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# VISIBILITY COMPLEXES AND THE BAUES PROBLEM FOR TRIANGULATIONS IN THE PLANE

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Abstract. We give a positive answer for the special case of the Generalized Baues Problem which asks whether the complex of triangulations of a point set $\mathcal{A}$ in general position in the plane has the homotopy type of a sphere. In the process, we are led to define the *visibility complex* for a simplicial complex whose vertices lie in $\mathcal{A}$, and prove that this visibility complex has the same homotopy type as $P$. The main technique is a variant of *deletion-contraction* from matroid theory.

## I. Introduction.

The subject of triangulations of a point set $\mathcal{A}$ in $\mathbb{R}^d$ has undergone a recent surge of interest, partly due to the theory of *secondary polytopes* and *$\mathcal{A}$-discriminants and resultants* defined by Gelfand, Kapranov, and Zelevinsky [GKZ], and also due to a problem raised by Baues [Ba] in homotopy theory. These two developments were unified by Billera and Sturmfels in their work on *fiber polytopes* [BS], and led Billera et. al [BKS], to formulate the Generalized Baues Problem (GBP) associated to a surjection of convex polytopes $P \to Q$. Given two polytopes $P$ and $Q$ and an affine surjection $P \to Q$, the problem asks whether a certain complex of polygonal decompositions of $Q$ induced by the faces of $P$ has the same homotopy type as the subcomplex of *coherent decompositions*, which are defined geometrically. This subcomplex is known to be isomorphic to the boundary complex of a convex polytope (the *fiber polytope* $\Sigma(P,Q)$) and hence is spherical.

The answer to the GBP is known to be positive when $dim(Q) = 1$ [Bj1], [BKS], and when $dim(P) - dim(Q) \leq 2$ [RZ], however counterexamples were given recently by Rambau and Ziegler [RZ] with $dim(Q) \geq 2$ and $dim(P) - dim(Q) \geq 3$ (see this reference for a more complete discussion of the GBP). The most interesting cases of the GBP are those where $P$ is either an $n$-cube or an $n$-simplex, since a positive answer in these instances would resolve the weaker question of whether all cubical subdivisions of a *zonotope $Z$* are connected by *mutations*, or all triangulations of a point set $\mathcal{A}$ are connected by *bistellar operations*, respectively. For $P$ an $n$-cube the answer to the GBP is known to be positive for $dim(Q) \leq 2$ [SZ]. For $P$ an $n$-simplex, it was proven that all triangulations of $\mathcal{A}$ in $\mathbb{R}^2$ are connected by bistellar operations by Lawson [La], which shows that the complex considered in this case

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of the GBP is at least connected. In this paper, we will give a positive answer to the GBP for triangulations of $\mathcal{A}$ in general position in $\mathbb{R}^2$.

In fact, we prove something stronger. For an arbitrary simplicial complex $P$ embedded in $\mathbb{R}^2$ and using only vertices in $\mathcal{A}$, we will show that there is a complex generalizing the complex considered by the GBP, and that when $\mathcal{A}$ is in general position in $\mathbb{R}^2$, this complex is contractible (Theorems 2 and 3). Our method is a variant of the *deletion-contraction* technique from matroid theory, in combination with Quillen's Fiber Lemma. An interesting by-product of the analysis is that it requires us to understand the homotopy type of a simplicial complex which we call the *visibility complex* for the pair $(P, \mathcal{A})$. We prove that this visibility complex has the same homotopy type as $P$ (Theorem 1).

The paper is organized as follows. Section II gives the basic definitions and states the main results (Theorems 1,2, and 3). Section III reduces Theorems 1 and 2 to the case where $P$ is a manifold with boundary. Section IV completes the proof of Theorem 1, and Section V uses this to complete the proof of Theorem 2. Theorem 3 is then deduced as a corollary to the proof of Theorem 2.

## II. Definitions and statement of results.

Let $\mathcal{A}$ be a finite set of points in $\mathbb{R}^d$. A set $P$ in $\mathbb{R}^d$ will be called $\mathcal{A}$-*triangulable* if, roughly speaking, it has a triangulation as a simplicial complex whose vertices are a subset of $\mathcal{A}$. To be more precise, let $\mathcal{A} = \{v_1, \dots, v_n\} \subseteq \mathbb{R}^d$ and say that $P \subseteq \mathbb{R}^d$ is $\mathcal{A}$-triangulable if $\mathcal{A} \subseteq P$ and there exists an abstract simplicial complex $\Delta$ on vertex set $\{1, 2, \dots, n\}$ (not necessarily using every index as a vertex) whose *geometric realization* $\|\Delta\|$ maps homeomorphically onto $P$ (endowed with the subspace topology) under the map from $\|\Delta\|$ into $\mathbb{R}^d$ which sends an abstract vertex $i$ to the point $v_i$. In what follows, we will often abuse notation by drawing no distinction among the topological space $P$, the homeomorphic space $\|\Delta\|$, and the simplicial complex $\Delta$, so that we can refer to vertices, edges in $P$, and so on.

For a set $A \subseteq \mathcal{A}$, let $conv(A)$ denote the convex hull of the points of $A$. Given an $\mathcal{A}$-triangulable set $P$, define the *visibility complex* $\Delta_{vis}(P, \mathcal{A})$ to be the abstract simplicial complex on vertex set $\mathcal{A}$ whose simplices are the subsets $A \subset \mathcal{A}$ with $conv(A) \subseteq P$. The *visibility graph* $G_{vis}(P, \mathcal{A})$ is the 1-skeleton (vertices and edges) of $\Delta_{vis}(P, \mathcal{A})$ (cf. [OR2])

A *polytopal decomposition* $\delta$ of the pair $(P, \mathcal{A})$ is a set of pairs $\{(P_i, \mathcal{A}_i)\}_{i=1}^k$ satisfying the following conditions:

(1) For each $i$, we have $\mathcal{A}_i \subseteq \mathcal{A}$ and $P_i = conv(\mathcal{A}_i)$.
(2) $P = \bigcup_{i=1}^k P_i$.
(3) For each $i \neq j$, the polytopes $P_i, P_j$ intersect in a common proper face $F$ of each (possibly the empty face, but not $P_i$ or $P_j$ itself), and $\mathcal{A}_i \cap F = \mathcal{A}_j \cap F$.

Note that an $\mathcal{A}$-triangulable space $P$ always has at least one polytopal decomposition coming from its $\mathcal{A}$-triangulation $\Delta$, namely $\delta = \{(P_i, \mathcal{A}_i)\}$ where $\{P_i\}$ are the *maximal faces* of $\Delta$, and $\mathcal{A}_i$ is the set of vertices of $P_i$. Furthermore, if $\delta = \{(P_i, \mathcal{A}_i)\}$ is a polytopal decomposition of $P$, then any union $\bigcup_\alpha P_{i_\alpha}$ is an $(\bigcup_\alpha P_{i_\alpha} \cap \mathcal{A})$-triangulable space: simply refine each $P_i$ in $\delta$ to a triangulation which introduces no new vertices and restrict this simplical complex to its faces lying in $\bigcup_\alpha P_{i_\alpha}$.

Define the *decomposition poset* $Dec(P, \mathcal{A})$ to be the set of all polytopal decompositions $\delta$ of $(P, \mathcal{A})$ ordered by *refinement*, i.e., if

$$\delta = \{(P_i, \mathcal{A}_i)\}_{i=1}^{k}$$
$$\delta' = \{(P'_j, \mathcal{A}'_j)\}_{j=1}^{k'}$$

then $\delta \leq \delta'$ means for each $i = 1, 2, \ldots, k$ there exists some $j$ so that $P_i \subseteq P'_j$ and $\mathcal{A}_i \subseteq \mathcal{A}'_j$. Check this!

We will prove the following results about the homotopy type of $\Delta_{vis}(P, \mathcal{A})$ and $Dec(P, \mathcal{A})$ (when referring to the homotopy type of a poset $Q$, we mean the homotopy of its *order complex* $\Delta(Q)$; see [Bj2, (9.3)]). All of our results will assume that $\mathcal{A}$ is in general position in $\mathbb{R}^2$, i.e. no three points of $\mathcal{A}$ are collinear. It is not clear whether this assumption is necessary for any of these results, but it will greatly simplify some of the proofs.

**Theorem 1.** *Assume $\mathcal{A}$ is in general position in $\mathbb{R}^2$, and $P$ is an $\mathcal{A}$-triangulable space. Then $\Delta_{vis}(P, \mathcal{A})$ is homotopy equivalent to $P$.* false in $\mathbb{R}^3$

**Theorem 2.** *Assume $\mathcal{A}$ is in general position in $\mathbb{R}^2$ and $P$ is an $\mathcal{A}$-triangulable space. Then $Dec(P, \mathcal{A})$ is contractible.*

**Theorem 3.** *Assume $\mathcal{A}$ is in general position in $\mathbb{R}^2$. If $P = conv(\mathcal{A})$, then $Dec(P, \mathcal{A})$ has a unique top element $\hat{1}$, and $Dec(P, \mathcal{A}) - \hat{1}$ is homotopy equivalent to a sphere of dimension $|\mathcal{A}| - 3$.*

When $P = conv(\mathcal{A})$, this poset $Dec(P, \mathcal{A}) - \hat{1}$ is isomorphic to the poset $\mathcal{S}(\Delta^{n-1}, P)$ considered in the Generalized Baues Problem [BKS, p. 554], where here $\Delta^{n-1}$ is a simplex with $n = |\mathcal{A}|$ vertices mapping onto $P$ by the canonical surjection.

Because we will need it frequently, we record here the following well-known lemma [OR1,Theorem 1.2]:

**Lemma 4.** *Let $C$ be an $\mathcal{A}$-triangulable Jordan curve in the plane, i.e., $C$ is a polygonal embedding of the circle $\mathbb{S}^1 \to \mathbb{R}^2$ using only vertices in $\mathcal{A}$. Then the closure $\overline{C}$ of the interior of $C$ has an $\mathcal{A}$-triangulation.* □

It is important to note that Lemma 4 does not generalize to three dimensions. That is, there exists a non-convex polyhedron in three dimensions whose interior cannot be partitioned into tetrahedra with vertices chosen from those of the polyhedron. See [OR1,§10.2.1] for a discussion of some examples.

## III. Reduction to manifold with boundary.

In this section we simultaneously reduce both Theorems 1 and 2 to the case where $P$ is an $\mathcal{A}$-triangulable manifold with boundary by a sequence of lemmas.

The first lemma allows us to assume that $P$ (and hence $\Delta_{vis}(P, \mathcal{A})$) is connected.

**Lemma 5.** *Let $P$ be an $\mathcal{A}$-triangulable set in $\mathbb{R}^d$, with connected components $\{P_i\}_{i=1}^{c}$. Then*

(1) *Each $P_i$ is $(\mathcal{A} \cap P_i)$-triangulable.*
(2) *$\Delta_{vis}(P, \mathcal{A})$ has connected components $\{\Delta_{vis}(P_i, \mathcal{A} \cap P_i)\}_{i=1}^{c}$.*

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(3) *$Dec(P,\mathcal{A})$ is isomorphic as a poset to the Cartesian product*

$$\prod_{i=1}^{c} Dec(P_i, \mathcal{A} \cap P_i)$$

*Proof.* Let $\Delta$ be an $\mathcal{A}$-triangulation of $P$. Since the connected components of $\Delta$ are identified with those of $P$, the first assertion is obvious.

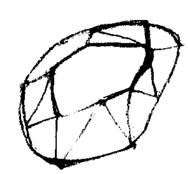
To prove the second assertion, we must show that $v, v' \in \mathcal{A}$ lie in the same connected component of $P$ if and only if they lie in the same connected component of $\Delta_{vis}(P,\mathcal{A})$. But this is clear, since $v, v'$ being connected in $P$ implies that there is a path in the 1-skeleton of $\Delta$ between them, which corresponds to a path in $\Delta_{vis}(P,\mathcal{A})$, and likewise if there is a path in $\Delta_{vis}(P,\mathcal{A})$ between them it leads to a path in $P$ between them.

To prove the third assertion, note that any decomposition $\delta$ of $(P,\mathcal{A})$ is uniquely defined by its restriction to each connected component of $P$. $\square$

The next lemma allows us to assume that the visibility graph $G_{vis}(P,\mathcal{A})$ is biconnected. Recall that for a graph $G$, a vertex $v$ is an *articulation point* if $G - v$ is disconnected, and a graph is *biconnected* if it has no articulation points. Equivalently, $G$ is biconnected if any two vertices $v, v'$ have two vertex-disjoint paths in $G$ between them.

**Lemma 6.** *Let $P$ be a connected $\mathcal{A}$-triangulable set in $\mathbb{R}^d$. Assume $v \in \mathcal{A}$ is an articulation point of $G_{vis}(P,\mathcal{A})$. Let $\{\mathcal{A}_i\}_{i=1}^c$ be the sets of vertices of the different connected components of $G_{vis}(P,\mathcal{A}) - v$, and let*

$$P_i = \bigcup_{\substack{A \subseteq \mathcal{A}_i \cup \{v\} \\ conv(A) \subseteq P}} conv(A).$$



(1) *Each $\{P_i\}$ is $\mathcal{A}_i \cup \{v\}$-triangulable.*
(2) *$P$ is homeomorphic to the one-point wedge $\bigvee_{i=1}^c P_i$, with wedge point $v$.*
(3) *$\Delta_{vis}(P,\mathcal{A})$ is homeomorphic to the one-point wedge*

$$\bigvee_{i=1}^{c} \Delta_{vis}(P_i, \mathcal{A}_i \cup v),$$

*with wedge point $v$.*

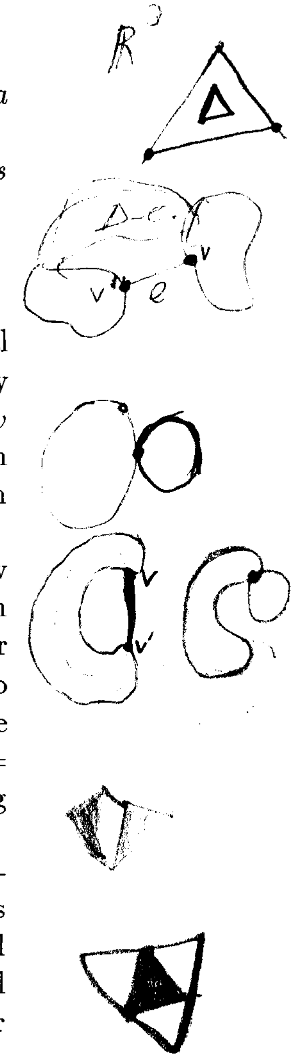
(4) *$Dec(P,\mathcal{A})$ is isomorphic as a poset to $\prod_{i=1}^c Dec(P_i, \mathcal{A}_i \cup v)$*

*Proof.* All four assertions follow immediately from the following observation: if $A \subseteq \mathcal{A}$ has $conv(A) \subseteq P$, then all the points of $A$ lie in the same biconnected component of $G_{vis}(P,\mathcal{A})$, so $A \subseteq \mathcal{A}_i \cup v$ for some $i$. $\square$

The next lemma allows us to assume that $P$ has no *maximal edges*, i.e., every edge $e$ in the triangulation of $P$ is contained in some higher dimensional face.

**Lemma 7.** *Let $P$ be a connected $\mathcal{A}$-triangulable set in $\mathbb{R}^2$. Assume $\mathcal{A}$ is in general position (no three points collinear) $G_{vis}(P, \mathcal{A})$ is biconnected, and let $e$ be a maximal edge in the triangulation $\Delta$ of $P$.*

1. *$P - e$ is $\mathcal{A}$-triangulable.*
2. *$P$ is homotopy equivalent to the one-point wedge of the spaces $P - e$ and a circle $\mathbb{S}^1$.*
3. *$\Delta_{vis}(P, \mathcal{A})$ is homotopy equivalent to the one-point wedge of the spaces $\Delta_{vis}(P - e, \mathcal{A})$ and a circle $\mathbb{S}^1$.*
4. *$Dec(P, \mathcal{A})$ is isomorphic as a poset to $Dec(P - e, \mathcal{A})$*



*Proof.* The first assertion is obvious.

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The second assertion follows from the following observation: if $e$ is a maximal edge in a finite simplicial complex $\Delta$ and $\Delta - e$ is connected, then $\Delta$ is homotopy equivalent to the one-point wedge of $\Delta - e$ and $\mathbb{S}^1$. To see this fix one endpoint $v$ of the edge $e$, and let the other endpoint $v'$ slide along a path connecting $v$ to $v'$ in $\Delta - e$. This gives the homotopy between the two spaces. Note that in the situation of the lemma, $\Delta - e$ is connected since $G_{vis}(P, \mathcal{A})$ is biconnected.

The third assertion will also follow from the same observation, if we can show that the edge $\hat{e}$ in $\Delta_{vis}(P, \mathcal{A})$ spanned by the endpoints $v, v'$ of $e$ is maximal in $\Delta_{vis}(P, \mathcal{A})$, since then we would have $\Delta_{vis}(P, \mathcal{A}) - \hat{e} = \Delta_{vis}(P - e, \mathcal{A})$ by our general position assumption. If $\hat{e}$ were not maximal, then there exists some $v''$ so that $conv(v, v', v'') \subseteq P$. Again by our general position assumption, $v, v', v''$ are not collinear, so $conv(v, v', v'')$ is a triangle inside $P$ which contains the edge $e = conv(v, v')$. But this implies that some 2-simplex in $\Delta$ must contain $e$, contradicting maximality of $e$ in $\Delta$.

The fourth assertion follows from the following claim: every polytopal decomposition $\delta = \{P_i, \mathcal{A}_i\}$ of $P$ must use the edge $e$ as one of the $P_i$. The reason is that the union of the $P_i$ must cover $P$ and hence contain $e$, but $e$ is not contained in any triangle, and there are no vertices $v''$ collinear with $v, v'$ (by our general position assumption) which could be used to subdivide $e$ or to produce a larger edge covering $e$. ∎

The next lemma allows us to assume that every vertex used in the triangulation $\Delta$ of $P$ has connected link. But first we recall some definitions from simplicial topology (see [Bj2, (9.9)]): The *link, star* and *deletion* of a face $F$ in a simplicial complex $\Delta$ are the subcomplexes defined by

$$\left.\begin{aligned} link_\Delta(F) &= \{G \in \Delta, G \cup F \in \Delta, G \cap F = \varnothing\} \\ star_\Delta(F) &= \{G \in \Delta, G \cup F \in \Delta\} \\ del_\Delta(F) &= \{G \in \Delta, G \cap F = \varnothing\} \end{aligned}\right\}$$

Note that these definitions satisfy

$$\Delta = star_\Delta(F) \cup del_\Delta(F)$$
$$link_\Delta(F) = star_\Delta(F) \cap del_\Delta(F)$$
$$star_\Delta(F) = \overline{F} * link_\Delta(F)$$

where here $\overline{F}$ means the simplical subcomplex generated by $F$, and $*$ denotes the operation of *simplicial join* [Bj2, (9.5)].

**Lemma 8.** *Let $P$ be a connected $\mathcal{A}$-triangulable set in $\mathbb{R}^2$. Assume $\mathcal{A}$ is in general position (no three points collinear) $G_{vis}(P, \mathcal{A})$ is biconnected, and no edge of $P$ is maximal. Let $v$ be a vertex in the triangulation $\Delta$ of $P$ which has disconnected link. Then there exists a point set $\mathcal{A}'$ and an $\mathcal{A}'$-triangulable space $P'$ satisfying the same hypotheses as $P, \mathcal{A}$ and furthermore:*

(1) *$P$ is homotopy equivalent to the one-point wedge of the spaces $P'$ and a circle $\mathbb{S}^1$.*
(2) *$\Delta_{vis}(P, \mathcal{A})$ is homotopy equivalent to the one-point wedge of the spaces $\Delta_{vis}(P', \mathcal{A}')$ and a circle $\mathbb{S}^1$.*
(3) *$Dec(P, \mathcal{A})$ is isomorphic as a poset to $Dec(P', \mathcal{A}')$.*
(4) *The quantity*
$$\sum_{vertices\ w} \hat{\beta}_0(link_\Delta(w))$$
*is smaller for $(P', \mathcal{A}')$ than for $(P, \mathcal{A})$, where $\hat{\beta}_0$ denotes rank of the reduced 0-homology, which is simply one less than the number of connected components.*

*Proof.* Construct $\mathcal{A}'$ from $\mathcal{A}$ and $P'$ from $P$ as in Figure 1. To be precise, pick a connected component $C$ in the link $link_\Delta(v)$, and double the vertex $v$ to create a "twin" vertex $v'$ very near to $v$ at a small but generic distance in a generic direction toward the middle of an edge of $C$. Note that this is possible by our general position assumption on $\mathcal{A}$, and also $C$ must contain an edge, for if $C$ were a single vertex $\{w\}$ then the edge $vw$ would be maximal in $P$. Let $\mathcal{A}' = \mathcal{A} \cup \{v'\}$, and obtain $P'$ from $P$ by replacing the subcomplex $v * C$ in $\Delta$ with $v' * C$.

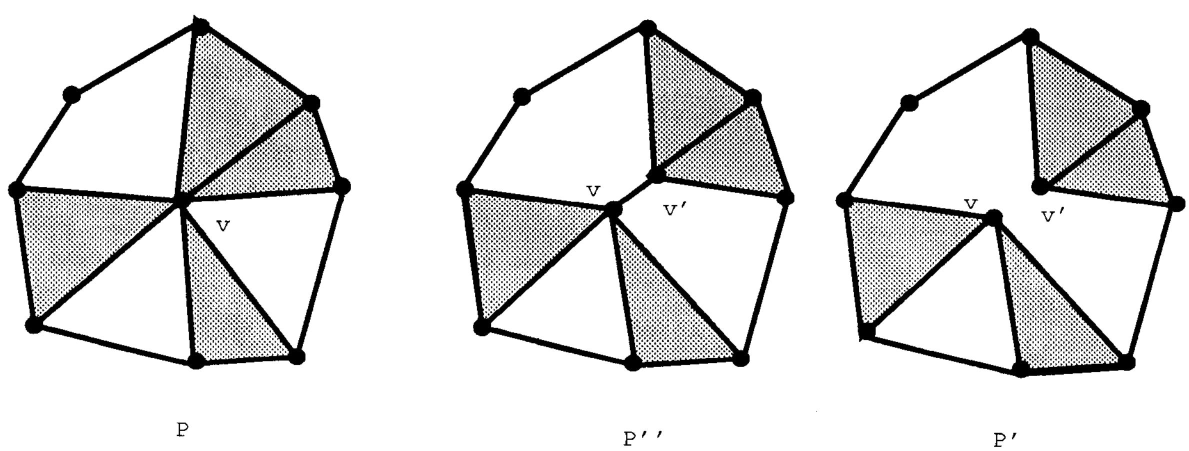



FIGURE 1. The process $P \rightsquigarrow P'' \rightsquigarrow P'$.

It is clear that $P', \mathcal{A}'$ satisfy the same hypotheses as $P, \mathcal{A}$. It will be helpful in proving the rest of the assertions of the theorem to imagine an intermediate step in this process
$$P \rightsquigarrow P'' \rightsquigarrow P'$$
where $P''$ is obtained from $P'$ by adding an edge from $v$ to $v'$ as shown in Figure 1.

To prove the first assertion, note that $P, P''$ are obviously homotopy equivalent (shrink $v'$ toward $v$ along the edge between them), and since $G_{vis}(P'', \mathcal{A}')$ is still biconnected, we can apply the second assertion of the previous lemma to show that $P'$ is homotopy equivalent to the one-point wedge of $P''$ with a circle.

To prove the second assertion, by the third assertion of the previous lemma, $\Delta_{vis}(P'', \mathcal{A}')$ is homotopy equivalent to the one-point wedge of $\Delta_{vis}(P', \mathcal{A}')$ with a circle. However, it is not quite as obvious that $\Delta_{vis}(P, \mathcal{A}), \Delta_{vis}(P'', \mathcal{A}')$ are homotopy equivalent. However this is still true, since we showed in the proof of the previous lemma that the edge $\hat{e}$ spanned by $v, v'$ in $\Delta_{vis}(P'', \mathcal{A}')$ is maximal, because the edge from $v$ to $v'$ was maximal in $P''$. Hence we can still "shrink down" $v'$ toward $v$ in $\Delta_{vis}(P'', \mathcal{A}')$ along the maximal edge $\hat{e}$, and the result is $\Delta_{vis}(P, \mathcal{A})$.

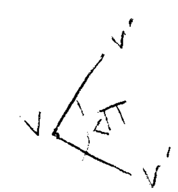
The third assertion is obvious.

The fourth assertion is easy, since the number of connected components of $link_\Delta(w)$ is unchanged for most vertices in going from $P$ to $P'$, is one smaller for $v$, while $v'$ contributes 0 to the sum for $P'$ since its link is connected. □

**Corollary 9.** *In proving Theorems 1 and 2, it suffices to consider the case where $P$ is a connected $\mathcal{A}$-triangulable 2-manifold with boundary in $\mathbb{R}^2$.*

*Proof.* By Lemmas 5-8, we may assume (using induction on various quantities) that $P$ is connected, has no maximal edges, and has the link of every vertex connected. We claim that this implies $P$ is a manifold with boundary.

To see this, choose some point $p \in P$, and we will show that it has a neighborhood homeomorphic to either $\mathbb{R}^2$ or the upper half-plane $\mathbb{R}^2_+$. If $p$ lies in the interior of a 2-simplex in the $\mathcal{A}$-triangulation $\Delta$ of $P$, then it has a neighborhood in this 2-simplex homeomorphic to $\mathbb{R}^2$. If $p$ lies in the interior of a 1-simplex of $\Delta$, then this 1-simplex lies either in one or two 2-simplices, since $P$ has no maximal edges. But this then implies $p$ has a neighborhood inside these 2-simplices homeomorphic to either $\mathbb{R}^2_+$ or $\mathbb{R}^2$ respectively. Lastly, if $p$ is a 0-simplex of $\Delta$, then its link $link_\Delta(v)$ is a connected 1-dimensional simplicial complex, in which every vertex has degree at most 2 (else $\Delta$ would not embed in $\mathbb{R}^2$). Therefore the link is either a path or a circle, so $p$ has a neighborhood homeomorphic to either $\mathbb{R}^2_+$ or $\mathbb{R}^2$, respectively, inside $star_\Delta(v) = v * link_\Delta(v)$. □




In preparation for the following two sections, we establish some notation and terminology about $(P, \mathcal{A})$ when $P$ is a connected 2-manifold with boundary in $\mathbb{R}^2$, and $\mathcal{A}$ is in general position.

Since $P$ is a 2-manifold with boundary embedded in $\mathbb{R}^2$, it follows that its boundary is a collection $\{C_i\}_{i=1}^k$ of polygonal $\mathcal{A}$-triangulable Jordan curves, and there will be one such curve $C$ containing all the rest inside it.

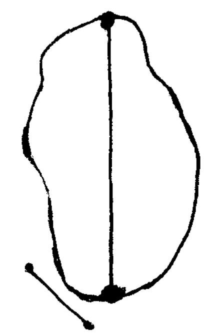
Given a vertex $v$ in $\mathcal{A} \cap C$ having nearest neighbors $v', v''$ on $C$, we will say $C$ *bends inward at $v$* if the *interior angle between $\overrightarrow{vv'}$* and $\overrightarrow{vv''}$ (i.e., the angle swept out by a ray emanating from $v$ and pointing toward the *inside* of the Jordan curve $C$) has measure less than $\pi$ radians. Note that any polygonal Jordan curve has at least one vertex $v$ at which it bends inward.

## IV. Proof of Theorem 1.

We recall here the statement of Theorem 1:

**Theorem 1.** *Let $\mathcal{A}$ be a finite set of points in general position in $\mathbb{R}^2$. Then $\Delta_{vis}(P, \mathcal{A})$ is homotopy equivalent to $P$.*

The proof of Theorem 1 is essentially a deletion-contraction argument, using induction on the cardinality $|\mathcal{A}|$. By Corollary 9, we may assume $P$ is a 2-manifold with boundary, and we let $C, v, v'$, and $v''$ be as in the end of the previous section, and assume that $C$ bends inward at $v$.



We define three other pairs of spaces and vertex sets

$$(P_{del}, \mathcal{A}_{del}),\ (P_{star}, \mathcal{A}_{star}),\ (P_{link}, \mathcal{A}_{link})$$

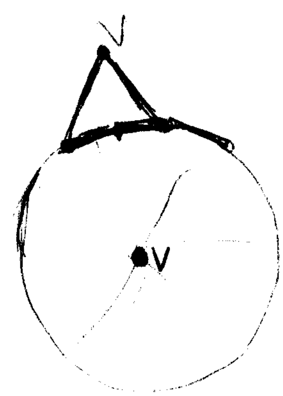
as follows:

$$P_{del} = \bigcup_{\substack{A \subseteq \mathcal{A} - v \\ conv(A) \subseteq P}} conv(A)$$

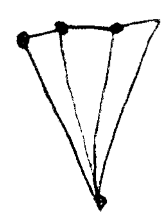
$$P_{star} = \bigcup_{\substack{v \in A \subseteq \mathcal{A} \\ conv(A) \subseteq P}} conv(A)$$

$$P_{link} = \bigcup_{\substack{v \in A \subseteq \mathcal{A} \\ conv(A) \subseteq P}} conv(A - v)$$

$$\mathcal{A}_{del} = \mathcal{A} - v$$
$$\mathcal{A}_{star} = \{v' \in \mathcal{A} : conv(v, v') \subseteq P\}$$
$$\mathcal{A}_{link} = \mathcal{A}_{star} - v.$$




Theorem 1 will follow from these two lemmas:



**Lemma 10.**

(1) *$P_{del}$ is an $\mathcal{A}_{del}$-triangulable space.*
(2) *$del_{\Delta_{vis}(P,\mathcal{A})}(v) \cong \Delta_{vis}(P_{del}, \mathcal{A}_{del})$.*
(3) *$P_{del}$ is homotopy equivalent to $P$.*

**Lemma 11.**

(1) *$P_{link}$ is an $\mathcal{A}_{link}$-triangulable space.*
(2) *$link_{\Delta_{vis}(P,\mathcal{A})}(v) \cong \Delta_{vis}(P_{link}, \mathcal{A}_{link})$.*
(3) *$P_{link}$ is contractible.*

To see why Theorem 1 follows from these lemmas (and for other purposes) we will make use of the following well-known fact about simplicial complexes (see e.g. [Bj2, Lemma 10.3]):

**Lemma 12.** *In a simplicial complex $\Delta$, if the link of a vertex $v$ is contractible, then the deletion $del_\Delta(v)$ is a deformation retract of $\Delta$.* ← HOW DO YOU PROVE THIS?

Using Lemma 11 assertion (2), we have that $link_{\Delta_{vis}(P,\mathcal{A})}(v) \cong \Delta_{vis}(P_{link}, \mathcal{A}_{link})$, and $\Delta_{vis}(P_{link}, \mathcal{A}_{link})$ is homotopy equivalent to the space $P_{link}$ by induction on $|\mathcal{A}|$, so it is contractible by Lemma 11 assertion (3). Hence $\Delta_{vis}(P, \mathcal{A})$ is homotopy equivalent to $del_{\Delta_{vis}(P,\mathcal{A})}(v)$ by Lemma 12, and by Lemma 10 assertion 2 this is isomorphic to $\Delta_{vis}(P_{del}, \mathcal{A}_{del})$. But this is homotopy equivalent to $P_{del}$ by induction, and hence to $P$ by Lemma 10 assertion (3).

Therefore it only remains to prove Lemmas 10 and 11.

*Proof of Lemma 10.* Our strategy will be to change the given $\mathcal{A}$-triangulation $\Delta$ of $P$ into another triangulation $\Delta'$ which restricts to a triangulation of $\mathcal{A}_{del}$. To this end, denote by $v' = v_1, v_2, \ldots, v_{k-1}, v_k = v''$ the sequence of vertices in the path $link_\Delta(v)$ (we know this link is a path from $v'$ to $v''$, since $v$ is a boundary vertex in the triangulation of a 2-manifold with boundary, and $v', v''$ are its nearest neighbors in the boundary). It follows that $star_\Delta(v)$ is the subcomplex generated by the triangles $conv(v, v_i, v_{i+1})$ for $i = 1, 2, \ldots, k-1$ (see Figure 2a), and we will only change $\Delta$ within this subcomplex to obtain $\Delta'$, leaving the rest of $\Delta$ alone. So for our purposes, one may as well excise the rest, replacing $P, \Delta$ with $\|star_\Delta(v)\|, star_\Delta(v)$, and replacing $\mathcal{A}$ with $\mathcal{A} \cap star_\Delta(v)$.

Now let $v' = w_1, w_2, \ldots, w_{k'-1}, w'_k = v''$ be the vertices in the path which forms the part of the boundary of $conv((\mathcal{A} - v) \cap star_\Delta(v))$ which is not "obscured" from $v$, i.e. those points $x$ in $conv((\mathcal{A} - v) \cap star_\Delta(v))$ for which $conv(x, v) \subseteq P$ and there are no other points of $conv((\mathcal{A} - v) \cap star_\Delta(v))$ on $conv(x, v)$ (see Figure 2a). b? By the definition of "obscuring" the two paths $v' = v_1, v_2, \ldots, v_{k-1}, v_k = v''$ and $v' = w_1, w_2, \ldots, w_{k'-1}, w'_k = v''$ fit together at their endpoints $v', v''$, with the latter path always weakly "farther" from $v$ than the first, and so together they bound a sequence of Jordan curves in the plane (Figure 2b). We then obtain $\Delta'$ by replacing the triangulation of $star_\Delta(v)$ with arbitrary triangulations of the interiors of these Jordan curves as in Lemma 4, and then adding the triangles $conv(v, w_i, w_{i+1})$.

We now claim that $del_{\Delta'}(v)$ triangulates $P_{del}$. To see this note that every simplex of $del_{\Delta'}(v)$ lies in $P_{del}$ since it lies in $P$ and doesn't involve the vertex $v$. Conversely, we need to show every point $x$ of $P_{del}$ lies in some simplex of $del_{\Delta'}(v)$. Such a point $x$ lies in some simplex $F$ of $\Delta$, and if $F$ does not involve $v$, then it is still a simplex of $\Delta'$. If $F$ does involve $v$, then we still must have that $x$ lies in $star_\Delta(v)$, but on the "other side" of the path $v' = w_1, w_2, \ldots, w_{k'-1}, w'_k = v''$ from $v$ (by the definition of "obscuring" and the fact that $x \in P_{del}$). So $x$ ends up inside one of the sequences of Jordan curves from the previous paragraph, which means it lies in $del_{\Delta'}(v)$.

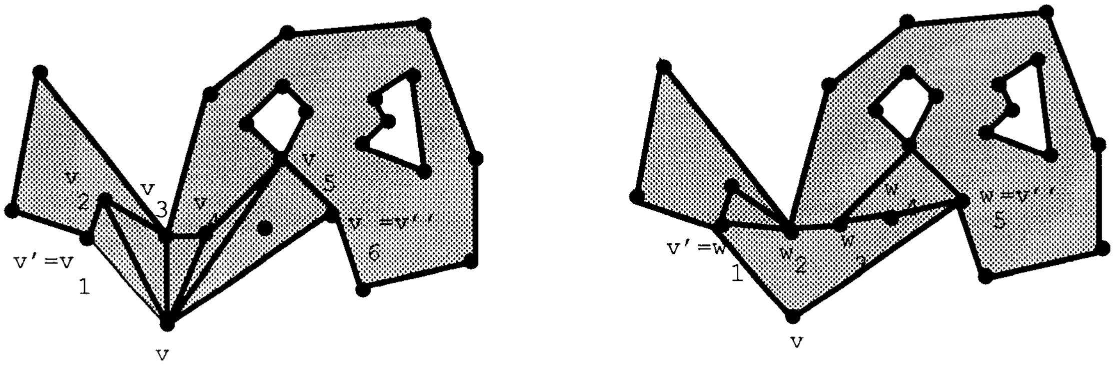
From this claim, assertion (1) of Lemma 10 is now clear, and assertion (3) follows from Lemma 12, since $link_{\Delta'}(v)$ is the contractible path $v' = w_1, w_2, \ldots, w_{k'-1}, w'_k = v''$. Assertion (2) is obvious once we know that $P_{del}$ is $\mathcal{A}_{del}$-triangulable. $\square$

*Proof of Lemma 11.* Rather than showing the assertions of Lemma 11 directly, note that if we can show $P_{star}$ is an $\mathcal{A}_{star}$-triangulable contractible space, then Lemma 11 immediately follows from Lemma 10 applied with $\mathcal{A} = \mathcal{A}_{star}$ and $P = P_{star}$. Contractibility of $P_{star}$ is obvious since it is star-shaped with respect to $v$.

Therefore our strategy will be to change the given $\mathcal{A}$-triangulation $\Delta$ of $P$ into another triangulation which restricts to a triangulation of $\mathcal{A}_{star}$. To do this, we proceed in two steps:

**Step 1**. In the first step, we change $\Delta$ into a triangulation $\Delta'$ which uses every edge $conv(v, w)$ for which $w \in \mathcal{A}$ and $conv(v, w) \subseteq P$. To see this can be done, assume that $w$ is such a vertex which is *not* in such an edge of $\Delta$, and we will show that one can alter $\Delta$ so as to increase the number of such edges.

Consider the set of triangles (2-simplices) $\{T_i\}$ in $\Delta$ which one crosses while traversing the line segment from $v$ to $w$. By our general position assumption, the




(a) (b)

FIGURE 2. Changing $\Delta$ to $\Delta'$.

line segment crosses the boundaries of each $T_i$ through the interior of an edge, and hence the union of the $T_i$'s generates a subcomplex $T$ of $\Delta$ whose boundary consists of two paths $L_1, L_2$ from $v$ to $w$, with the two paths separated by the line segment $conv(v, w)$ (see Figure 3a). Lemma 4

Therefore $conv(v, w) \cup L_1, conv(v, w) \cup L_2$ form two polygonal Jordan curves, and if we triangulate their interiors arbitrarily, we can replace the subcomplex $T$ in $\Delta$ with this new triangulation to obtain $\Delta'$. Since we did not remove any edges between $v$ and other vertices, but added the edge from $v$ to $w$, this does what was promised, and Step 1 is complete.

**Step 2.** Now let $\Delta'$ be any $\mathcal{A}$-triangulation of $P$ satisfying these two properties:

(1) Every vertex $w$ in $\mathcal{A}$ which is visible from $v$ lies inside $||star_{\Delta'}(v)||$.
(2) The set of vertices contained in $link_{\Delta'}(v)$ is minimal under inclusion among all triangulations satisfying (1).

We know from Step 1 that one can produce a $\Delta'$ satisfying (1), and hence one can find one that also satisfies (2).

We now claim, that such a $\Delta'$ has $star_{\Delta'}(v)$ triangulating $\mathcal{A}_{star}$. To prove this, note that every simplex in $star_{\Delta'}(v)$ lies in $\mathcal{A}_{star}$ by definition. Conversely, we need to show every point $x$ of $P_{star}$ lies in some simplex of $star_{\Delta'}(v)$.

By definition such a point $x$ lies in some set $conv(A) \subseteq P$ where $v \in A$ and so it is easy to see that $x$ must lie in some triangle $conv(v, w, w')$ where $w, w' \in A$ . Since both $w, w'$ are visible from $v$, they lie inside $||star_{\Delta'}(v)||$. If $x$ also lies inside $||star_{\Delta'}(v)||$ we are done, so assume not and we will reach a contradiction. It is easy to see (Figure 3b) that if $x \notin ||star_{\Delta'}(v)||$, there must be some triple of vertices $u', u, u''$ which are consecutive (in this order) on the path $link_{\Delta'}(v)$, and for which $u \in conv(v, u', u'')$. But we claim then that there are no points in $\mathcal{A} \cap conv(u, u', u'')$, since they would be visible from $v$ but not contained in $||star_{\Delta'}(v)||$. This then allows us to do an argument like the one in the second paragraph of the proof of Lemma 10 (replacing $v, v', v''$ by $u, u', u''$) to show that we can alter $\Delta'$ outside of $star_{\Delta'}(v)$ so that $conv(u, u', u'')$ forms a triangle in the triangulation. Then we can replace the three triangles $conv(v, u, u'), conv(v, u, u''), conv(u, u', u'')$ in this triangulation with the single triangle $conv(v, u', u'')$, eliminating $u$ as a vertex in $link_{\Delta'}(v)$, and contradicting the minimality of $\Delta'$. $\square$

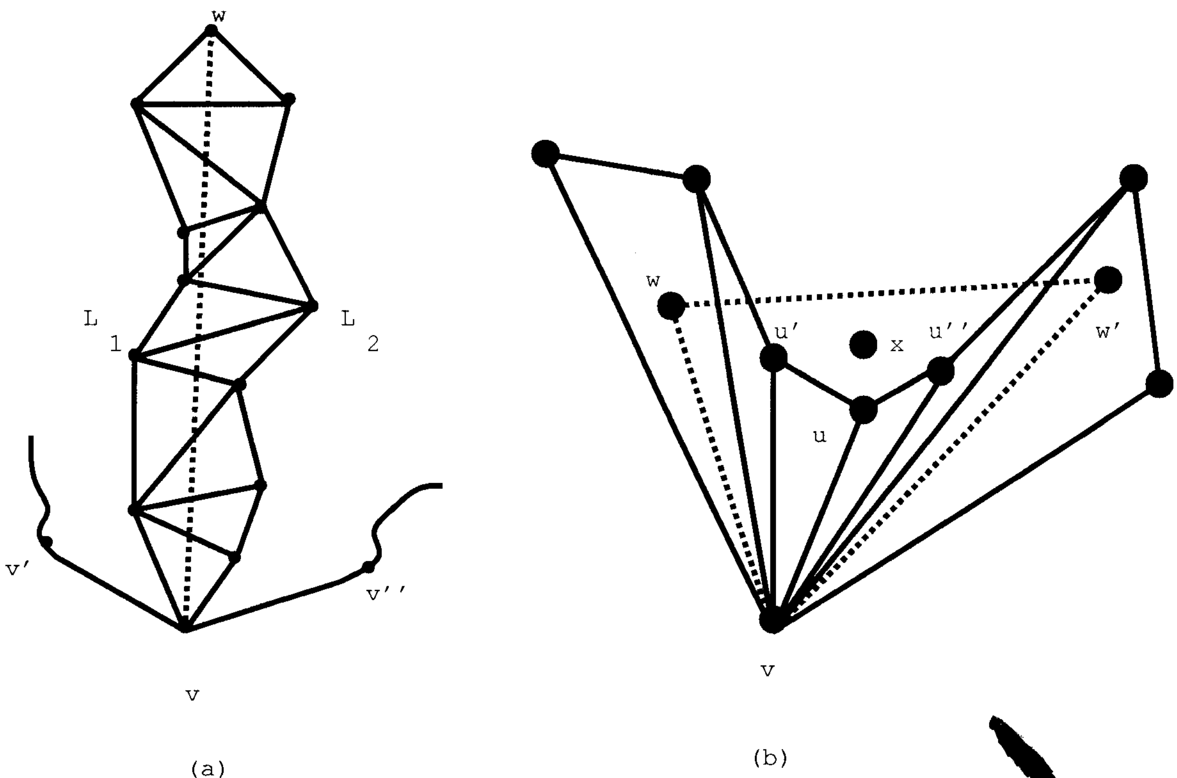



FIGURE 3. (a) Step 1 (b) Step 2

This completes the proof of Theorem 1. Before closing this section, we mention a stronger statement which we were not able to prove. Note that for any $\mathcal{A}$-triangulable space $P$ in $\mathbb{R}^d$, there is an obvious surjection

$$\phi : \|\Delta_{vis}(P, \mathcal{A})\| \to P$$

induced by the map which sends a vertex $v \in \mathcal{A}$ to the point $v \in \mathbb{R}^d$ which it represents.

**Conjecture 13.** *For $\mathcal{A}$ in general position in $\mathbb{R}^2$, the map $\phi$ induces the homotopy equivalence between $\Delta_{vis}(P, \mathcal{A})$ and $P$.*

## V. Proof of Theorem 2.

In this section we prove Theorems 2 and 3, again using a variant of the deletion-contraction method, in combination with Quillen's Fiber Lemma. We should mention that the relation between deletion and contraction and *regular* triangulations of $\mathcal{A}$ (when $P = conv(\mathcal{A})$) was studied by Billera, Gel'fand and Sturmfels [BGS] in the context of secondary polytopes. However the application of these ideas to an arbitrary $\mathcal{A}$-triangulable space $P$ and the Baues problem appears to be new. We should perhaps also mention that this argument bears a slight resemblance to the proof of Theorem 1.2 in [BKS].

We recall here the statement of Theorem 2.

**Theorem 2.** *Let $\mathcal{A}$ be a finite set of points in general position in $\mathbb{R}^2$. Then $Dec(P, \mathcal{A})$ is contractible.*

As shown in Section III it suffices to prove Theorem 2 when $P$ is a 2-manifold with boundary. We let $v, v', v''$ and $C$ have the same meaning as in the end of that section.

Let $v'/v, v''/v$ be points chosen close enough to $v$ on the rays $\overrightarrow{vv'}, \overrightarrow{vv''}$ respectively, so that the triangle $conv(v, v'/v, v''/v)$ lies in $P$ and contains no other points of $\mathcal{A}$. Define the *vertex figure* $P/v$ of $P$ at $v$ to be the line segment $conv(v'/v, v''/v)$. When $P = conv(\mathcal{A})$ this notion of vertex figure coincides with the classical notion of the vertex figure of $v$ on the boundary of the polygon $P$ (see [Gr, page 49]). Also define the *contraction* $\mathcal{A}/v$ to be the collection of points $\{w/v\}$ on the line segment $P/v$ obtained by taking each point $w \in \mathcal{A}$ visible from $v$, and intersecting the line segment $conv(v, w)$ with $P/v$. Note that because of our general position assumption on $\mathcal{A}$, all the points $\{w/v\}_{w \in \mathcal{A}}$ are distinct. Clearly $P/v$ is an $\mathcal{A}/v$-triangulable space. Therefore we have the notion of its decomposition poset $Dec(P/v, \mathcal{A}/v)$. is this a cube?

Define a map

$$f : Dec(P, \mathcal{A}) \to Dec(P/v, \mathcal{A}/v)$$

as follows: given $\delta = \{(P_i, \mathcal{A}_i)\} \in Dec(P, \mathcal{A})$, let $f(\delta) = \{(P_i/v, \mathcal{A}_i/v) : v \in P_i\}$, where $P_i/v$ is the line segment connecting the two neighbors of $v$ in the boundary of $P_i$, and $\mathcal{A}_i/v = \{w/v : w \in \mathcal{A}_i\}$. It is trivial to check this map is order-preserving.

yes ... almost.

We will make use of Quillen's Fiber Lemma [Bj2, (10.5)]:

**Lemma 14.** *Let $f : Q \to Q'$ be an order-preserving map of posets. If $f^{-1}(Q'_{\leq q'})$ is contractible for all $q' \in Q'$, then $f$ induces a homotopy equivalence of the associated order complexes $\Delta(Q) \to \Delta(Q')$.*

We will refer to the subposet $f^{-1}(Q'_{\leq q'})$ in the theorem as the *Quillen fiber* of $f$ at $q'$. Our immediate goal is to show that for the map $f$ defined above, the non-empty Quillen fibers are indeed contractible, so we need to describe these fibers more concretely.

For $\delta = \{(P_i, \mathcal{A}_i)\} \in Dec(P, \mathcal{A})$, define the *deletion* $del_\delta(v)$ to be the union of all polygons $P_i$ which do *not* contain $v$. By our remarks in Section 2, this is an $(\mathcal{A} \cap del_\delta(v))$-triangulable space, and hence by induction has contractible decomposition poset $Dec(del_\delta(v), \mathcal{A} \cap del_\delta(v))$.

**Lemma 15.** *Let $\delta'$ be in the image of the map $f : Dec(P, A) \to Dec(P/v, \mathcal{A}/v)$, and let $\delta$ be any pre-image of $\delta'$, i.e., $f(\delta) = \delta'$. Then the Quillen fiber*

$$f^{-1}(Dec(P/v, \mathcal{A}/v)_{\leq \delta'})$$

*is isomorphic to*

$$Dec(P/v, \mathcal{A}/v)_{\leq \delta'} \times Dec(del_\delta(v), \mathcal{A} \cap del_\delta(v)).$$

*Proof.* Every $\gamma = \{(P_i, \mathcal{A}_i)\}$ in $f^{-1}(Dec(P/v, \mathcal{A}/v)_{\leq \delta'})$ breaks into two pieces: the $(P_i, \mathcal{A}_i)$ for which $v \in P_i$, giving rise to an element of $Dec(P/v, \mathcal{A}/v)$ which refines $\delta'$, and the $(P_i, \mathcal{A}_i)$ for which $v \notin P_i$, giving rise to an element of $Dec(del_\delta(v), \mathcal{A} \cap del_\delta(v))$. Conversely, any such pair in the above Cartesian product gives rise to an element $\gamma$ in $f^{-1}(Dec(P/v, \mathcal{A}/v)_{\leq \delta'})$. $\square$

The two factors of the Quillen Fiber described in Lemma 15 are both contractible; the first is contractible since it is a principal order ideal in the poset $Dec(P/v, \mathcal{A}/v)$, and the second is contractible by induction on $|\mathcal{A}|$. Therefore by Lemma 15 we conclude that $Dec(P, \mathcal{A})$ is homotopy equivalent to the image of the map $f$. It therefore only remains to identify the image of $f$, and prove that it is contractible. We first identify the image.

Let $Dec_{vis}(P/v, \mathcal{A}/v)$ be the subposet of $Dec(P/v, \mathcal{A}/v)$ consisting of those $\delta = \{(P_i/v, \mathcal{A}_i/v)\}$ in which each 1-polygon (line segment) $P_i/v$ occurring satisfies

$$conv(\mathcal{A}_i \cup v) \subseteq P,$$

OK ✓

where $\mathcal{A}_i$ is the set $\{w\}_{w/v \in \mathcal{A}_i/v}$. It is easy to see that this subposet is an order ideal in $Dec(P/v, \mathcal{A}/v)$, and it follows immediately from the definition of the map $f$ that its image lies in $Dec_{vis}(P/v, \mathcal{A}/v)$. It is not quite as obvious that this is *exactly* the image of $f$:

surjective

**Lemma 16.** *The image of $f : Dec(P, \mathcal{A}) \to Dec(P/v, \mathcal{A}/v)$ is $Dec_{vis}(P/v, \mathcal{A}/v)$.*

*Proof.* By the previous remarks, it suffices to show that for any

$$\delta' = \{(P_i/v, \mathcal{A}_i/v)\}_{i=1}^{k} \in Dec_{vis}(P/v, \mathcal{A}/v),$$

there is some $\delta \in Dec(P, \mathcal{A})$ with $f(\delta) = \delta'$. For each $i = 1, 2, \ldots, k$, let $\mathcal{A}_i$ be the set $\{w\}_{w/v \in \mathcal{A}_i/v}$ and $P_i = conv(\mathcal{A}_i)$. This yields a set $\{(P_i, \mathcal{A}_i)\}_{i=1}^{k}$, and we wish to extend this set by more polygons $\{(P_i, \mathcal{A}_i)\}_{i=k+1}^{l}$ to obtain a polytopal decomposition $\delta = \{(P_i, \mathcal{A}_i)\}_{i=1}^{l}$ of $P$. If this can be done, then clearly $f(\delta) = \delta'$. +v !

To do this, we need only show that $P$ has an $\mathcal{A}$-triangulation $\Delta'$ in which $star_{\Delta'}(v)$ is a triangulation of $\bigcup_{i=1}^{k} P_i$, since then the maximal simplices in $del_{\Delta'}(v)$ will give us the remaining $\{(P_i, \mathcal{A}_i)\}_{i=k+1}^{l}$ that are needed to make up $\delta$. So start with the triangulation $\Delta$ that was constructed in the proof of Lemma 11, i.e. one such that $star_{\Delta}(v)$ is an $\mathcal{A}_{star}$-triangulation of $P_{star}$. Since $\bigcup_{i=1}^{k} P_i \subseteq P_{star}$, the paths $link_{\Delta}(v)$ and $\bigcup_{i=1}^{k} link_{P_i}(v)$ (where $link_{P_i}(v)$ means the boundary segments of $P_i$ which do not contain $v$ as a vertex) both connect $v'$ to $v''$, and bound a sequence of polygonal Jordan curves between them (see Figure 4a). Therefore if we replace $star_{\Delta}(v)$ in $\Delta$ with any triangulation of these Jordan curves (as in Lemma 4) and any refinement of $\bigcup_{i=1}^{k} link_{P_i}(v)$ to a triangulation, we get what we want. $\square$

It therefore only remains in the proof of Theorem 2 to show that $Dec_{vis}(P/v, \mathcal{A}/v)$ is contractible, which we will do using a second deletion-contraction/Quillen fiber argument, via induction on $|\mathcal{A}|$.

First of all, it should be fairly clear that when considering $Dec_{vis}(P/v, \mathcal{A}/v)$, the part of $P$ outside $P_{star}$ is irrelevant, and therefore we may as well excise it and assume $P = P_{star}$. Define a map

$$g : Dec_{vis}(P/v, \mathcal{A}/v) \to 2^{\mathcal{A}}$$

as follows: if $\delta = \{(P_i/v, \mathcal{A}_i/v)\}_{i=1}^{k} \in Dec_{vis}(P/v, \mathcal{A}/v)$ then there is a unique line segment $P_1/v$ containing the vertex $v'/v \in P/v$, and we set

$$g(\delta) = \{w \in \mathcal{A} : w/v \in \mathcal{A}_1/v, w \neq v'\}.$$

It is easy to check this map is order-preserving, if we order $2^{\mathcal{A}}$ by inclusion of sets.

Let $(\tilde{P}, \tilde{\mathcal{A}})$ denote the pair obtained from $(P, \mathcal{A})$ by doing the $(P_{link}, \mathcal{A}_{link})$ construction with respect to $v$. Then, let $(\hat{P}, \hat{\mathcal{A}})$ denote the pair obtained from $(\tilde{P}, \tilde{\mathcal{A}})$ by doing the $(\tilde{P}_{link}, \tilde{\mathcal{A}}_{link})$ construction with respect to $v'$ (since one can easily check that the boundary curve of $\tilde{P}$ must bend inward at $v'$). Notice that the sets $A$ which are in the image of $g$ must have the property that $conv(A \cup \{v, v'\}) \subseteq P$). Therefore the image of the map $g$ lies in $\Delta_{vis}(\hat{P}, \hat{\mathcal{A}})$. In fact, this characterizes the image:

**Lemma 17.** *The image of $g$ is $\Delta_{vis}(\hat{P}, \hat{\mathcal{A}})$.*

*Proof.* By the previous remarks, we need to show that any set $A \subseteq \mathcal{A} - \{v, v'\}$ for which $conv(A \cup \{v, v'\}) \subseteq P$ is in the image of $g$. Tracing this back through the definition of $g$, we need to show there is a $\delta' \in Dec(P/v, \mathcal{A}/v)$ having $conv(A/v)$ as one of its line segments, which is equivalent by Lemma 16 to showing there is a $\delta \in Dec(P, \mathcal{A})$ having $conv(A \cup \{v, v'\})$ as one of its polygons. It would suffice then to show that we can extend some triangulation of $conv(A \cup \{v, v'\})$ to a triangulation of $P$. This follows by the usual argument. The area $P - conv(A \cup \{v, v'\})$ is bounded by a sequence of Jordan curves formed by all the boundary edges of $conv(A \cup \{v, v'\})$ except $conv(v, v')$, and the edges of the link of $v$ in the triangulation $\Delta$ constructed in Lemma 11 which has $star_{\Delta}(v)$ triangulating $P = P_{star}$ (Figure 4b). Therefore one can triangulate the inside of each of these Jordan curves arbitrarily by Lemma 4. $\square$

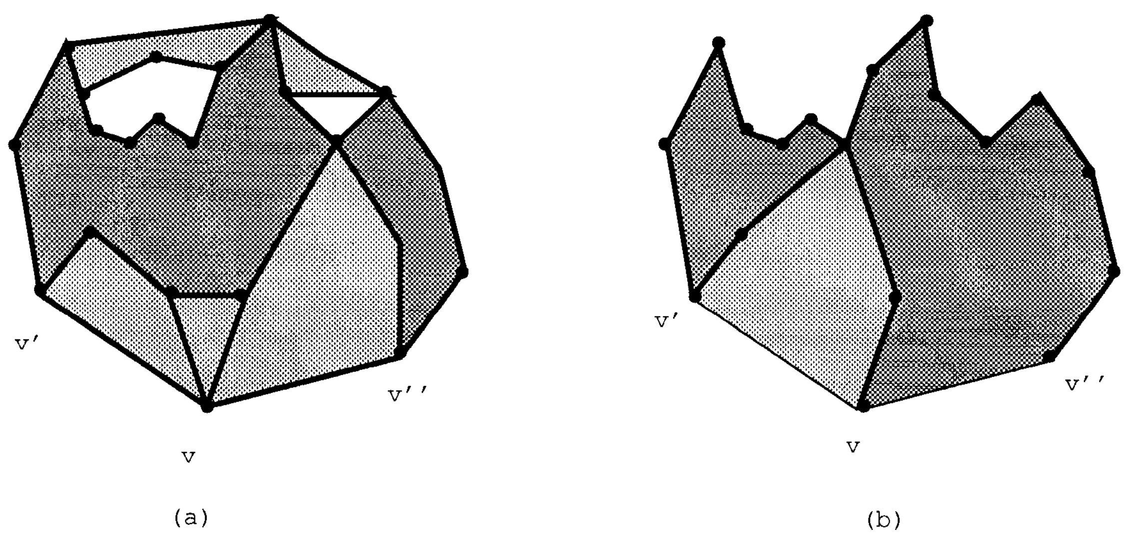



FIGURE 4. Sequences of Jordan curves

Since $\hat{P}$ comes from applying the $P \rightsquigarrow P_{link}$ construction, it is contractible by Lemma 11, and hence $\Delta_{vis}(\hat{P}, \hat{\mathcal{A}})$ is contractible by Theorem 1. Therefore the contractibility of $Dec_{vis}(P/v, \mathcal{A}/v)$ will follow from Lemma 14, if we can show that the Quillen fibers of $g$ are contractible.

**Lemma 18.** *Let $A \in \Delta_{vis}(\hat{P}, \hat{\mathcal{A}})$, i.e.,*

$$A \subseteq \mathcal{A} - \{v, v'\} \text{ and } conv(A \cup \{v, v'\}) \subseteq P$$

*and let $w \in A$ be such that $w/v$ is the farthest vertex from $v'/v$ in $A/v$ on the line segment $P/v$. Then the Quillen Fiber*

$$g^{-1}(\Delta_{vis}(\hat{P},\hat{A})_{\leq A}) \cong \begin{cases} 2^{A-w} \times Dec_{vis}(P'/v, \mathcal{A}'/v) & \text{if } w \neq v'' \\ 2^{A-w} & \text{if } w = v'' \end{cases}$$

*where $P'$ is an $\mathcal{A}'$-triangulable 2-manifold with boundary bending inward at $v$, and $|\mathcal{A}'| < |\mathcal{A}|$.*

*Proof.* Any $\delta \in g^{-1}(\Delta_{vis}(\hat{P},\hat{A})_{\leq A})$ is completely determined by two pieces of data: (1) the set of its endpoints (other than $v'/v, w/v$) used in the line segments with which it subdivides the line segment $conv(v'/v, w/v)$, and (2) its restriction to a polytopal decomposition of $P/v - conv(v'/v, w/v)$. If $w \neq v''$, then the latter restriction is an element of $Dec_{vis}(P'/v, \mathcal{A}'/v)$, where $P'$ is the closure of the interior of the unique Jordan curve containing $v''$ in the proof of Lemma 17, and $\mathcal{A}' = \mathcal{A} \cap P'$. Conversely any such pair in $2^{A-w} \times Dec_{vis}(P'/v, \mathcal{A}'/v)$ gives rise to an element in the fiber. □

The contractibility of $Dec_{vis}(P/v, \mathcal{A}/v)$ now follows, since both factors in the Quillen fiber are contractible: $2^{A-w}$ is a poset with a maximum element, and $Dec_{vis}(P'/v, \mathcal{A}'/v)$ is contractible by induction on $\mathcal{A}$.

The proof of Theorem 2 is now complete. From the proof we deduce Theorem 3.

**Theorem 3.** *Assume $\mathcal{A}$ is in general position in $\mathbb{R}^2$. If $P = conv(\mathcal{A})$, then $Dec(P, \mathcal{A})$ has a unique top element $\hat{1}$, and $Dec(P, \mathcal{A}) - \hat{1}$ is homotopy equivalent to a sphere of dimension $|\mathcal{A}| - 3$.*

*Proof.* When $P$ is convex, i.e. $P = conv(\mathcal{A})$, the map $f : Dec(P, \mathcal{A}) \to Dec(P/v, \mathcal{A}/v)$ is a surjection since $Dec(P/v, \mathcal{A}/v) = Dec_{vis}(P/v, \mathcal{A}/v)$. Furthermore, the top element $\hat{1}$ in $Dec(P, \mathcal{A})$ is the polytopal decomposition $\{(P, \mathcal{A})\}$, having a single polygon, and this is the unique pre-image of the top element $\hat{1}/v$ of $Dec(P/v, \mathcal{A}/v)$ consisting of the single pair $\{(P/v, \mathcal{A}/v)\}$. Therefore $Dec(P, \mathcal{A}) - \hat{1}$ surjects onto $Dec(P/v, \mathcal{A}/v) - \hat{1}/v$, and induces a homotopy equivalence since we have already shown the fibers are contractible. But $Dec(P/v, \mathcal{A}/v) - \hat{1}/v$ is well-known (see e.g. [BKS, Example 3.1]) to be the face poset of the boundary of an $(|\mathcal{A}/v| - 2)$-cube: identify the vertices of the cube with the subsets of vertices (other than $v'/v, v''/v$) used in the triangulations of $P/v$, which are the minimal elements of $Dec(P/v, \mathcal{A}/v) - \hat{1}/v$. Since $|\mathcal{A}/v| = |\mathcal{A}| - 1$, Theorem 3 follows. □

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