SOME CONCRETE ASPECTS OF HILBERT'S 17TH PROBLEM

Bruce Reznick

University of Illinois

This paper is dedicated to the memory of Raphael M. Robinson and Olga Taussky Todd.

1. Introduction

Hilbert's 17th Problem asks whether a real positive semidefinite (psd) polynomial in several variables must be a sum of squares of rational functions. This paper gives a survey of the literature on two closely related questions: **What can be said about a psd polynomial which is *not* a sum of squares of polynomials? *How* can one write a given psd polynomial as a sum of squares of rational functions?** These questions go back to Hilbert himself, and his interest in them predated the 1900 Paris Congress.

My original presentation in the seminar consisted of two parts: the first was a summary of the history of the answers to the first question, the second was a detailed exposition of my recent contribution towards understanding the second question for positive definite forms. The paper on which I based the second part has now appeared in print [63], and so I have emphasized the first part in this paper.

Sadly, two mathematicians influential to the development of this subject have passed away this year. Raphael M. Robinson (1911–1995) died on January 25 and Olga Taussky Todd (1906–1995) died on October 7. The reader will see below the vital contributions made by Professors Robinson and Taussky.

I thank Danielle Gondard for the opportunity to speak (in English!) in her seminar. I would also like to express my gratitude to the members of the seminar (and to many other patient audiences) for their insights, tolerance and good humor during my presentations. Finally, I thank Man-Duen Choi, Chip Delzell, Tsit-Yuen Lam, David Leep, Lou van den Dries and Beate Zimmer for their assistance in preparing this paper.

2. Notations

It is becoming standard to let $H_d(K^n)$ denote the set of homogeneous forms of degree d in n variables ("n-ary d-ics") with coefficients from the field K. By identifying $p \in H_d(K^n)$ with its $N = \binom{n+d-1}{n-1}$-tuple of coefficients, we see that $H_d(K^n) \approx K^N$. Suppose m is an even integer. A form $p \in H_m(\mathbf{R}^n)$ is called *positive semidefinite* or *psd* if $p(x_1, \dots, x_n) \geq 0$ for all $(x_1, \dots, x_n) \in \mathbf{R}^n$. Following [12], we denote the set of psd forms in $H_m(\mathbf{R}^n)$ by $P_{n,m}$. Since $P_{n,m}$ is closed under addition and closed under multiplication by positive

The author was supported in part by NSF grant DMS 95-00507 and NATO grant CRG 930760

scalars, it is a convex cone. In fact, $P_{n,m}$ is a *closed* convex cone: if $p_n \to p$ coefficientwise, and each p_n is psd, then so is p. A psd form is called *positive definite* if $p(x_1, \dots, x_n) = 0$ implies $(x_1, \dots, x_n) = 0$. It is not difficult to see that the positive definite forms constitute the interior of the cone $P_{n,m}$.

A form $p \in H_m(\mathbf{R}^n)$ is called a *sum of squares* or *sos* if it can be written as a sum of squares of polynomials. It is easy to show that if $p \in H_m(\mathbf{R}^n)$ and $p = \sum_k h_k^2$ with $h_k \in \mathbf{R}[x_1, \dots, x_n]$, then each $h_k \in H_{m/2}(\mathbf{R}^n)$. Again following [12], we denote the set of sos forms in $H_m(\mathbf{R}^n)$ by $\Sigma_{n,m}$. It is easy to see that $\Sigma_{n,m}$ is a convex cone; less so that it is closed; this was first proved by R. M. Robinson [68]. Finally, we note the inclusion $\Sigma_{n,m} \subseteq P_{n,m}$ and define $\Delta_{n,m} = P_{n,m} \setminus \Sigma_{n,m}$. If $p \in \Delta_{n,m}$, then p can be construed as lying in $\Delta_{n_1,m}$ for $n_1 \ge n$; for even $m_1 \ge m$, it is easy to show that $x_1^{m_1 - m} p \in \Delta_{n,m_1}$.

One may, of course, dehomogenize a forms into a polynomial by setting, say, $x_n = 1$, and in this way reduce the number of variables by 1. Any polynomial $f(x_1, \dots, x_n)$ of degree d over K can be homogenized into a form $p \in H_e(K^{n+1})$ with $e \ge d$, by adding a new variable y, and defining $p(x_1, \dots, x_n, y) = y^e f(x_1/y, \dots, x_n/y)$. The properties of being psd and sos are inherited under dehomogenization, and conversely, are preserved when a polynomial is homogenized into a form of even degree. **However, the property of being positive definite is not preserved upon homogenization.** For example, $f(x,y) = x^2 + (1 - xy)^2$ takes only positive values for real (x,y), but its homogenized form $p(x,y,z) = x^2z^2 + (z^2 - xy)^2$ has a non-trivial zero at $(1,0,0)$ owing to f's "zero at infinity". Many of the results in the literature were originally presented in the non-homogeneous case, and in most contexts, it does not matter much which is used. We prefer to phrase all results in terms of forms.

3. Hilbert's 17th Problem

> "It is a truth universally acknowledged, that a mathematical object whose orderings are non-negative must be in want of a representation as a sum of squares."
> – after Jane Austen

a. Before 1900. It was well-known by the late 19th century that $P_{n,m} = \Sigma_{n,m}$ when $n = 2$ or $m = 2$. This is easy for $m = 2$: any psd n-ary quadratic form can be diagonalized as a sum of n squares of linear forms. If $p(x,y) \in P_{2,m}$, then $f(t) = p(t,1) \ge 0$ for all real t, so the roots of f are either real (with even multiplicity) or appear in complex conjugate pairs, and the leading coefficient of f is positive. Thus, we have the expression

$$\begin{aligned} f(t) &= c^2 \prod_{j=1}^{r} (t - t_j)^{2a_j} \prod_{k=1}^{s} \big(t - (\alpha_k + i\beta_k)\big) \prod_{k=1}^{s} \big(t - (\alpha_k - i\beta_k)\big) \\ &= P(t)^2 \big(Q(t) + iR(t)\big)\big(Q(t) - iR(t)\big) = (P(t)Q(t))^2 + (P(t)R(t))^2. \end{aligned}$$

It follows upon homogenizing f that p is also a sum of two polynomial squares. Note that, if f is a product of m distinct linear factors in $\mathbf{C}[t]$, then there are $2^{m/2-1}$ possible distinct pairs $\{Q + iR, Q - iR\}$, and that many inequivalent representations of p as a sum of two squares. For example,

$$\begin{aligned} x^6 + y^6 &= (x^3)^2 + (y^3)^2 = (x^3 - 2xy^2)^2 + (2x^2y - y^3)^2 \\ &= \left(x^3 - \tfrac{1}{2}xy^2 \pm \tfrac{\sqrt{3}}{2}y^3\right)^2 + \left(x^2y \mp \tfrac{\sqrt{3}}{2}xy^2 - \tfrac{1}{2}y^3\right)^2. \end{aligned}$$

In 1888, the 26-year old David Hilbert proved two remarkable results in one paper, [32]. First, he showed that $\Sigma_{3,4} = P_{3,4}$; in fact, he showed that $p \in P_{3,4}$ is a sum of three squares of quadratic forms. (For an elementary proof, with "three" replaced by "five", see [13]; for a modern exposition of Hilbert's proof by Cassels, see [58,pp.89-93].) Hilbert's second result is that the preceding are the *only* cases for which $\Delta_{n,m} = \emptyset$. That is, if $n \geq 3$ and $m \geq 6$ or $n \geq 4$ and $m \geq 4$, then there exist forms $p \in P_{n,m}$ which are not sos. These can be derived as noted above from forms in $\Delta_{3,6}$ and $\Delta_{4,4}$. Hilbert's proofs used the techniques of 19th century algebraic geometry. Since we shall describe shorter and simpler examples, we state Hilbert's construction of $p \in \Delta_{3,6}$ without complete proof. It is presented in the non-homogeneous setting: we obtain a polynomial $F(x,y) \geq 0$ of degree six which is not a sum of squares of polynomials. The final step in the proof is key to these constructions: if $p = \sum_k h_k^2$ and $p(u) = 0$, then $0 = \sum_k h_k^2(u)$, hence $h_k(u) = 0$ for all k. Hilbert had already isolated one of the essential principles of real algebra.

Let $\phi(x,y)$ and $\psi(x,y)$ be two cubic polynomials with no common factor which have common zeros at $\{P_1, \ldots, P_9\} \subset \mathbf{R}^2$. (This is the maximum number of zeros by Bezout's Theorem; ϕ and ψ might have complex zeros in common.) It is (or used to be) commonly known that any cubic $h(x,y)$ which vanishes at eight of the P_j's must vanish at the ninth. Choose a quadratic polynomial $0 \neq f(x,y)$ which vanishes at P_1, P_2, P_3, P_4 and P_5 and a quartic polynomial $0 \neq g(x,y)$ which vanishes at P_1, P_2, P_3, P_4 and P_5 and is singular at P_6, P_7 and P_8. (Such curves exist by constant-counting arguments since there are 5 conditions on f and $\binom{4}{2} = 6$ coefficients in a quadratic, and $5 + 3 \cdot 3 = 14$ conditions on g and $\binom{6}{2} = 15$ coefficients in a quartic.) It can then be shown that there exists λ so that

$$F(x,y) = \phi^2(x,y) + \psi^2(x,y) + \lambda f(x,y) g(x,y) \geq 0$$

for all real (x,y), and that $F(P_j) = 0$ for $1 \leq j \leq 8$ but $F(P_9) > 0$. If $F = \sum_k h_k^2$, then each h_k is a cubic and $h_k(P_j) = 0$ for $1 \leq j \leq 8$, hence $h_k(P_9) = 0$ for all k, contradicting $\sum_k h_k^2(P_9) = F(P_9) > 0$.

The most complete exposition of Hilbert's method seems to be by **Gel'fand and Vilenkin** [23,pp.232-235], which **established the connection between forms in $\Delta_{n,m}$ and the Hamburger moment problem in $n-1$ variables.** For more on this connection, see [62] and the references contained within. Robinson [68] greatly simplified Hilbert's methods and cited an unpublished example of Ellison using the original construction (see below). Ellison also generalized a key step of Hilbert's construction in [22,p.668].

The earliest published reference to [32] seems to be [40], by Hilbert's close friend Adolf Hurwitz. Hurwitz proves the arithmetic-geometric inequality by showing that for even m, the form $x_1^m + \cdots + x_m^m - m x_1 \cdots x_m$ is a sum of squares of forms. He remarks in a footnote (p. 507) that "Die Möglichkeit einer solchen Darstellung ist freilich nicht von vornherein klar. Es giebt nämlich, wie Herr Hilbert gezeigt hat, positive Formen, welche *nicht* als Summen von Formenquadraten darstellbar sind." The Hurwitz construction, which can also be found in [29,p.55], is simplified somewhat in [60].

In 1893, Hilbert [33] generalized his earlier theorem on $P_{3,4}$; his proof seems to be non-constructive and lacks a modern exposition. He proves that if $p \in P_{3,m}$, then there exists $p_1 \in P_{3,m-4}$ so that $pp_1 = h_1^2 + h_2^2 + h_3^2$, with $h_k \in H_{m-2}(\mathbf{R}^3)$. Similarly, there exists $p_2 \in P_{3,m-8}$ so that $p_1 p_2$ is a sum of three squares of forms, etc. After $s = \lfloor \frac{m}{4} \rfloor$ steps,

$p_{s-1}p_s$ is a sum of three squares, but now p_s is psd with degree 0 or 2, and so is a sum of squares. Thus, $p(p_1 \cdots p_s)^2 = (pp_1)(p_1p_2)\cdots(p_{s-1}p_s)p_s$ is a product of sums of squares of forms, and so is a sum of squares of forms. It follows that p is a sum of squares of rational functions with denominator $p_1 \cdots p_s$. Landau [45] observed that Hilbert's proof implies that $p \in P_{3,m}$ is a sum of *four* squares of rational functions. This was generalized by Pfister [52] in 1967, who proved in a celebrated theorem that every $p \in P_{n,m}$ is a sum of at most 2^{n-1} squares of rational functions. The further studies in this direction are beyond the scope of this paper.

b. Hilbert's "Hilbert's 17th Problem". In his 1900 address to the International Congress of Mathematicians in Paris [35], Hilbert posed a generalization of his results as the 17th Problem: Must every psd form p be a sum of squares of rational functions? We quote from the contemporary English translation [6,p.24] of Hilbert's address:

"A rational integral function or form in any number of variables with real coefficients such that it becomes negative for no real values of these variables, is said to be *definite*. The system of all definite forms is invariant with respect to the operations of addition and multiplication, but the quotient of two definite forms — in case it should be an integral function of the variables — is also a definite form. The square of any form is evidently always a definite form. But since, as I have shown ([32]), not every definite form can be compounded by addition from squares of forms, the question arises — which I have answered affirmatively for ternary forms ([33]) — whether every definite form may not be expressed as a quotient of sums of squares of forms. At the same time it is desirable, for certain questions as to the possibility of certain geometrical constructions, to know whether the coefficients of the forms to be used in the expression may always be taken from the realm of rationality given by the coefficients of the form represented ([34,§38])."

That is, suppose $p \in P_{n,m} \cap H_m(K^n)$ where $K \subseteq \mathbf{R}$. Hilbert's 17th Problem asks whether it true that there exist $0 < \lambda_k \in K$ and $h_k \in K(x_1, \ldots, x_n)$ so that $p = \sum_k \lambda_k h_k^2$. Upon clearing denominators, we get an equivalent formulation: do there exist $0 < \lambda_k \in K$, $q \in H_r(K^n)$ (for some r) and $g_k \in H_{m/2+r}(K^n)$ so that $pq^2 = \sum_k \lambda_k g_k^2$? The geometrical roots of Hilbert's 17th Problem will not be discussed here, but see [57].

The requirement that we allow coefficients from K might seem odd to those unfamiliar with real algebra, but it is essential. For an example of its necessity, let $K = \mathbf{Q}(\sqrt{2})$, $(n,m) = (1,0)$ and $p(x) = \sqrt{2}$, which is certainly psd as a form. If there were a representation $p(x) = \sum_k h_k^2(x)$, with $h_k \in K[x]$, then each h_k would be a constant; write $h_k = \alpha_k + \beta_k\sqrt{2}$, with $\alpha_k, \beta_k \in \mathbf{Q}$. Then $\sqrt{2} = \sum_k(\alpha_k + \beta_k\sqrt{2})^2$, hence $-\sqrt{2} = \sum_k(\alpha_k - \beta_k\sqrt{2})^2$, a contradiction to the order in $\mathbf{R}$. Initiates will recognize that $\sqrt{2}$ is negative in one ordering of K, and so is not a sum of squares in K. Lam [43,pp.16–18] discusses three aspects of Hilbert's work which might have motivated a study of formally real or ordered fields. The first is geometrical and mentioned above. The second is the 17th Problem. The third is the study of totally positive elements in number fields, which are sums of four squares by the Hilbert-Landau Theorem. Elements which are not totally positive, such as $\sqrt{2}$ above, are negative in at least one embedding into $\mathbf{R}$, and so cannot be sums of squares.

c. After 1920. In 1927, Emil Artin [1] used the Artin-Schreier theory of real closed fields to answer Hilbert's 17th Problem in the affirmative. However, given a particular psd form

$p \in P_{n,m}$, Artin's proof gives no information about any specific representation of p as a sum of squares of rational functions.

Among the many generalizations of the 17th Problem, we mention one in detail. In 1981, Becker [2,3] gave necessary and sufficient conditions for a rational function p over a formally real field to be a sum of $2k$-th powers of rational functions. For such functions over $\mathbf{R}$, the criterion is, roughly speaking, that p must be psd, its degree must be a multiple of $2k$ and all zeros must have "$2k$-th order". A concrete application [3,p.144] is that for all $k \geq 1$, there exist $0 < \lambda_{j,k} \in \mathbf{Q}$ and $f_{j,k}$, $g_{j,k}$ in $\mathbf{Q}[t]$ so that

$$B(t) = \frac{1+t^2}{2+t^2} = \sum_j \lambda_{j,k} \left(\frac{f_{j,k}(t)}{g_{j,k}(t)} \right)^{2k}.$$

As with Artin's result, one does not obtain an explicit representation of $B(t)$ as a sum of $2k$-th powers of rational functions. These are not hard to find for small k, and there was some interest in finding them for all k. We give such an expression at the end of the paper, but one in which $\lambda_{j,k}$ and the coefficients of $f_{j,k}$ and $g_{j,k}$ are not, in general, rational. One can deduce from recent work of Becker and Powers [4] that there is a representation of $B(t)$ as a sum of $2k$-th powers in which each $g_{j,k}$ is positive definite. Schmid has also recently shown [70] that if f and g are real positive definite polynomials in one variable with the same degree, then $(f/g)(t)$ can be written as above, but where $f_{j,k}$ and $g_{j,k}$ are positive definite polynomials of the *same* degree.

There are now many expositions of Artin's proof in the literature, e.g., [5,41,43,44]. Ribenboim [65] and Pfister [53] wrote surveys on Hilbert's 17th Problem in the 1970s; two more recent surveys are by Gondard [24] (in a previous collection of this seminar) and Scheiderer [69]. The deep connections of Hilbert's 17th Problem with logic were initiated by A. Robinson [66,67] in the mid-1950's; Delzell [20] has a very recent survey which includes the history of logicians' interest in Hilbert's 17th Problem.

The spectacular development of real algebra and real algebraic geometry is well-known (see e.g. [5]) and will not be further discussed here. Rajwade [58] contains detailed expositions of much of the material discussed and alluded to here, and should be read with care (see [64,72]). Lam has written two wonderful expository articles on real algebra: [43,44]. In 1982, he was awarded the Steele Prize by the AMS, in part for [43] ([44] had not yet appeared). Taussky wrote two survey articles ([74,75]) on sums of squares in algebra. The first one was particularly influential in calling attention to the ubiquitous role of sums of squares in algebra, and was awarded the Ford Prize by the MAA in 1971. Olga Taussky was always supportive and encouraging to all of us interested in sums of squares, and, as a direct link to Hilbert, embodied the intellectual continuity of mathematics.

4. Examples from the 1960s and 1970s

For reasons that may be more psychological than mathematical, it took nearly 80 years for explicit forms in $\Delta_{n,m}$ to appear in the literature. When they appeared, they were much simpler than those in [32]. Interestingly, different authors produced different examples.

a. Examples of Motzkin. The way the first one arose is described in the introduction to Theodore S. Motzkin's collected works [50,pp.xvi–xvii]: "During many of his years at

UCLA, Motzkin conducted seminars that were very exciting to the students and faculty members who participated in them. Some of Motzkin's most beautiful and important work made its first appearance here ... [D]uring a seminar on inequalities, a colleague presented Artin's solution of Hilbert's 17th [P]roblem ... Motzkin wondered out loud what would happen if the classical inequalities of the type $f(x_1, \dots, x_n) \ge 0$ (such as the arithmetic-geometric inequality, when suitably formulated) were proved by expressing f in the form $f = \sum_{i=1}^m \rho_i^2$, and in particular if the ρ_i would turn out to be polynomials. At the next meeting of the seminar he carried out this program and presented for the first time the now celebrated Motzkin polynomial ... Although some results of the seminar were published in the proceedings of a symposium at Dayton, Ohio [49], the polynomial was still not as widely known as it became after O. Taussky-Todd mentioned its existence to A. Pfister who, along with J. W. S. Cassels and W. J. Ellison, did further work in this area."

Motzkin proved [49,p.217] that for all $n \ge 3$,

$$(t_1^2 + \cdots + t_{n-1}^2 - nu^2)t_1^2 \cdots t_{n-1}^2 + u^{2n} \in \Delta_{n,2n}.$$

The form in the special case $n = 3$ was denoted S' in [13] and M in [59]. The proofs for general n are very similar to that we give here for $n = 3$. Let

$$M(x,y,z) = (x^2+y^2-3z^2)x^2y^2 + z^6 = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.$$

The fact that M is psd follows from the arithmetic-geometric inequality $\frac{a+b+c}{3} \ge (abc)^{1/3}$ applied to $(a,b,c) = (x^4y^2, x^2y^4, z^6)$. If M were sos, then the equation $M(x,y,z) = \sum_k h_k^2(x,y,z)$ would hold for suitable $h_k \in H_3(\mathbf{R}^3)$. Write out M as a ternary sextic, using all potential monomials:

$$\begin{array}{c}
0x^6 + 0x^5y + 1x^4y^2 + 0x^3y^3 + 1x^2y^4 + 0xy^5 + 0y^6 \\
+ 0x^5z + 0x^4yz + 0x^3y^2z + 0x^2y^3z + 0xy^4z + 0y^5z \\
+ 0x^4z^2 + 0x^3yz^2 - 3x^2y^2z^2 + 0xy^3z^2 + 0y^4z^2 \\
+ 0x^3z^3 + 0x^2yz^3 + 0xy^2z^3 + 0y^3z^3 \\
+ 0x^2z^4 + 0xyz^4 + 0y^2z^4 \\
+ 0xz^5 + 0yz^5 \\
+ 1z^6.
\end{array}$$

Now write out $h_k(x,y,z)$, utilizing the same geometric scheme:

$$\begin{array}{c}
A_kx^3 + B_kx^2y + C_kxy^2 + D_ky^3 \\
+ E_kx^2z + F_kxyz + G_ky^2z \\
+ H_kxz^2 + I_kyz^2 \\
+ J_kz^3.
\end{array}$$

Since the coefficient of x^6 in M is 0, $\sum_k A_k^2 = 0$, hence $A_k = 0$ for all k. This can also be seen directly: $M(1,0,0) = 0$ and $h_k(1,0,0) = A_k$. Now look at the coefficient of x^4z^2 in $\sum_k h_k^2$: it is $\sum_k(E_k^2 + 2A_kH_k)$. Since $A_k = 0$ and the coefficient of x^4z^2 in M is 0, it follows that $E_k = 0$ for all k as well. Continuing down the xz edge, we compare the coefficients of x^2z^4 in $\sum_k h_k^2$ and M: $\sum_k 2E_kJ_k + H_k^2 = 0$. Since $E_k = 0$, it follows that $H_k = 0$. (These also follow from M vanishing to 5th order at (1,0,0) in the direction of (0,0,1).) A similar argument applied to y^6, y^4z^2 and y^2z^4 shows that $D_k = G_k = I_k = 0$.

At this point there are two paths to our conclusion. We have already reduced our task to drawing a contradiction from the equation

$$x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 = \sum_k (B_kx^2y + C_kxy^2 + F_kxyz + J_kz^3)^2.$$

Since $M(1,\pm1,\pm1) = 0$, we have $h_k(1,\pm1,\pm1) = 0$; that is,

$$B_k + C_k + F_k + J_k = B_k + C_k - F_k - J_k = -B_k + C_k - F_k + J_k = -B_k + C_k + F_k - J_k = 0.$$

Thus, $B_k = C_k = F_k = J_k = 0$, hence each $h_k = 0$, and this contradicts $\sum h_k^2 = M$.

It is more telling to consider the coefficient of $x^2y^2z^2$ in M and $\sum_k h_k^2$; the contradiction is immediate from $-3 = \sum_k F_k^2$! This second argument is more powerful. Let $N(x,y,z) = M(x,y,z) + x^2y^2z^2$. Then N is evidently psd. If $N = \sum_k h_k^2$, then an identical argument to that for M shows that each h_k can only use the same monomials as before. However, the zeros of N are just $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, and these give no additional linear equations on B_k, C_k, F_k and L_k. On the other hand, a consideration of the coefficient of $x^2y^2z^2$ in N and $\sum_k h_k^2$ gives the contradiction: $-2 = \sum_k F_k^2$.

b. Examples of Robinson. In the late 1960s, Raphael M. Robinson [68,p.264] saw "an unpublished example of a ternary sextic worked out recently by W. J. Ellison using Hilbert's method. It is, as would be expected, very complicated. After seeing this, I discovered that an astonishing simplification would be possible by dropping some unnecessary assumptions made by Hilbert." He adds in a footnote: "When I submitted this paper for publication, I did not think that any such example had ever appeared in print. However, shortly thereafter, T. S. Motzkin called my attention to the fact that he had published a counterexample for the case of ternary sextics in 1967. I have added an Appendix which discusses Motzkin's result."

Motzkin replaced the set $\{P_j\}$ of nine points formed by the intersections of the two cubics with the square array $\{-1,0,1\}^2$, which is the intersection of $\phi(x,y) = x^3 - x = 0$ and $\psi(x,y) = y^3 - y = 0$. Where Hilbert had argued that some λ makes the perturbed form positive, Robinson took the maximal perturbation, and defined

$$R(x,y,z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2.$$

(Robinson primarily discussed $R(x,y,1)$ and [12] introduced the notation R.)

Robinson proved that R is psd by writing $(x^2+y^2)R(x,y,1)$ as a sum of squares of polynomials. In fact, the inequality $R \geq 0$ is a special case of an inequality which also appears in Motzkin [49,p.211] (!) and is due to Schur, see [29,p.64]:

$$u^r(u-v)(u-w) + v^r(v-u)(v-w) + w^r(w-u)(w-v) \geq 0 \quad \text{if } r,u,v,w \geq 0.$$

(Take $r = 1$ and $(u, v, w) = (x^2, y^2, z^2)$ to obtain R.) For much more on Schur's inequalities and related sextic forms, see [16].

It is easy to see that $R = 0$ on the set

$$\mathcal{Z} = \{(1, \pm 1, \pm 1), (1, \pm 1, 0), (1, 0, \pm 1), (0, 1, \pm 1)\}.$$

If $R = \sum_k h_k^2$, where each h_k is a ternary cubic, then h_k vanishes on $\mathcal{Z}$. This gives ten equations on the ten coefficients of h_k, which together imply that $h_k = 0$, a contradiction, since $R \neq 0$.

Robinson also gave the first explicit example in $\Delta_{4,4}$:

$$f(x, y, z, w) = x^2(x - w)^2 + y^2(y - w)^2 + z^2(z - w)^2 + 2xyz(x + y + z - 2w).$$

The proof that $f \in \Delta_{4,4}$ is not quite as simple as the proof for $R \in \Delta_{3,6}$, and f has been replaced as the exemplary element of $\Delta_{4,4}$ by Q (see below). In the Appendix of [68], Robinson gives a method for generalizing Motzkin's example: if f is a real polynomial in n variables with degree $d < 2n$ which is not sos, then neither is

$$g(x_1, \dots, x_n) = x_1^2 \cdots x_n^2 f(x_1, \dots, x_n) + 1.$$

When $n = 2$ and $f(x_1, x_2) = x_1^2 + x_2^2 - 3$, this construction produces $M(x_1, x_2, 1)$.

c. Examples of Choi and Lam. Man-Duen Choi was trying in 1973 to classify positive linear mappings; mappings between matrix algebras which preserve the cone of positive semidefinite matrices. In the real case, this reduces to the cone of psd biquadratic forms; quartic forms which are quadratic forms in two different sets of variables. Choi learned of a paper by an electrical engineer [42], which purported to show (p.14) that every psd biquadratic form is sos. A recent paper of Calderón [7] had covered some low-dimensional cases and convinced Choi that the result could not be extended. He tried to find the flaw in the proof, and, in doing so, constructed a counterexample (in [9]). He writes [10]: "Without Koga's false proof, I would have dared not construct a counterexample. Actually, I had been haunted by Hilbert's non-constructive example (in [23]) as a graduate student."

Choi's example is

$$F(x_1, x_2, x_3, y_1, y_2, y_3) = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + 2(x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2)$$
$$-2x_1 x_2 y_1 y_2 - 2x_1 x_3 y_1 y_3 - 2x_2 x_3 y_2 y_3.$$

Choi also specializes F in [9] to give some other forms in $\Delta_{4,4}$ and $\Delta_{3,6}$.

Choi had a lectureship in Berkeley from 1973–1976, and started working with Tsit-Yuen Lam, who had already written extensively on quadratic forms. The following year, Choi and Lam wrote the first two papers devoted to a systematic study of our subject: [12] and [13]. They made monomial substitutions in $B = F - (x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2)$, which they prove to lie in $\Delta_{6,6}$, and gave two more simple explicit elements of $\Delta_{4,4}$ and $\Delta_{3,6}$:

$$Q(x, y, z, w) := B(x, w, z, y, z, w) = x^2 y^2 + x^2 z^2 + y^2 z^2 + w^4 - 4wzyz,$$
$$S(x, y, z) := B(yz, xz, xy, x, y, z) = x^4 y^2 + y^4 z^2 + z^4 x^2 - 3x^2 y^2 z^2.$$

In each case, the fact that these forms are psd can be demonstrated from first principles as a consequence of the arithmetic-geometric inequality, and the fact that these forms are not sos follow in a manner similar to that shown above for M, using what Choi and Lam now called the "term-inspection method".

Choi and Lam constructed several other examples of psd forms which are not sos. One is a symmetric quaternary quartic: $\sum x_i^2 x_j^2 + \sum x_i^2 x_j x_k - 2x_1x_2x_3x_4$; another arises from making the substitution $x_1 \to x_i^2$ in a quadratic form studied by A. Horn:

$$H(x_1, \ldots, x_5) = (x_1^2 + \cdots + x_5^2)^2 - 4(x_1^2x_2^2 + \cdots + x_4^2x_5^2 + x_5^2x_1^2).$$

(Observe that $x_i^2x_j^2$ has coefficient ∓ 2 in H depending on whether or not i and j are adjacent in $\{1, 2, 3, 4, 5\}$, viewed cyclically.) It can be shown that any psd *even* symmetric quartic form in n variables must be sos — see [15,30]; for higher degrees, see below.

Let C be a closed convex cone; $x \in C$ is called *extremal* if $x = y_1 + y_2$, $y_i \in C$ implies that $y_i = \lambda_i x$ for some $\lambda_i > 0$. Every element in a closed convex cone C for which $C \cap -C = \{0\}$ can be written as a finite sum of extremal elements. Choi and Lam studied extremal elements in the convex cones $P_{n,m}$ and $\Sigma_{n,m}$. (Calderón had used extremality in studying psd biquadratic forms in [7].) If $p \in P_{n,m}$ is extremal, then $p \geq q \geq 0$ for $q \in P_{n,m}$ implies $q = \lambda p$. If $p \in \Sigma_{n,m}$ is extremal, then $p = h^2$ for some $h \in H_{m/2}(\mathbf{R}^n)$. (Some sufficient conditions on h are discussed in [11]; it is *not* true that if h^2 is extremal in $\Sigma_{n,m}$, then it is also extremal in $P_{n,m}$.)

Perhaps the most significant results in [12] and [13] were the proofs that the forms M, R, S and Q are all extremal elements in their respective $P_{n,m}$'s. It is remarkable that these early examples were also best-possible, in this sense. The method of proof is an extension of the "zero argument" used in sums of squares. For example, $R \geq f \geq 0$ implies that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ vanish on $\mathcal{Z}$. This imposes 30 conditions on the 28 coefficients of f, implying that $f = \lambda R$. These paragraphs do not completely describe the contents of [12] and [13]; many of the ideas in these papers have yet to be fully developed.

My entry into the subject came in late 1976. I was studying the two-dimensional Hamburger moment problem as it applied to an embedding problem in functional analysis which had earlier arisen in my thesis – see [62,pp.117-120] for details. I found a reference to the abstract of [68] and was immediately captivated. My colleague Leonard Carlitz gave me his reprint of [68] and I wrote Prof. Robinson about other papers in the area. He directed me to the then-new [12] and [13]. I first met Lam at the 1977 Winter Meetings in St. Louis, and showed him a counterexample to a minor conjecture in the preprint of [13]. I visited him at his Berkeley home that summer and had the shortest four-hour conversation of my life. I enjoyed a post-doctoral year at Berkeley in 1978–1979, and Choi, Lam and I have worked together (with occasional fourth authors) ever since.

d. Examples of Lax, Lax and Schmüdgen. Two other forms in $\Delta_{n,m}$ were discovered independently in the 1970's. Anneli and Peter Lax [46] showed that the form

$$A(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} \prod_{j \neq i} (x_i - x_j),$$

A graph polynomial!!!

which appeared on the 1971 International Mathematical Olympiad, is psd and not sos. They observe that A is a polynomial in the $x_i - x_j$'s, so it is "really" a form in four variables. (The proof that A is psd was an Olympiad question!)

Konrad Schmüdgen [71], following the program of Gel'fand and Vilenkin, produced a sextic polynomial which homogenizes to a form in $\Delta_{3,6}$:

$$q(x,y,z) = 200(x^3 - 4xz^2)^2 + 200(y^3 - 4yz^2)^2 + (y^2 - x^2)x(x+2z)(x^2 - 2xz + 2y^2 - 8z^2).$$

The proof that q is psd involves decomposing $\mathbf{R}^3$ into ten regions; the proof that q is not sos involves the eight (surprise!) zeros of q.

5. Some later developments

a. Zeros of psd forms and multiforms. The first Choi-Lam-Reznick paper [14] was largely concerned with the number of zeros of psd forms. Viewed projectively, R has the ten zeros of $\mathcal{Z}$. We show that if $p \in P_{3,6}$ and p has more than ten zeros, then p is divisible by the square of an indefinite form and p is a sum of three squares of ternary cubics. If $p \in P_{3,6}$ has exactly ten zeros, then it is cannot be sos. In fact, if $p \in P_{3,m}$ has more than $m^2/4$ zeros, then it is either not sos or is divisible by the square of an indefinite form. If $p \in P_{4,4}$ has more than eleven zeros, then it has infinitely many, and is a sum of six squares of quaternary quadratics.

The ten zeros for $p \in P_{3,6}$ above cannot be in general position, but $\mathcal{Z}$ is not the only possible set. Here is a previously unpublished example. For the real parameter $a \geq 0$, let

$$f_a(x,y,z) =$$
$$a^4(x^6 + y^6 + z^6) + (1 - 2a^6)(x^4y^2 + y^4z^2 + z^4x^2) + (a^8 - 2a^2)(x^2y^4 + y^2z^4 + z^2x^4)$$
$$-3(1 - 2a^2 + a^4 - 2a^6 + a^8)x^2y^2z^2.$$

Then $f_0 = S$, $f_1 = R$, and it can be shown that $f_a \in \Delta_{3,6}$ for $0 < a < 1$ with the following ten zeros: $\{(1, \pm 1, \pm 1), (\pm a, 1, 0), (0, \pm a, 1), (1, 0, \pm a)\}$.

A second concern of [14] was multiforms. A *multiform* of type $(n_1, \dots, n_r; m_1, \dots, m_r)$ is a form in $\sum n_k$ variables in r blocks $\{x_{11}, \dots, x_{1,n_1}; x_{2,1}, \dots x_{r,n_r}\}$ so that each term is homogeneous of degree m_k in the $x_{k,j}$'s. Hilbert's characterization of $P_{2,m}, P_{n,2}$ and $P_{3,4}$ is generalized by showing that a psd multiform of type $(n_1, \dots, n_r; m_1, \dots, m_r)$ must be sos if and only if the type is $(2, n; m, 2)$ (up to permutation). The counterexamples were closely based on Q and S. The fact that a psd $(2, n; m, 2)$ multiform is sos was already known: it is the assertion that an n-ary quadratic form $\sum_{ij} f_{ij}(y_1, y_2)x_ix_j$ ($f_{ij} \in H_m(\mathbf{R}^2)$) which is psd for every fixed (y_1, y_2) is a sum of squares of forms linear in the x_i's. This had been proved by D. Ž. Djoković, V. A. Jakubovič, V. M. Popov and M. Rosenblum and J. Rovnyak in somewhat different contexts (see [14] for details and references). Calderón [7] had also proved the special case $m = 2$.

d. Generalizations of the term-inspection method. We now adopt multinomial notation: if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n$, write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \sum_k \alpha_k$. Suppose

$$f(x_1, \dots, x_n) = \sum_{|\alpha| = d} c(\alpha)x^\alpha \in H_d(\mathbf{R}^n),$$

and let $C(f) = cvx(\{\alpha : c(\alpha) \neq 0\}) \subset \mathbf{R}^n$ denote the *Newton polytope* of f; $C(f)$ is a subset of the simplex whose vertices are de_k. It is proved in [59] that, if $p = \sum_k q_k^2$, then $\frac{1}{2}C(p) \supseteq C(q_k)$. (In the Motzkin example, $C(M)$ is the triangle whose vertices are $(4,2,0), (2,4,0), (0,0,6)$, so if $M = \sum_k h_k^2$, the monomials in h_k must come from the lattice points contained in the triangle with vertices $(2,1,0)$, $(1,2,0)$ and $(0,0,3)$. There is one non-vertex lattice point, (1,1,1), and the corresponding monomials are x^2y, xy^2, z^3 and xyz. This result automates the first part of the term-inspection method.)

The rest of the term-inspection method is formalized in [17] into the "Gram matrix" method. Suppose $p(x) = \sum_\alpha a_\alpha x^\alpha \in P_{n,2d}$ and $p(x) = \sum_{i=1}^t h_i^2(x)$, where $h_i(x) = \sum_\beta u_\beta^{(i)} x^\beta \in H_d(\mathbf{R}^n)$ for $1 \leq i \leq t$. Let $U_\beta = (u_\beta^{(1)}, \ldots, u_\beta^{(t)}) \in \mathbf{R}^t$. Then

$$\sum_{|\alpha|=2d} a_\alpha x^\alpha = p(x) = \sum_{i=1}^t h_i^2(x) = \sum_{i=1}^t \Big(\sum_{|\beta|=d} u_\beta^{(i)} x^\beta \Big) \Big(\sum_{|\beta'|=d} u_{\beta'}^{(i)} x^{\beta'} \Big)$$
$$= \sum_{|\beta|,|\beta'|=d} (U_\beta \cdot U_{\beta'}) x^{\beta+\beta'}.$$

By comparing the coefficients of x^α in p and $\sum_i h_i^2$, we get

$$a_\alpha = \sum_{\beta+\beta'=\alpha} U_\beta \cdot U_{\beta'}.$$

Conversely, if there exist vectors $\{U_\beta\} \subset \mathbf{R}^t$ which satisfy these equations for all α, then we can write p as a sum of t squares by using the coordinates of the U_β's as the coefficients of the h_i's. The dot product matrix $(U_\beta \cdot U_{\beta'})$ is called the *Gram matrix* associated to the expression $p = \sum_i h_i^2$. In order to state the Gram matrix method, we first recall that a symmetric matrix can serve as the set of dot products of vectors in $\mathbf{R}^t$ if and only if the corresponding quadratic form is psd with rank $\leq t$. We also define the *length* of an sos form p to be the smallest number of forms $\{h_k\}$ required to write $p = \sum h_k^2$. The following results are proved in [17,p.106]; very recently, Powers and Wörmann [56] have written an algorithm which implements them.

(1) Let $p(x) = \sum_\alpha a_\alpha x^\alpha$, and $V = (v_{\beta\beta'})$ be a real symmetric matrix. The following statements are equivalent:

(A) p is a sum of squares and V is a Gram matrix associated to p (with respect to some sums of squares expression $p = \sum h_i^2$);

(B) V is psd and $\sum_{\beta+\beta'=\alpha} v_{\beta\beta'} = a_\alpha$ for all α.

(2) If p is a sum of squares, then the length of p is equal to the minimum rank of V, where V ranges over all Gram matrices associated to p.

c. Generalizations of M and S. The quadrinomial property of M and S has also been generalized. It is proved in [59] that if p is in any $\Delta_{n,m}$, then it has at least four terms. If $p \in \Delta_{n,m}$ has four terms, then it is extremal in $P_{n,m}$ if and only if, after a scaling $x_j \to \sigma_j x_j$, we have $p(x) = x^{2a} + x^{2b} + x^{2c} - 3x^{2d}$, where $a, b, c \in \mathbf{Z}_+^n$ have a certain geometric property. To be specific, if T is the triangle with vertices $\{a, b, c\}$, then $T \cap \mathbf{Z}^n$ must equal $\{a, b, c, d\}$, where $d = \frac{1}{3}(a + b + c)$ is the median of T. This implies the extremality of the forms M (with $a = (2, 1, 0)$, $b = (1, 2, 0)$, $c = (0, 0, 3)$, $d = (1, 1, 1)$) and S (with $a = (2, 1, 0)$, $b = (0, 2, 1)$, $c = (1, 0, 2)$, $d = (1, 1, 1)$).

An *agiform* (see [61]) is derived by even monomial substitutions into the arithmetic-geometric inequality. Suppose $a_i \in (2\mathbf{Z})^n_+$ and $0 < \lambda_i$, $\sum_i \lambda_i = 1$ are such that $\sum_i \lambda_i a_i = b \in \mathbf{Z}^n$. Then the agiform $\lambda_1 x^{a_1} + \cdots + \lambda_n x^{a_n} - x^b$ is psd as a consequence of the arithmetic-geometric inequality. Necessary and sufficient conditions are given in [61] for this form to be sos and necessary and sufficient conditions are given for this form to be an extremal psd form. These conditions depend heavily on the combinatorial structure of the lattice points contained in the simplex with vertices $\{a_i\}$. This paper also contains six explicit families of extremal psd forms in n variables, two each generalizing M, S and Q; three of these families were defined in [13] and proved there to be psd but not sos.

One consequence of the sos property for agiforms is that, if $p(x_1, \ldots, x_n)$ is an agiform and $r \geq n$, then $p(x_1^r, \ldots, x_n^r)$ is sos, so p is a sum of squares of forms in the variables $\{x_k^{1/r}\}$. This property is not true for all psd forms. If $H \in \Delta_{5,4}$ denotes the Horn form, then it can be shown that $H(x_1^r, \ldots, x_5^r)$ is not sos for every $r \geq 1$. A related question involves taking odd powers of psd forms. Stengle [73] proved in 1979 that for $k \geq 0$ and $m \geq 1$, every odd power of

$$x^{2m+1} z^{2m+1} + (y^2 z^{2m-1} - x^{2m+1} - x z^{2m})^2$$

is psd and not sos. This is also true for $S(x, y, z)$ and $M(x, y, z)$.

d. Symmetric Examples. One obstacle to understanding the geometry of $P_{n,m}$ and $\Sigma_{n,m}$ is these cones lie in R^N for $N = \binom{n+m-1}{n-1}$; if $\Delta_{n,m} \neq \emptyset$, then $N \geq 28$. One way to overcome this obstacle is to take sections of lower dimension, and one of the simplest ways to do this is to consider even symmetric forms. This is done in [15] for $m = 6$. A typical even symmetric n-ary sextic is:

$$p(x_1, \ldots, x_n) = \alpha \sum_{i=1}^{n} x_i^6 + \beta \sum_{i \neq j} x_i^4 x_j^2 + \gamma \sum_{i<j<k} x_i^2 x_j^2 x_k^2.$$

The ultimate conditions on p are more easily expressed in terms of power-sums. Write

$$p(x_1, \ldots, x_n) = a \sum_{i=1}^{n} x_i^6 + b \left(\sum_{i=1}^{n} x_i^4\right)\left(\sum_{i=1}^{n} x_i^2\right) + c\left(\sum_{i=1}^{n} x_i^2\right)^3.$$

(The two expressions for p are related by $\alpha = a + b + c$, $\beta = b + 3c$, $\gamma = 6c$.) Let $p^*(t) = a + bt + ct^2$ and let $v^{(k)}$ denote any n-tuple whose coordinates consist of k 1's and $n - k$ 0's, then $p(v^{(k)}) = ak + bk^2 + ck^3 = kp^*(k)$. A clear necessary condition that p be psd is that $p^*(k) \geq 0$ for $k = 1, 2, \ldots, n$. This is also a sufficient condition: p is psd if and only if it is psd on the "test set" $\{v_1, \ldots, v_n\}$. The necessary and sufficient condition that p be sos is that $p^*(t) \geq 0$ for $t \in \{1\} \cup [2, n]$. (One does not have a clear idea what $p^*(t)$

means when t is not an integer.) For the Robinson form, $R^*(t) = \frac{1}{2}(2-t)(3-t)$; since $R^*(t) < 0$ for $2 < t < 3$, this gives another proof that R is not sos.

William Harris [30] has generalized this study to even symmetric octics ($m = 8$) and to even symmetric ternary forms ($n = 3$). One surprising result is that every psd even symmetric ternary octic is sos. Harris gives test sets which determine whether an even n-ary symmetric octic or ternary decic ($m = 10$) is psd, and a complete list of the extremal even ternary symmetric octics, as well as many new examples in $\Delta_{3,10}$ and $\Delta_{4,8}$.

We mentioned [16] earlier, in the context of showing that R is psd. This paper contains an extensive discussion of the Robinson form, and many other explicit examples. The possible sides of a triangle can be parameterized $(a,b,c) = (x^2+y^2, x^2+z^2, y^2+z^2)$ in view of the triangle inequality, so any polynomial inequality satisfied by the sides of a triangle can be interpreted as a psd even ternary symmetric polynomial and vice versa. The inequality $R \geq 0$ is equivalent in this way to an 1820 theorem of Lehmus. Harris gives all symmetric polynomial inequalities of degree ≤ 5 satisfied by the sides of a triangle.

6. PÓLYA'S THEOREM

In 1928, George Pólya [55] (see also [29,pp.57-59]) gave an explicit solution to Hilbert's 17th Problem for even positive definite forms $p \in P_{n,2d}$; that is, for those positive definite forms p which can be written $p(x_1,\dots,x_n) = f(x_1^2,\dots,x_n^2)$ for some $f \in H_d(\mathbf{R}^n)$.

Suppose $f(y_1,\dots,y_n) > 0$ for y on the simplex $\Delta_n = \{\sum_j y_j = 1,\ y_j \geq 0, 1 \leq j \leq n\}$. Pólya first constructs a sequence of functions $\{f_t\}$ which converges uniformly to f on the compact set Δ_n; it follows that for $t \geq t_0 = t_0(f)$ and $y \in \Delta_n$, we have $f_t(y) \geq 0$. Elementary combinatorial manipulations give

$$\left(\sum_{i=1}^{n} y_i\right)^r f(y_1,\dots,y_n) = r!(r+d)^d \sum_{|\alpha|=r+d} \frac{f_{r+d}\left(\frac{\alpha_1}{r+d},\dots,\frac{\alpha_n}{r+d}\right)}{\alpha_1!\cdots\alpha_n!} y^\alpha.$$

Since $\left(\frac{\alpha_1}{r+d},\dots,\frac{\alpha_n}{r+d}\right) \in \Delta_n$, the above has positive coefficients when $r \geq t_0(f) - d$.

Another way of viewing this result is that any form f which is positive on Δ_n can be written as the quotient of two polynomials with positive coefficients, where the denominator is a power of $\sum_i y_i$. Without this last specification on the denominator, this result had been proved by Poincaré [54] in 1883 (the date is wrong in [29]) for $n = 2$ and by Meissner [48] in 1911 for $n = 3$.

Upon replacing y_i by x_i^2, Δ_n becomes the unit sphere, and Pólya's Theorem becomes the statement that if p is positive definite and even, then for sufficiently large r,

$$\left(\sum_{i=1}^{n} x_i^2\right)^r p(x_1,\dots,x_n)$$

is a sum of monomials with positive coefficients. Since each monomial in the product involves only even exponents, it follows that $\left(\sum x_i^2\right)^r p(x_1,\dots,x_n)$ is in fact a sum of squares of monomials. And since the coefficients in the product evidently come from the same field as the coefficients of p, Pólya's Theorem solves Hilbert's 17th Problem in the special case that p is even and positive definite. Pólya remarks on the significance of his result [55,p.144]: "Es kann schliesslich bemerkt werden, dass die Darstellung einigermassen in Zusammenhang mit einer Fragestellung von Hilbert steht, die kürzlich durch E. Artin mit tiefgehenden Mittleln gelöst wurde."

In 1940, Habicht [25] (see also [29,pp.300-304]) used Pólya's theorem to prove directly that a positive definite form (not necessarily even) is a sum of squares of rational functions. The denominators are positive definite, but are no longer necessarily powers of $\sum_i x_i^2$; the coefficients are still in the original field. A key step in reducing to Pólya's Theorem is the observation that, if $p \in P_{n,2d}$ is positive definite, then

$$q(x_1, \dots, x_n) = \prod_{\epsilon_k = \pm 1} p(x_1, \epsilon_2 x_2, \dots, \epsilon_n x_n), \in P_{n,2^n d},$$

which is clearly positive definite, is also even.

There have been several generalizations of Pólya's theorem. Motzkin and Straus [51] extended it (without denominator information) to polynomials in several complex variables and to power series. Very recently, Catlin and D'Angelo [8] have extended it (with denominator information) to Hermitian forms in several complex variables. Handelman (see [26,27]) has completely solved a related question. Suppose a polynomial p in several variables has non-negative coefficients. For which f does there always exist an r so that $p^r f$ has non-negative coefficients? Recently, De Loera and Santos [47] have made an explicit algorithm out of Pólya's theorem, and made quantitative estimates for $t_0(f)$.

The restriction to positive definite forms is necessary. There exist positive semidefinite forms p which have the remarkable property that, in any representation $p = \sum_k \phi_k^2$, where $\phi_k = f_k/g_k$ is a rational function, each g_k must have a specified non-trivial zero. The existence of these so-called "bad points" insures that $p \cdot (\sum x_i^2)^r$ can never be a sum of squares of forms for *any* r. Habicht's theorem implies that no positive definite form can have a bad point. Bad points were first noted by E. G. Straus in an unpublished 1956 letter to G. Kreisel. An extensive history of bad points can be found in Delzell's thesis [18], and in his forthcoming [20]. An example from [18] is: $w^4z^6 + w^2x^6y^2 + y^{10} - 3w^2x^2y^4z^2$, which has a bad point at $(w, x, y, z) = (1, 0, 0, 0)$.

7. A NEW THEOREM

The final section of this paper is devoted to a sketch of the main theorem in [63]: if **$p \in P_{n,m}$ is positive definite, then for sufficiently large r, $(\sum x_i^2)^r p$ is a sum of squares**; that is, Pólya's conclusion holds with Habicht's hypotheses. Moreover, if $p \in H_m(K^n)$ is positive definite, then for large enough r, $(\sum x_i^2)^r p$ is a positive linear combination over K of a set of $(2r+m)$-th powers of linear forms, in which the linear forms depend only on m, r and n, not p. (Ellison [22] showed in 1969 that for all (n, m), $m \geq 4$, there are forms in $\Sigma_{n,m}$ which are not a sum of powers of linear forms, so the conclusion about $(\sum x_i^2)^r p$ is stronger than that it is sos.) For much more on this subject, see [62,63]. The construction is specific enough to give an explicit representation for Becker's $B(t)$ as a sum of $2k$-th powers over $\mathbf{R}$, but not, unfortunately, over $\mathbf{Q}$. Finally, it can be argued that each component of our proof is, or could have been, familiar to Hilbert.

We introduce two new notations: write $G_n(x_1,\ldots,x_n) = x_1^2+\cdots+x_n^2$, and let $(x)_t$ denote the falling factorial $x(x-1)\cdots(x-(t-1))$.

The proof depends on the existence of Hilbert identities. These arose as part of Hilbert's solution of Waring's Problem [36]: for every n and s, there exist $0<\lambda_k=\lambda_k(n,s)\in\mathbf{Q}$ and $\alpha_{kj}=\alpha_{kj}(n,s)\in\mathbf{Q}$ for $1\le k\le N=\binom{n+2s-1}{n-1}$ so that

$$G_n^s(x_1,\ldots,x_n)=(x_1^2+\cdots+x_n^2)^s=\sum_{k=1}^{N}\lambda_k(\alpha_{k1}x_1+\cdots+\alpha_{kn}x_n)^{2s}.$$

There are no known *explicit* Hilbert identities for arbitrary n, s. For $s=1,2,3$, they are far from unique and not hard to find (see [62,§8,9]). If we relax the conditions that the coefficients be rational, then Hausdorff [31] gives explicit Hilbert identities involving the roots of the Hermite polynomials. A simple non-rational trigonometric formula for $n=2$ will be used below. A self-contained proof of the existence of Hilbert identities (and more), which dots all i's and crosses all t's, is contained in [63].

The idea of the proof is to differentiate both sides of a Hilbert identity. To be specific, if $h\in H_d(\mathbf{R}^n)$, define the associated d-th order differential operator $h(D)$ by replacing each appearance of x_j by $\frac{\partial}{\partial x_j}$; thus, $G_n(D)=\sum_j\frac{\partial^2}{\partial x_j^2}=\Delta$, the Laplacian. It turns out that there are old formulas to describe the effect of $h(D)$ on both sides of a Hilbert identity.

In the 19th century, Sylvester and Clifford developed the method of "contravariant differentiation". If $h\in H_d(K^n)$ and $d\le m$, then there is a simple representation for $h(D)$ applied to a sum of m-th powers:

$$h(D)\sum_k(\alpha_{k1}x_1+\cdots+\alpha_{kn}x_n)^m=(m)_d\sum_k h(\alpha_{k1},\ldots,\alpha_{kn})(\alpha_{k1}x_1+\cdots+\alpha_{kn}x_n)^{m-d}.$$

This identity is not hard to prove. It suffices consider a single m-th power, and by linearity, it suffices to consider $h(x)=x^i$. But then $h(D)$ is a product of successive $\frac{\partial}{\partial x_j}$'s, and the formula is immediate by the chain rule.

It is somewhat more difficult to evaluate $h(D)G_n^s$. Each differentiation reduces the exponent of G_n by at most one, so G_n^{s-d} divides $h(D)G_n^s$. This suggests the notation

$$h(D)G_n^s=\Phi_s(h)G_n^{s-d},$$

where $\Phi_s(h)$ has degree $2s-d-2(s-d)=d$. Thus Φ_s is a linear map from $H_d(\mathbf{R}^n)$ to itself. An explicit formula for Φ_s follows from a theorem of E. W. Hobson. If $h\in H_d(\mathbf{R}^n)$, and F is a sufficiently differentiable function of one variable, then

$$h(D)F(G_n)=\sum_{k\ge0}\frac{2^d}{2^{2k}k!}\Delta^k(h)F^{(d-k)}(G_n).$$

Observe that h on the right-hand side appears only in the sequence of iterated Laplacians. (This is proved in [37,38], see also [39]; the formula was lauded by Hardy [28] as an "elegant theorem in formal differentiation".) Set $F(t)=t^s$, so $F^{(j)}(t)=(s)_jt^{s-j}$ and

$$h(D)G_n^s=\sum_{k\ge0}\frac{(s)_{d-k}}{2^{2k-d}k!}\Delta^k(h)G_n^{s-d+k}=\left(\sum_{k\ge0}\frac{(s)_{d-k}}{2^{2k-d}k!}\Delta^k(h)G_n^k\right)\cdot G_n^{s-d}.$$

Thus,

$$\Phi_s(h) = \sum_{k\geq 0} \frac{(s)_{d-k}}{2^{2k-d}k!} \Delta^k(h) G_n^k.$$

Putting this all together, we see that $h(D)$ applied to a Hilbert identity gives:

$$h(D)G_n^s = h(D)\left(\sum_{k=1}^{N} \lambda_k (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2s}\right);$$

$$\Phi_s(h)G_n^{s-d} = (2s)_d \sum_{k=1}^{N} \lambda_k h(\alpha_{k1}, \ldots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2s-d}.$$

Set $r = s - d$. This equation states that $\Phi_s(h)G_n^r$ is a linear combination of $(2r+d)$-th powers of linear forms. If h happens to be psd with coefficients in K, so d is even, then $\Phi_s(h)G_n^r$ is a positive linear combination over K of $(2r+d)$-th powers of linear forms, each of which is, *per se* a square. Thus Hilbert's 17th Problem is solved, but for $\Phi_s(h)$!

The final step is to invert Φ_s. The formula given below is apparently new, but is well within the grasp of Hobson's techniques. If $s > d$, then

$$\Phi_s^{-1}(p) = \frac{1}{(s)_d 2^d} \sum_{\ell\geq 0} \frac{(-1)^\ell}{2^{2\ell}\ell!(\frac{n}{2}+s-1)_\ell} \Delta^\ell(p) G_n^\ell$$

$$= \frac{1}{(s)_d 2^d}\left(p - \frac{\Delta(p)G_n}{2(n+2s-2)} + \frac{\Delta^2(p)G_n^2}{8(n+2s-2)(n+2s-4)} - + \cdots\right).$$

We see that, if $p \in H_d(K^n)$, then so is $\Phi_s^{-1}(p)$. It is not hard to prove that

$$\lim_{s\to\infty} (s)_d 2^d \Phi_s^{-1}(p) = p,$$

and it follows that if p is positive definite, then so is $\Phi_s^{-1}(p)$ for sufficiently large s.

The preceding can be made quantitative. If p is positive definite, let

$$\epsilon(p) = \frac{\inf\{p(u) : u \in S^{n-1}\}}{\sup\{p(u) : u \in S^{n-1}\}}$$

measure how "close" p is to having a zero. After some pleasant analytical estimates, omitted here, including one comparing the L_∞ norms of p and Δp on S^{n-1}, we prove that if $p \in P_{n,m}$ is positive definite and $s \geq \frac{nm(m-1)}{(4\log 2)\epsilon(p)} - \frac{n-m}{2}$, then $\Phi_s^{-1}(p) \in P_{n,m}$.

Theorem. *Suppose $p \in H_m(K^n)$ is positive definite. If $r \geq \frac{nm(m-1)}{(4\log 2)\epsilon(p)} - \frac{n+m}{2}$, then pG_n^r is a non-negative K-linear combination of a set of $(m+2r)$-th powers of linear forms in $\mathbf{Q}[x_1, \ldots, x_n]$. This set depends only on n, m and r.*

The last sentence above is based on the fact that the linear forms come from the Hilbert identities. Interestingly, the analysis of Pólya's theorem in [47] also shows a dependence of $t_0(f)$ on $\epsilon(f)^{-1}$.

Let $P_{n,m}^{(\epsilon)}$ be the set of $p \in P_{n,m}$ so that $\epsilon(p) \geq \epsilon$; observe that $P_{n,m} = \bigcup_{\epsilon \geq 0} P_{n,m}^{(\epsilon)}$. For each $\epsilon > 0$, the Theorem implies that if $p \in P_{n,m}^{(\epsilon)}$, $r \geq \frac{nm(m-1)}{(4\log 2)\epsilon} - \frac{n+m}{2}$ is even and $G_n^{m+r} = \sum(\alpha_k \cdot x)^{2m+2r}$, then after applying $(\Phi_{r+m}^{-1}(p))(D)$ and clearing fractions,

$$p(x_1, \dots, x_n) = \sum_k \lambda_k(p) \left(\frac{(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{m/2+r}}{(x_1^2 + \cdots + x_n^2)^{r/2}} \right)^2,$$

where $\lambda_k(p) \geq 0$ is linear in p. There has been considerable interest in the representations of p as a sum of squares of rational functions with continuous dependence on p; see [19,21]. Such a formula cannot hold over all of $P_{n,m}$. It is not hard to prove that if $p \in P_{n,m}$ has a non-trivial zero, then pG_n^r cannot be written as a positive linear combination of $(2r+m)$-th powers. The reader is referred to [63] for details, as well as many explicit examples.

The Theorem also gives new, concrete information about representations as a sum of $2k$-th powers of rational functions. The following corollary (without the specification of the denominators) can be given an abstract proof using Becker's methods.

Corollary. *If $p \in K[x_1, \dots, x_n]$ is a positive definite form of degree $m = 2kt$, then p is a non-negative K-linear combination of $2k$-th powers of rational functions in $\mathbf{Q}[x_1, \dots, x_n]$ whose denominators are powers of G_n. If p and q are positive definite forms in $K[x_1, \dots, x_n]$ and the degree of the rational function p/q is a multiple of $2k$, then p/q is a non-negative K-linear combination of $2k$-th powers of rational functions whose numerators are in $\mathbf{Q}[x_1, \dots, x_n]$ and whose denominators are products of powers of G_n and q.*

We conclude with a sketch of an explicit formula for $B(t)$ as a sum of $2k$-th powers. We start with the familiar observation that

$$B(t) = \frac{1+t^2}{2+t^2} = \frac{(1+t^2)(2+t^2)^{2k-1}}{(2+t^2)^{2k}},$$

hence if we can write $(1+t^2)(2+t^2)^{k-1}$ and $(2+t^2)^k$ as a sum of $2k$-th powers of linear forms, then their product is a sum of $2k$-th powers of quadratics, each of which can be divided by $(2+t^2)$ to give $B(t)$ as a sum of $2k$-th powers of rational functions.

Although there are no known families of Hilbert identities for $(x^2+y^2)^s$ over $\mathbf{Q}$, they are easy to find over $\mathbf{R}$. The simplest one holds for $v \geq s+1$:

$$(x^2+y^2)^s = \frac{2^{2s}}{v\binom{2s}{s}} \sum_{j=0}^{v-1} \left(\cos(\tfrac{j\pi}{v})x + \sin(\tfrac{j\pi}{v})y\right)^{2s}.$$

By taking $s = k$, $v = k+2$, $x = \sqrt{2}$, $y = t$ and applying the explicit formula for Φ_s^{-1} given above, we obtain (after several pages of computation) a formula for $B(t)$. Let $L_j(x,y) = (\cos\frac{j\pi}{k+2})x + (\sin\frac{j\pi}{k+2})y$ and $\lambda_j = 3k - (k+1)\cos(\frac{2j\pi}{k+2})$ for $0 \leq j \leq k+1$. Then

$$B(t) = \frac{1+t^2}{2+t^2} = \frac{2^{4k-2}}{k(k+2)^2\binom{2k}{k}^2} \sum_{i=0}^{k+1}\sum_{j=0}^{k+1} \lambda_j \left(\frac{L_i(\sqrt{2},t)L_j(\sqrt{2},t)}{2+t^2} \right)^{2k}.$$

Although this gives $B(t)$ as a sum of $2k$-th powers in $\mathbf{R}(t)$, the summands are not in $\mathbf{Q}(t)$. Such a representation cannot yet be found by our methods, because there is no known family of Hilbert identities with coefficients in $\mathbf{Q}$.

References

[1] E. Artin, *Über die Zerlegung definiter Funktionen in Quadrate.*, Hamb. Abh. **5** (1927), 100–115; See Collected Papers (S. Lang, J. Tate, eds.), Addison-Wesley 1965, reprinted by Springer-Verlag 1982, pp. 273–288.

[2] E. Becker, *Valuations and real places in the theory of formally real fields*, Géométrie Algébrique Réelle et Formes Quadratiques, Proceedings, Rennes 1981 (J.-L. Colliot Thélène, M. Coste, L. Mahé et M.-F. Roy, eds.), Lecture Notes in Math., vol. 959, Springer-Verlag, Berlin, Heidelberg, New York, 1982, pp. 1–40.

[3] E. Becker, *The real holomorphy ring and sums of 2n-th powers*, Géométrie Algébrique Réelle et Formes Quadratiques, Proceedings, Rennes 1981 (J.-L. Colliot Thélène, M. Coste, L. Mahé et M.-F. Roy, eds.), Lecture Notes in Math., vol. 959, Springer-Verlag, Berlin, Heidelberg, New York, 1982, pp. 139–181.

[4] E. Becker and V. Powers, *Sums of powers in rings and the real holomorphy ring*, preprint.

[5] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie algébrique réelle*, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[6] F. Browder (ed.), *Mathematical developments arising from Hilbert problems*, Proc. Symp. Pure Math., vol. 28, Amer. Math. Soc., Providence, R.I., 1976.

[7] A. P. Calderón, *A note on biquadratic forms*, Linear Algebra Appl. **7** (1973), 175–177.

[8] D. W. Catlin and J. P. D'Angelo, *A stabilization theorem for Hermitian forms and applications to holomorphic mappings*, in preparation.

[9] M. D. Choi, *Positive semidefinite biquadratic forms*, Linear Algebra Appl. **12** (1975), 95–100.

[10] M. D. Choi, *Personal communication*, Oct. 1995.

[11] M. D. Choi, M. Knebusch, T. Y. Lam and B. Reznick, *Transversal zeros and positive semidefinite forms*, Géométrie Algébrique Réelle et Formes Quadratiques, Proceedings, Rennes 1981 (J.-L. Colliot Thélène, M. Coste, L. Mahé et M.-F. Roy, eds.), Lecture Notes in Math., vol. 959, Springer-Verlag, Berlin, Heidelberg, New York, 1982, pp. 273–298.

[12] M. D. Choi and T. Y. Lam, *An old question of Hilbert*, Queen's Papers in Pure and Appl. Math. (Proceedings of Quadratic Forms Conference, Queen's University (G. Orzech ed.)) **46** (1976), 385–405.

[13] M. D. Choi and T. Y. Lam, *Extremal positive semidefinite forms*, Math. Ann. **231** (1977), 1–18.

[14] M. D. Choi, T. Y. Lam and B. Reznick, *Real zeros of positive semidefinite forms, I*, Math. Z. (1980), 1–26.

[15] M. D. Choi, T. Y. Lam and B. Reznick, *Even symmetric sextics*, Math. Z. **195** (1987), 559–580.

[16] M. D. Choi, T. Y. Lam and B. Reznick, *Positive sextics and Schur's inequalities*, J. Algebra **141** (1991), 36–77.

[17] M. D. Choi, T. Y. Lam and B. Reznick, *Sums of squares of real polynomials*, Proc. Symp. Pure Math. **58.2** (1995), 103–126.

[18] C. N. Delzell, *A constructive, continuous solution to Hilbert's 17th problem, and other results in semi-algebraic geometry*, Ph.D. thesis, Stanford University, 1980.

[19] C. N. Delzell, *Nonexistence of analytically varying solutions to Hilbert's 17th problem*, Recent Advances in Real Algebraic Geometry and Quadratic Forms, Proc. RAGSQUAD Year, Berkeley, 1990–1991, Contemp. Math., 155 (W. B. Jacob, T. Y. Lam, R. O. Robson, eds.), Amer. Math. Soc., 1994, pp. 107–117.

[20] C. N. Delzell, *Kreisel's unwinding of Artin's proof*, About and around Georg Kreisel (P. Odifreddi, ed.), to appear, A. K. Peters, Wellesley.

[21] C. N. Delzell, L. González-Vega, H. Lombardi, *A continuous and rational solution to Hilbert's 17th problem and several cases of the Positivstellensatz*, Computational algebraic geometry, Progr. Math. 109 (F. Eyssette, A. Galligo, eds.), Birkhäuser, Basel, 1993, pp. 61–75.

[22] W. J. Ellison, *A 'Waring's problem' for homogeneous functions*, Proc. Camb. Phil. Soc. **65** (1969), 663–672.

[23] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Academic Press, New York, 1964; Translated by Amiel Feinstein from the Russian edition, Moscow, 1961.

[24] D. Gondard-Cozette, *Le 17éme probléme de Hilbert et ses développements récents*, Seminaire sur les structures algébriques ordonnées: Sélection d'exposés 1984–1987, Volume II (F. Delon, M. Dickmann, D. Gondard, eds.), Publ. Math. Univ. Paris VII, 1990, pp. 21–49.

[25] W. Habicht, *Über die Zerlegung strikte definiter Formen in Quadrate*, Comment. Math. Helv. **12** (1940), 317–322.

[26] D. Handelman, *Positive polynomials and product type actions of compact groups*, Mem. Amer. Math. Soc. **54** (1985), no. 320.

[27] D. Handelman, *Deciding eventual positivity of polynomials*, Ergodic Theory Dynam. Systems **6** (1986), 57–79.

[28] G. H. Hardy, *E. W. Hobson*, J. London Math. Soc. **9** (1934), 225–237; See Collected Papers, Vol. VII, Clarendon Press, Oxford, 1979, pp. 749–761.

[29] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1967.

[30] W. Harris, *Real even symmetric forms*, Ph.D. thesis, University of Illinois, Urbana, 1992.

[31] F. Hausdorff, *Zur Hilbertschen Lösung des Waringschen Problems*, Math. Ann. **67** (1909), 301–305.

[32] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Ann. **32** (1888), 342–350; See Ges. Abh. 2, 154–161, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1981.

[33] D. Hilbert, *Über Ternäre definite Formen*, Acta Math. **17** (1893), 169–197; See Ges. Abh. 2, 345–366, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1981.

[34] D. Hilbert, *Grundlagen der Geometrie*, B.G. Teubner, Leipzig, 1899; Reprinted many times, most recently 1977. English translation by E. J. Townsend, Foundations of Geometry, Open Court, La Salle, Illinois, 1902, reprinted 1965.

[35] D. Hilbert, *Mathematische Probleme*, Arch. f. Math. u. Phys. **3** (1901), 44–63, 213–237; See Ges. Abh. 3, 290–329, Springer, Berlin, 1935, reprinted by Chelsea, New York, 1981; English translation by Mary Winston Newson in Bull. Amer. Math. Soc. **8** (1902), 437–479, see [6,pp.1–34].

[36] D. Hilbert, *Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches Problem)*, Math. Ann. **67** (1909), 281–300; See Ges. Abh. 1, 510–527, Springer, Berlin, 1932, reprinted by Chelsea, New York, 1981.

[37] E. W. Hobson, *On a theorem in differentiation, and its application to spherical harmonics*, Proc. London Math. Soc. **24** (1892–1893), 55–67.

[38] E. W. Hobson, *On a theorem in the differential calculus*, Messenger of Math. **23** (1893–1894), 115–119.

[39] E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Univ. Press, Cambridge, 1931.

[40] A. Hurwitz, *Über den Vergleich des arithmetischen und des geometrischen Mittels*, J. Reine Angew. Math. **108** (1891), 266–268; See Math. Werke II Basel 1933 pp. 505–507.

[41] N. Jacobson, *Basic Algebra* I, Second edition, W. H. Freeman, New York, 1985; Basic Algebra II, W. H. Freeman, 1980.

[42] T. Koga, *Synthesis of finite passive n-ports with prescribed positive real matrices of several variables*, IEEE Trans. Circ. Theory **CT-15** (1968), 2–23.

[43] T. Y. Lam, *The theory of ordered fields*, Ring Theory and Algebra III (Proc. Third Oklahoma Conf., 1979), Lecture Notes in Pure and Appl. Math., vol. 55, Marcel Dekker, New York, 1980, pp. 1 - 152.

[44] T. Y. Lam, *An introduction to real algebra*, Rocky Mountain J. Math. **14** (1984), 767–814.

[45] E. Landau, *Über die Darstellung definiter Funktionen durch Quadrate*, Math. Ann. **62** (1906), 272–285; See Collected Works, vol. 2, pp. 237–250, Thales-Verlag, Essen, 1986.

[46] A. Lax and P. D. Lax, *On sums of squares*, Linear Algebra Appl. **20** (1978), 71–75.

[47] J. A. de Loera and F. Santos, *An effective version of Pólya's theorem on positive definite forms*, to appear in J. Pure Appl. Algebra.

[48] E. Meissner, *Über positive Darstellungen von Polynomen*, Math. Ann. **70** (1910-1911), 223–235.

[49] T. S. Motzkin, *The arithmetic-geometric inequality*, Inequalities (O. Shisha, ed.) Proc. of Sympos. at Wright-Patterson AFB, August 19–27, 1965, Academic Press, New York, 1967, pp. 205–224; See [50,pp.203–222].

[50] T. S. Motzkin, *Theodore S. Motzkin: Selected Papers*, D. Cantor, B. Gordon and B. Rothschild, eds., Birkhäuser, Boston, Basel, Stuttgart, 1983.

[51] T. S. Motzkin and E. G. Straus, *Divisors of polynomials and power series with positive coefficients*, Pacific J. Math. (1969), 641–652; See [50,pp.349–360].
[52] A. Pfister, *Zur Darstellung definiter Funktionen als Summe von Quadraten*, Invent. Math. **4** (1967), 229–237.
[53] A. Pfister, *Hilbert's Seventeenth Problem and related problems on definite forms*, [6], 1976, pp. 483–489.
[54] H. Poincaré, *Sur les équations algébriques*, Comptes Rendus **97** (1883), 1418–1419; See Oeuvres, vol. 5, Gauthier-Villars, 1950, pp. 81–83.
[55] G. Pólya, *Über positive Darstellung von Polynomen*, Vierteljschr. Naturforsch. Ges. Zürich. **73** (1928), 141–145; See Collected Papers, Vol. 2, pp. 309–313, MIT Press, Cambridge, 1974.
[56] V. Powers and T. Wörmann, *An algorithm for sums of squares of real polynomials*, preprint.
[57] A. Prestel, *Sums of squares over fields*, Proc. of 5th School of Algebra (Rio de Janeiro, 1978), Soc. Brasil. Mat., Rio de Janeiro, 1979, pp. 33–44.
[58] A. R. Rajwade, *Squares*, London Math. Soc. Lecture Note Series 171, Cambridge Univ. Press, Cambridge, 1993.
[59] B. Reznick, *Extremal psd forms with few terms*, Duke Math. J. **45** (1978), 363–374.
[60] B. Reznick, *A quantitative version of Hurwitz' theorem on the arithmetic-geometric inequality*, J. Reine Angew. Math. **377** (1987), 108–112.
[61] B. Reznick, *Forms derived from the arithmetic-geometric inequality*, Math. Ann. **283** (1989), 431–464.
[62] B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc. **96** (1992), no. 463.
[63] B. Reznick, *Uniform denominators in Hilbert's Seventeenth Problem*, Math. Z. **220** (1995), 75–98.
[64] B. Reznick, *Review of [58]*, in preparation, Bull. Amer. Math. Soc..
[65] P. Ribenboim, *17ème problème de Hilbert*, Queen's Papers in Pure and Appl. Math. **41** (1974), 149–164.
[66] A. Robinson, *On ordered fields and definite forms*, Math. Ann. **130** (1955), 257–271; See Selected Papers, vol. 1, pp. 355–369, Yale U. Press, New Haven 1979.
[67] A. Robinson, *Further remarks on ordered fields and definite forms*, Math. Ann. **130** (1956), 405–409; See Selected Papers, vol. 1, pp. 370–374, Yale U. Press, New Haven 1979.
[68] R. M. Robinson, *Some definite polynomials which are not sums of squares of real polynomials*, Selected questions of algebra and logic (a collection dedicated to the memory of A. I. Mal'cev), Izdat. "Nauka" Sibirsk. Otdel. Novosibirsk (1973), 264–282, Abstract in Notices Amer. Math. Soc. 16 (1969), p. 554.
[69] C. Scheiderer, *Real algebra and its applications to geometry in the last ten years: some major developments and results*, Real Algebraic Geometry, Proceedings, Rennes 1991 (M. Coste, L. Mahé et M.-F. Roy, eds.), Lecture Notes in Math., vol. 1524, Springer-Verlag, Berlin, Heidelberg, New York, 1992, pp. 75–96.
[70] J. Schmid, *On totally positive units of real holomorphy rings*, Israel J. Math. **85** (1994), 339–350.
[71] K. Schmüdgen, *An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional*, Math. Nachr. **88** (1979), 385–390.
[72] D. B. Shapiro, *Review of [58]*, Amer. Math. Monthly **102** (1995), 281–284.
[73] G. Stengle, *Integral solution of Hilbert's Seventeenth Problem*, Math. Ann. **246** (1979), 33–39.
[74] O. Taussky, *Sums of squares*, Amer. Math. Monthly **77** (1970), 805–830; Reprinted in Selected Papers on Algebra (S. Montgomery, E. W. Ralston, S. R. Gordon, G. J. Janusz, M. M. Schacher, M. K. Smith, eds.), Volume Three of the Raymond W. Brink Selected Mathematical Papers, MAA, 1977, pp. 487–513.
[75] O. Taussky, *History of sums of squares in algebra*, American mathematical heritage: algebra and applied mathematics, Math. Ser. No. 13, Texas Tech Univ., 1981, pp. 73–90.

1409 W. Green Street Urbana, IL 61801
E-mail address: reznick@math.uiuc.edu