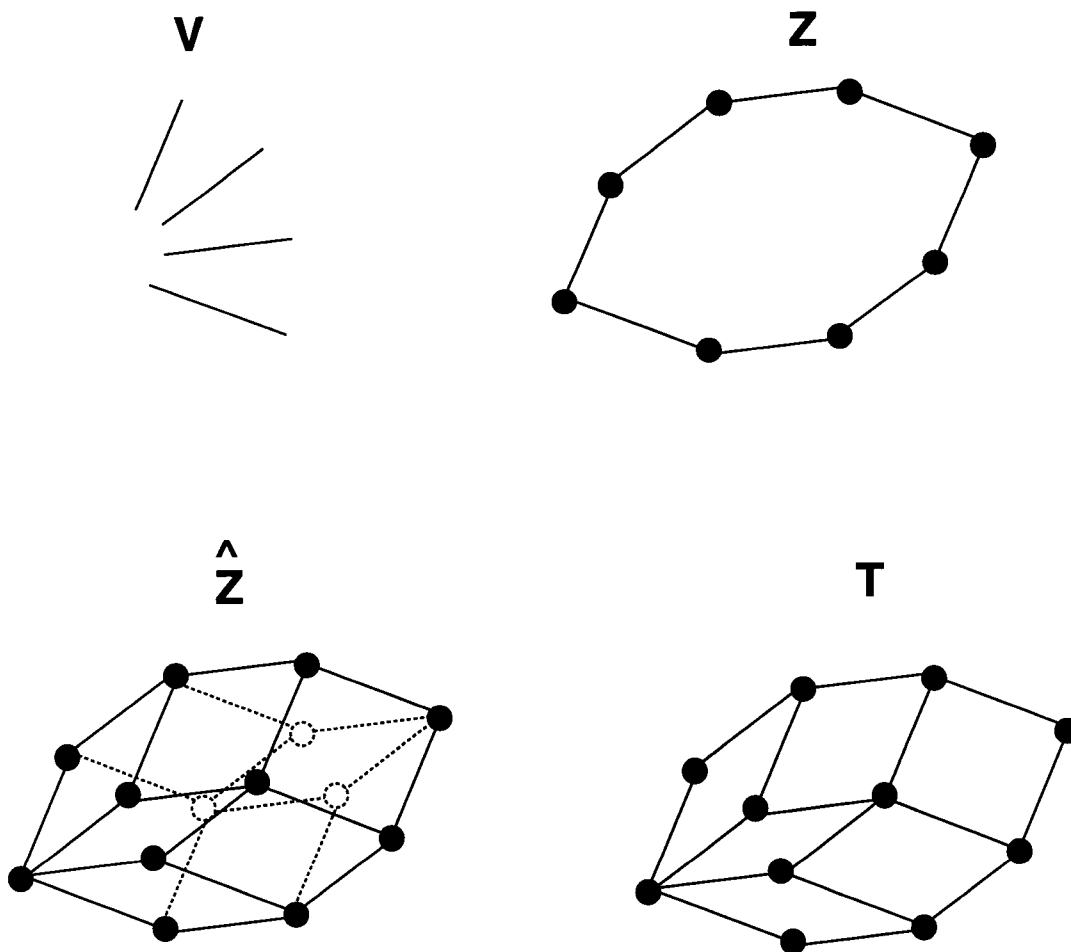


FIGURE 10. A coherent tiling T , induced by lifting into \mathbb{R}^3 the generating segments V of the 2-dimensional zonotope Z , then projecting the upper facets of the resulting 3-dimensional zonotope \hat{Z} back into the plane.

generalizing earlier work of Manin and Schechtman [49]. The discriminantal arrangement associated to Z in [8] is nothing more than the hyperplane arrangement which is the polar dual of the fiber zonotope associated to Z in [14].

Monotone paths

Let P be a polytope in \mathbb{R}^d , and f a linear functional in $(\mathbb{R}^d)^*$ that achieves its minimum and maximum values uniquely on P , say at two vertices v_{min}, v_{max} . A path from v_{min} to v_{max} in the 1-skeleton of the boundary of P will be called *f-monotone* if every step in the path is along an edge which strictly increases the value of f (as in the paths produced by the *simplex algorithm* for linear programming- see [83, Lecture 3.2]). We wish to consider the structure of the set of all *f-monotone* paths. Note that these paths are exactly the subject of Ziegler's *strict monotone Hirsch conjecture* [83, Conjecture 3.9]:

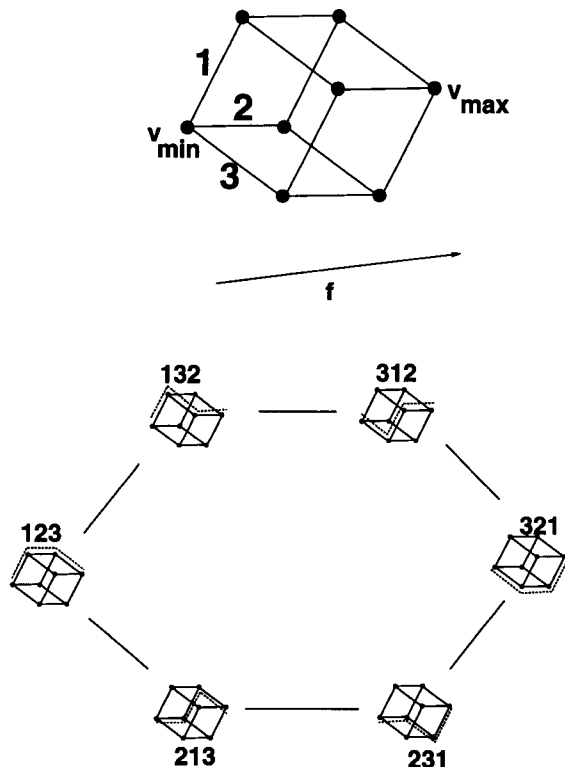


FIGURE 11. Monotone paths in the 3-cube. Edges pointing in the original three coordinate directions have been labelled 1, 2, 3, and then monotone paths are labelled by the sequence of directions of the steps

Conjecture 4. *If P has n facets, then there always exists an f -monotone path from v_{\min} to v_{\max} of length at most $n - d$.*

This is stronger than the usual Hirsch conjecture, and it is hoped that further understanding of f -monotone paths may eventually shed light on this notoriously difficult question.

Just as in the case of triangulations of \mathcal{A} or tilings of a zonotope Z , there is a natural set of moves which connect f -monotone paths: if two paths agree in most of their steps and differ only by following opposite paths around some 2-dimensional face of P , we say that the two paths differ by a *polygon move*. Figure 11 illustrates the graph of f -monotone paths and polygon moves where P is the 3-cube $[0, 1]^3 \subset \mathbb{R}^3$ and $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$.

In general if P is the n -cube $[0, 1]^n \subset \mathbb{R}^n$ and $f(\mathbf{x}) = \sum_{i=1}^n x_i$, the f -monotone paths biject with permutations of $\{1, 2, \dots, n\}$: one obtains a permutation by recording which coordinate axis is parallel to each step of the path in sequence, as in Figure 11. The polygon moves across square faces of the n -cube then correspond to *adjacent transpositions* of the permutations, and the whole graph is isomorphic to the 1-skeleton of the well-known *permutohedron* [83, Example 0.10].

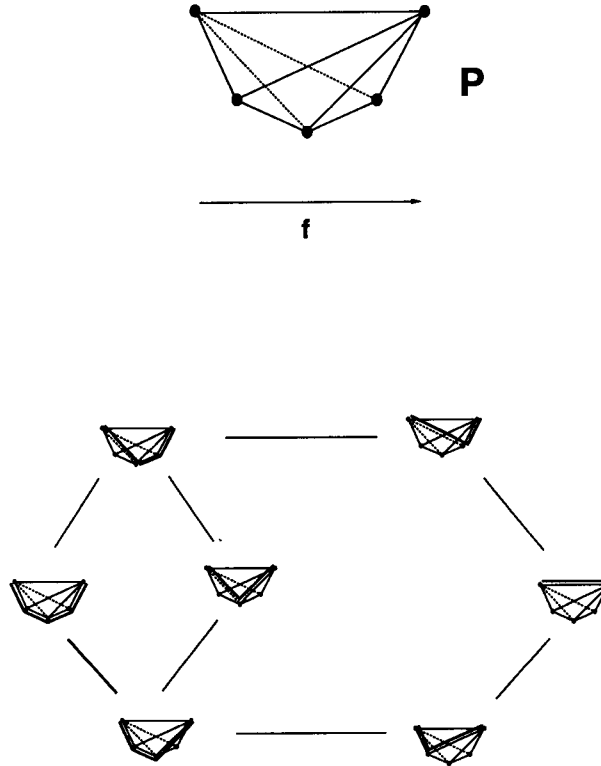


FIGURE 12. Monotone paths in a cyclic 3-polytope with 5-vertices

What happens for other polytopes P and functionals f ? Figure 12 shows the graph of f -monotone paths in a (cyclic) 3-polytope with 5 vertices in \mathbb{R}^3 which is the convex hull of the points $\{(t, t^2, t^3) : t = -2, -1, 0, 1, 2\}$, and $f(x_1, x_2, x_3) = x_1$. Although this graph is connected, it is perhaps disappointing that it is not circular as in the case of Figure 11. Once again, geometry comes to the rescue in singling out a well-behaved subset of f -monotone paths. Say that an f -monotone path γ on P is *coherent* if there exists some linear functional $g \in (\mathbb{R}^d)$ which induces γ in the following way: each point of γ is the g -maximal point among all those points of P with the same f value, or in other words, γ is the union over all points x in $f(P) \subset \mathbb{R}$ of the g -maximal points in the fibers $f^{-1}(x)$. With this definition, the monotone path in the middle of the graph in Figure 12 is incoherent. To see this, assume there is some functional g inducing this monotone path, and identify g as the dot product with some fixed vector. Then this vector must point roughly toward the front (the visible side) of the polytope P in order to induce the right portion of the path, but also point toward the back (the invisible side) in order to induce the left portion of the path; contradiction. The remaining six paths in Figure 12 are easily seen to be coherent (by imagining appropriate functionals g) and the subgraph on the corresponding six vertices is indeed circular.

In general, it follows as a special case of the fiber polytope construction [14, §7] that the graph of f -monotone paths in a polytope P is the 1-skeleton of a polytope called the *monotone path polytope*. Higher dimensional faces of the monotone path polytope correspond to objects called *coherent cellular strings* on P with respect to

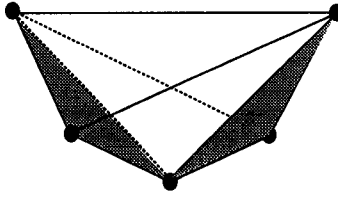


FIGURE 13. An incoherent cellular string.

f . A *cellular string* on P with respect to f is a sequence (F_1, \dots, F_r) of boundary faces of P with the property that $v_{\min} \in F_1, v_{\max} \in F_r$ and for each i the f -maximizing face of F_i is the f -minimizing face of F_{i+1} . A cellular string (F_1, \dots, F_r) is *coherent* if there exists some functional $g \in (\mathbb{R}^d)^*$ such that union $\bigcup_{i=1}^r F_i$ equals the union over all points x in $f(P)$ of the g -maximal points in the fibers $f^{-1}(x)$.

As an exercise to get a feeling for how the set of all cellular strings fit into the graph of f -monotone paths, and to further motivate the Baues problem, we invite the reader to try the following labelling exercise. Label each edge in the graph of Figure 12 by a cellular string containing mostly 1-faces along with exactly one triangular 2-face corresponding to the polygon move for that edge. Having done this, there is only one other possible cellular string, consisting of two triangles and pictured in Figure 13. This cellular string should label a square 2-cell attached to the four leftmost vertices and edges in Figure 12 (see also the middle picture in Figure 14). Notice that the resulting 2-dimensional cell complex is homotopy equivalent to the circular subgraph indexed by coherent cellular strings. This raises the following question.

Question 5.

For a polytope P and functional f , is the graph of cellular strings and their polygon moves connected? Is it part of a complex homotopy equivalent to a $(d-1)$ -sphere?

This question includes the original question asked by Baues [7] as a special case—see [60, §1.2] for a nice sketch of the ideas involved. Specifically, Baues asked if the poset of cellular strings on the *permutohedron* with respect to a generic linear functional f has the homotopy type of a sphere (after endowing the poset with a certain topology—see Section 2). As we will see in Section 4, this original problem was resolved positively for cellular strings more generally, both by the work of Björner [16] and by Billera, Kapranov, and Sturmfels [13].

2. FIBER POLYTOPES AND THE BAUES PROBLEM

The theory of fiber polytopes [14] provides a common framework in which to discuss triangulations, tilings, and monotone paths, and also a common notion of coherence for these objects. The fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$, is a polytope naturally associated to any linear projection of polytopes $\pi : P \rightarrow Q$. Let P be a d' -dimensional polytope in $\mathbb{R}^{d'}$, Q a d -dimensional polytope in \mathbb{R}^d and $\pi : \mathbb{R}^{d'} \rightarrow \mathbb{R}^d$ a linear map with $\pi(P) = Q$. A *polytopal subdivision* of Q is a polytopal complex which subdivides Q . A polytopal subdivision of Q is π -*induced* if

- (i) it is of the form $\{\pi(F) : F \in \mathcal{F}\}$ for some specified collection \mathcal{F} of faces of P having all $\pi(F)$ distinct, and
- (ii) $\pi(F) \subseteq \pi(F')$ implies $F = F' \cap \pi^{-1}(\pi(F))$, and in particular $F \subseteq F'$.

It is possible that different collections \mathcal{F} of faces of P project to the same subdivision $\{\pi(F) : F \in \mathcal{F}\}$ of Q , so we distinguish these subdivisions by labelling them with the family \mathcal{F} . We partially order the π -induced subdivisions of Q by $\mathcal{F}_1 \leq \mathcal{F}_2$ if and only if $\bigcup \mathcal{F}_1 \subseteq \bigcup \mathcal{F}_2$. The resulting partially ordered set is denoted by $\omega(P \xrightarrow{\pi} Q)$ and called the *Baues poset*. The minimal elements in this poset are the *tight* subdivisions, that is those for which F and $\pi(F)$ have the same dimension for all F in \mathcal{F} .

We claim that π -induced subdivisions of Q generalize triangulations, tilings, and monotone paths. This is perhaps easiest to see for monotone paths and cellular strings. Given a polytope P and functional f , let Q be the 1-dimensional polytope $f(P)$ in \mathbb{R}^1 . Then a cellular string (F_1, \dots, F_r) on P with respect to f gives a family $\mathcal{F} = \{F_i\}_{i=1}^r$ satisfying the definition for a π -induced subdivision of Q . Tight π -induced subdivisions of Q correspond to monotone paths in P .

For triangulations and tilings, there is a concealed projection of polytopes lurking in the background. Given a point set \mathcal{A} in \mathbb{R}^d with cardinality n , let Q denote its convex hull. There is a natural surjection $\pi : \Delta^{n-1} \rightarrow Q$ from a simplex Δ^{n-1} having n vertices, which sends each vertex of the simplex to one of the points of \mathcal{A} . Given a triangulation, or more generally a subdivision of \mathcal{A} , each polytope in the subdivision is the projection of some face F of the simplex, and the collection \mathcal{F} of these faces gives a π -induced polytopal subdivision of Q . What we have just said is not quite precise, since it is possible for different faces of the simplex to project to the same polytope inside Q . To remedy this, we revise our original naive definition of a *subdivision* of \mathcal{A} ; subdivisions of \mathcal{A} should be defined to be a collection of pairs $\{(Q_\alpha, \mathcal{A}_\alpha)\}$ where

- \mathcal{A}_α are subsets of \mathcal{A} ,
- each Q_α is the convex hull of \mathcal{A}_α and is d -dimensional,
- the union of the Q_α covers Q ,
- for any α, β , the intersection of Q_α, Q_β is a face F (possibly empty) of each, and $\mathcal{A}_\alpha \cap F = \mathcal{A}_\beta \cap F$.

Then the collection $\mathcal{F} = \{F_\alpha\}$ of faces of the simplex spanned by the vertices corresponding to the \mathcal{A}_α satisfies the definition of a π -induced subdivision of Q . Tight π -induced subdivisions of Q correspond to triangulations of \mathcal{A} . Furthermore, the Baues poset corresponds to the natural *refinement* ordering on subdivisions of \mathcal{A} : $\{(Q_\alpha, \mathcal{A}_\alpha)\} \leq \{(Q'_\beta, \mathcal{A}'_\beta)\}$ if and only if for every α there exists some β with $\mathcal{A}_\alpha \subseteq \mathcal{A}'_\beta$ (and hence also $Q_\alpha \subseteq Q'_\beta$).

Given a zonotope Z in \mathbb{R}^d generated by n line segments V , without loss of generality we may assume that these segments all have one endpoint at the origin, and we can think of them as vectors pointing in a certain direction rather than segments. There is then a natural surjection $\pi : I^n \rightarrow Z$ of the n -cube I^n in \mathbb{R}^n onto Z which sends the standard basis vectors in \mathbb{R}^n onto the vectors V . Given a cubical tiling, or more generally a zonotopal subdivision of Z , each zonotope in the subdivision is the projection of some face F of the cube, and the collection \mathcal{F} of these faces gives a π -induced polytopal subdivision of Z . Just as in the previous paragraph, what we have just said is not quite precise, since it is possible for different faces of the cube to project to the same zonotope inside Z . However, one can again remedy this by revising the naive notion of a zonotopal subdivision- one nice approach by Richter-Gebert and Ziegler uses the equivalent notions of *strong and weak zonotopal tilings* (see [64, Definitions 1.3, 1.4]). Tight π -induced subdivisions

then correspond to cubical tilings of Z , and the Baues poset corresponds to a natural *refinement* ordering on zonotopal subdivisions. As was remarked in Section 1, the Bohne-Dress Theorem shows that the zonotopal subdivisions, or equivalently π -induced subdivisions of Z , are the same as single-element liftings of the realized oriented matroid \mathcal{M} corresponding to the vectors V . It is furthermore true that the Baues poset corresponds to the usual *weak map* ordering [18, §7.2] on single-element liftings of \mathcal{M} , or equivalently on the single-element extensions of the dual \mathcal{M}^* .

Returning to our general set-up of a projection $\pi : P \rightarrow Q$, we wish to define when a π -induced subdivision is π -coherent, generalizing the notion of coherence for triangulations, tilings, and monotone paths. There is more than one way to do this, and we start with the original definition from [14]. Choose a linear functional $g \in (\mathbb{R}^d)^*$. For each point q in Q , the fiber $\pi^{-1}(q)$ is a convex polytope which has a unique face \overline{F}_q on which the value of f is minimized. This face lies in the relative interior of a unique face F_q of P and the collection of faces $\mathcal{F} = \{F_q\}_{q \in Q}$ projects under π to a subdivision of Q . Subdivisions of Q which arise from a functional g in this fashion are called π -coherent. Note that this definition of π -coherence clearly generalizes our earlier notion of coherence for cellular strings on P .

In [83, §9.1], Ziegler defines π -coherent subdivisions in the following equivalent fashion. Having chosen the functional $g \in (\mathbb{R}^d)^*$ as above, form the graph of the linear map $\hat{\pi} : P \rightarrow \mathbb{R}^{d+1}$ given by $p \mapsto (\pi(p), g(p))$. The image of this map is a polytope \hat{Q} in \mathbb{R}^{d+1} which maps onto Q under the forgetful map $\mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$. Therefore, the set of *lower faces* of \hat{Q} (those faces whose normal cone contains a vector with negative last coordinate) form a polytopal subdivision of Q . We identify this subdivision of Q with the family of faces $\mathcal{F} = \{F\}$ in P which are the inverse images under $\hat{\pi}$ of the lower faces of \hat{Q} . Under this identification, it is not hard to check that the subdivision of Q is exactly the same as the π -coherent subdivision induced by g , described in the previous paragraph. A glance at the definitions shows that this second definition of π -coherence generalizes the ones we gave for coherent subdivisions of a point set \mathcal{A} and for coherent zonotopal subdivisions of a zonotope Z .

Let $\omega_{coh}(P \xrightarrow{\pi} Q)$ denote the subposet of the Baues poset $\omega(P \xrightarrow{\pi} Q)$ consisting of the π -coherent subdivisions of Q . The beautiful result of Billera and Sturmfels which explains all of our pretty polytopal pictures is the following:

Theorem 6. [14, Theorem 3.1] *Let P be a d' -polytope, Q a d -polytope, and $\pi : P \rightarrow Q$ a linear surjection. Then the poset $\omega_{coh}(P \xrightarrow{\pi} Q)$ is the face poset of a $(d' - d)$ -polytope $\Sigma(P \xrightarrow{\pi} Q)$.*

In particular, the tight π -coherent subdivisions of Q correspond to the vertices of $\Sigma(P \xrightarrow{\pi} Q)$.

The $(d' - d)$ -polytope $\Sigma(P \xrightarrow{\pi} Q)$ is called the *fiber polytope* of the surjection π . It generalizes the *secondary polytopes* $\Sigma(\mathcal{A})$, *fiber zonotopes*, and *monotone path polytopes* encountered in Section 1. A striking feature of $\Sigma(P \xrightarrow{\pi} Q)$ is that it can be constructed as the ‘‘Minkowski average’’ over points $q \in Q$ (in a well-defined sense; see [14, §2]) of all of the polytopal fibers $\pi^{-1}(q)$.

As a consequence of Theorem 6, if one removes the top element $\hat{1}$ from $\omega_{coh}(P \xrightarrow{\pi} Q)$, corresponding to the improper π -coherent subdivision $\mathcal{F} = \{P\}$, one obtains the face poset of a polytopal $(d' - d - 1)$ -sphere, that is the boundary of $\Sigma(P \xrightarrow{\pi} Q)$. The generalized Baues problem asks roughly how close the whole Baues poset

$\omega(P \xrightarrow{\pi} Q) - \hat{1}$ is to this sphere. Before phrasing the problem precisely, we must first give the poset $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ a topology. The standard way to do this is to consider its *order complex*, the abstract simplicial complex of chains in the poset [17, (9.3)]. We will abuse notation and use the name of any poset also to refer to the topological space which is the geometric realization of its order complex.

We can now state the *Generalized Baues Problem*, in at least two forms, one stronger than the other.

Question 7. (Weak GBP)

Is $\omega_{coh}(P \xrightarrow{\pi} Q) - \hat{1}$ homotopy equivalent to a $(d' - d - 1)$ -sphere ?

Question 8. (Strong GBP)

Is the inclusion

$$\omega_{coh}(P \xrightarrow{\pi} Q) - \hat{1} \hookrightarrow \omega(P \xrightarrow{\pi} Q) - \hat{1}$$

a strong deformation retraction?

The strong GBP captures the sense we had from Figures 2 and 12 that the incoherent subdivisions were nothing more than “warts” attached to the spherical subcomplex indexed by the coherent subdivisions, and that these warts could be easily retracted onto this subcomplex. We should beware however that these pictures of small examples can be deceptive. In particular, we mention a vague meta-conjecture that has several examples of empirical evidence (see [2], [43], and [3, Remark 3.6]):

Vague meta-conjecture: *In most “naturally occurring” infinite families of polytope surjections $P_n \rightarrow Q_n$ for which either*

- $\dim(Q) \rightarrow \infty$, or
- $\dim(P) - \dim(Q) \rightarrow \infty$,

the fraction of the number of π -coherent subdivisions out of the total number of π -induced subdivisions approaches 0.

In other words, the warts take over eventually.

Besides the weak and strong versions, one can imagine other intermediate versions of the GBP. For example, one might ask whether the inclusion referred to in the strong GBP induces only a homotopy equivalence, rather than the stronger property of being a deformation retraction. We will resist naming these other versions, since they seem not to have been addressed in the literature.

Knowing that $\omega_{coh}(P \xrightarrow{\pi} Q)$ is the poset of faces of a (polytopal) cell complex, the reader may be disappointed that we have not defined the entire Baues poset $\omega(P \xrightarrow{\pi} Q)$ to be the poset of faces in some cell complex, since it appears to be so in all of our small examples. For example, Figure 14 shows the order complex of $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ for the example in Figure 12, which turns out to be the *barycentric subdivision* of the cell complex one would have liked to call “the Baues complex”. Whenever such a cell complex exists, then of course, the order complex of $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ will be its barycentric subdivision, and hence be homeomorphic. Unfortunately, such a cell complex does not exist in general; relatively small examples show that lower intervals in $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ need not be *homeomorphic* to spheres, which is the necessary condition for a poset to be the poset of faces of a regular cell complex [17, (12.5)].

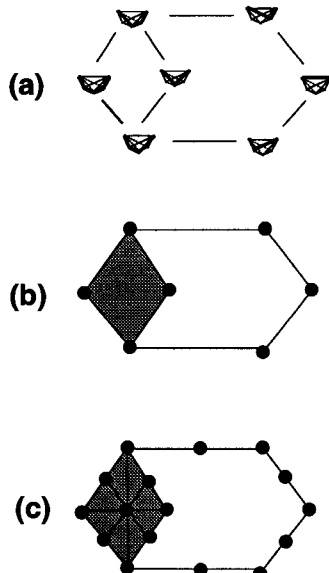


FIGURE 14. (a) Monotone paths as in Figure 12. (b) A cell complex which happens to have $\omega(P \xrightarrow{\pi} Q) - \hat{1}$ as its poset of faces. (c) The order complex of $\omega(P \xrightarrow{\pi} Q) - \hat{1}$

3. RELATIONS TO OTHER PROBLEMS

Having stated the GBP, we can now explain how it relates to some of our previous questions, and to other problems in discrete geometry and topological combinatorics.

Connectivity questions

Questions 2, 3, 5 are clearly related to the GBP, and appear at first glance to be weaker, in that they only ask for connectivity of a certain graph rather than homotopy sphericity of a complex. However, a positive answer to the strong GBP does not quite imply a positive answer to either of these questions. There are at least two difficulties with this conclusion, which we will now attempt to make precise.

For an element of a finite poset, let its *rank* be the length of the shortest saturated chain below it in the poset, so that minimal elements have rank 0. Let A_π, B_π denote the elements at rank 0 and rank 1 respectively in the Baues poset $\omega(P \xrightarrow{\pi} Q)$, and let G_π be the graph on the union $A_\pi \cup B_\pi$ obtained by restricting the Hasse diagram for $\omega(P \xrightarrow{\pi} Q)$ to this union of bottom two ranks. Given a point set \mathcal{A} , let $G_{\mathcal{A}}$ denote its graph of triangulations and bistellar operations. Similarly, for a zonotope Z , let G_Z be its graph of cubical tilings and cube-flips, and for a polytope P with a linear functional f , let $G_{P,f}$ be the graph of monotone paths and polygon moves.

The first difficulty we encounter is the relation between the graphs $G_{\mathcal{A}}, G_Z, G_{P,f}$ and the graph G_π . It is tempting to say that the barycentric subdivision of $G_{\mathcal{A}}, G_Z$, or $G_{P,f}$ is the same as G_π for the appropriate map π , since these barycentric subdivisions will certainly be edge-subgraphs of G_π . However, it is not known that these graphs coincide. In the case of zonotopes Z , it is known that every zonotopal

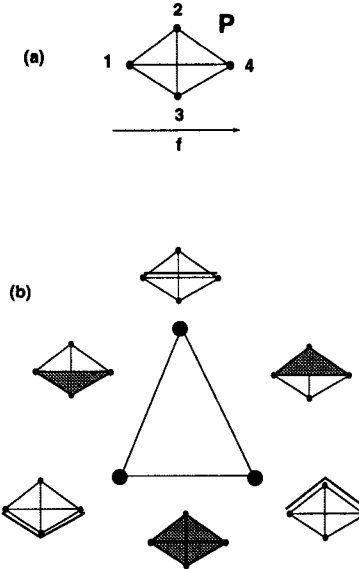


FIGURE 15. (a) A tetrahedron P with a non-generic functional f for which vertices 2,3 have the same f -value. (b) The boundary complex of the fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$, whose face poset coincides with the poset of all cellular strings (all are coherent). Note that the cellular string $\mathcal{F} = \{1, 123, 23, 234, 4\}$ labelling the bottom edge of this complex does not correspond to a polygon move.

subdivision can be refined to a cubical tiling [18, Corollary 7.7.9], and in the case of a polytope P and functional f , one can check directly that every cellular string can be refined to a monotone path. Therefore A_π really does correspond to the set of cubical tilings or monotone paths in these special cases. However, in the case of a point set \mathcal{A} , it is not known that every subdivision can be refined to a triangulation, so that A_π could conceivably contain subdivisions which are not triangulations! Beyond this, one still has the question of whether the elements B_π at rank 1 always correspond to bistellar operations, cube-flips, and polygon moves, respectively. However, under certain genericity assumptions (for triangulations: the points of \mathcal{A} lie in general position in \mathbb{R}^d ; for monotone paths: the functional f is not constant along any edge of P) it is easy to check that union $A_\pi \cup B_\pi$ is exactly the graph of bistellar moves or polygon moves respectively. Note that this is certainly false for the case of cellular strings when we make no genericity assumption, as illustrated by the example in Figure 15.

We summarize our lack of knowledge here:

Question 9.

- (a) Can every subdivision of a point set be refined to a triangulation?
- (b) Are the elements of rank 1 in the subdivision poset for \mathcal{A} always bistellar operations?
- (c) Are the elements of rank 1 in the poset of zonotopal tilings of a zonotope always cube-flips?

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The second difficulty arises from the fact that even in cases where the GBP has a positive answer, connectivity of $\omega(P \xrightarrow{\pi} Q)$ does not necessarily imply connectivity of the graph G_π , since the 1-skeleton of $\omega(P \xrightarrow{\pi} Q)$ (or rather its order complex) contains some vertices corresponding to poset elements with ranks higher than 0, 1. On the other hand, one would like to be able to apply the following easily verified lemma to $\omega(P \xrightarrow{\pi} Q)$:

Lemma 10. *Let X be a poset with a top element $\hat{1}$, and assume that X has the property that every strict principal order ideal $X_{<x} := \{x' \in X : x' < x\}$ is connected if it is non-empty.*

Then the graph obtained by restricting X to its elements at rank 0 and 1 is connected.

Of course the GBP only implies the hypotheses of this lemma are satisfied with $X = \omega(P \xrightarrow{\pi} Q)$ for the strict principal order ideal $X_{<\hat{1}}$. But there is some hope that if one could prove the weak GBP in some case, then one can also prove homotopy sparsity for the rest of the order ideals $X_{<x}$, and hence use the lemma. Under the genericity assumptions which were mentioned above for triangulations and monotone paths, one can check that these principal order ideals are Cartesian products of Baues posets for smaller polytopes, and hence their connectivity follows from positive answers to the GBP for these smaller polytopes. In particular, the positive answer for the strong GBP for monotone paths [13] (to be discussed in the next section) implies a positive answer to all of Question 5 under the assumption that the functional f is generic. Without such genericity assumptions, the structure of these principal order ideals may be more complicated. A specific study of these principal order ideals in the case of triangulations of a point set \mathcal{A} was initiated by Santos [68].

Flip deficiency.

While we are discussing Questions 2, 3, and 5, it is appropriate to mention questions about the number of bistellar neighbors of a triangulation, the number of cube-flip neighbors of a tiling, and the number of polygon-move neighbors of a monotone path. In the general setting of $\pi : P \rightarrow Q$, every tight π -coherent subdivision of Q represents a vertex of the $(d' - d)$ -polytope $\Sigma(P \xrightarrow{\pi} Q)$, and therefore will have at least $d' - d$ neighboring tight π -coherent subdivisions lying along the edges of the polytope. On the other hand, π -induced subdivisions which are not π -coherent may have fewer neighbors, in which case we will say that the subdivision in question has *flip deficiency*. If the subdivision has no neighbors we say that it is *isolated*, which of course gives a negative answer to the GBP if $d' - d > 1$ in that case. Note that the example with all coherent cellular strings in Figure 15 shows that for monotone paths, we must either be careful to restrict ourselves to the case of a generic functional f , or else redefine what is meant by a “polygon-move” in talking about flip-deficiency.

Flip deficiency has been very well-explored for cubical tilings of zonotopes in the guise of counting *simplicial regions* of hyperplane arrangements or *mutations* in oriented matroids- see [63, Introduction §3] for a nice summary. For triangulations, flip-deficiency has been explored only more recently (see [45, 66]). For monotone paths, the question of flip deficiency appears not to have been considered much at all.

A related question concerns the level of connectivity of the graphs $G_A, G_Z, G_{P,f}$ of triangulations and bistellar moves, tilings and cube-flips, monotone paths and polygon moves respectively. For each of these graphs, the edge-subgraph on the coherent elements is the 1-skeleton of $(d' - d)$ -polytope and hence is $(d' - d)$ -vertex-connected in the graph-theoretic sense by Balinski's Theorem [83, §3.5]. One can ask whether the entire graphs $G_A, G_Z, G_{P,f}$ share the same level of vertex-connectivity, which is stronger than saying that every vertex has at least $(d' - d)$ neighbors. So far this question has only been resolved negatively in some cases where flip-deficiency exists. No non-trivial cases where the graph is $(d' - d)$ -vertex-connected have been proved.

Extension spaces, MacPhersonians and OM-Grassmannians.

Let Z be a d -dimensional zonotope generated by a set of n vectors V . As mentioned in Section 1, one can associate to V its oriented matroid \mathcal{M} . The the Bohné-Dress Theorem [20, 64] then implies that the Baues poset $\omega(P \xrightarrow{\pi} Q)$ is isomorphic to the extension poset $\mathcal{E}(\mathcal{M}^*)$, consisting of all single-element extensions of the dual oriented matroid \mathcal{M}^* ordered by weak maps. The following *Extension Space Conjecture* [18, §7.2] appears not to be attributable to any single source:

Conjecture 11. *For a realizable oriented matroid \mathcal{N} , the order complex of the extension poset $\mathcal{E}(\mathcal{N}) - \hat{1}$ is homotopy equivalent to a $(\text{rank}(\mathcal{N}) - 1)$ -sphere.*

Hence the extension space conjecture is equivalent to the special case of the weak GBP dealing with zonotopal subdivisions. We will discuss positive cases of this conjecture (mostly due to Sturmfels and Ziegler [78]) in Section 4, but we mention that the results of Mnëv and Richter-Gebert [51] show that one cannot remove the assumption that \mathcal{N} is realizable. They cleverly construct non-realizable oriented matroids \mathcal{N} of rank 4 for which $\mathcal{E}(\mathcal{N}) - \hat{1}$ is disconnected!

The extension space $\mathcal{E}(\mathcal{N})$ is also closely related to certain combinatorial models of Grassmannians called *OM-Grassmannians* (see [63, Introduction §4], [60, §1.2], [52] for fuller discussions.) Briefly, given an oriented matroid \mathcal{M} , the *OM-Grassmannian* $\mathbb{G}_k(\mathcal{M})$ is the poset of rank k oriented matroids which are *strong images* [18, §7.7] of \mathcal{M} , ordered by weak maps. If \mathcal{M} has rank d , the order complex of $\mathbb{G}_k(\mathcal{M})$ is intended as a combinatorial model for the Grassmannian of k -planes in \mathbb{R}^d . The following is conjectured by MacPherson, Mnëv and Ziegler [63, Conjecture 4.2]

Conjecture 12. *When \mathcal{M} is a realizable oriented matroid of rank d , $\mathbb{G}_k(\mathcal{M})$ is homotopy equivalent to the Grassmannian of k -planes in \mathbb{R}^d .*

In the special case where \mathcal{M} is the *Boolean* or *free* oriented matroid on d elements, $\mathbb{G}_k(\mathcal{M})$ is called the *MacPhersonian* $MacP(d, k)$, due to its occurrence in the work of Gelfand and MacPherson [30, 48] on combinatorial formulas for characteristic classes. It was proven by Babson [4] (see [52]) that Conjecture 12 is true for $k \leq 2$, and that $MacP(d, 3)$ is homotopy equivalent to the appropriate Grassmannian.

The relation to extension spaces and the Baues problem is that $\mathbb{G}_{d-1}(\mathcal{M})$ is a double cover of the extension poset $\mathcal{E}(\mathcal{M})$, in the sense that there is a two-to-one order-preserving map $\mathcal{E}(\mathcal{M}) \rightarrow \mathbb{G}_{d-1}(\mathcal{M})$. As a consequence, one can view the conjecture that $\mathbb{G}_{d-1}(\mathcal{M})$ is homotopy equivalent to the Grassmannian of $(d - 1)$ -planes in \mathbb{R}^d (or $(d - 1)$ -dimensional real projective space) as a projectivized version of the Extension Space Conjecture. This also implies that the positive results of

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Sturmfels and Ziegler [78] on the Extension Space Conjecture 11 give some special cases of Conjecture 12.

4. POSITIVE RESULTS

In this section we review results which give a positive answer to the weak or strong GBP. The methods used tend to segregate into the three paradigms described below, where we have indicated the references whose proofs exemplify these paradigms:

Retraction: A proof of the strong GBP, by exhibiting an explicit homotopy retracting $\omega_{coh}(P \xrightarrow{\pi} Q)$ onto $\omega_{coh}(P \xrightarrow{\pi} Q)$ ([13, Theorem 2.3], [62, Theorem 1.4], [3, Theorem 1.2]),

Homotopies: A proof of the weak GBP by a short chain of homotopy equivalences leading from between $\omega_{coh}(P \xrightarrow{\pi} Q)$ and some poset that is well-known to have spherical homotopy type. ([16, Theorem 2], [26, Theorem 1.2]),

Deletion-Contraction: An inductive proof of the weak GBP using (sometimes implicitly) the notion of *deletion-contraction* from matroid theory ([13, Theorem 1.2], [78, Theorem 1.2], [25, Theorem 3], [61, Theorem 1.1]).

Recall the general set-up: we consider a linear surjection of polytopes $\pi : P \rightarrow Q$ with P, Q being d', d -dimensional, respectively, and with P having n vertices. We divide our discussion of positive results into the following categories: $d = 1$ (monotone paths), $d' - d = 2$ (low codimension), $P = \text{cube}$ (zonotopal tilings), $n - d' = 1$ or $P = \text{simplex}$ (triangulations), cyclic polytopes.

d = 1: monotone paths

The original paper of Billera, Kapranov and Sturmfels [13] that posed the GBP proves both the weak and strong GBP for monotone paths and cellular strings, under our usual genericity assumption that f is nonconstant along each edge of P (although Rambau and Ziegler [62] claim that their proofs can be adapted to remove this assumption). Interestingly, there are two proofs given in [13], one which follows the retraction paradigm in proving the strong GBP (their Theorem 2.3) and one which uses (implicitly) the deletion-contraction paradigm (their Theorem 1.2) to prove the weak GBP.

extra This settles the original problem of Baues [7, Conjecture 7.4], which is the special case in which the polytope P is a permutohedron and f is a generic functional. The weak GBP for of cellular strings on zonotopes (as in Baues' special case) also follows from work of Björner [16, Theorem 2], which is a good example of the homotopies paradigm. Björner observes that cellular strings on a zonotope Z are the same as what he calls the *essential chains* in the *poset of regions* [22] of the hyperplane arrangement which is the polar dual [83, §7.3] to Z . An essential chain in a poset is a chain in which every step corresponds to a non-contractible (open) interval, and he shows that the subposet of essential chains order by refinement is homotopy equivalent to the entire order complex of all chains in the poset. For the poset of regions of a hyperplane arrangement, the homotopy type is known to be spherical by work of Edelman and Walker [23]. Interestingly, Björner actually proves his result not just for zonotopes or realized oriented matroids, but for an arbitrary

oriented matroid, where the notion of a cellular string and the poset of regions still make sense.

$d' - d \leq 2$: *low codimension.*

In very low codimension there is not much to say. If $d - d' = 0$ then $P = Q$ and the only π -induced subdivision of Q is the improper one. In the case $d' - d = 1$, there are exactly two proper π -induced subdivisions of Q , one coming from the “top” faces of P with respect to the projection π , the other coming from the “bottom” faces. Both of these subdivisions are coherent, and hence $\omega_{coh}(P \xrightarrow{\pi} Q)$, $\omega(P \xrightarrow{\pi} Q)$ are both 0-spheres.

In the case $d' - d \leq 2$, the fiber polytope $\Sigma(P \xrightarrow{\pi} Q)$ is a polygon, and hence $\omega_{coh}(P \xrightarrow{\pi} Q)$ is its boundary circle. Rambau and Ziegler used the retraction paradigm to prove the strong GBP in this case.

$P = \text{cube}$: *zonotopal tilings.*

We saw in Section 2 that the case when P is an d' -cube corresponds to the case of zonotopal subdivisions and tilings of the zonotope $Z = Q = \pi(P)$. Furthermore, if \mathcal{M} denotes the oriented matroid associated to the generating segments V of Z , then we saw that the Baues poset is the same as the poset of *single-element extensions* $\mathcal{E}(\mathcal{M}^*)$ for the dual oriented matroid \mathcal{M}^* , and the weak GBP is the same as the Extension Space Conjecture (Conjecture 11) for \mathcal{M}^* .

The extension space conjecture was investigated by Sturmfels and Ziegler [78], who proved most of the strongest positive results at present. They showed that an inductively defined technical hypothesis called *strong Euclideaness* on the oriented matroid \mathcal{M} implies that the extension space conjecture holds, using the deletion-contraction paradigm. They then showed that an oriented matroid on n elements with rank r is strongly Euclidean under various hypotheses: if $r \leq 3$, or $n - r \leq 2$, or when \mathcal{M} is the *alternating* oriented matroid $C^{n,r}$ that comes from a cyclic arrangement of vectors [18, §9.4]. Since oriented matroid duality exchanges r for $n - r$ and keeps n fixed, and since the alternating oriented matroids satisfy $(C^{n,r})^* \cong C^{n,n-r}$, their results imply the weak GBP when P is a d' -cube and $Q = Z$ is a d -dimensional zonotope under the following conditions:

- $d' - d \leq 3$, or
- $d \leq 2$, or
- Z is a cyclic zonotope.

It was also shown by Bailey [6] that the hypothesis of strong Euclideaness holds for \mathcal{M}^* when \mathcal{M} is the oriented matroid associated to a d -dimensional zonotope having $d+1$ generic generating segments, but with arbitrary multiple copies of each segment. Hence the weak GBP also holds for tilings of such zonotopes. We remark that for $d = 2$, the cubical tilings of these zonotopes (hexagons) were enumerated by MacMahon [47, Vol II, §X] in 1899.

Before closing our discussion of the Baues problem for tilings, we would like to mention an important result of Santos which shows that the GBP for zonotopal tilings is a special case of the GBP for triangulations. To any realized oriented matroid \mathcal{M} one can associate a polytope $\Lambda(\mathcal{M})$ known as its *Lawrence polytope* [9], [18, §9.3], [67, Chapter 4] using the technique of *Gale transforms*. The face lattice of the polytope $\Lambda(\mathcal{M})$ encodes all the information of the oriented matroid \mathcal{M} , and is useful for transferring matroid constructions and examples into the world of polytopes.

Theorem 13. [67, Theorem 4.14] *Let Z be zonotope with associated oriented matroid \mathcal{M} . There is a natural bijection between the subdivisions of $\Lambda(\mathcal{M})$ and the zonotopal subdivisions of Z (=single-element liftings of \mathcal{M}) which induces an isomorphism between the associated Baues posets.*

Consequently, a negative answer to the GBP for zonotopal tilings produces a negative answer for triangulations. We remark that the Lawrence construction applies more generally to oriented matroids \mathcal{M} which are not necessarily realizable, yielding a *matroid polytope* $\Lambda(\mathcal{M})$ [18, §9.1] rather than a polytope. Santos' result also applies in this situation, where one defines the triangulation of a matroid polytope via his definition of a *triangulation of an oriented matroid*. This definition unifies previous notions of such triangulations that had been proposed by Billera and Munson [12] and Anderson [1].

$n - d' = 1$ or $P = \text{simplex}$: triangulations.

When $n - d' = 1$ the polytope P must be an n -dimensional simplex Δ^{n-1} and then we have seen that the Baues poset is the poset of subdivisions of the point set \mathcal{A} , where \mathcal{A} is the image under π of the vertices of Δ^{n-1} . This case, which deals with triangulations and subdivisions of point sets, is the one which has perhaps received the most attention.

When the dimension d of \mathcal{A} 's ambient space is very small, as with the case of tilings there is not much to say. For $d \leq 1$ every subdivision is coherent, and the secondary (or fiber) polytope $\Sigma(\mathcal{A})$ is a Cartesian product of simplices whose dimensions are given by the multiplicities of the points in \mathcal{A} .

For $d = 2$, things start to get interesting. The fact that the graph of triangulations and bistellar operations is connected follows from work of Lawson [40], who gave an algorithm which starts with any triangulation and moves it toward a particular coherent triangulation called the *Delaunay* triangulation. Edelsbrunner and Shah [27] observed that a generalization of this flipping procedure works to move any coherent triangulation to the particular coherent triangulation which is induced by a some chosen set of heights. In fact, this procedure amounts to nothing more than linear programming on the secondary polytope $\Sigma(\mathcal{A})$.

It is claimed at the end of [13] that one can positively answer the GBP for \mathcal{A} in \mathbb{R}^2 , and this was justified under the extra assumption that the points lie in general position by Edelman and Reiner [25] using the deletion-contraction paradigm. The question of flip-deficiency for \mathcal{A} in \mathbb{R}^2 was resolved by deLoera, Santos and Urrutia [45], who gave a clever counting argument for why every triangulation has at least $|\mathcal{A}| - 3$ bistellar neighbors.

For \mathcal{A} in \mathbb{R}^3 and higher dimensions, our ignorance of bistellar connectivity and the GBP for triangulations is astounding. Question 2 is completely open. The only non-trivial general result here to guide us is by deLoera, Santos and Urrutia [45] who used a similar counting argument as in the $d = 2$ case to show that for \mathcal{A} in \mathbb{R}^3 in general position *and convex position* (i.e. no point of \mathcal{A} is in the convex hull of the rest) there can be no flip deficiency. They also show that the assumptions of convex position and $d = 3$ are crucial, by exhibiting an example having only 4 flips consisting of 9 points in general position in \mathbb{R}^3 with one of the points interior, and an example having only 4 flips consisting of 10 points in convex general position in \mathbb{R}^4 .

Joe



There are relatively few families of polytopes in higher dimensions whose triangulations have been well-studied, other than the cyclic polytopes which will be discussed in the next heading. We briefly discuss some of these other families here.

Triangulations of the d -cube which use few maximal simplices are desirable for the purposes of fixed point algorithms [81]. Therefore one would be interested in algorithms which enumerate the triangulations, such as deLoera's program PUNTOS [44] which enumerates all the triangulations lying in the same connected component of the graph of bistellar operations as the coherent triangulations. Unfortunately, deLoera [44, Theorem 2.3.20] has shown that incoherent triangulations of the d -cube exist for $d \geq 4$, so it is not known whether one can produce all triangulations of the cube by this method.

Cartesian products of simplices $\Delta^m \times \Delta^n$ were conjectured to have only coherent triangulations (see [83, Problem 5.3]). This is true when m or n is equal to 1, as the secondary polytope in this case is known to be the *permutohedron* [31, p. 243] (in fact $\Delta^1 \times \Delta^n$ happens to be the Lawrence polytope associated to the zonotope generated by n line segments in \mathbb{R}^1). However deLoera [44, Theorem 2.2.17] showed that there are incoherent triangulations whenever $m, n \geq 3$, and Sturmfels [76, Theorem 10.15] showed that they exist when $m = 2$ and $n \geq 5$. A close study of the secondary polytope $\Sigma(\Delta^m \times \Delta^n)$ and its facets was initiated by Billera and Babson [5], whose point of departure was the fact that a typical fiber of the map $\Delta^{(m+1)(n+1)-1} \rightarrow \Delta^m \times \Delta^n$ is a *transportation polytope*, i.e. the polytope of nonnegative $(m + 1) \times (n + 1)$ matrices with some prescribed row and column sums.

Another interesting family of polytopes are the (k, n) -*hypersimplices* $\Delta(k, n)$ defined by Gelfand and MacPherson [30] as the convex hull of all sums of k distinct standard basis vectors $e_{i_1} + \dots + e_{i_k}$ in \mathbb{R}^n . Particular triangulations of the second hypersimplex $\Delta(2, n)$ were studied by deLoera, Thomas and Sturmfels [46], and by Gelfand, Kapranov, and Zelevinsky (see [44, §2.5]). The triangulation of $\Delta(2, n)$ described in [46] turns out to be related to a triangulation of the slice of the cube $[0, 1]^d$ lying between $\sum_i x_i = 1$ and $\sum_i x_i = 2$ described by Stanley [72].

Cyclic polytopes $C(n, d)$.

The cyclic polytope $C(n, d)$ is defined to be the convex hull of any n distinct points on the d -dimensional *moment curve* $\{(t, t^2, \dots, t^d); t \in \mathbb{R}\}$. Cyclic polytopes play an important role in polytope theory because of the *Upper Bound Theorem* of McMullen [83, §8.4]: for any i , the cyclic polytope $C(n, d)$ achieves the maximum number of i -dimensional faces possible for a d -dimensional polytope with n vertices. Although the definition of $C(n, d)$ implicitly depends upon the parameters $t_1 < \dots < t_n$ which are the x_1 -coordinates of the points chosen on the moment curve, much of the combinatorial structure of $C(n, d)$ (including its face lattice, and its set of triangulations and subdivisions) does not depend upon this choice. Therefore we will omit the reference to these parameters except when necessary.

Note that the moment curve in $\mathbb{R}^{d'}$ maps to the moment curve in \mathbb{R}^d under the natural surjection $\pi : \mathbb{R}^{d'} \rightarrow \mathbb{R}^d$ which forgets the last $d' - d$ coordinates. This equips the cyclic polytopes with natural maps $\pi : C(n, d') \rightarrow C(n, d)$. Much has been said recently about the fiber polytopes and Baues problem associated with these natural maps, which include as special cases the study of triangulations of $C(n, d)$ when $d' = n - 1$, and the monotone paths on $C(n, d')$ with respect to the functional $f(\mathbf{x}) = x_1$ when $d = 1$.

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In [3], the authors determine when the fiber polytope $\Sigma(C(n, d') \xrightarrow{\pi} C(n, d))$ is canonical in either of the following two ways:

- all π -induced subdivisions of $C(n, d)$ are coherent (this happens only when $d - d' \leq 2$, or $n - d' = 1$ and $d = 2$, excepting a few sporadic cases), or
- not all are coherent but the subset of π -coherent subdivisions does not depend upon the choice of parameters $t_1 < \cdots < t_n$ (this happens exactly if $d = 1$ and $d' - d, n - d \geq 2$).

This paper also give a positive answer to the strong GBP for $C(n, n - 2) \rightarrow C(n, 2)$ using the retraction paradigm. In fact, they prove more generally that the strong GBP has positive answer for $\pi : P \rightarrow Q$ whenever P is a polytope with exactly 2 more vertices than its dimension, and the point configuration \mathcal{A} which is the image of the vertices of P under π has only coherent subdivisions.

The remaining results about cyclic polytopes deal exclusively with the case of triangulations, that is $n - d' = 1$. The philosophy here has been to try and generalize as many things as possible from the case $d = 2$, where the cyclic polytope $C(n, 2)$ is a convex polygon as in Figure 3. For $d = 2$ we know almost everything about its triangulations and subdivisions, as was described in Section 1. All these subdivisions of $C(n, 2)$ are coherent, so the poset of subdivisions is the face poset of the secondary polytope $\Sigma(\mathcal{A})$, the $(n - 3)$ -dimensional associahedron. The 1-skeleton of the associahedron is the Hasse diagram for the Tamari poset, and this poset turns out to be a lattice.

In contrast to the $d = 2$ case, not every triangulation of $C(n, d)$ is coherent for $d \geq 3$ (starting with $C(9, 3), C(9, 4), C(9, 5)$; see [3]). On the other hand, Rambau and Santos [61] prove that triangulations of $C(n, d)$ have the somewhat weaker property that they are always *lifting* triangulations (see [67, Definition 3.4], [18, p. 410]). On the other hand, triangulations of $C(n, d)$ can have flip-deficiency [61]. Rambau [58] also proves the interesting fact that triangulations of $C(n, d)$ are always *shellable* (see [17, §11.1]) as simplicial complexes.

In a somewhat cryptic paper, Kapranov and Voevodsky [39] suggested a generalization of the Tamari poset on triangulations of $C(n, 2)$ to a partial order on triangulations of $C(n, d)$, which they called the *higher Stasheff orders*, and which were studied by Edelman and Reiner [24] under the name of *higher Stasheff-Tamari orders*. Actually, [24] defines two possible such orders which are related to each other, and it is not quite clear (but presumably true) that one of these orders is the same as that considered in [39]. In [24] it was proved for $d \leq 3$ that these two partial orders coincide and both are lattices, and also that for $d \leq 5$ the graph of bistellar operations on triangulations of $C(n, d)$ is connected. This last result was greatly improved by Rambau [58], who showed that the graph is connected for all d , and even further improved by Rambau and Santos [61] who give a positive answer to the weak GBP using the deletion-contraction paradigm. Previously Edelman, Rambau and Reiner [26] had used the lattice structure on the poset of triangulations and the homotopy paradigm to positively answer the weak GBP for triangulations of $C(n, d)$ with $d \leq 3$. In that same paper, the authors show that for arbitrary d , both higher Stasheff-Tamari orders on the set of triangulations have proper parts which are homotopy equivalent to $(d - 4)$ -spheres.

5. NEGATIVE RESULTS

Recall from the previous section that we had a positive answer to the strong GBP by Billera, Kapranov, and Sturmfels [13] for $d = 1$, and by Rambau and Ziegler [62] for $d' - d \leq 2$. Together these imply that a negative answer to the weak GBP would require a surjection $\pi : P \rightarrow Q$ with $d' \geq 2$ and $d' - d \geq 3$ so that $d' \geq 5$. In the same paper as their positive result, Rambau and Ziegler cleverly construct such a counterexample $\pi : P \rightarrow Q$ with the minimum possible dimensions $d' = 5, d = 2$. In fact, they show that the Baues poset $\omega(P \xrightarrow{\pi} Q)$ in this case is not homotopy equivalent to a 2-sphere by showing that it has an isolated element: a π -coherent subdivision of Q which lies below no other element of the poset! Their counterexample is also quite small and uncomplicated, in the sense that the number of n vertices of P is only 10, and the point configuration \mathcal{A} in \mathbb{R}^2 which is the image of these 10 vertices under π is relatively simple: it consists of a triangle with each corner vertex tripled, along with one interior point of the triangle. They also give a perturbed version of this same example in which the point configuration \mathcal{A} lies in general position in \mathbb{R}^2 , and the Baues poset is again disconnected (although it does not have any isolated points). These counterexamples can also be used to produce negative answers to the weak GBP for all d', d with $d \geq 2$ and $d' - d \geq 3$.

In light of this counterexample, attention has shifted to the motivating special cases of the GBP dealing with triangulations of point sets \mathcal{A} and zonotopal tilings of a zonotope Z . Here no counterexamples have been found. The closest thing to a counterexample was provided by the previously mentioned work of Mněv and Richter-Gebert [51]. They produce (by two different methods) examples of rank 4 oriented matroids \mathcal{M} whose extension posets $\mathcal{E}(\mathcal{M})$ contain isolated points. These examples do not give a counterexample to the Extension Space Conjecture or to the weak GBP because the oriented matroids in question are not *realizable*, that is they do not come from a zonotope Z . However, they do settle in the negative an earlier extension space conjecture which did not assume realizability of \mathcal{M} .

We should also view the instances of flip deficiency for triangulations found in [45] as negative results, even though they are far from settling the GBP.

We summarize the main open cases of the (weak) GBP here:

Question 14.

1. Is the poset of zonotopal subdivisions of a d -dimensional zonotope with d' generators homotopy equivalent to a $(d' - d - 1)$ -dimensional sphere?
2. Is the poset of subdivisions of a point set \mathcal{A} in \mathbb{R}^d homotopy equivalent to a $(|\mathcal{A}| - d - 2)$ -dimensional sphere?

As was mentioned earlier, the work of Santos [67] shows that the first question is a special case of the second, and therefore a counterexample for the first would also settle the second, as well as the Extension Space Conjecture 11 and Conjecture 12.

6. THE QUESTIONS, PROBLEMS, CONJECTURES

Most people in the area agree that the major open problems related to the GBP are Questions 2, 3, 14. In this section, we collect some other problems and questions, some of which address more specifically the expected frontier between the cases of $\pi : P \rightarrow Q$ for which the GBP has positive and negative answer. In some cases, we go out on a limb by offering our predictions, but we warn the reader that many of

these opinions are not based on very much data, and are only the opinion of this author.

We begin by conjecturing the frontier for triangulations, partly inspired by the positive and negative results of deLoera, Santos, and Urrutia [45].

Conjecture 15. *Let \mathcal{A} be a point configuration in \mathbb{R}^d with $d \leq 2$, or $d = 3$ and in convex position, or $|\mathcal{A}| - d \leq 4$. Then*

- (a) *The strong GBP has positive answer for subdivisions of \mathcal{A} , (without the general position assumption needed in [25]).*
- (b) *Furthermore, the graph of triangulations and bistellar operations is $(|\mathcal{A}| - d - 1)$ -vertex-connected, so in particular every triangulation of \mathcal{A} has at least $|\mathcal{A}| - d - 1$ bistellar neighbors.*
- (c) *On the other side, there exists a point configuration in convex position in \mathbb{R}^4 and also one not in convex position in \mathbb{R}^3 , each of which has an isolated triangulation which refines no other subdivision.*

In fact, it would be nice to have a simpler proof of the weak GBP for \mathcal{A} in \mathbb{R}^2 , even assuming general position, or perhaps a proof of the strong GBP by retraction. Although the idea of the deletion-contraction in [25] is relatively simple, the details are somewhat technical.

For zonotopal tilings, one wonders whether the non-realizable oriented matroid counterexamples of Mněv and Richter-Gebert [51] can be made realizable.

Question 16. *Does there exist a zonotope with a cubical tiling that refines no other zonotopal tiling?*

We suspect that the answer to this is “Yes”, but admittedly with little basis.

For monotone paths we know that the strong GBP has a positive answer, so the interesting question remaining is about connectivity via polygon moves. As we saw in Figure 15, one cannot expect good behavior without a genericity assumption.

Conjecture 17. *For a d -dimensional polytope P and a generic functional f , (f is not constant on any edge of P) the graph of f -monotone paths and polygon moves is $(d - 1)$ -vertex-connected.*

In particular, every f -monotone path has at least $d - 1$ neighbors in the graph $G_{P,f}$ of polygon moves.

We next discuss several conjectures related to cyclic polytopes and the natural maps $\pi : C(n, d') \rightarrow C(n, d)$ between them. The recent positive results of Rambau and Santos [61] on triangulations of cyclic polytopes, and [3, Corollary 6.3] relating to $\pi : C(n, n - 2) \rightarrow C(n, 2)$ prompt the following conjecture:

Conjecture 18. *The strong GBP has positive answer for $\pi : C(n, d') \rightarrow C(n, d)$.*

We mention related a conjecture which would settle the case $\pi : C(n, d') \rightarrow C(n, 2)$, and is motivated by [3, Theorem 1.2].

Conjecture 19. *The strong GBP has positive answer for $\pi : P \rightarrow Q$ if Q lies in \mathbb{R}^2 and all vertices of P project under π to the boundary of Q .*

Note that this conjecture would explain the importance of the interior point of Q present in the Rambau-Ziegler counterexample [62].

The proof of the weak GBP for triangulations of cyclic polytopes in [61] uses the deletion-contraction paradigm. We next discuss some other conjectural approaches

to this result, involving the relation of cyclic polytopes to *cyclic zonotopes* and *alternating matroids* [18, §9.4].

For any point configuration \mathcal{A} in \mathbb{R}^d , the *dual point configuration* or *Gale transform* \mathcal{A}^* lives in $\mathbb{R}^{|\mathcal{A}|-d-1}$. A single element extension of the oriented matroid \mathcal{M} corresponding to \mathcal{A}^* gives rise to a subdivision of \mathcal{A} called a *lifting subdivision*, and hence gives a map from the extension poset $\mathcal{E}(\mathcal{M})$ to the poset of subdivisions of \mathcal{A} . In the case where \mathcal{A} is the set of vertices of a cyclic polytope $C(n, d)$, \mathcal{A}^* is a cyclic arrangement of vectors [83, Problem 6.13] so that \mathcal{M} is the alternating matroid $C^{n, n-d-1}$. Therefore the results of [78] imply that $\mathcal{E}(\mathcal{M})$ is homotopy equivalent to an $(n-d-2)$ -sphere. Furthermore, [61] shows that every subdivision of $C(n, d)$ is a lifting subdivision, so that this map is surjective.

Conjecture 20. *When \mathcal{A} is the set of vertices of a cyclic polytope $C(n, d)$, the map described in the previous paragraph is a homotopy equivalence from the extension space $\mathcal{E}(\mathcal{M})$ to the poset of subdivisions.*

A different approach relates the triangulations of cyclic polytopes to cyclic hyperplane arrangements and the work of Manin and Schechtman [49], Ziegler [84], and Kapranov and Voevodsky [39] on *cyclic hyperplane arrangements and zonotopes* and the *higher Bruhat orders*.

Let $Z(n, d)$ be the d -dimensional *cyclic zonotope* with n generating segments in the directions $\{(1, t_i, t_i^2, \dots, t_i^{d-1})\}_{i=1}^n$ for any n distinct values of the parameters $t_1 < \dots < t_n$. The higher Bruhat orders $B(n, d)$ were defined in [49], and may be thought of as a natural poset structure on the cubical tilings of $Z(n, d)$. For $d = 1$, $B(n, 1)$ is the *weak Bruhat order* [18, §2.3(b)] on the symmetric group. Ziegler [84] observed that there were actually two natural and related (but different!) definitions for higher Bruhat orders, which he called $B(n, d)$ and $B_{\subseteq}(n, d)$. Among other things, he showed that the homotopy type of the second of these posets $B_{\subseteq}(n, d)$ was spherical. Rambau [59] later showed that $B(n, d)$ also has spherical homotopy type.

As was mentioned in Section 4 Kapranov and Voevodsky [39] define a partial order on triangulations of $C(n, d)$, and in [24], the authors consider two such related partial orders $S_1(n, d)$ and $S_2(n, d)$, generalizing the Tamari poset on triangulations of $C(n, 2)$. It is not quite clear, although presumably true, that the order $S_1(n, d)$ coincides with the order defined in [39]. In [26], it is shown that both posets $S_1(n, d)$ and $S_2(n, d)$ have spherical homotopy type. Kapranov and Voevodsky also define an order-preserving map $B(n, d) \rightarrow S_1(n+2, d+1)$, and a similar map was given two definitions by Rambau in [58]. Rambau shows that his two definitions give the same map, but it is not clear that his map is the same as the one in [39]. For $d = 1$, this map coincides with a map from permutations to triangulations of an n -gon studied by Björner and Wachs [19, §9].

Conjecture 21. *The maps $B(n, d) \rightarrow S_1(n+2, d+1)$ defined by Kapranov-Voevodsky and Rambau are the same, and induce a homotopy equivalence between the proper parts of these posets (True for $d = 1$ [19, §9]).*

The fact that these poset have homotopy equivalent proper parts already follows from the sphericity results previously mentioned. What does this have to do with the GBP? It is easy to see that for any zonotopal subdivision of $Z(n, d)$, the set of all cubical tilings which refine it forms an interval in either $B(n, d)$ or $B_{\subseteq}(n, d)$. This gives a very natural order-preserving map from the Baues poset $\omega(I^n \rightarrow$

$Z(n, d)$ to the poset of proper intervals in $B(n, d)$ or $B_{\subseteq}(n, d)$. Similarly, for any subdivision of $C(n, d)$, the set of triangulations which refine it forms an interval in either $S_1(n, d)$ or $S_2(n, d)$, giving an order-preserving map from the Baues poset $\omega(\Delta^{n-1} \rightarrow C(n, d))$ to the poset of proper intervals in $S_1(n, d)$ or $S_2(n, d)$.

- Conjecture 22.**
1. *The image of the map from $\omega(I^n \rightarrow Z(n, d))$ to the poset of proper intervals in either $B(n, d)$ or $B_{\subseteq}(n, d)$ is exactly the set of non-contractible (open) intervals (True for $d = 1$ [19, §9]).*
 2. *The image of the map from $\omega(\Delta^{n-1} \rightarrow C(n, d))$ to the poset of proper intervals in either $S_1(n, d)$ or $S_2(n, d)$ is exactly the set of non-contractible intervals (True for $d \leq 3$ [26, Lemma 6.3]).*

The previous conjecture would have two nice consequences:

- (i) It would completely describe the homotopy type of all intervals (and hence compute the *Möbius function*) in the posets $B(n, d)$, $B_{\subseteq}(n, d)$ and $S_1(n, d)$, $S_2(n, d)$. The intervals which are the images of the above maps are always isomorphic to Cartesian products of posets $B(n', d)$, $B_{\subseteq}(n', d)$ or $S_1(n', d)$, $S_2(n', d)$ for smaller values $n' < n$, and hence by the known sphericity results, are also homotopy spherical.
- (ii) It would imply that $\omega(I^n \rightarrow Z(n, d))$ is homotopy equivalent to the suspension of the proper part of $B(n, d)$ or $B_{\subseteq}(n, d)$, and similarly $\omega(\Delta^{n-1} \rightarrow C(n, d))$ is homotopy equivalent to the suspension of the proper part of $S_1(n, d)$ or $S_2(n, d)$. This uses the fact observed by Walker [82] that the poset of proper intervals in a bounded poset P is homeomorphic to the suspension of the proper part of P , and the fact that the poset of proper noncontractible intervals in P is a deformation retract of the poset of all proper intervals in P [26, Lemma 6.5].

Note that if Conjectures 21 and 22 hold, one could then string together three homotopy equivalences relating the Baues posets for triangulations of $C(n, d)$ to the one for zonotopal tilings of $Z(n, d)$:

$$\begin{aligned} \omega(I^n \rightarrow Z(n, d)) &\cong \text{Susp}(B(n, d) - \hat{0}, \hat{1}) \\ &\cong \text{Susp}(S_1(n+2, d+1) - \hat{0}, \hat{1}) \\ &\cong \omega(\Delta^{n+1} \rightarrow C(n+2, d+1)) \end{aligned}$$

Here the middle map is the suspension of the Kapranov-Voevodsky-Rambau map described above. Note that by results previously cited, each of the spaces occurring here is known to be homotopy equivalent to a $(n-d-1)$ -sphere.

Another challenging problem is to count the number of triangulations of $C(n, d)$ -see [3, Table 4] for some data, produced by deLoera's program PUNTOS [44] and by software of Rambau dedicated to this task. As was said in the introduction, almost the only non-trivial known formula is the Catalan number $\frac{1}{n-1} \binom{2n-4}{n-2}$ counting triangulations of the n -gon $C(n, 2)$. We also have the following mostly trivial results:

- $C(n, 1)$ has 2^{n-2} triangulations,
- $C(d+1, d)$ has 1 triangulation,
- $C(d+2, d)$ has 2 triangulations, and
- $C(d+3, d)$ has $d+3$ triangulations by the results of [42].

Santos [69] recently provided (and partly proved) the following non-trivial conjecture, based on the known data:

Conjecture 23. *Let a_d be the number of triangulations of $C(d+4, d)$. Then the second difference $a_d - 2a_{d-1} + a_{d-2}$ has the following form:*

$$a_{2k} - 2a_{2k-1} + a_{2k-2} = 2^k \text{ (proven by Santos)}$$

$$a_{2k+1} - 2a_{2k} + a_{2k-1} = 2^{k+1} + k2^{k-1}.$$

Note that Santos' conjecture easily leads to a closed formula for the number of triangulations of $C(n, d)$.

In addition to the previous specific conjectures, we would also like to describe some more general problems related to the GBP.

The first of these relates to the concept of an *iterated fiber polytope* introduced by Billera and Sturmfels [15]. Given a tower $P \xrightarrow{\pi} Q \xrightarrow{\rho} R$ of linear surjections of polytopes, it was shown in [14] that the map π induces a surjection of the fiber polytopes

$$\pi : \Sigma(P \xrightarrow{\rho \circ \pi} R) \rightarrow \Sigma(Q \xrightarrow{\rho} R).$$

They called the fiber polytope of this surjection the *iterated fiber polytope*

$$\Sigma(P \xrightarrow{\pi} Q \xrightarrow{\rho} R).$$

It is also clear how one can iterate the construction to define higher iterated fiber polytopes if one has longer towers of projections.

Question 24. *Study the iterated fiber polytopes for subsequences of the tower of natural projections*

$$\Delta^{n-1} = C(n, n-1) \rightarrow C(n, n-2) \rightarrow \cdots \rightarrow C(n, 2) \rightarrow C(n, 1)$$

between cyclic polytopes. Are there any cases (like those classified in [3] for $C(n, d') \rightarrow C(n, d)$) where the structure of the iterated fiber polytope does not depend upon the choice of points on the moment curve defining $C(n, d)$?

Another line of inquiry is suggested by the first part of Conjecture 22. Ziegler [84] considers the higher Bruhat orders $B(n, d), B_{\subseteq}(n, d)$ as posets of *uniform extensions* of the affine oriented matroid corresponding to a cyclic hyperplane arrangement of hyperplanes. More generally, he introduces these *uniform extension posets* $\mathcal{U}(\mathcal{M}, g), \mathcal{U}_{\subseteq}(\mathcal{M}, g)$ for any affine oriented matroid (\mathcal{M}, g) . The map considered in Conjecture 22 can be generalized to a map from the extension poset $\mathcal{E}(\mathcal{M})$ to the set of proper intervals in $\mathcal{U}_{\subseteq}(\mathcal{M}, g)$: send a single element extension of \mathcal{M} to the set of uniform extensions which lie below it in $\mathcal{E}(\mathcal{M})$.

Question 25. *Study the map from the extension space $\mathcal{E}(\mathcal{M})$ to the poset of intervals in $\mathcal{U}(\mathcal{M}, g)$. Are there any nice classes of affine oriented matroids where the image of the map is exactly the set of non-contractible proper intervals?*

As was the case in Conjecture 22, whenever the above question has a positive answer, the map in question induces a homotopy equivalence between $\mathcal{E}(\mathcal{M})$ and the suspension of the proper part of $\mathcal{U}(\mathcal{M}, g)$. The examples of Mnëv and Richter-Gebert [51] show that $\mathcal{E}(\mathcal{M})$ does not always have spherical homotopy type, but it is still possible that such a homotopy equivalence may exist even in cases where sphericity fails.

Our last (??) question relates to Stembridge's $q = -1$ phenomenon (see [75]) occurring in the context of cubical tilings of zonotopes. A zonotope Z is a centrally-symmetric polytope, and hence the antipodal map induces a natural involution ω

on its set of cubical tilings. Say that a tiling of Z is *centrally symmetric* if it is fixed by this involution ω . Consider also the graph G_Z of cubical tilings and cube-flips on Z . This graph will always be bipartite, due to the fact that cubical tilings are the same as uniform single-element liftings, which can be specified by a circuit signature [18, §7.1]- the bipartition roughly speaking comes from the parity of the number of +’s in this circuit signature. Say that the $q = -1$ phenomenon holds for Z if the number of centrally symmetric tilings of Z is the same as the difference in cardinality of the two sides of the bipartition of G_Z .

Stembridge observed in [75] that known formulas counting symmetry classes of plane partitions implied the $q = -1$ phenomenon for zonotopal hexagons in the plane (with multiple copies of the 3 line segments which generate the hexagon as a zonotope), and further examples involving certain zonotopal octagons were found by Elnitsky [28] and Bailey [6]. However, one can check that the phenomenon does not hold for all zonotopes Z , as there are already examples of zonotopal octagons for which it fails.

Question 26. *For which zonotopes Z does the $q = -1$ phenomenon hold?*

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