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Even Symmetric Sextics

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1. Introduction and Rationale

A form (real homogeneous polynomial) f is called positive semidefinite (psd) if $f(\mathbf{x}) \geqq 0$ for all $\mathbf{x}=(x_1, \ldots, x_n) \in \mathbf{R}^n$; f is a *sum of squares* (sos) if there exist forms h_k such that $f=\sum h_k^2$. In this paper we study even symmetric sextic forms in $n \geqq 3$ variables. A typical even symmetric sextic is

$$f(\mathbf{x})=\alpha \sum_{i=1}^{n} x_i^6+\beta \sum_{i \neq j} x_i^4 x_j^2+\gamma \sum_{i<j<k} x_i^2 x_j^2 x_k^2. \tag{1.1}$$

We determine necessary and sufficient conditions for such a form to be psd and for such a form to be sos. These conditions are easy to check for any given form f. They provide an explicit description of the cones of psd and sos even symmetric sextics as semialgebraic sets and they settle some open questions of R.M. Robinson. As $f(\mathbf{x}) \geqq 0$ is an inequality, we also obtain an interesting (and largely new) family of symmetric polynomial inequalities. The remainder of this introduction is devoted to describing the psd and sos conditions and placing our work in the context of previous work on psd and sos forms.

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An *n-ary m-ic* is a form $f(x_1, \ldots, x_n)$ of degree m. For fixed n and even m the set of psd (resp. sos) n-ary m-ics, denoted $P_{n,m}$ (resp. $\Sigma_{n,m}$), comprises a closed convex cone (see [CL₁]). Since every sum of squares is psd, $\Sigma_{n,m} \subseteq P_{n,m}$. Hilbert [H] proved that this inclusion is strict unless $n=2$ or $m=2$ or $(n,m)=(3,4)$. Recall that an element u in a closed convex cone C is called *extremal* if the equation $u=u_1+u_2$, $u_i \in C$, implies that $u_i=\lambda_i u$ for suitable real λ_i. If $C \subseteq \mathbf{R}^d$ then every element of C is the sum of finitely many extremal elements. We may associate the n-ary m-ic $f(\mathbf{x})=\sum a(\alpha)\mathbf{x}^\alpha$ with the d-tuple of its coefficients, $d=d(n,m)=\binom{n+m-1}{n-1}$. In this way, the cones $P_{n,m}$ and $\Sigma_{n,m}$ may be thought of as lying in $\mathbf{R}^d$. If the cones $P_{n,6}$ and $\Sigma_{n,6}$ are intersected with the (closed) cone of even symmetric sextics then the resulting cones lie in a three-dimensional subspace in $\mathbf{R}^d$. The coefficients of x_1^6, $x_1^4x_2^2$ and $x_1^2x_2^2x_3^2$ determine f.

Intuitively speaking, the cone $\Sigma_{n,m}$ is defined *internally:* every extremal element is a perfect square, though it is difficult to decide whether a given form f is sos (see [CLR₁]). On the other hand, the cone $P_{n,m}$ is defined *externally:* we know, in principle, whether a given form f is psd (calculate $f(\mathbf{x})$ for all $\mathbf{x}$!); it is difficult, however, to find the extremal elements. (Extremal psd forms are discussed in [CL₁], [CL₂] and [Re].) What is appealing about even symmetric sextics is that cones in $\mathbf{R}^3$ are easily visualized, and these difficult general questions can be answered in complete detail in this special case.

There is a representation for the even symmetric sextic which is somewhat more useful than (1.1). Let

$$M_r(\mathbf{x})=\sum_{i=1}^{n} x_i^r, \tag{1.2}$$

then the form f from (1.1) can also be written

$$f=aM_6+bM_2M_4+cM_2^3, \tag{1.3}$$

where (α, β, γ) and (a,b,c) are related by

$$\alpha=a+b+c, \quad \beta=b+3c, \quad \gamma=6c. \tag{1.4}$$

It is also useful to isolate a crucial set of points: for an integer m, $1 \leq m \leq n$, let

$$\mathbf{v}_m=(1,1,\ldots,1,0,\ldots,0) \tag{1.5}$$

denote the n-tuple consisting of m 1's followed by $n-m$ 0's. Observe that $M_r(\mathbf{v}_m)=m$, independently of r, hence

$$f(\mathbf{v}_m)=am+bm^2+cm^3. \tag{1.6}$$

The simplicity of (1.6) suggests the following definition of the auxiliary quadratic f^*: let

$$f^*(t)=a+bt+ct^2 \tag{1.7}$$

so that

$$f(\mathbf{v}_m)=mf^*(m), \quad m=1,2,\ldots,n. \tag{1.8}$$

Note that $f^*=g^*$ implies $f=g$, hence $\sum c_i f_i^*=0$ implies $\sum c_i f_i=0$. It is often convenient to work with the auxiliary quadratics, rather than with the sextics themselves.

If f is psd then $f(\mathbf{v}_m) \geq 0$ and so $f^*(m) \geq 0$ for $m=1,2,\ldots,n$. Theorem (3.7) states that the converse is true: if $f^*(t) \geq 0$ for $t \in \{1,2,\ldots,n\}$ then f is psd. Although the techniques of proof are quite different, the conclusion is similar in the sos case. Theorem (4.25) states that f is sos if and only if $f^*(t) \geq 0$ for $t \in \{1\} \cup [2,n]$. We may rephrase this criterion in a somewhat surrealistic way. Let t be a positive real number and let $\mathbf{v}_t$ denote a "pseudo-point" in $\mathbf{R}^n$ with t components "equal" to 1 and $n-t$ components "equal" to 0. Extend the domain of definition of M_r to pseudo-points in the natural way: $M_r(\mathbf{v}_t)=t$. (If t is an integer, this is consistent with our previous notation.) Using pseudo-points, the criteria for f to be psd and sos can be given a parallel structure: *f is psd if and only if $f(\mathbf{v}_n) \geq 0$ for the points $\mathbf{v}_1, \ldots, \mathbf{v}_n$; f is sos if and only if "$f(\mathbf{v}_t) \geq 0$" for the pseudo-points $\mathbf{v}_t$, $t \in \{1\} \cup [2,n]$.*

Let P_n (resp. Σ_n) denote the set of (a,b,c) such that the form $aM_6+bM_2M_4+cM_2^3$ is psd (resp. sos). As noted above, these are both closed cones in $\mathbf{R}^3$. It is well-known that $P_{n,m}$ is a semialgebraic set in $\mathbf{R}^d$ for all (n,m). Hence the intersection of $P_{n,6}$ with the cone of symmetric forms is also semialgebraic. (See [L] for the definitions and theorems in this paragraph.) It is not immediately obvious but Σ_n is also a semialgebraic set. Theorems (3.7) and (4.25) can be used to give an explicit decomposition of P_n and Σ_n as a finite union of closed basic semialgebraic sets. It is worth noting that P_n is actually a polyhedron and Σ_n can be defined using linear inequalities and exactly one quadratic inequality.

At least one of the extremal sextics we discuss has already appeared in the literature. R.M. Robinson [Ro] studied the ternary sextic

$$S(x,y,z) = x^6+y^6+z^6-(x^4y^2+x^4z^2+y^4x^2+y^4z^2+z^4x^2+z^4y^2)+3x^2y^2z^2. \tag{1.9}$$

He proved that S is psd (and extremal in $P_{3,6}$ by [CL₂]) but not sos; this form was the first psd symmetric form shown to be not sos. Robinson also showed that there exists β_0 such that $S+\beta M_6$ is sos if and only if $\beta \geq \beta_0$ and gave the estimate

$$1 \geq \beta_0 \geq 2.5-\sqrt{6} \approx 0.05.$$

We use the sos criterion to show that the true value for β_0 is $1/8=0.125$. Robinson also asked when $S+\beta M_6$ is a sum of squares of monomials and binomials; we answer this question as well. The ternary form S represents the vanguard of a family of extremal psd even symmetric sextics which are not sos: there are $n-2$ of them for the sextics in n variables. The sextic S has also surfaced in another context. Make the substitution $(x^2,y^2,z^2)=(a,b,c)$ in (1.9), then we obtain the classical Lehmus inequality:

$$(1.10)\quad S(x,y,z)=\Gamma(a,b,c)=abc-(b+c-a)(c+a-b)(a+b-c)\geqq 0 \quad \text{for } a,b,c\geqq 0.$$

In 1820 Lehmus showed that $\Gamma(a,b,c)\geqq 0$ when a, b and c are the sides of a triangle; in 1920 Peano showed that $\Gamma(a,b,c)\geqq 0$ for all non-negative a, b and c. Hobson has shown that, if a triangle has sides a, b and c and area Δ, then $\Gamma(a,b,c)$ is 8Δ times the distance between the centers of the inscribed circle and the nine-point circle, while $S(a,b,c)^{1/2}$ is 8Δ times the distance between the centers of the circumscribed circle and the nine-point circle ([Ho, pp. 198–200]). See also Coxeter's survey article [Co] for references and more details. Finally, as noted in [CL_2],

$$\Gamma(a,b,c)=a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b)$$

and so $\Gamma(a,b,c)\geqq 0$ for $a,b,c\geqq 0$ is a special case of Schur's inequality. (See [HLP, p. 64]). See note added in proof.

H.D. Ursell [U] studied the mapping

$$T:\ (x_1,\dots,x_n)\to(M_{p_1}(\mathbf{x}),\dots,M_{p_r}(\mathbf{x}))$$

where $0<p_1<\dots<p_r$ and $x_i\geqq 0$. He determined the set $\mathscr{S}(p_1,\dots,p_r)$ of points $\mathbf{z}\in\mathbb{R}^n_+$ for which $T(\mathbf{z})$ is on the topological boundary of $T(\mathbb{R}^n_+)$. Let $p_1=1$, $p_2=2$ and $p_3=3$; one can show that every zero of the psd even symmetric sextic $aM_6+bM_2M_4+cM_2^3$ is contained in $\mathscr{S}(1,2,3)$, but not vice versa. Ursell's Theorem states in this case that the elements of $\mathscr{S}(1,2,3)$ have, projectively, and up to permutation, the shape $\mathbf{z}=(t,1,\dots,1,0,\dots,0)$ with $t\geqq 1$. As we shall see (Theorem (3.6)), such a $\mathbf{z}$ can be a zero of a psd even symmetric sextic if and only if $t=1$ or $\mathbf{z}$ has the form $(t,1,0,\dots,0)$.

This paper is organized as follows. Section two introduces the several families of extremal even symmetric sextics. It contains a crucial technical proposition analyzing the set of quadratics q which are non-negative on certain subsets of $\mathbb{R}$. For completeness, the psd and sos even symmetric sextics in $n=2$ variables are also determined. Section three considers the psd case. By means of a surprising algebraic identity, we are able to determine the possible zeros of a psd even symmetric sextic. This leads directly to Theorem (3.7) on the criteria for f to be psd. Section four considers the sos case. We use the evenness and symmetry of f to "even out" and "symmetrize" any given representation $f=\sum h_k^2$, obtaining a small family of extremal sums of squares. The technical proposition of section two allows us to phrase the sos criterion for f in terms of f^*. Section five discusses the answers to Robinson's questions on $S+\beta M_6$ and the sums of binomial squares; section six presents pictures of the psd and sos cones for $n=3$, 4 and 5 which reflect facts about semialgebraic sets. We conclude in section seven with some open questions.

We wish to thank Professor R.M. Robinson for suggesting the inclusion of the diagrams in section six and for other valuable comments.

2. Machinery and Preliminaries

We now introduce the *dramatis personae* for the rest of the paper. We shall assume that f is an even symmetric sextic in a fixed number ($n\geqq 3$) of variables, and shall use the representations (1.1) and (1.3) interchangeably (always keeping (1.4) in mind). The extremal forms turn out to be those f whose auxiliary quadratics f^* have a simple zero structure. This should not be too surprising: if $f^*(m)=0$ for an integer m, $1\leqq m\leqq n$, then $f(\mathbf{v}_m)=0$. By taking all permutations (f is symmetric) and sign choices (f is even) for the components, f has at least $2^{m-1}\binom{n}{m}$ zeros, viewed projectively. Further, if $f^*(m_1)=f^*(m_2)=0$, $m_1\neq m_2$, then $f^*(t)=\lambda(t-m_1)(t-m_2)$ is determined up to a scalar multiple, and hence so is f.

The first class of forms is suggested by the quadratics $(t-k)(t-(k+1))$, $1\leqq k\leqq n-1$, which are non-negative on the set $\{1,2,\dots,n\}$. Define forms f_k for integers k, $1\leqq k\leqq n-1$, by any of the following three representations:

$$(2.1)(a)\qquad f_k=(k^2+k)M_6-(2k+1)M_2M_4+M_2^3,$$

$$(2.1)(b)\qquad f_k=(k^2-k)\sum x_i^6-2(k-1)\sum x_i^4x_j^2+6\sum x_i^2x_j^2x_k^2,$$

$$(2.1)(c)\qquad f_k^*(t)=k^2+k-(2k+1)t+t^2=(t-k)(t-(k+1)).$$

(See (1.4) for the equivalence of (a) and (b) in this and the other representations.) It is certainly not obvious (and we have not yet proved) that f_k is psd, although evidently $f_k(\mathbf{v}_m)\geqq 0$ with $f_k(\mathbf{v}_k)=f_k(\mathbf{v}_{k+1})=0$. It turns out that f_k is sos only when $k=1$:

$$(2.2)\qquad f_1=6\sum x_i^2x_j^2x_k^2$$

is obviously sos and so is psd. A comparison of (1.9) with (2.1)(b) shows that Robinson's form $S(x,y,z)$ equals $\frac{1}{2}f_2(x,y,z)$. *Thus, the forms $f_2,\dots,f_{n-1}$ may be viewed as generalizations of Robinson's form $S(x,y,z)$ to the case of any arbitrary number ($\geqq 3$) of variables.*

Another quadratic which is non-negative on $\{1,2,\dots,n\}$ is $-(t-1)(t-n)$. Accordingly, we define f_0 by any of these three representations:

$$(2.3)(a)\qquad f_0=-nM_6+(n+1)M_2M_4-M_2^3,$$

$$(2.3)(b)\qquad f_0=(n-2)\sum x_i^4x_j^2-6\sum x_i^2x_j^2x_k^2,$$

$$(2.3)(c)\qquad f_0^*(t)=-n+(n+1)t-t^2=-(t-1)(t-n).$$

The following expression of f_0, easily checked against (2.3)(b) by expansion, shows that f_0 is a sum of squares, and hence also psd:

$$(2.4)\qquad f_0(\mathbf{x})=\sum_{i=1}^{n}\Big(\sum_{\substack{j<k\\ j,k\neq i}}x_i^2(x_j^2-x_k^2)^2\Big).$$

The last special forms are suggested by the quadratics $(t-\lambda)^2$; for all real λ define g_λ by any of the following:

(2.5)(a) $$g_\lambda = \lambda^2 M_6 - 2\lambda M_2 M_4 + M_2^3,$$

(2.5)(b) $$g_\lambda = (\lambda-1)^2 \sum x_i^6 + (3-2\lambda)\sum x_i^4 x_j^2 + 6\sum x_i^2 x_j^2 x_k^2,$$

(2.5)(c) $$g_\lambda^*(t) = \lambda^2 - 2\lambda t + t^2 = (t-\lambda)^2.$$

The following expression of g_λ, easily checked against (2.5)(a) by expansion, shows that g_λ is a sum of squares, and hence also psd:

(2.6) $$g_\lambda(\mathbf{x}) = \sum_{i=1}^n x_i^2(\lambda x_i^2 - M_2)^2 = \sum_{i=1}^n x_i^2[(\lambda-1)x_i^2 - \sum_{j\neq i} x_j^2]^2.$$

Observe that $M_6 = \lim_{\lambda\to\infty} \lambda^{-2} g_\lambda$ is a limiting case for the g_λ's. However,

(2.7) $$M_6 = \left(\frac{2}{n-1}\right)^2 (f_0 + g_{(n+1)/2})$$

is not extremal as an sos even symmetric sextic.

We now turn to a more technical matter. Suppose I is a compact subset of $\mathbf{R}$ with at least 3 elements, and let

(2.8) $$P(I) = \{p(t) = a + bt + ct^2 \colon p(t) \geq 0 \text{ for all } t \in I\}.$$

Intuitively, either p or $-p$ belongs to $P(I)$ unless two components of I are separated by a single zero of p. It is clear from (2.8) that $P(I)$ is a closed convex cone. We are interested in representing the elements of $P(I)$ in terms of its extremal elements in the special cases $I = \{1, 2, \ldots, n\}$ and $I = \{1\} \cup [2, n]$.

(2.9) **Lemma.** *Suppose $p(t) = (t-u_1)(t-u_2) \in P(I)$ and $u_j \in I$. If $u_1 \neq u_2$ or $u_1 = u_2 = u$ and a one-sided neighborhood of u lies in I, then p is extremal.*

Proof. In the first case, if $p = q_1 + q_2$, $q_i \in P(I)$, then $q_i(u_j) \geq 0$, hence $q_i(u_j) = 0$. Thus $q_i = a_i p$; that is, p is extremal. In the second case, keeping the previous notation, as q_i is quadratic and $(t-u)^2 \geq q_i(t) \geq 0$ in a one-sided neighborhood of u, it again follows that each q_i is a multiple of p. □

(2.10) **Theorem.** (i) *If $I = \{1\} \cup [2, n]$ and $p \in P(I)$, then there exists $u \in [2, n]$ and non-negative c_i so that*

(2.11) $$p(t) = c_1(t-u)^2 + c_2(t-1)(t-2) + c_3(t-1)(n-t).$$

Conversely, any such p belongs to $P(I)$. The extremal elements of $P(I)$ are, up to a scalar multiple, $(t-u)^2$, $(u \in [2, n])$, $(t-1)(t-2)$ and $(t-1)(n-1)$.

(ii) *If $I = \{1, 2, \ldots, n\}$ and $p \in P(I)$, then there exists an integer k, $2 \leq k \leq n-1$, and non-negative c_i so that*

(2.12) $$p(t) = c_1(t-k)(t-(k-1)) + c_2(t-k)(t-(k+1)) + c_3(t-1)(n-t).$$

Conversely, any such p belongs to $P(I)$. The extremal elements of $P(I)$ are, up to a scalar multiple, $(t-1)(n-t)$ and $\{(t-k)(t-(k-1))\colon 2 \leq k \leq n\}$.

Proof. (i) We separate two cases. If $p(n) = 0$, then $p(t) = (n-t)r(t)$, where r is linear and non-negative on $I \setminus \{n\}$ and so on I by continuity. Since the functions $(n-t)$ and $(t-1)$ are independent, there exist real c_i so that $r(t) = c_1(n-t) + c_3(t-1)$. As $r(1) = c_1(n-1) \geq 0$ and $r(n) = c_3(n-1) \geq 0$, it follows that $c_i \geq 0$; that is, p has shape (2.11). If $p(n) > 0$, let $c_3 = \inf\{p(t)/[(t-1)(n-t)]\colon t \in [2, n]\}$. Then $q(t) = p(t) - c_3(t-1)(n-t) \in P(I)$ and $q(u) = 0$ for some $u \in [2, n)$. If $u \neq 2$, then $q \geq 0$ in a neighborhood of u, hence $q(t) = c_1(t-u)^2$. If $u = 2$, then $q(t) = (t-2)r(t)$, where r is linear, $r(1) \leq 0$ and $r(t) \geq 0$ on $(2, n]$, and so on $[2, n]$ by continuity. As above, $r(t) = c_1(t-2) + c_2(t-1)$ for real c_i, and evaluating r at $t = 1, 2$ gives $c_i \geq 0$. Thus, in any event, p can be put into the form (2.11). It is easy to see that any such polynomial belongs to $P(I)$ and that the only possible extremal elements are therefore $(t-u)^2$, $(u \in [2, n])$, $(t-1)(t-2)$ and $(t-1)(n-t)$. By Lemma (2.9) they are all extremal.

(ii) Let $c_3 = \min\{p(t)/[(t-1)(n-t)]\colon t \in \{2, \ldots, n-1\}\}$ and $q(t) = p(t) - c_3(t-1)(n-t)$. Then $q \in P(I)$ and $q(k) = 0$ for some $k \in \{2, \ldots, n-1\}$. As before, $q(t) = (t-k)r(t)$ and $r(t) = c_1(t-(k-1)) + c_2(t-(k+1))$ for real c_i, and evaluation of q at $k \pm 1$ shows that $c_i \geq 0$. The extremality statements again follow as in (i). □

For completeness, we include the following general proposition about the structure of $P(I)$ if I is compact and $|I| \geq 3$. Let $l = \inf(I)$ and $r = \sup(I)$; for $u \in I$, $l < u \leq r$, let $L(u) = \sup\{(-\infty, u) \cap I\}$ and for $u \in I$, $l \leq u < r$, let $R(u) = \inf\{(u, \infty) \cap I\}$. Intuitively, $L(u)$ (resp. $R(u)$) is the "nearest point" of I to the left (resp. right) of u; $L(u) = u$ ($R(u) = u$) if there are points of I arbitrarily close and to the left (right) of u. Since I is closed, $L(u)$ and $R(u)$ belong to I, and since I has at least three elements, $l < L(r) \leq r$ and $l \leq R(l) < r$. The proof of the following proposition is similar to that of the last theorem and is omitted.

(2.13) **Proposition.** *If $p \in P(I)$, then there exists $u \in I$ and non-negative c_i so that*

(2.14) $$p(t) = c_1(t-u)(t-L(u)) + c_2(t-u)(t-R(u)) + c_3(t-l)(r-t).$$

(If $u = l$ set $c_1 = 0$ as $L(l)$ is undefined; if $u = r$ set $c_2 = 0$ as $R(r)$ is undefined). Every such p lies in $P(I)$ and the extremal elements of $P(I)$ are, up to a scalar multiple, $(t-l)(r-t)$, $(t-u)(t-L(u))$ $(l \neq u \in I)$ and $(t-u)(t-R(u))$ $(r \neq u \in I)$.

We conclude this section with a brief but complete discussion of even symmetric sextics in two variables.

These form a degenerate case: we should omit the empty sum $\sum_{i<j<k} x_i^2 x_j^2 x_k^2$, and there is the dependence relation $M_2^3 - 3M_2M_4 + 2M_6 = 0$.

(2.15) **Proposition.** *Let $f(x, y) = \alpha(x^6 + y^6) + \beta(x^4y^2 + x^2y^4)$ be an even symmetric sextic in two variables. The following statements are equivalent:*

(i) *f is sos.*
(ii) *f is psd.*
(iii) *$f(1, 0) = \alpha \geq 0$ and $f(1, 1) = 2(\alpha + \beta) \geq 0$.*

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are immediate; (iii)⇒(i) follows from the factorization

$$f(x, y) = (x^2 + y^2)[\alpha(x^2 - y^2)^2 + (\alpha + \beta)x^2y^2]. \quad □$$

We note in passing that this proposition is consistent with Theorems (3.7)(i), (ii), (iii) and (4.25)(i), (iii) with $n=2$, and that the extremal forms are $x^2y^2(x^2+y^2)$ and $(x^2-y^2)^2(x^2+y^2)$.

3. Psd Even Symmetric Sextics

Suppose f is a psd even symmetric sextic in $n \geqq 3$ variables and let $\lambda = \inf\{f(\mathbf{x}): \sum x_i^2 = 1\}$. If $\lambda = 0$ then $f(\mathbf{y}) = 0$ for some $\mathbf{y} \neq 0$. If $\lambda > 0$ then $\bar{f} = f - \lambda M_2^3$ is also psd with $\bar{f}(\mathbf{y}) = 0$ for some $\mathbf{y} \neq 0$. In either case we see the importance of studying psd even symmetric sextics which are not strictly definite. Our analysis is greatly aided by the following intriguing identity for even symmetric sextics.

(3.1) **Lemma.** *Fix real numbers $w, x_3, \ldots, x_n$ and define*

(3.2) $$\mathbf{x}(\theta) = (w\cos\theta, w\sin\theta, x_3, \ldots, x_n).$$

Then for any even symmetric sextic f and all real θ,

(3.3) $$f(\mathbf{x}(\theta)) = (\cos^2 2\theta)\, f(\mathbf{x}(0)) + (\sin^2 2\theta)\, f(\mathbf{x}(\pi/4)).$$

Proof. It suffices to prove (3.3) for the basic sextics M_6, M_2M_4 and M_2^3. Since $M_2(\mathbf{x}(\theta)) = w^2 + \sum_{i=3}^n x_i^2$ is independent of θ, we need only show

(3.4) $$(w\cos\theta)^r + (w\sin\theta)^r + \sum_{i=3}^n x_i^r = \cos^2 2\theta\left(w^r + \sum_{i=3}^n x_i^r\right) + \sin^2 2\theta\left(\frac{2}{(\sqrt{2})^r} w^r + \sum_{i=3}^n x_i^r\right)$$

for $r=4$ and 6. Observe that $\sum_{i=3}^n x_i^r$ cancels and then w^r factors out. Hence (3.4) is equivalent to

$$\cos^r\theta + \sin^r\theta = \cos^2 2\theta + \frac{2}{2^{r/2}}\sin^2 2\theta, \quad r=4 \text{ or } 6.$$

Indeed, for $r=4$, RHS $= (\cos^2\theta - \sin^2\theta)^2 + \frac{1}{2}(2\sin\theta\cos\theta)^2 = \cos^4\theta + \sin^4\theta$, and for $r=6$,

$$\text{RHS} = (\cos^2\theta - \sin^2\theta)^2 + \tfrac{1}{4}(2\sin\theta\cos\theta)^2 = \cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta = \cos^6\theta + \sin^6\theta. \quad \square$$

Using this lemma we obtain a powerful condition on the zero-set of a psd even symmetric sextic.

(3.5) **Lemma.** *Suppose f is a psd even symmetric sextic and $f(\mathbf{y}) = 0$ where $y_1 \neq 0$, $y_2 \neq 0$ and $y_1^2 \neq y_2^2$. Let $w = (y_1^2 + y_2^2)^{1/2}$ and define $\mathbf{y}(\theta)$ by (3.2). Then $f(\mathbf{y}(\theta)) = 0$ for all real θ.*

Proof. Since f is even and symmetric we may assume $y_1 > y_2 > 0$ and define ϕ, $0 < \phi < \frac{\pi}{4}$, such that $\mathbf{y} = \mathbf{y}(\phi)$. By (3.3), $0 = f(\mathbf{y}) = f(\mathbf{y}(\phi)) = (\cos^2 2\phi)\, f(\mathbf{y}(0)) + (\sin^2 2\phi)\, f\left(\mathbf{y}\left(\frac{\pi}{4}\right)\right)$. As f is psd and $\sin^2 2\phi$, $\cos^2 2\phi$ are both positive, it follows that $f(\mathbf{y}(0)) = f\left(\mathbf{y}\left(\frac{\pi}{4}\right)\right) = 0$. But then (3.3) for an arbitrary θ implies that $f(\mathbf{y}(\theta)) = 0$. $\square$

We are now able to characterize the zeros of a psd even symmetric sextic. We shall use the symmetry of f implicitly in permuting the components of a given zero. Recall that $\mathbf{v}_m = (1, \ldots, 1, 0, \ldots, 0)$ is the n-tuple with m components equal to 1 and $n-m$ equal to 0.

(3.6) **Theorem.** *Suppose f is a nonzero psd even symmetric sextic and $f(\mathbf{y}) = 0$ for some $\mathbf{y} \neq 0$: $y_1 \geqq y_2 \geqq \ldots \geqq y_m > 0 = y_{m+1} = \ldots = y_n$, $1 \leqq m \leqq n$. Then either $y_1 = y_m$ (and $\mathbf{y} = \lambda\mathbf{v}_m$) or $m=2$ and $f = \gamma\sum x_i^2x_j^2x_k^2$. In either case, $f(\mathbf{v}_k) = 0$ for some k, $1 \leqq k \leqq n$.*

Proof. If $y_1 = y_m$ then $\mathbf{y} = \lambda\mathbf{v}_m$ and we are done. Otherwise, $m \geqq 2$ and $y_1 > y_m$. By Lemma (3.5) with $\theta = 0$,

$$f((y_1^2+y_2^2)^{1/2}, 0, y_3, \ldots, y_m, 0, \ldots, 0) = f((y_1^2+y_2^2)^{1/2}, y_3, \ldots, y_m, 0, 0, \ldots, 0) = 0.$$

Repeating this argument, we eventually obtain

$$f((y_1^2 + \ldots + y_{m-1}^2)^{1/2}, y_m, 0, \ldots, 0) = 0,$$

and by Lemma (3.5) and the homogeneity of f, $f(\mathbf{z}) = 0$ for any $\mathbf{z}$ with at most two non-zero components. In particular, $f(\mathbf{v}_1) = f(\mathbf{v}_2) = 0$ so by (1.8), $f^*(1) = f^*(2) = 0$. Thus $f^* = \lambda f_1^*$ and by (2.2), $f = \gamma\sum x_i^2x_j^2x_k^2$. As $f(\mathbf{x}) \geqq \gamma x_i^2x_j^2x_k^2$, $f(\mathbf{y}) = 0$ if and only if at most two components of $\mathbf{y}$ are non-zero. $\square$

H.W. Schülting has pojnted out to us that it is also possible to prove the Lemma above by using Bezout's Theorem. Keeping the notations in (3.5), let C_1 be the affine curve defined by $0 = F(x_1, x_2) := f(x_1, x_2, y_3, \ldots, y_n)$. (We may, of course, assume that $F \not\equiv 0$.) This curve passes through the eight distinct points $P_1, \ldots, P_8$ given by $(\pm y_1, \pm y_2)$, $(\pm y_2, \pm y_1)$. Since $F(x_1, x_2)$ is psd, $\partial F/\partial x_1$, $\partial F/\partial x_2$ must vanish on $P_1, \ldots, P_8$, so these are *singular* points of C_1. If C_2 denotes the affine circle $x_1^2 + x_2^2 = y_1^2 + y_2^2$, the intersection multiplicities $m_{P_i}(C_1, C_2)$ are thus $\geqq 2$. If $C_2 \nsubseteq C_1$, Bezout's Theorem would give the contradiction

$$12 \geqq 2(\deg F) \geqq \sum_{i=1}^8 m_{P_i}(C_1, C_2) \geqq 16.$$

Thus, $C_2 \subseteq C_1$, i.e. $f(\mathbf{y}(\theta)) = 0$ for all real θ.

(3.7) **Theorem.** *Let f be an even symmetric sextic in $n \geqq 3$ variables. The following statements are equivalent:*

(i) f *is psd.*
(ii) $f(\mathbf{v}_m) \geqq 0$ *for* $m=1,2,\ldots,n$.
(iii) $f^*(t) \geqq 0$ *for* $t \in \{1,2,\ldots,n\}$.
(iv) *There exists* k, $2 \leqq k \leqq n-1$, *and non-negative* c_i *such that*

(3.8) $$f = c_1 f_{k-1} + c_2 f_k + c_3 f_0,$$

where the f_i*'s are given in* (2.1) *and* (2.3).

Proof. The implications (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii) are immediate from the definitions and (1.8). Suppose (ii) holds and f is not psd. Let

$$-\mu = \inf\left\{ f(\mathbf{x}) \colon \sum_{i=1}^{n} x_i^2 = 1 \right\}.$$

By construction $\mu > 0$, $g = f + \mu M_2^3$ is psd and $g(\mathbf{y}) = 0$ for some $\mathbf{y} \neq \mathbf{0}$. By Theorem (3.6), $g(\mathbf{v}_m) = 0$ for some m, $1 \leqq m \leqq n$, but $g(\mathbf{v}_m) = f(\mathbf{v}_m) + \mu m^3 \geqq \mu m^3$, a contradiction.

In the notation of Theorem (2.10)(ii), we have now shown that f is psd if and only if $f^* \in P(J)$ for $J = \{1,2,\ldots,n\}$. So, for such f there exist k, $2 \leqq k \leqq n-1$, and $c_i \geqq 0$ such that

(3.9) $$f^*(t) = c_1(t-k)(t-(k-1)) + c_2(t-k)(t-(k+1)) + c_3(t-1)(n-t),$$

thus f has the shape (3.8). □

We conclude this section with a precise description of the extremal psd even symmetric sextics.

(3.10) **Theorem.** *The forms* $f_0, f_1, \ldots, f_{n-1}$ *are psd, and, up to scalar multiple, they give precisely all the extremal elements in the cone of psd even symmetric sextics.*

Proof. Use Theorem (2.10)(ii) and Theorem (3.7). □

We have pointed out before that the forms $f_2, \ldots, f_{n-1}$ are analogues of Robinson's form $S(x,y,z)$ in the case of n variables. According to the theorem above, these forms, together with the sos forms f_0 and f_1, form a "spanning set" for all the psd even symmetric sextics. (We shall show in section four below that the forms $f_2, \ldots, f_{n-1}$ are also *not* sums of squares, for any $n \geqq 3$.)

4. Sos Even Symmetric Sextics

In this section we determine necessary and sufficient conditions for an even symmetric sextic f to be a sum of squares. Throughout we shall use the multi-index notation $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, etc. The first result is a generalization of the "polarization" identity $(u+v)^2 + (u-v)^2 = 2u^2 + 2v^2$ valid for all even forms.

(4.1) **Theorem.** *Suppose* $p = \sum_{i=1}^{r} h_i^2$ *is an even form. Then we may write* $p = \sum_{j=1}^{s} q_j^2$, *where each form* q_j^2 *is even. In particular,* $q_j(\mathbf{x}) = \sum c_j(\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\alpha}}$, *where the sum is taken over* $\boldsymbol{\alpha}$*'s in one congruence class* mod 2 *componentwise.*

Proof. Let p be an n-ary form and for $\boldsymbol{\varepsilon} \in \{-1,1\}^n$ let $\boldsymbol{\varepsilon}\mathbf{x}$ denote the n-tuple $(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$. As p is even, $p(\boldsymbol{\varepsilon}\mathbf{x}) = p(\mathbf{x})$ for all choices of $\boldsymbol{\varepsilon}$, so

$$p(\mathbf{x}) = 2^{-n} \sum_{\boldsymbol{\varepsilon}} p(\boldsymbol{\varepsilon}\mathbf{x}) = \sum_{i=1}^{r} 2^{-n} \sum_{\boldsymbol{\varepsilon}} h_i^2(\boldsymbol{\varepsilon}\mathbf{x}).$$

It suffices then to write $2^{-n} \sum_{\boldsymbol{\varepsilon}} h^2(\boldsymbol{\varepsilon}\mathbf{x})$ as a sum of squares, each of which is even. Let $h(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} a(\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\alpha}}$, then $h(\boldsymbol{\varepsilon}\mathbf{x}) = \sum_{\boldsymbol{\alpha}} a(\boldsymbol{\alpha}) \boldsymbol{\varepsilon}^{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ and

(4.2) $$2^{-n} \sum_{\boldsymbol{\varepsilon}} h^2(\boldsymbol{\varepsilon}\mathbf{x}) = 2^{-n} \sum_{\boldsymbol{\varepsilon}} \sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}} a(\boldsymbol{\alpha}) a(\boldsymbol{\beta}) \boldsymbol{\varepsilon}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}.$$

For any n-tuple of integers $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$,

(4.3) $$\sum_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}^{\boldsymbol{\gamma}} = \sum_{\varepsilon_1 = \pm 1} \cdots \sum_{\varepsilon_n = \pm 1} \varepsilon_1^{\gamma_1} \ldots \varepsilon_n^{\gamma_n} = \prod_{i=1}^{n} (1^{\gamma_i} + (-1)^{\gamma_i}).$$

Thus $\sum_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}^{\boldsymbol{\gamma}} = 0$ unless each γ_i is even, in which case it equals 2^n. Putting this into (4.2),

(4.4) $$2^{-n} \sum_{\boldsymbol{\varepsilon}} h^2(\boldsymbol{\varepsilon}\mathbf{x}) = \sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}} a(\boldsymbol{\alpha}) a(\boldsymbol{\beta}) \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}},$$

where the term $a(\boldsymbol{\alpha}) a(\boldsymbol{\beta}) \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ appears if and only if each component of $\boldsymbol{\alpha}+\boldsymbol{\beta}$ is even. Alternatively, $\alpha_i \equiv \beta_i \bmod 2$ for $i = 1, \ldots, n$. Let $A(1), \ldots, A(N)$, $N \leqq 2^n$, denote the congruence classes of $\boldsymbol{\alpha}$'s mod 2 for which a $(\boldsymbol{\alpha}) \neq 0$ and let

$$q_j(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in A(j)} a(\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\alpha}}.$$

Then (4.4) becomes

$$2^{-n} \sum_{\boldsymbol{\varepsilon}} h^2(\boldsymbol{\varepsilon}\mathbf{x}) = \sum_{j=1}^{N} q_j^2(\mathbf{x});$$

each exponent in $q_j^2(\mathbf{x})$ has the shape $\boldsymbol{\alpha}+\boldsymbol{\beta}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A(j)$. Thus q_j^2 is an even function, completing the proof. □

(4.5) **Corollary.** *If* f *is an sos even sextic then* $f = \sum q_j^2$ *where each* q_j *has one of the following two shapes:*

(4.6) $$d x_r x_s x_t, \quad 1 \leqq r < s < t \leqq n,$$

(4.7) $$d x_r^3 + \sum_{s \neq r} e_s x_r x_s^2, \quad 1 \leqq r \leqq n.$$

Proof. If $f = \sum h_i^2$ then h_i is a cubic and $h_i(\mathbf{x}) = \sum a_i(\boldsymbol{\alpha}) \mathbf{x}^{\boldsymbol{\alpha}}$ implies that $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $\sum \alpha_i = 3$. There are two types of congruence classes for such

α's mod 2: either 3 or 1 of the components are odd. These classes generate forms of the shape (4.6) and (4.7) respectively. □

Having exploited the fact that f is even, we turn to its symmetry. For any permutation $\sigma \in S_n$ and for $\mathbf{x} \in \mathbb{R}^n$ define $\sigma(\mathbf{x}) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. If a form p is symmetric then $p(\mathbf{x}) = p(\sigma(\mathbf{x}))$ for all σ and $\mathbf{x}$. It is not necessarily true that an sos symmetric form is a sum of squares of *symmetric* forms: $x^2 + y^2 \neq \sum_i (\lambda_i x + \lambda_i y)^2$. However sums of squares can be symmetrized in a way analogous to Theorem (4.1). If q is an n-ary form, the *symmetrization* of q is

(4.8) $$S(q)(\mathbf{x}) = \frac{1}{n!} \sum_{\sigma \in S_n} q(\sigma(\mathbf{x})).$$

Suppose $p = \sum_{i=1}^{r} h_i^2$ is symmetric, then

(4.9) $$p(\mathbf{x}) = \frac{1}{n!} \sum_{i=1}^{r} \sum_{\sigma} h_i^2(\sigma(\mathbf{x})) = \sum_{i=1}^{r} S(h_i^2).$$

It is possible to treat symmetrizations systematically; in this paper we confine ourselves to $S(h^2)$ where h has shape (4.6) or (4.7).

If $h(x) = dx_r x_s x_t$ then it is easy to see that

(4.10) $$S(h^2)(\mathbf{x}) = \frac{d^2}{\binom{n}{3}} \sum_{i<j<k} x_i^2 x_j^2 x_k^2 = \frac{d^2}{n(n-1)(n-2)} f_1(x).$$

If h has shape (4.7) then $S(h^2)$ is more complicated. It is convenient to first sum over those σ which fix r. In this way we obtain a form which is symmetric in the $n-1$ variables other than x_r. Having done this, we then decompose S_n into cosets depending on $\sigma(r)$, and sum over these cosets.

(4.11) **Lemma.**

(4.12) $$\frac{1}{m!} \sum_{\sigma \in S_m} \Big(dy + \sum_{k=1}^{m} e_k z_{\sigma(k)}\Big)^2 = \Big(dy + e \sum_{k=1}^{m} z_k\Big)^2 + w \sum_{k<l} (z_k - z_l)^2,$$

where

(4.13) $$e = \frac{1}{m} \sum_{k=1}^{m} e_k$$

and

(4.14) $$w = \frac{1}{m(m-1)} \Big(\sum_{k=1}^{m} e_k^2 - m e^2\Big) = \frac{1}{m^2(m-1)} \sum_{k<l} (e_k - e_l)^2 \geq 0.$$

Proof. By exploiting its symmetry, we compute the left hand side of (4.12):

(4.15) $$d^2 y^2 + 2de \cdot y \cdot \sum_{k=1}^{m} z_k + \Big(\frac{1}{m} \sum_{k=1}^{m} e_k^2\Big) \Big(\sum_{k=1}^{m} z_k^2\Big) + \frac{2}{m(m-1)} \Big(\sum_{k<l} 2e_k e_l\Big)\Big(\sum_{k<l} z_k z_l\Big).$$

The right hand side of (4.12) equals

(4.16) $$d^2 y^2 + 2de \cdot y \cdot \sum_{k=1}^{m} z_k + e^2 \sum_{k=1}^{m} z_k^2 + 2e^2 \sum_{k<l} z_k z_l + w(m-1) \sum_{k=1}^{m} z_k^2 - 2w \sum_{k<l} z_k z_l.$$

Comparing (4.15) and (4.16) we see that they are equal, provided

(4.17) $$\frac{1}{m} \sum_{k=1}^{m} e_k^2 = e^2 + w(m-1)$$

and

(4.18) $$\frac{2}{m(m-1)} \sum_{k<l} 2e_k e_l = 2e^2 - 2w.$$

The first relation is immediate from (4.14), and the second follows by noting that

$$\frac{2}{m(m-1)} \sum_{k<l} 2e_k e_l = \frac{2}{m(m-1)} \Big(m^2 e^2 - \sum_{k=1}^{m} e_k^2\Big) = 2e^2 - \frac{2}{m(m-1)} \Big(\sum_{k=1}^{m} e_k^2 - m e^2\Big). \quad \square$$

(4.19) **Lemma.** *Fix r, $1 \leq r \leq n$ and let $h(\mathbf{x}) = dx_r^3 + \sum_{s \neq r} e_s x_r x_s^2$. Then*

(4.20) $$S(h^2(\mathbf{x})) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \Big(dx_i^2 + e \sum_{j \neq i} x_j^2\Big)^2 + \frac{w}{n} \sum_{i=1}^{n} x_i^2 \Big(\sum_{\substack{j<k \\ j,k \neq i}} (x_j^2 - x_k^2)^2\Big),$$

where e and w are as above, with $m = n-1$.

Proof. Let $T_i = \{\sigma \in S_n : \sigma(r) = i\}$ so that $S_n = \bigcup_{i=1}^{n} T_i$. By Lemma (4.11),

(4.21) $$\frac{1}{(n-1)!} \sum_{\sigma \in T_i} h^2(\sigma(\mathbf{x})) = \Big(dx_i^3 + e \sum_{\substack{j=1 \\ j \neq i}}^{n} x_i x_j^2\Big)^2 + w \sum_{\substack{j<k \\ j,k \neq i}} (x_i x_j^2 - x_i x_k^2)^2.$$

Since $S(h^2(\mathbf{x})) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(n-1)!} \sum_{\sigma \in T_i} h^2(\sigma(\mathbf{x}))$, (4.20) follows from (4.21). □

(4.22) **Theorem.** *If f is an sos even symmetric sextic then*

(4.23) $$f = c_0 f_0 + c_1 f_1 + \sum_{i=2}^{T} c_i g_{\lambda_i}$$

for appropriate $c_i \geq 0$ and real λ_i.

Proof. By Corollary (4.5), f is a sum of symmetrizations of the squares of (4.6) and (4.7), which we have shown to be a non-negative combination of f_1 ((4.10)), f_0 ((2.4) and (4.20)) and

(4.24) $$\sum_{i=1}^{n} x_i^2 (d x_i^2 + e \sum_{j \neq i} x_j^2)^2.$$

If $e=0$ then (4.24) reduces to $d^2 M_6$ which has shape (4.23) by (2.7). If $e \neq 0$ then (4.24) equals

$$e^2 \sum_{i=1}^{n} x_i^2 \left(-\frac{d}{e} x_i^2 - \sum_{j \neq i} x_j^2\right)^2 = e^2 g_{(e-d)/e}.$$

This completes the proof. □

(4.25) **Theorem.** *Let f be an even symmetric sextic in $n \geq 3$ variables. The following statements are equivalent:*

(i) *f is sos.*

(ii) *$f^*(t) \geq 0$ for $t \in J := \{1\} \cup [2, n]$* (i.e. *$f^* \in P(J)$, in the notation of* § 2).

(iii) *There exist non-negative c_i and u, $2 \leq u \leq n$, such that $f = c_0 f_0 + c_1 f_1 + c_2 g_u$.*

Proof. (iii)⇒(i) is clear since f_0, f_1 and g_u are all sos.

(i)⇒(ii) If f is sos then (4.23) implies that

(4.26) $$f^*(t) = c_0(t-1)(n-t) + c_1(t-1)(t-2) + \sum_{i=2}^{T} c_i (t - \lambda_i)^2,$$

hence $f^*(t) \geq 0$ for $t \in J := \{1\} \cup [2, n]$.

(ii)⇒(iii) As $f^* \in P(J)$, we can apply Theorem (2.10)(i); f^* is then a non-negative combination of $(t-1)(t-2)$, $(t-1)(n-t)$ and $(t-u)^2$ for some $u \in [2, n]$. Thus, f is a nonnegative combination of f_1, f_0 and g_u for some $u \in [2, n]$. (Note that this is an improvement of the earlier representation in (4.23).) □

Capping the results in this section, we have:

(4.27) **Theorem.** (a) *The extremal sos even symmetric sextics are f_0, f_1 and g_u, $2 \leq u \leq n$.*

(b) *The psd forms $f_2, \ldots, f_{n-1}$ are not sos.*

Proof. (a) follows from (i)⇔(iii) in the theorem above; (b) follows from (i)⇔(ii) in the same theorem, in view of the fact that $f_k^*(t) = (t-k)(t-(k+1))$ is negative on $(k, k+1) \subseteq [2, n]$ for $2 \leq k \leq n-1$. □

Note that, even in the case of ternary forms ($n=3$), our proof for the fact that f_2 (a scalar multiple of Robinson's form) is not a sum of squares is drastically different from Robinson's original proof in [Ro], as well as from the proof in [CL_2].

5. Robinson's Questions and Sbs Even Symmetric Sextics

In 1969, Robinson proved [Ro: pp. 268–269] that if $G(x_1, \ldots, x_n)$ is any real $2d$–ic form then there exists β_0 such that the form $G(x_1, \ldots, x_n) + \beta(x_1^{2d} + \ldots + x_n^{2d})$ is a sum of squares of real forms if and only if $\beta \geq \beta_0$. His method of proof also showed that, for β sufficiently large, these forms may be chosen to be binomials. After showing that $S(x, y, z)$ $(= f_2(x, y, z)/2$, see (2.1)(b)) is psd and not sos, Robinson showed that $S + \beta M_6$ is sos (in fact, a sum of squares of binomials) at least for $\beta \geq 1$:

(5.1) $$S(x, y, z) + x^6 + y^6 + z^6 = (x^3 - x y^2)^2 + (x^3 - x z^2)^2 + (y^3 - y x^2)^2 + (y^3 - y z^2)^2 + (z^3 - z x^2)^2 + (z^3 - z y^2)^2 + 3(x y z)^2.$$

Robinson also proved that $S + \beta M_6$ is *not* sos for $\beta < 5/2 - \sqrt{6} \approx 0.0505$. He asked for the exact value of β_0 and for the "division point ... if we insist on using squares of binomials".

Let us say that a form f is a *sum of binomial squares* (sbs) if $f = \sum h_i^2$, where each h_i is a monomial or binomial. *We shall prove in this section that, for the form S, $\beta_0 = 1/8$ and that $S + \beta M_6$ is sbs if and only if $\beta \geq 1$.* More generally, we shall answer these questions for the full set of extremal forms $f_2, \ldots, f_{n-1}$ in n variables, which are not sos. We shall also use the techniques of section four to provide a complete description of the sbs even symmetric sextics.

By (2.2) and (2.4), *the forms f_0 and f_1 are both sbs.* There is exactly one other extremal sbs form. Define F by any of the following three representations:

(5.2)(a) $$F = n M_6 - M_2 M_4.$$

(5.2)(b) $$F = (n-1) \sum x_i^6 - \sum x_i^4 x_j^2.$$

(5.2)(c) $$F^*(t) = n - t.$$

It is easy to verify that $F(x)$ is sbs; in fact, using (5.2)(b), we see that

(5.3) $$F(x) = \sum_{i \neq j} (x_i^3 - x_i x_j^2)^2 = \sum_{i \neq j} x_i^2 (x_i^2 - x_j^2)^2.$$

In this notation, (5.1) becomes $S + M_6 = F + f_1/2$.

Robinson's first question is answered by the following easy consequence of Theorem (4.25). The representation (5.5) follows from (2.5) and (2.6).

(5.4) **Theorem.** *Suppose $n \geq 3$ and $k = 2, 3, \ldots, n-1$. Then $f_k + \beta M_6$ is sos if and only if $\beta \geq 1/4$. Further,*

(5.5) $$f_k + \tfrac{1}{4} M_6 = g_{k+\frac{1}{2}} = \sum_{i=1}^{n} x_i^2 [(k - \tfrac{1}{2}) x_i^2 - \sum_{j \neq i} x_j^2]^2.$$

Robinson's second question is answered by the following theorem, which is the main result of this section.

(5.6) **Theorem.** *Let f be an even symmetric sextic. The following statements are equivalent:*

(i) *f is sbs.*

(ii) *There exist $c_i \geq 0$ such that $f = c_0 f_0 + c_1 f_1 + c_2 F$.*

(iii) *$f^*(1) \geq 0$, $f^*(n) \geq 0$ and $(n-1) f^*(2) - (n-2) f^*(1) \geq 0$.*

Proof. We have already remarked that (ii)$\Rightarrow$(i). In the following, we show that (i)$\Rightarrow$(ii) and that (ii)$\Leftrightarrow$(iii).

Suppose $f=\sum h_i^2$ and each h_i is monomial or binomial; we shall apply the polarization and symmetrization procedures of the last section to h_i^2. Briefly, as before, if $h(\mathbf{x})=a\mathbf{x}^{\alpha}+b\mathbf{x}^{\beta}$, then $2^{-n}\sum_{\varepsilon} h^2(\varepsilon\mathbf{x})=a^2\mathbf{x}^{2\alpha}+b^2\mathbf{x}^{2\beta}$ unless $\alpha\equiv\beta \pmod 2$. Therefore, we may assume that each h_i is monomial, or has the form $dx_r^3+ex_rx_s^2$ $(r\neq s)$ or $dx_rx_s^2+ex_rx_t^2$ $(r,s,t$ distinct). We leave the details of the computations of the symmetrized monomial squares to the reader.

$$S((x_r^3)^2)=\frac{1}{n}\sum x_i^6=\frac{1}{n(n-1)(n-2)}(f_0+f_1+(n-2)F),$$

(5.7)
$$S((x_r^2x_s)^2)=\frac{1}{n(n-1)}\sum x_i^4x_j^2=\frac{1}{n(n-1)(n-2)}(f_0+f_1),$$

$$S((x_rx_sx_t)^2)=\frac{6}{n(n-1)(n-2)}\sum x_i^2x_j^2x_k^2=\frac{1}{n(n-1)(n-2)}f_1.$$

For the binomial squares, we may apply Lemma (4.19) (or (5.7)) and obtain the following:

(5.8) $S((dx_r^3+ex_rx_s^2)^2)=d^2S(x_r^2(x_r^2-x_s^2)^2)+(d+e)^2S(x_r^4x_s^2)$,

(5.9) $S((dx_rx_s^2+ex_rx_t^2)^2)=\frac{1}{2}(d^2+e^2)S(x_r^2(x_s^2-x_t^2)^2)+(d+e)^2S(x_r^2x_s^2x_t^2)$.

Observe that $S((dx_r^3+ex_rx_s^2)^2)$ is a nonnegative combination of F and $\sum x_i^4x_j^2$, and $S((dx_rx_s^2+ex_rx_t^2)^2)$ is a nonnegative combination of f and $\sum x_i^2x_j^2x_k^2$. Taken together, (5.7), (5.8) and (5.9) show that (i)$\Rightarrow$(ii).

Suppose $f^*(t)=a+bt+ct^2$. Since f_0^*, f_1^* and F^* are linearly independent, there exist unique real c_i such that $f^*=c_0f_0^*+c_1f_1^*+c_2F^*$. Solving for c_i in terms of (a,b,c), the inequalities $c_i\geq 0$ amount to

$$c_0=\frac{a+nb+(3n-2)c}{(n-1)(n-2)}=\frac{(n-1)f^*(2)-(n-2)f^*(1)}{(n-1)(n-2)}\geq 0,$$

(5.10)
$$c_1=\frac{a+nb+n^2c}{(n-1)(n-2)}=\frac{f^*(n)}{(n-1)(n-2)}\geq 0,$$

$$c_2=\frac{a+b+c}{n-1}=\frac{f^*(1)}{n-1}\geq 0.$$

Thus (ii)$\Leftrightarrow$(iii) and we are done. □

Of course, we could have also proved (ii)$\Rightarrow$(iii) more easily by directly verifying the inequalities in (iii) for the three forms f_0, f_1 and F.

The following corollary follows from a straightforward application of Theorem (5.6)(iii); the proof is omitted.

(5.11) **Corollary.** *For $1\leq k\leq n-1$, the n-ary form $f_k+\beta M_6$ is sbs if and only if $\beta\geq(k-1)(2n-(k+2))$.*

By putting $n=3$ and $k=2$ in Corollary (5.11), we see that $S(x,y,z)+\alpha(x^6+y^6+z^6)=(f_2+2\alpha M_6)/2$ is sbs if and only if $\alpha\geq 1$.

Capping the results in this section, we have:

(5.12) **Theorem.** (i) *The extremal forms among n-ary sbs even symmetric sextics are precisely the forms f_0, f_1 and F.*

(ii) *The sos form g_λ defined in (2.5) is sbs if and only if $\lambda\notin(n-\sqrt{(n-1)(n-2)}, n+\sqrt{(n-1)(n-2)})$. In particular, none of the extremal sos forms g_λ, $2\leq\lambda\leq n$, is sbs.*

Proof. (i) follows easily from Theorem (5.6). To test whether g_λ is sbs, we need only evaluate $(n-1)g_\lambda^*(2)-(n-2)g_\lambda^*(1)$ as $g_\lambda^*(t)=(t-\lambda)^2\geq 0$ for all t. By a straightforward computation, we obtain

(5.13) $(n-1)g_\lambda^*(2)-(n-2)g_\lambda^*(1)=(\lambda-n)^2-(n-1)(n-2)$.

Thus, by (5.6), g_λ is sbs if and only if the distance from λ to n is no less than $\sqrt{(n-1)(n-2)}$. It follows that none of the extremal sos forms g_λ, $2\leq\lambda\leq n$, is sbs. (On the other hand, $g_1=\sum x_i^4x_j^2+6\sum x_i^2x_j^2x_k^2$ is indeed sbs.) □

Note that by (5.12)(i) the cone B_n of sbs even symmetric sextics is not only a polyhedron - it is, in fact, a closed orthant in 3-space. The sos cone Σ_n is not a polyhedron, but is "squeezed between" the two polyhedra B_n and P_n for $n\geq 3$.

6. Interpretations in Real Semialgebraic Geometry and Illustrations

Recall that P_n (resp. Σ_n) is the set of (a,b,c) such that $aM_6+bM_2M_4+cM_2^3$ is psd (resp. sos). The sets P_n and Σ_n are both semialgebraic by an application of the Tarski Principle. By the Finiteness Theorem (cf. [L: Thm. (8.11)]), any closed semialgebraic subset $W\subseteq\mathbb{R}^d$ is a finite union of the form

(6.1) $$W=\{\mathbf{x}\in\mathbb{R}^d:\ g_i(\mathbf{x})\geq 0,\ i=1,\dots,m\},$$

where each g_i is a polynomial. A set such as (6.1) is called a *basic closed semialgebraic set*. Theorem (3.7) presents P_n as a single basic closed semialgebraic set:

(6.2) $$P_n=\{(a,b,c):\ a+kb+k^2c\geq 0,\ k=1,2,\dots,n\}.$$

The explicit description of Σ_n as a closed semialgebraic set is given by the following result.

(6.3) **Proposition.**

(6.4) $$\Sigma_n=W_1^{(n)}\cup W_2\cup W_3^{(n)},$$

where

(6.5)(a) $W_1^{(n)}=\{c\leq 0\}\cap\{a+b+c\geq 0\}\cap\{a+nb+n^2c\geq 0\}$,

(6.5)(b) $W_2=\{c\geq 0\}\cap\{4ac\geq b^2\}\cap\{a\geq 0\}$,

(6.5)(c) $W_3^{(n)}=\{c\geq 0\}\cap\{a+b+c\geq 0\}\cap\{a+2b+4c\geq 0\}$
$$\cap\{a+nb+n^2c\geq 0\}\cap\{(b+4c)(b+2nc)\geq 0\}.$$

This presentation of Σ_n can be easily deduced from Theorem (4.25); we leave the details to the reader.

Note that, in (6.3) we used only linear and quadratic inequalities. If we allow inequalities of higher degree, we may write Σ_n as a union of *two* closed basic semialgebraic sets: $\Sigma_n=W_0^{(n)}\cup W_3^{(n)}$, where

(6.6) $W_0^{(n)}=\{c(4ac-b^2)\geq 0\}\cap\{a+b+c\geq 0\}\cap\{a+nb+n^2c\geq 0\}.$

The question naturally arises: is Σ_n itself a single closed basic semialgebraic set? The answer is "no"; regrettably, for reasons of space, we must omit the proof. (See also [D, p. 100].)

The preceding discussion can be illuminated by considering some pictures. We have now discussed three cones of even symmetric sextics: the psd forms, the sos forms and the sbs forms. Since these cones lie in $\mathbb{R}^3$, they are fully described by their non-trivial two-dimensional sections. Throughout, let f be an even symmetric sextic with $f^*(t)=a+bt+ct^2$. If f is psd, then $f^*(t)\geq 0$ for $t\in\{1,2,\dots,n\}$ and $f^*(t)=0$ for at most two t's, unless $f\equiv 0$. Thus, if f is psd and non-trivial, then $\Sigma_1^n f^*(j)>0$. It is computationally convenient to take the cross-section for which

(6.7) $$\Sigma_1^n f^*(j)=\Sigma_1^n(s-1)(s-2)=n(n-1)(n-2)/3,$$

and to plot $(x,y)=(b+(n+1)c,c)$. To be specific, the form f is represented in the section by

$$P(f):=\alpha(b+(n+1)c,c),$$

(6.8) $$\alpha=\Sigma_1^n(s-1)(s-2)/\Sigma_1^n f^*(j)$$
$$=2n(n-1)(n-2)/\{6an+3bn(n+1)+cn(n+1)(2n+1)\}.$$

It is not hard to show that, if g is related to f by $g^*(t)=f^*(n+1-t)$, and $P(f)=(x,y)$ in this sense, then $P(g)=(-x,y)$. Since the mapping $t\to n+1-t$ preserves the set $\{1,2,\dots,n\}$, f and g are jointly psd. It follows that the section of psd forms is symmetric about the y-axis. This property is *not* shared by the sos and sbs sections. The following table gives the (x,y) coordinates for the normalized extremal forms: $P(f_0)$, $P(f_k)$ $(1\leq k\leq n-1)$, $P(g_\lambda)$ $(\lambda\in[2,n])$, and $P(F)$.

f	$\Sigma_1^n f^*(j)$	$P(f)$
f_0	$n(n-1)(n-2)/6$	$(0,-2)$
f_k	$n(3k^2-3kn+n^2-1)/3$	$((n-2k)r_k,r_k)$
g_λ	$6\lambda^2n-6\lambda n(n+1)+n(n+1)(2n+1)$	$((n+1-2\lambda)s_\lambda,s_\lambda)$
F	$n(n-1)/2$	$(-2(n-2)/3,0)$

$$r_k=(n-1)(n-2)/(3k^2-3kn+n^2-1)$$
$$s_\lambda=2(n-1)(n-2)/(6\lambda^2-6\lambda(n+1)+(n+1)(2n+1)).$$

Note that $r_k=r_{n-k}$ and $r_1=1$, so that $P(f_1)=(n-2,1)$. It is not hard to check that $P(f_k)$ lies on the ellipse $3x^2+(n^2-4)y^2=4(n-1)(n-2)y$ and that $P(g_\lambda)$ lies on the ellipse $3x^2+(n^2-1)y^2=4(n-1)(n-2)y$. These ellipses are both tangent to the x-axis and lie in the upper half-plane. Further, $P(F)=(2/3)P(f_1)+(1/3)P(f_0)$ and, since $g_k=(f_{k-1}+f_k)/2$, $P(g_k)$ is on the line segment connecting $P(f_k)$ and $P(f_{k-1})$. We may describe the sections as follows. The psd section is an n-gon, symmetric with respect to the y-axis, in which $n-1$ vertices ($P(f_k)$, $1\leq k\leq n-1$) lie symmetrically on an ellipse in the upper half-plane and the other, $P(f_0)$, is $(0,-2)$. The sos section is bounded by an elliptical arc ($P(g_\lambda)$ for $\lambda\in[2,n]$), the ends of which are tangent to line segments which terminate in the two corners of the section ($P(f_1)$ and $P(f_0)$). The sbs section is simply a triangle with vertices $P(f_0)$, $P(f_1)$, and $P(F)$. We now show the three simplest cases: $n=3$, 4 and 5 (Figs. 1-3).

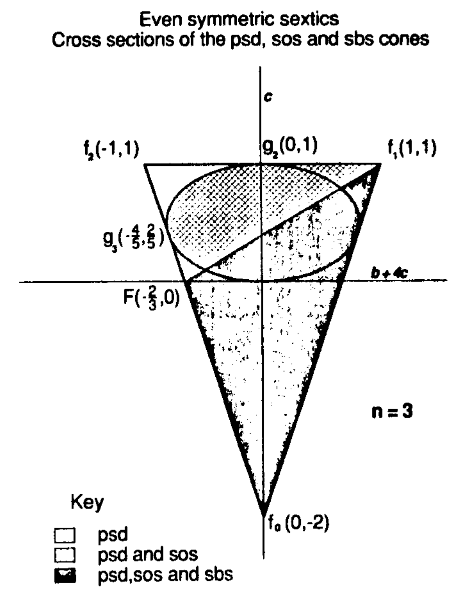

Fig. 1

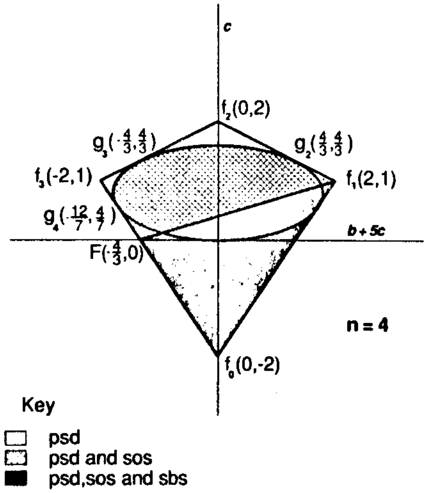

Fig. 2

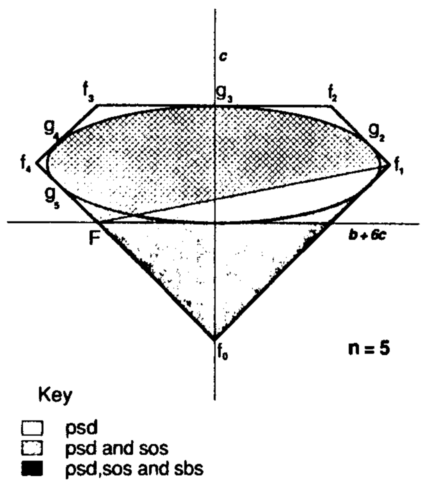

Fig. 3

7. Open Questions

Our results for even symmetric sextics suggest several possible generalizations, which we put in the form of questions.

(7.1) **Question.** For which cones of (even) symmetric forms can we conclude that f is psd if and only if $f(\mathbf{y}_i)\geqq 0$ for a "small" set of points $\{\mathbf{y}_i\}$?

(7.2) **Question.** More generally, for which of these cones is the corresponding set of psd coefficients a single closed basic semialgebraic set?

Regarding Question (7.2), Delzell has shown [D] that the full psd cone $P_{n,m}$ is *not* a single closed basic semialgebraic set for $m\geqq 4$.

(7.3) **Question.** For which cones of (even) symmetric forms is there an *easily expressed* set of pseudo-points $\{\mathbf{z}\}$ (see Introduction) so that f is sos if and only if "$f(\mathbf{z})\geqq 0$"?

This question is deliberately vague. Let $C\subseteq\mathbf{R}^d$ be a closed cone, then there exists a dual set of functionals $\{\mathbf{v}_t\colon t\in T\}\subseteq\mathbf{R}^d$ so that $\mathbf{u}\in C$ if and only if $\mathbf{u}\cdot\mathbf{v}_t\geqq 0$ for $t\in T$. Suppose the elements of C are the coefficients of a set of forms: $f(\mathbf{x})=\sum_{i=1}^{d} a(\alpha_i)\mathbf{x}^{\alpha_i}$, $\mathbf{u}=(a(\alpha_1),\dots,a(\alpha_d))$. Then the dot product can be written as

$$\sum_{i=1}^{d} a(\alpha_i)\,\mathbf{v}_t(i)\geqq 0.$$

We are interested in those cases in which $\mathbf{v}_t(i)$ bears some *natural* relation to $\mathbf{z}_t^{\alpha_i}$ for points or pseudopoints $\{\mathbf{z}_t\}$.

We now turn to a different sort of question. Suppose f is a strictly definite even form, then for a suitable r, $(x_1^2+\dots+x_n^2)^r f(x_1,\dots,x_n)$ is a form with positive coefficients [HLP, p. 57] and hence a sum of squares of monomials. The sextics f_k, $2\leqq k\leqq n-1$, are not sos and not strictly definite; perhaps $M_2^r f_k$ is sos for r sufficiently large. Define $r_k(n)$ to be the smallest r (if it exists) such that

$$M_2^r f_k=(x_1^2+\dots+x_n^2)^r f_k(x_1,\dots,x_n)$$

is sos, and let $r(n)=\max\{r_k(n)\colon 2\leqq k\leqq n-1\}$. If this number is indeed defined, then $M_2^{r(n)}f$ is sos for *every* psd n-ary even symmetric sextic f. By Corollary (4.27), $r_k(n)\geqq 1$ for $n\geqq 3$ and $2\leqq k\leqq n-1$. In [Ro], Robinson showed that $r_2(3)=1$ (and hence $r(3)=1$) by exhibiting the equation

(7.4) $$M_2\cdot f_2(x_1,x_2,x_3)=2\sum_{i<j}x_i^2x_j^2(x_i^2-x_j^2)^2+\sum_{\substack{i<j\\ k\neq i,j}}(x_i^2-x_j^2)^2(x_i^2+x_j^2-x_k^2)^2.$$

This equation seems to be peculiar to the ternary case. On the other hand, for $n\geqq 4$, the form f_{n-1} satisfies a more elaborate equation:

(7.5) $$\begin{aligned}M_2\cdot f_{n-1}(x_1,\dots,x_n)&=(n-1)(n-2)\sum_{i<j}x_i^2x_j^2(x_i^2-x_j^2)^2+\frac{2n-5}{n-3}\sum(x_i^2-x_j^2)^2(x_k^2-x_l^2)^2\\&\quad+\frac{2}{n-3}\sum((x_i^4+x_j^4-x_i^2x_j^2)-(x_k^4+x_l^4-x_k^2x_l^2))^2.\end{aligned}$$

Here the second sum is taken over distinct indices (i,j,k,l) such that $i<j$, $k<l$ and $i<k$, and the third sum is taken over distinct indices (i,j,k,l) such that $i<j$ and $k<l$. Note that (7.5) makes sense only when $n\geq 4$: aside from using four indices, we also need $n-3$ in the denominator! This equation shows that $r_{n-1}(n)=1$. On the other hand, we can show that, for $2\leq k\leq n-2$, the forms f_k are "wilder" than f_{n-1} in that $M_2\cdot f_k(x_1,\ldots,x_n)$ is still *not* sos; thus, $r_k(n)\geq 2$ and $r(n)\geq 2$ for $n\geq 4$. For $n=4$, an explicit computation shows that $M_2^2\cdot f_2(x_1,\ldots,x_4)$ *is* sos, and so $r_2(4)=2$, $r(4)=2$. We close with the following:

(7.6) **Question.** Determine $r_k(n)$ and $r(n)$ for $n\geq 5$ and $2\leq k\leq n-2$.

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Note Added in Proof (June 1987)

Stated in another form, our determination of P_n $(n\geq 3)$ in (5.2) amounts to the fact that the symmetric cubic $\alpha\sum x_i^3+\beta\sum x_i^2x_j+\gamma\sum x_ix_jx_k$ is psd for $x_i\geq 0$ iff

$$(*)\qquad \left(\alpha-\beta+\frac{\gamma}{3}\right)+k\left(\beta-\frac{\gamma}{2}\right)+k^2\cdot\frac{\gamma}{6}\geq 0 \quad \text{for } k=1,2,\ldots,n.$$

In the special case when $n=3$, this has been proved earlier by J.F. Rigby (Univ. Beograd. Publ. Elektrotehn. Fak. Sci. Mat. Fiz. No. 412-460 (1973), pp. 217-226). Thus our, complete result on P_n can be viewed as a generalization of Rigby's result from 3 variables to any number of variables. In comparing our results with Rigby's (Theorem 2, *loc. cit.*), note that the first three conditions in $(*)$ are: $\alpha\geq 0$, $\alpha+\beta\geq 0$, and $\alpha+2\beta+\gamma/3\geq 0$.