Short algebraic proofs of theorems of Schmüdgen and Pólya

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Abstract: The aim of this note is to give short algebraic proofs of theorems of Schmüdgen and Pólya by means of the representation theorem of Kadison-Dubois. The proof of the latter is elementary and algebraic but tricky. Due to its use no theory from functional analysis is needed. For the notions used here see [BS] and [KS]. For an overview of this and related questions see [R].

1) Prelimenaries: All Rings R are commutative with 1 and contain $\mathbb{R}$. A subset $P \subset R$ is called a preprime iff

$$P + P \subset P,\ P \cdot P \subset P,\ \mathbb{R}_+ \subset P,\ -1 \notin P,$$

where $\mathbb{R}_+$ denotes the nonnegative real numbers. A preprime P is a preorder iff $a^2 \in P$ for all $a \in R$. For $g_1, \dots, g_m \in R$ we denote by $< g_1, \dots, g_m >_{PP}$ ($< g_1, \dots, g_m >_{PO}$) the preprime (preorder) generated by the g_i, i.e.

$$< g_1, \dots, g_m >_{PP} = \{ \sum_{finite} a_i g_1^{i_1} \cdot \ldots \cdot g_m^{i_m} \mid a_i \in \mathbb{R}_+ \}$$

$$< g_1, \dots . g_m >_{PO} = \{ \sum_{\epsilon \in \{0,1\}^m} h_\epsilon g_1^{\epsilon_1} \cdot \ldots \cdot g_m^{\epsilon_m} \mid h_\epsilon \text{ sums of squares in } R \}.$$

Finally a preprime P is called archimedean iff for all $a \in R$ there exists an $n \in \mathbb{N}$, such that

$$n - a \in P.$$

Theorem 1 (Kadison-Dubois): Let R be a ring, $P \subset R$ an archimedean preprime.

$$X(P) := \{ \phi \in Hom(R, \mathbb{R}) \mid \phi(P) \subset \mathbb{R}_+ \}$$

the representation space. If for $f \in R$ $\phi(f) \geq 0\ \forall\ \phi \in X(P)$, then

$$\forall n \in \mathbb{N}: \quad 1 + nf \in P.$$

Proof: See [BS].

Remark: In the case of an affine $\mathbb{R}$-algebra $R = \mathbb{R}[X_1, \ldots, X_n]/\mathfrak{a}$ and a finitely generated preprime/preorder $P =< g_1, \ldots, g_m >_{PP/PO}$ the representation space $X(P)$ ca be identified with the semialgebraic subset $\{g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ of the real variety $Hom(R, \mathbb{R})$.

Lemma 1: Let $R = \mathbb{R}[y_1, \ldots, y_n]$ be an affine $\mathbb{R}$-algebra, $P \subset R$ a preprime. Then

$$P \text{ is archimedean } \iff \exists N \in \mathbb{N}: \; N + y_i, N - y_i \in P \; \forall i = 1 \ldots n.$$

Lemma2: $R = \mathbb{R}[y_1, \ldots, y_n]$ an affine $\mathbb{R}$-algebra, $P \subset R$ a preorder. Then

$$P \text{ is archimedean } \iff \exists N \in \mathbb{N}: \; N - \sum y_i^2 \in P.$$

Proofs: straightforward

2) The theorems

Theorem 2 (Pólya): Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be homogeneous, $f(x) > 0 \; \forall x \in S' := \{x \in \mathbb{R}^n | \; x_1 \geq 0, \ldots x_n \geq 0, \sum x_i \neq 0\}$. Then for some $N \in \mathbb{N}$ $(\sum X_i)^N \cdot f$ is a positive linear combination of power products of the X_i. In other words $(\sum X_i)^N \cdot f$ belongs to the preprime generated by the X_i.

Proof: Consider the ring $R := \mathbb{R}[Y_1, \ldots, Y_n]/(1 - \sum Y_i)$ and the preprime $P :=< y_1, \ldots, y_n >_{PP}$ generated by the residues y_i of the Y_i. By lemma 1 P is archimedean. The set $X(P)$ consists of all evaluations at points of the simplex $S = \{y \in \mathbb{R}^n | y_1 \geq 0, \ldots, y_n \geq 0, \sum y_i = 1\}$. Obviously there exists an $n \in \mathbb{N}$ such that $f - 1/n$ is still positive on S, hence by theorem 1

$$n \cdot f(y_1, \ldots, y_n) = 1 + n \cdot (f(y_1, \ldots, y_n) - 1/n) \in P \Rightarrow f(y_1, \ldots, y_n) \in P.$$

This means that as functions on S we have the equality

$$f(y_1, \ldots, y_n) = \sum a_i y_1^{i_1} \cdot \ldots \cdot y_n^{i_n} \; (a_i \in \mathbb{R}_+).$$

Since $x/\sum x_i \in S$ for all $x \in S'$ by plugging in this for y and multiplying denominators, we get:

$$(\sum x_i)^N \cdot f(x) = \sum b_i x_1^{i_1} \cdot \ldots x_n^{i_n} \text{ (for some } b_i \geq 0)$$

as functions on S'. Since S' is a Zariski dense subset of $\mathbb{R}^n$, this is also an identity of polynomials, hence Pólyas theorem.

Remark: Since Pólyas theorem is wrong for arbitrary real closed fields (see [DS]) this implies, that theorem 1 is also not true for general real closed fields.

Theorem 3 (Schmüdgen): Assume the basic closed semialgebraic set $S := \{x \in \mathbb{R}^n | \; g_1(x) \geq 0, \dots, g_m \geq 0\}$ to be compact. Then any $f \in \mathbb{R}[X_1, \dots, X_n] =: R$ with $f_{|S} > 0$ belongs to the preorder $P :=< g_1, \dots, g_n >_{PO}$ generated by the $g_i \in R$.

Proof: Assume w.l.o.g. that $S \subset \{x \in \mathbb{R}^n | \; \sum x_i^2 < 1\}$. The extended preorder

$$P' := P + (1 - \sum X_i^2)P$$

is archimedean by lemma 2. With the same reasoning as above we conclude from theorem 1 that any f positive on S belongs to P'. Due to the Positivstellensatz there exist $p, p' \in P$ with

$$(1 + p) \cdot (1 - \sum X_i^2) = 1 + p'.$$

By adding $p \cdot \sum X_i^2$ to the left hand side we see that

$$(*) \quad 1 - \sum X_i^2 + p \in P.$$

On the other hand we have seen, that any polynomial positive on S belongs to P'. This shows that $(1 + p)g$ belongs to P for any such polynomial g. Since S is compact there exists $N \in \mathbb{N}$ such that $(N - p)_{|S} > 0$ which yields

$$(**) \quad (1 + p)(N - p) = N + (N - 1)p - p^2 \in P.$$

Finally the follwing square lies in P

$$(***) \quad (N/2 - p)^2 = N^2/4 - Np + p^2 \in P.$$

Adding these three elements of P, we get

$$(*) + (**) + (***) \;=\; (1 + N + N^2/4) - \sum X_i^2 \in P.$$

This shows that P itself is archimedean und hence by theorem 1 any polynomial positive on S belongs to P.

Remark: The technique used in these two proofs allow to state a large variety of Schmüdgen/Pólya type theorems, e.g. it can be used to generalize a theorem of Habicht (see[H]). This will be subject to a forthcoming article.

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