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Real Zeros of Positive Semidefinite Forms. I

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1. Introduction

A real polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is called *positive semidefinite* (psd) if $p(a_1, \ldots, a_n) \geqq 0$ for all real a_i (we write $p \geqq 0$ for short). A form (homogeneous polynomial) in n variables of degree m is called an n-ary m-ic. This work (and its sequel) will be occupied with the study of psd n-ary m-ics, and the properties of their real zeros. Note that if an m-ic $p \neq 0$ is psd, then $\deg p = m$ must be even. Throughout this paper, the cone of all psd n-ary m-ics will be denoted by $P_{n,m}$. An important subcone is $\Sigma_{n,m}$, the set of all n-ary m-ics which are sums of squares of real polynomials $\left(\text{necessarily } n\text{-ary } \frac{m}{2}\text{-ics}\right)$. The set theoretic difference $P_{n,m} - \Sigma_{n,m}$ will be denoted by $\Delta_{n,m}$. (To avoid trivial cases, one usually assumes $n, m \geqq 2$.)

In [13], Hilbert showed that $\Delta_{n,m} = \varnothing$ if and only if $n=2$, or $m=2$, or $(n,m) = (3,4)$. Artin, in solving Hilbert's 17th Problem (see [1]), showed that if $p \in P_{n,m}$, then there is an n-ary d-ic h such that $h^2 p \in \Sigma_{n,m+2d}$. Several recent papers [20, 4, 5, 19] have dealt with various aspects of $\Delta_{n,m}$.

For any n-ary m-ic p, the zero set of p:

$$\mathfrak{Z}(p) := \{(a_1, \ldots, a_n) \in \mathbb{R}^n : p(a_1, \ldots, a_n) = 0\}$$

may be viewed as a subset in real projective space. (In particular, we regard $(0, \ldots, 0) \notin \mathfrak{Z}(p)$ for purposes of counting $|\mathfrak{Z}(p)|$.) In this paper, we shall be interested in psd forms p for which $|\mathfrak{Z}(p)|$ is infinite, or, say, relatively large. Recall that the "basic" cases for $\Delta_{n,m} \neq \varnothing$ are given by $(n,m) = (3,6)$ (ternary sextics) and $(n,m) = (4,4)$ (quaternary quartics). All *known* examples [20, 3, 4] of forms p in $\Delta_{3,6}$ and $\Delta_{4,4}$ happen to have $|\mathfrak{Z}(p)| < \infty$. A main result of this paper

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states that this is the case for *all* forms p in $\Delta_{3,6}$ and $\Delta_{4,4}$. In fact, our result can be stated more quantitatively, as follows:

Main Theorem. (A) *If* $p \in P_{3,6}$ *and* $|\mathfrak{Z}(p)| > 10$, *then* $p \in \Sigma_{3,6}$. *In fact, p is a sum of three squares of cubics.*

(B) *If* $p \in P_{4,4}$ *and* $|\mathfrak{Z}(p)| > 11$, *then* $p \in \Sigma_{4,4}$. *In fact, p is a sum of six squares of quadratics.*

In both cases, we must have $|\mathfrak{Z}(p)| = \infty$.

The results (A), (B) above are peculiar to ternary sextics and quaternary quartics. In fact, beyond these two cases, it is easy to see that, if $\Delta_{n,m} \neq \varnothing$, then forms with infinitely many zeros abound in $\Delta_{n,m}$.

The proof of (A) above is relatively easy, using the classical theory of plane curves and Hilbert's Theorem [13] for ternary quartics (cf. Sect. 3). The proof of (B) is, however, much harder, and occupies thè entirety of Sects. 5–6.

In Sect. 4, we study the numbers $B_{n,m} = \sup|\mathfrak{Z}(p)|$ where p ranges over all forms in $P_{n,m}$ with $|\mathfrak{Z}(p)| < \infty$. It is shown that $B_{3,m}$ is always finite; in fact $B_{3,4} = 4$, $B_{3,6} = 10$ and for $m \geqq 6$, $m^2/4 \leqq B_{3,m} \leqq (m-1)(m-2)/2$. Asymptotically, we show that $\beta = \lim_{m \to \infty} B_{3,m}/m^2$ exists, with $5/18 \leqq \beta \leqq 1/2$. For quartic forms, we show that $B_{4,4}$ is either 10 or 11, but for $n \geqq 5$, we do not know if $B_{n,4}$ need be finite in general. The study of $B_{3,m}$ and $B_{4,4}$ appears to be related to Harnack's Theorem on real curves and Hilbert's 16th Problem; see the discussion at the end of Sect. 4.

One of the main ingredients of the proof of the Main Theorem is the following important fact about biforms: *If a psd biform* $p(y,z;x_1,\ldots,x_n)$ *is quadratic in* $x_1,\ldots,x_n$, *then p is a sum of squares of biforms.* This result has appeared in the literature (see [18, 17, 21] or [10]), but, for the sake of completeness, we offer a new proof in Sect. 7. (In fact, we shall prove a stronger, inhomogeneous version.) Another reason for re-proving this result is that it plays a very special role in the study of psd "multiforms". In Sect. 8, we show that the type of biforms described above constitutes essentially the only kind of multi-forms for which "psd" is equivalent to being a sum of squares of polynomials (see Theorem 8.4). This provides a further generalization of Hilbert's work [13].

Several of the results in this paper have been announced by two of the present authors in [5]. We would like to thank Professors R.M. Robinson and R. Hartshorne for making several valuable comments on the manuscript.

2. Preliminaries

We adopt a few notational conventions. When the number of variables is small, we shall denote them by $x, y, z, w, \ldots$; otherwise, the vector $\underline{x}$ will abbreviate the variables $(x_1,\ldots,x_n)$. The symbol $\underline{y}$ will often denote a specific n-tuple $(y_1,\ldots,y_n) \in \mathbb{R}^n$. If we use $(x,y,z,\ldots)$ for the variables instead, then a specific tuple will be denoted by $(\bar{x},\bar{y},\bar{z},\ldots)$.

The phrase "linear change" will be short for "invertible (homogeneous) linear change of variables". If T is a linear change in $(x_1,\ldots,x_n)$, then membership in $P_{n,m}$, $\Sigma_{n,m}$ or $\Delta_{n,m}$ is unaffected by T, and $|\mathfrak{Z}(p \circ T)| = |\mathfrak{Z}(p)|$; such a linear change will often be used to put zeros of p at convenient places.

For tuples of natural numbers, define a partial ordering by $(a_1,\ldots,a_r) \geqq (b_1,\ldots,b_r)$ if $a_i \geqq b_i$ for all i, and write $(a_1,\ldots,a_r) > (b_1,\ldots,b_r)$ if at least one of these inequalities is strict. These notations will be used in (2.2), as well as later in Section 8.

We now begin the analysis of $|\mathfrak{Z}(p)|$. This is rather straightforward in a situation where $P_{n,m} = \Sigma_{n,m}$. If $p \in P_{2,m}$ and $p \not\equiv 0$, then clearly $|\mathfrak{Z}(p)| \leqq \frac{1}{2}m$ since any (real) linear factor must occur in p to an even multiplicity. If $p \in P_{n,2}$, then up to a linear change, $p(\underline{x}) = x_1^2 + \ldots + x_r^2$. If $r = n$, $\mathfrak{Z}(p) = \varnothing$; if $r = n-1$, $|\mathfrak{Z}(p)| = 1$; otherwise $|\mathfrak{Z}(p)| = \infty$. Finally, if $p \in P_{3,4}$, we shall show in (4.3) (1) that either $|\mathfrak{Z}(p)| = \infty$ or else $|\mathfrak{Z}(p)| \leqq 4$ (and 4 is the best bound).

Consider now the situation where $P_{n,m} \neq \Sigma_{n,m}$. The "simplest" forms in $\Delta_{n,m}$ in the basic cases $(n,m) = (3,6)$ or $(4,4)$ are given by

$$S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2, \tag{2.1}$$
$$Q(x,y,z,w) = w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw;$$

see [4] for a quick proof that they are in Δ. Using the well-known condition for the arithmetic mean to equal the geometric mean, the zeros of S and Q can be explicitly determined; in particular, $|\mathfrak{Z}(S)| = |\mathfrak{Z}(Q)| = 7$. A large portion of this paper will be devoted to proving that $\mathfrak{Z}(p)$ is *always* finite for $p \in \Delta_{3,6}$ and $p \in \Delta_{4,4}$. This contrasts with the following easy proposition:

Proposition 2.2. *If* $(n,m) > (3,6)$ *or* $(n,m) > (4,4)$, *then there exists* $p \in \Delta_{n,m}$ *with* $|\mathfrak{Z}(p)| = \infty$.

Proof. Suppose $(n,m) > (3,6)$. Then clearly $p(x_1,\ldots,x_n) = x_1^{m-6}S(x_1,x_2,x_3) \in \Delta_{n,m}$. If $m = 6$, then $n > 3$ so $(1,1,1,x_4,\ldots,x_n) \in \mathfrak{Z}(p)$ yields $|\mathfrak{Z}(p)| = \infty$. If $m > 6$, then $(0,x_2,\ldots,x_n) \in \mathfrak{Z}(p)$ yields $|\mathfrak{Z}(p)| = \infty$. Finally, if $(n,m) > (4,4)$, use instead $p(x_1,\ldots,x_n) = x_1^{m-4}Q(x_1,\ldots,x_4)$. Q.E.D.

Via the basic forms Q and S, the above constructions show, in particular, that $\Delta_{n,m} \neq \varnothing$ for $(n,m) \geqq (3,6)$ or $(n,m) \geqq (4,4)$. A similar procedure will be used later in Theorem 3.9, as well as in Sect. 8 for investigating sums of squares properties of multiforms.

We close this section by recording a few basic facts to be used in later sections (sometimes implicitly). The first conclusion of the proposition below has already been used in the proof of (2.2):

Proposition 2.3. *Let p be an n-ary polynomial and* $\lambda \neq 0$ *be an n-ary linear form. If* $\mathfrak{Z}(\lambda) \subseteq \mathfrak{Z}(p)$, *then* $\lambda | p$. *If, moreover, p is psd, then* $\lambda^2 | p$.

(Vertical line indicates divisibility in $\mathbb{R}[x_1,\ldots,x_n]$).

Proof. We may assume, after a linear change, that $\lambda = x_1$. Write $p(x_1,\ldots,x_n) = \sum_{i=0}^{m} h_i(x_2,\ldots,x_n)x_1^i$, then $0 = p(0,x_2,\ldots,x_n) = h_0(x_2,\ldots,x_n)$ so $x_1 | p$. If $p \geqq 0$, then for fixed $\bar{x}_2, \ldots, \bar{x}_n$, $p(x_1, \bar{x}_2, \ldots, \bar{x}_n)$ is psd in x_1 so we must have $h_1(\bar{x}_2, \ldots, \bar{x}_n) = 0$, i.e. $h_1 \equiv 0$ so $\lambda^2 | p$. Q.E.D.

Remark 2.4. The Proposition remains valid if we replace λ by any irreducible indefinite form; for a proof, see [7].

Proposition 2.5. *Let f be an n-ary form with $n \geqq 3$. If $|\mathfrak{Z}(f)| < \infty$, then one of $\pm f$ is psd. (In other words, any indefinite n-ary form must have infinitely many real zeros).*

Proof. To prove the contrapositive form, assume f is negative on an open set A and positive on an open set B. After shrinking A and B, we may assume that A, B are disjoint from the hyperplane $x_1 = 0$. Thus, there exist (non-empty) open sets A_0, B_0 in $\mathbb{R}^{n-1}$ such that $f(1, A_0) < 0 < f(1, B_0)$. For any $a_0 \in A_0$ and $b_0 \in B_0$, we can find, by the Intermediate Value Theorem, a vector $c_0 \in \mathbb{R}^{n-1}$ such that $f(1, c_0) = 0$. This enables us to construct uncountably many points $(1, c_0) \in \mathfrak{Z}(f)$ (if $n \geqq 3$). Q.E.D.

For a polynomial

$$p(x_1, \ldots, x_n, x) = x^2 f(x_1, \ldots, x_n) + 2xg(x_1, \ldots, x_n) + h(x_1, \ldots, x_n)$$

which is quadratic in x, the expression $D = fh - g^2 \in \mathbb{R}[x_1, \ldots, x_n]$ is called the *discriminant* of p with respect to x. Viewing p as a polynomial in x, it is easy to see that $p \geqq 0$ iff $f \geqq 0$ *and* $D \geqq 0$.

Lemma 2.6. *Let p be as above and $\lambda(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$. Let $p'(x_1, \ldots, x_n, x') = p(x_1, \ldots, x_n, x' + \lambda(x_1, \ldots, x_n))$. Then the discriminant of p' w.r.t. x' is the same as the discriminant of p w.r.t. x.*

Proof. By definition, $p' = (x' + \lambda)^2 f + 2(x' + \lambda) g + h = x'^2 f + 2(g + \lambda f) x' + h + 2\lambda g + \lambda^2 f$. Thus, p' has discriminant $f(h + 2\lambda g + \lambda^2 f) - (g + \lambda f)^2 = fh - g^2$. Q.E.D.

Lemma 2.7. *Let p be as above. If p is a sum of squares of polynomials, then so is the discriminant $D = fh - g^2$.*

(The converse is false; for example, the discriminant of the form Q in (2.1) w.r.t. x is easily checked to be in $\Sigma_{3,6}$, but $Q \notin \Sigma_{4,4}$.)

Proof. Say $p = \sum (u_i x + v_i)^2$ where $u_i, v_i \in \mathbb{R}[x_1, \ldots, x_n]$. Then $f = \sum u_i^2$, $h = \sum v_i^2$ and $g = \sum u_i v_i$, so

$$D = (\sum u_i^2)(\sum v_i^2) - (\sum u_i v_i)^2$$
$$= \sum_{i<j} (u_i v_j - u_j v_i)^2. \quad \text{Q.E.D.}$$

The last lemma is, in fact, a special case of a much more general result: *Let A be any commutative ring with a formally real quotient field, and $p(y_1, \ldots, y_t) = \sum_{i,j} a_{ij} y_i y_j$ be a quadratic form over A, with $a_{ij} = a_{ji} \in A$. If p is a sum of squares in $A[y_1, \ldots, y_t]$, then* $\det(a_{ij})$ *is a sum of squares in A.* (Lemma 2.7 is the special case of this for $t = 2$ and $A = \mathbb{R}[x_1, \ldots, x_n]$.) The proof of this will be left to the reader, who may want to get a hint from the book of Hardy, Littlewood and Polya [16, p. 16].

3. Ternary Forms

In order to treat ternary sextics, we shall first derive some key results about ternary forms in general. Each ternary form, when set equal to zero, defines a curve in the complex projective plane, so the classical theory of plane algebraic curves can be used to get insight into the structure of the form itself. This algebro-geometric viewpoint will be important for the formulation of the results in this section.

We start with an easy lemma which is a special case of [8]. We record a proof here for the sake of completeness.

Lemma 3.1. *Let $p(x_1, \ldots, x_n)$ be an irreducible polynomial in $\mathbb{R}[x_1, \ldots, x_n]$. Then p becomes reducible in $\mathbb{C}[x_1, \ldots, x_n]$ iff one of $\pm p$ is a sum of two squares in $\mathbb{R}[x_1, \ldots, x_n]$.*

Proof. The "if" part is obvious. For the converse, assume p factors nontrivially into $p = (r_1 + ir_2)(s_1 + is_2)$ ($i = \sqrt{-1}$, $r_1, r_2, s_1, s_2 \in \mathbb{R}[x_1, \ldots, x_n]$). Since p is real, taking complex conjugate yields $p = (r_1 - ir_2)(s_1 - is_2)$ so $p^2 = (r_1^2 + r_2^2)(s_1^2 + s_2^2)$. By the Unique Factorization Theorem, we have $p = a(r_1^2 + r_2^2)$ for some $a \in \mathbb{R}$. Since one of $\pm a$ is a square in $\mathbb{R}$, the desired conclusion follows. Q.E.D.

Proposition 3.2. *Suppose $p \in P_{3,m}$ is irreducible in $\mathbb{R}[x, y, z]$. Then*

$$|\mathfrak{Z}(p)| \leqq \max\left(\frac{m^2}{4}, \frac{(m-1)(m-2)}{2}\right).$$

Proof. First, assume that p becomes *reducible* in $\mathbb{C}[x, y, z]$. Then, by the above lemma, $\pm p = r_1^2 + r_2^2$ for suitable $r_1, r_2 \in \mathbb{R}[x, y, z]$, which are necessarily *forms* of degree $m/2$. We now have $\mathfrak{Z}(p) = \mathfrak{Z}(r_1) \cap \mathfrak{Z}(r_2)$ (recalling that "$\mathfrak{Z}$" denotes *real* zeros only). Since p is irreducible, r_1, r_2 must be relatively prime, so the plane curves $r_1 = 0$, $r_2 = 0$ cannot have any common component. According to Bezout's Theorem [23, p. 59], these two curves can intersect in at most $\frac{m}{2} \cdot \frac{m}{2}$ points in the complex projective plane. In particular, $|\mathfrak{Z}(p)| = |\mathfrak{Z}(r_1) \cap \mathfrak{Z}(r_2)| \leqq m^2/4$.

For the remaining case, we assume that p remains irreducible in $\mathbb{C}(x, y, z]$, so p defines an irreducible plane algebraic curve, say C. Since p is psd, each real zero $(\bar{x}, \bar{y}, \bar{z}) \in \mathfrak{Z}(p)$ must *also* be a zero of $\partial p/\partial x$, $\partial p/\partial y$ and $\partial p/\partial z$, so $(\bar{x}, \bar{y}, \bar{z})$ is a singular point on C. By classical curve theory, the *irreducible* curve C has at most $(m-1)(m-2)/2$ singular points [23, p. 65], so $|\mathfrak{Z}(p)| \leqq (m-1)(m-2)/2$. Q.E.D.

An interesting corollary can be drawn from the above arguments in the case $m = 4$:

Corollary 3.3. *If an irreducible psd ternary quartic p has $|\mathfrak{Z}(p)| \geqq 4$, then p is a sum of two squares in $\mathbb{R}[x, y, z]$.*

(Recall that Hilbert [13] has shown that any $q \in P_{3,4}$ is a sum of *three* squares in $\mathbb{R}[x, y, z]$.)

Proof. The form p cannot remain irreducible in $\mathbb{C}[x,y,z]$, for an irreducible plane quartic curve can have at most three singular points. Thus, p is reducible in $\mathbb{C}[x,y,z]$ and we can invoke Lemma 3.1. Q.E.D.

For convenience, let us write, in the following, $\alpha(m) = \max\left(\frac{m^2}{4}, \frac{(m-1)(m-2)}{2}\right)$. We shall be interested in $\alpha(m)$ only for positive integers m. As is easily verified, $\alpha(m)=m^2/4$ when $m \leqq 5$, and $\alpha(m)=(m-1)\cdot(m-2)/2$ when $m \geqq 6$. By a straightforward computation, one checks that the function $\frac{\alpha(m)}{m}$ is monotonically increasing. From this, we can deduce the "superadditive" property of the function α:

$$\alpha(m_1)+\alpha(m_2) \leqq \alpha(m_1+m_2) \tag{3.4}$$

The argument is standard: we have $\frac{\alpha(m_i)}{m_i} \leqq \frac{\alpha(m_1+m_2)}{m_1+m_2}$ for $i=1,2$, so

$$\alpha(m_1)+\alpha(m_2) \leqq \left(\frac{m_1}{m_1+m_2}+\frac{m_2}{m_1+m_2}\right)\cdot \alpha(m_1+m_2)=\alpha(m_1+m_2).$$

We are now in a position to prove the following result for psd ternary forms:

Theorem 3.5. *For any $p \in P_{3,m}$, the following are equivalent:*

(1) $|\mathfrak{Z}(p)| > \alpha(m)$.

(2) $|\mathfrak{Z}(p)| = \infty$.

(3) *p is divisible by the square of some indefinite form.*

Proof. (2) $\Rightarrow$ (1) is trivial, and (3) $\Rightarrow$ (2) follows from (2.5). Thus, it suffices to show that (1) $\Rightarrow$ (3). We shall do this by induction on m. For $m=2$, $\alpha(m)=1$; the form p in question is a psd ternary quadratic with at least two zeros. Up to a linear change, $p(x,y,z)=x^2$, as asserted in (3).

In general, suppose $|\mathfrak{Z}(p)|>\alpha(m)$. Then by Proposition 3.2, p is reducible in $\mathbb{R}[x,y,z]$, say $p=q_1 q_2 \dots q_r$ $(r \geqq 2)$ where each $q_i \in \mathbb{R}[x,y,z]$ is irreducible. *First, assume that all q_i are semidefinite.* Adjusting them by ± 1 if necessary, we may assume they are all psd. There must exist an index i such that $|\mathfrak{Z}(q_i)|>\alpha(m_i)$, where $m_i := \deg q_i$. For otherwise, by (3.4), we would get

$$|\mathfrak{Z}(p)| \leqq \sum_i |\mathfrak{Z}(q_i)| \leqq \sum_i \alpha(m_i) \leqq \alpha(\sum_i m_i)=\alpha(m),$$

a contradiction. Say $|\mathfrak{Z}(q_1)|>\alpha(m_1)$; since $m_1<m$, we are done by invoking the inductive hypothesis for q_1.

We may now assume that one of the q_i's is indefinite; in particular, by (2.5), $|\mathfrak{Z}(p)|=\infty$. If we assume the truth of Remark 2.4, then the desired conclusion (3) follows. Since we have *not* proved (2.4), however, we shall present an alternative argument to get (3). Assuming that $p(x,y,z)$ involves x, consider the plane curves defined by $p=0$ and $\partial p/\partial x=0$. The intersection of these curves contains the infinite set $\mathfrak{Z}(p)$, so they must have a common component [23, p. 59]. Thus, p and $\partial p/\partial x$ have an irreducible *real* common factor, say h. Write $p=h\cdot g$; we have $\partial p/\partial x = h\cdot \partial g/\partial x + g\cdot \partial h/\partial x$, so h divides $g\cdot \partial h/\partial x$. There are the following two cases:

Case 1. $\partial h/\partial x=0$. Then $h=h(y,z)$. If $h(y,z)$ has a zero, then p is divisible by some $ay+bz \neq 0$ and hence by $(ay+bz)^2$ according to (2.3). Now assume $h(y,z)$ has no zeros; upon a sign change, we may assume h is psd, so g is also psd. As a ternary form, $h(y,z)$ has a unique zero $(1,0,0)$, so $|\mathfrak{Z}(p)|=\infty$ implies $|\mathfrak{Z}(g)|=\infty$, and we are done by induction.

Case 2. $\partial h/\partial x \neq 0$. Since h was chosen to be irreducible, we have $h | g\cdot \partial h/\partial x \Rightarrow h|g$, so we may write $p=h^2 q$. If h is indefinite we are done, so assume h is (say positive) semidefinite. By (3.2), we have $|\mathfrak{Z}(h)|<\infty$ so we must have $|\mathfrak{Z}(q)|=\infty$. Since q is also psd, the induction proceeds. Q.E.D.

The above factorization theorem is, of course, peculiar to ternary forms. For four variables or more, simply note that $x^2y^2+z^2w^2$ is irreducible, but vanishes on all $(x,0,z,0,\dots)$.

The corollary below suggests that the study of psd ternary forms can be reduced, in some sense, to the case where the ternary form has only *finitely many* real zeros.

Corollary 3.6. *For any ternary form p, the following are equivalent:*

(1) *p is semi-definite* (*i.e. one of $\pm p$ is psd*);

(2) *p has a factorization $p=h^2 q$ where $|\mathfrak{Z}(q)|<\infty$ and h is a product of indefinite forms.* (*This product may be empty, in which case we agree that $h=1$.*)

Proof. (2) $\Rightarrow$ (1) follows from (2.5). For (1) $\Rightarrow$ (2), we assume p is psd and again induct on $\deg p$. If $|\mathfrak{Z}(p)|<\infty$, we are done by choosing $h=1$. If $|\mathfrak{Z}(p)|=\infty$, the theorem yields a factorization $p=h^2 q$ where h is indefinite. Now apply the inductive hypothesis to q. Q.E.D.

We can now prove our Main Theorem for ternary sextics:

Theorem 3.7. *If $p \in P_{3,6}$ and $|\mathfrak{Z}(p)|>10$, then $|\mathfrak{Z}(p)|=\infty$ and p is a sum of three squares of cubics.*

Proof. Since $\alpha(6)=10$, the first conclusion follows from (3.5). From the same result, we have a factorization $p=h^2 q (\deg h \geqq 1)$. If $\deg h \geqq 2$, then $\deg q \leqq 2$, and so q is a sum of three squares of forms of degree $\leqq 1$. If $\deg h=1$, we have $q \in P_{3,4}$, and, according to Hilbert [13], q is a sum of three squares of quadratics. Q.E.D.

To show an application of Theorem 3.7, let us consider the following question for n-ary m-ic forms:

Question 3.8. Suppose $p \in P_{n,m}$ and there exists an n-ary $\frac{m}{2}$-ic h such that $p \leqq h^2$. Does it follow that $p \in \Sigma_{n,m}$?

Theorem 3.9. *The answer to* (3.8) *is YES iff $n=2$ or $m=2$, or $(n,m)=(3,4)$ or $(n,m)=(3,6)$.*

Proof. For the "if" part, we need only consider the case $(n,m)=(3,6)$. From $0\leqq p\leqq h^2$, we have $\mathfrak{Z}(h)\subset\mathfrak{Z}(p)$. The cubic h is clearly indefinite, so $|\mathfrak{Z}(p)|\geqq|\mathfrak{Z}(h)|=\infty$ by (2.5). From (3.7), it follows that $p\in\Sigma_{3,6}$.

For the "only if" part, assume first that $n,m\geqq 4$. Take the form Q defined in (2.1), and let $N=\max Q(x_1,\dots,x_4)$ on the sphere $x_1^2+\dots+x_4^2=1$. By homogeneity, we have $Q\leqq N(x_1^2+\dots+x_4^2)^2$, so

$$x_1^{m-4}Q(x_1,x_2,x_3,x_4)\leqq[\sqrt{N}\cdot x^{\frac{m}{2}-2}(x_1^2+x_2^2+x_3^2+x_4^2)]^2.$$

But $x_1^{m-4}Q(x_1,\dots,x_4)\notin\Sigma_{n,m}$. Thus, (3.8) has a negative answer for $n,m\geqq 4$. Now assume $n=3$, but $m\geqq 8$. Using the form S in (2.1) and repeating the above argument, we have $S(x,y,z)\leqq N'(x^2+y^2+z^2)^3$ for some $N'>0$. Thus,

$$\begin{aligned}x^{m-6}S(x,y,z)&\leqq x^{m-8}(x^2+y^2+z^2)S(x,y,z)\\&\leqq[\sqrt{N'}x^{\frac{m}{2}-4}(x^2+y^2+z^2)^2]^2,\end{aligned}$$

but $x^{m-6}S(x,y,z)\notin\Sigma_{3,m}$. Q.E.D.

Remark 3.10. Although it is unimportant for the above proof, the maxima N, N' of Q and S on the respective unit spheres can be explicitly determined. The answers are: $N=1$, and $N'=4/27$. In fact, it is not difficult to check directly that

$$(x^2+y^2+z^2+w^2)^2-Q(x,y,z,w)\in\Sigma_{4,4},$$

and

$$4(x^2+y^2+z^2)^3-27\,S(x,y,z)\in\Sigma_{3,6},$$

while $Q(0,0,0,1)=1$, $S(0,\sqrt{2}/\sqrt{3},1/\sqrt{3})=4/27$.

4. The numbers $B_{n,m}$ and $B'_{n,m}$

We define $B_{n,m}$ (resp. $B'_{n,m}$) by $\sup|\mathfrak{Z}(p)|$ where p ranges over all forms in $P_{n,m}$ (resp. in $\Sigma_{n,m}$) with $|\mathfrak{Z}(p)|<\infty$. The determination of these numbers seems to be a rather awesome task. In this section, we shall present some partial results.

To begin with, one has $B_{2,m}=B'_{2,m}=m/2$ and $B_{n,2}=B'_{n,2}=1$, as we have already noted in Sect. 2. For general m,n, however, we do not know if $B_{n,m}$ need be always finite. The following lower bound shows, in any case, that $B_{n,m}\geqq B'_{n,m}\to\infty$ if either $n\to\infty$ or $m\to\infty$.

Proposition 4.1. $B'_{n,m}\geqq\left(\frac{m}{2}\right)^{n-1}$.

Proof. Let $m=2r$ and consider the following n-ary m-ic:

$$p=\sum_{i=1}^{n-1}\prod_{j=1}^{r}(x_i-jx_n)^2\in\Sigma_{n,m}.$$

Clearly, $\mathfrak{Z}(p)=\{(y_1,\dots,y_{n-1},1)\colon y_i\in\{1,\dots,r\}\}$, so $B'_{n,m}\geqq|\mathfrak{Z}(p)|=\left(\frac{m}{2}\right)^{n-1}$. Q.E.D.

We can also make the following simple observations about the relationship between the various numbers $B_{n,m}$ (resp. $B'_{n,m}$):

Proposition 4.2. (1) $B_{n_1+n_2-1,m}\geqq B_{n_1,m}\cdot B_{n_2,m}$.

(2) $B_{n,m_1+m_2}\geqq B_{n,m_1}+B_{n,m_2}$.

(3) *For any natural number* k, $B_{n,km}\geqq k^{n-1}B_{n,m}$.

These three inequalities also hold if B is replaced everywhere by B'.

Proof. It will suffice to prove the inequalities for the B-numbers. (The arguments below carry over *verbatim* for the B'-numbers.) Also, in the following, we shall assume that $B_{n_i,m}$, B_{n,m_i} and $B_{n,m}$ are all *finite*; the argument for the infinite cases will be completely similar.

(1) Take $p_i\in P_{n_i,m}$ with $|\mathfrak{Z}(p_i)|=B_{n_i,m}$. Write $p_1=p_1(x_1,\dots,x_{n_1})$ and $p_2=p_2(x_{n_1},\dots,x_{n_1+n_2-1})$. After a linear change (one for p_1 and one for p_2), we may assume that $(\bar{x}_1,\dots,\bar{x}_{n_1})\in\mathfrak{Z}(p_1)\Rightarrow\bar{x}_{n_1}\neq 0$, and that $(\bar{x}_{n_1},\dots,\bar{x}_{n_1+n_2-1})\in\mathfrak{Z}(p_2)\Rightarrow\bar{x}_{n_1}\neq 0$. Then $\mathfrak{Z}(p_1+p_2)$ consists of all $(\bar{x}_1,\dots,1,\dots,\bar{x}_{n_1+n_2-1})$ where $(\bar{x}_1,\dots,1)\in\mathfrak{Z}(p_1)$ and $(1,\dots,\bar{x}_{n_1+n_2-1})\in\mathfrak{Z}(p_2)$. This yields $|\mathfrak{Z}(p_1+p_2)|=|\mathfrak{Z}(p_1)|\cdot|\mathfrak{Z}(p_2)|=B_{n_1,m}\cdot B_{n_2,m}$ with $p_1+p_2\in P_{n_1+n_2-1,m}$.

(2) Take $p_i\in P_{n,m_i}$ with $|\mathfrak{Z}(p_i)|=B_{n,m_i}$. Replacing $p_1(\underline{x})$ by $p_1(T\underline{x})$ (for a suitable linear change T), we may assume that $\mathfrak{Z}(p_1)\cap\mathfrak{Z}(p_2)=\varnothing$. Then $\mathfrak{Z}(p_1\cdot p_2)=\mathfrak{Z}(p_1)\cup\mathfrak{Z}(p_2)$ implies $|\mathfrak{Z}(p_1p_2)|=B_{n,m_1}+B_{n,m_2}$.

(3) The estimate in (2) yields only $B_{n,km}\geqq kB_{n,m}$, so we need a different argument to get the better lower bound $k^{n-1}B_{n,m}$. For a fixed $p\in P_{n,m}$ with $|\mathfrak{Z}(p)|=B_{n,m}$, we may assume, as in (1) above, that $(\bar{x}_1,\dots,\bar{x}_n)\in\mathfrak{Z}(p)\Rightarrow\bar{x}_n\neq 0$. By scaling the variables, we may further assume that $\mathfrak{Z}(p)=\{(\alpha_1^{(i)},\dots,\alpha_{n-1}^{(i)},1)\colon 1\leqq i\leqq B_{n,m}\}$ where all $|\alpha_j^{(i)}|<1$. Let $T_k(t)$ be the k^{th} Chebyshev polynomial ($\deg T_k=k$), defined for $-1\leqq t\leqq 1$ by $T_k(t)=\cos(k\cdot\arccos t)$ and extended for all t. For $-1<u<1$, the equation $T_k(t)=u$ has exactly k distinct solutions (see, e.g. [2, Sect. 15.15]). We homogenize $T_k(t)$ by taking $T_k(t,u)=u^kT_k\left(\frac{t}{u}\right)$, a binary k-ic. Now define

$$\hat{p}(x_1,\dots,x_n)=p(T_k(x_1,x_n),\dots,T_k(x_{n-1},x_n),x_n^k)\in P_{n,km}.$$

Clearly, $(\bar{x}_1,\dots,\bar{x}_n)\in\mathfrak{Z}(\hat{p})$ implies $\bar{x}_n^k\neq 0$, so $\bar{x}_n\neq 0$ and we may assume that $\bar{x}_n=1$. For each i $(1\leqq i\leqq B_{n,m})$ and each $j(1\leqq j\leqq n-1)$ there are exactly k values of x_j such that $T_k(x_j,1)=\alpha_j^{(i)}$. Accordingly, $|\mathfrak{Z}(\hat{p})|=k^{n-1}B_{n,m}$. Q.E.D.

We shall now study the number of zeros of psd *ternary* forms. The results above, together with the results in the last section, lead to the following facts about $B_{3,m}$:

Theorem 4.3. (1) $B_{3,4}=4$, (2) $B_{3,6}=10$.

(3) *For* $m\geqq 6$, $m^2/4\leqq B_{3,m}\leqq(m-1)(m-2)/2$.

(4a) $B_{3,6k}\geqq 10k^2$, (4b) $B_{3,6k+2}\geqq 10k^2+1$ *and* (4c) $B_{3,6k+4}\geqq 10k^2+4$.

(5) *Let* $\beta(m)=B_{3,m}/m^2$. *Then* $\beta=\lim_{m\to\infty}\beta(m)$ *exists. Moreover,* $\beta(m)\leqq\beta$ *for all* m, *and* $5/18\leqq\beta\leqq 1/2$.

Proof. For any m, (4.1) gives $B_{3,m} \geqq B'_{3,m} \geqq m^2/4$ and (3.5) gives $B_{3,m} \leqq \alpha(m) = \max(m^2/4, (m-1)(m-2)/2)$. For $m=4$, this proves (1). For $m \geqq 6$, the maximum is given by $(m-1)(m-2)/2$, so we have (3). For $m=6$, the bounds in (3) give $9 \leqq B_{3,6} \leqq 10$, but it is known that Robinson's psd sextic

$$(4.4) \quad R(x,y,z) = x^6+y^6+z^6-x^4(y^2+z^2)-y^4(z^2+x^2)-z^4(x^2+y^2)+3x^2y^2z^2$$

has $|\mathfrak{Z}(R)|=10$ (see [20, 4]). Thus, $B_{3,6}=10$. From (4.2)(3), we have then $B_{3,6k} \geqq k^2 B_{3,6} = 10k^2$, and (4b), (4c) follow similarly from (4.2)(2), (4.2)(3), and $B_{3,2}=1$, $B_{3,4}=4$. It can be readily checked that (4a), (4b), (4c) give *better* lower bounds than (3) in case (a) for all k, (b) for all $k \geqq 6$ and (c) for all $k \geqq 12$.

Finally, to study the asymptotic behavior of $B_{3,m}$, we look at $\beta(m) := B_{3,m}/m^2$ ($\leqq 1/2$ by (1) and (3)). For $k, s \geqq 0$, we have, by (4.2):

$$(*) \quad \begin{aligned} \beta(km+s) &= \frac{B_{3,km+s}}{(km+s)^2} \geqq \frac{B_{3,km}}{(km+s)^2} \\ &\geqq \frac{k^2m^2}{(km+s)^2} \cdot \frac{B_{3,m}}{m^2} = \left(\frac{km}{km+s}\right)^2 \beta(m). \end{aligned}$$

Let $\beta = \overline{\lim_{n\to\infty}}\, \beta(m)$ and $\beta' = \underline{\lim_{n\to\infty}}\, \beta(m)$. Consider any positive $\varepsilon < \beta/2$, and fix an m_0 such that $\beta(m_0) \geqq \beta - \varepsilon$. Pick a large integer k_0 such that $\left(\frac{k}{k+1}\right)^2 \geqq \frac{\beta-2\varepsilon}{\beta-\varepsilon}$ for all $k \geqq k_0$. For any $m \geqq k_0 m_0$, let $m = km_0 + s$ where $0 \leqq s \leqq m_0 - 1$. Then $k \geqq k_0$ so by $(*)$ above, $\beta(m) \geqq \left(\frac{km_0}{km_0+m_0}\right)^2 \beta(m_0) \geqq \beta - 2\varepsilon$. This shows that $\beta' \geqq \beta - 2\varepsilon$ (for every $\varepsilon > 0$) and so $\beta' = \beta$. From $(*)$, we also have $\beta(km) \geqq \beta(m)$, so $\beta(m) \leqq \beta$ for all m. Finally, from $\beta(6k) \geqq 10k^2/(6k)^2$, we get $\beta \geqq 5/18$. Q.E.D.

While the above analysis does not pinpoint the exact values of $B_{3,m}$, a complete determination of the numbers $B'_{3,m}$ is possible. The answer is simply: $B'_{3,m} = m^2/4$ for all (even) m. Note that we already have $B'_{3,m} \geqq m^2/4$ from (4.1), so we shall only need the reversed inequality. This will be shown in (4.6), which requires the following lemma.

Lemma 4.5. *Let $f_i \in \mathbb{R}[x_1, \ldots, x_n]$ $(0 \leqq i \leqq t)$ be a set of $t+1$ polynomials which have no common factor (other than scalars). If $f_0 \not\equiv 0$, then there exists $a_1, \ldots, a_t \in \mathbb{R}$ such that there is no common factor between f_0 and $a_1 f_1 + \ldots + a_t f_t$.*

Proof. Let $f_0 = p_1 p_2 \ldots p_k$ be a factorization of f_0 into irreducible real polynomials. Let

$$H_i = \{(b_1, \ldots, b_t) \in \mathbb{R}^t : b_1 f_1 + \ldots + b_t f_t \text{ is divisible by } p_i\}.$$

Each H_i is a linear subspace in $\mathbb{R}^t$, and $H_i \subsetneqq \mathbb{R}^t$ for otherwise p_i will divide $f_1, \ldots, f_t$ as well as f_0. Since $\mathbb{R}^t$ cannot be covered by a finite number of proper linear subspaces, there exists a vector $(a_1, \ldots, a_t)$ in $\mathbb{R}^t - (H_1 \cup \ldots \cup H_k)$. Now $a_1 f_1 + \ldots + a_t f_t$ is relatively prime to *each* p_i, and therefore to f_0. Q.E.D.

Theorem 4.6. $B'_{3,m} = m^2/4$.

Proof. As we have observed already, it is sufficient to show that, *if $p \in \Sigma_{3,m}$ and $|\mathfrak{Z}(p)| < \infty$, then $|\mathfrak{Z}(p)| \leqq r^2$, where $r = m/2$.* This will be deduced below from the Theorem of Bezout.

(A) Write the given p as $f_0^2 + f_1^2 + \ldots + f_t^2$ where each f_i is a (non-zero) ternary r-ic. *We first assume that $\{f_0, f_1, \ldots, f_t\}$ have no common factor.* By the lemma above, we can choose $a_i \in \mathbb{R}$ such that f_0 is relatively prime to $g := a_1 f_1 + \ldots + a_t f_t$. By Bezout's Theorem, $|\mathfrak{Z}(f_0) \cap \mathfrak{Z}(g)| \leqq (\deg f_0)(\deg g) = r^2$. But $\mathfrak{Z}(p) = \bigcap_{i \geqq 0} \mathfrak{Z}(f_i) \subseteq \mathfrak{Z}(f_0) \cap \mathfrak{Z}(g)$, so $|\mathfrak{Z}(p)| \leqq r^2$, as desired.

(B) For the remaining case, we assume that $\{f_0, f_1, \ldots, f_t\}$ have a greatest common divisor f of *positive* degree $k \leqq r$. Write $f_i = g_i f$, where $\deg g_i = r - k$. Setting $\tilde{p} = g_0^2 + \ldots + g_t^2$, we have $p = \tilde{p} f^2$. By the analysis in the last paragraph, $|\mathfrak{Z}(\tilde{p})| \leqq (r-k)^2$. On the other hand, $|\mathfrak{Z}(f)| \leqq |\mathfrak{Z}(p)| < \infty$, so by (2.5), f is semidefinite. Using now (3.5) (and its notation), $|\mathfrak{Z}(f)| \leqq \alpha(k)$. From $p = \tilde{p} f^2$, we get

$$\begin{aligned} |\mathfrak{Z}(p)| \leqq |\mathfrak{Z}(\tilde{p})| + |\mathfrak{Z}(f)| &\leqq r^2 - 2rk + k^2 + \alpha(k) \\ &\leqq r^2 - k^2 + \alpha(k) \end{aligned}$$

since $r \geqq k > 0$. But certainly $\alpha(k) = \max(k^2/4, (k-1)(k-2)/2) < k^2$, so we get a *strict* inequality $|\mathfrak{Z}(p)| < r^2$. Q.E.D.

Note. In case $t=0$, i.e. $p = f_0^2$, we can skip the argument in (A) and use a small subset of the arguments in (B). Here, $f = f_0$ and so $|\mathfrak{Z}(p)| = |\mathfrak{Z}(f)| \leqq \alpha(r) < r^2$.

Corollary 4.7. (1) *If $p \in P_{3,m}$ and $|\mathfrak{Z}(p)| > m^2/4$, then either p is not a sum of squares of forms, or p is divisible by the square of some indefinite form.* (2) *Any $p \in \Sigma_{3,m}$ with $|\mathfrak{Z}(p)| = m^2/4$ is a sum of two squares of $m/2$-ics.* (For $m=4$, *cf.* (3.3)).

Proof. (1) follows from the Theorem above and (3.5). The proof of (2) will be an application of Max Noether's "Fundamental Theorem" for algebraic curves (see, e.g. [11, p. 120] or [23, p. 120]). Let p be as in (2), say $p = f_0^2 + \ldots + f_t^2$; we may assume that $t \geqq 2$. As in the proof of (4.6), choose $g = a_1 f_1 + \ldots + a_t f_t$ relatively prime to f_0, so $|\mathfrak{Z}(f_0) \cap \mathfrak{Z}(g)| \leqq m^2/4$. But the subset $\mathfrak{Z}(p) \subseteq \mathfrak{Z}(f_0) \cap \mathfrak{Z}(g)$ already has cardinality $m^2/4$, so

$$|\mathfrak{Z}(f_0) \cap \mathfrak{Z}(g)| = m^2/4, \quad \text{and} \quad \mathfrak{Z}(f_0) \cap \mathfrak{Z}(g) = \mathfrak{Z}(p) \subseteq \mathfrak{Z}(f_i) \qquad \text{(for all i)}.$$

By Noether's Theorem, each f_i is a $\mathbb{C}$-linear combination (hence also $\mathbb{R}$-linear combination) of f_0 and g. Say $f_i = b_i f_0 + c_i g$ ($b_i, c_i \in \mathbb{R}$, $i \geqq 0$), so $p = \sum_{i=0}^{t} (b_i f_0 + c_i g)^2$. The psd binary quadratic form $\sum_{i=0}^{t} (b_i u + c_i v)^2$ can be rewritten as $(au + bv)^2 + (cu + dv)^2$, so by substitution, $p = (af_0 + bg)^2 + (cf_0 + dg)^2$. Q.E.D.

Corollary 4.8. *If a ternary sextic p has exactly 10 zeros, then one of $\pm p$ is psd but not a sum of squares of cubics.*

Proof. This follows from (2.5), and the calculations $B'_{3,6} = 9$, $B_{3,6} = 10$. Q.E.D.

Recall Robinson's ternary sextic R in (4.4). In view of the last corollary, the fact that $|\mathfrak{Z}(R)|=10$ proved in [20]) already precludes R from being a sum of squares of cubics. In [4, Th. 3.8], two of the authors have shown that R is (up to a scalar multiple) the *only* form in $P_{3,6}$ vanishing at the 10-point set

$$\mathfrak{Z}(R)=\{(0,\pm1,1),(1,0,\pm1),(\pm1,1,0),(1,1,1),(-1,1,1),(1,-1,1),(1,1,-1)\}. \tag{4.9}$$

It would be of interest to determine, if possible, *all* forms $p\in P_{3,6}$ with exactly 10 zeros. From a combinatorial point of view, it would already be of interest to determine (or classify) all configurations of 10-point sets $S\subset\mathbb{P}^2$ for which there exist $p\in P_{3,6}$ such that $S=\mathfrak{Z}(p)$. Once one solves the combinatorial problem, one can consider, for each legitimate configuration S, the cone of all $p\in P_{3,6}$ vanishing on S, and then study the extremal rays of this cone in attempt to classify the forms p with $\mathfrak{Z}(p)=S$. We suspect (on the basis of [4, Th. 3.8]) that these cones are always of very small dimension, and that their extremal rays are few in number. We should not fail to observe, however, that the *only* 10-point configuration known to be possible so far is the one in (4.9)!

What can be said about $B_{n,m}$ and $B'_{n,m}$ for $m=4$, i.e. for quartics? If a quartic p has a zero, we may assume that this zero is at $(1,0,0,\ldots)$ after a linear change. Then x_1^4 does not appear in p, and, if p happens to be psd, p will be at most quadratic in x_1. The following convenient lemma will be used repeatedly to "control" the zeros of p in terms of the zeros of its discriminant (w.r.t. x_1):

Lemma 4.10. *Let $p(x_1,\ldots,x_n,x)=x^2f+2xg+h\geqq0$, where f,g,h are (real) polynomials in $x_1,\ldots,x_n$. Let $D=fh-g^2\geqq0$ be its discriminant w.r.t. x. If $p(\bar{x}_1,\ldots,\bar{x}_n,\bar{x})=0$, then $D(\bar{x}_1,\ldots,\bar{x}_n)=0$. If $D(\bar{x}_1,\ldots,\bar{x}_n)=0$ and $f(\bar{x}_1,\ldots,\bar{x}_n)\neq0$, then there is a unique $\bar{x}$ such that $p(\bar{x}_1,\ldots,\bar{x}_n,\bar{x})=0$. If $D(\bar{x}_1,\ldots,\bar{x}_n)=0=f(\bar{x}_1,\ldots,\bar{x}_n)$, then $g(\bar{x}_1,\ldots,\bar{x}_n)=0$, so $p(\bar{x}_1,\ldots,\bar{x}_n,x)=h(\bar{x}_1,\ldots,\bar{x}_n)=0$ either for all x or for no x.*

Proof. This is clear from the "completion of square" identity:

$$f\cdot p=(xf+g)^2+D. \tag{4.11}$$

Corollary 4.12. *Assume that p above is a form, and that $|\mathfrak{Z}(p)|<\infty$. Let $(\bar{x}_1,\ldots,\bar{x}_n)\neq0$. Then there exists $\bar{x}$ with $(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n,\bar{x})\in\mathfrak{Z}(p)$ iff $(\bar{x}_1,\ldots,\bar{x}_n)\in\mathfrak{Z}(D)$ and $f(\bar{x}_1,\ldots,\bar{x}_n)>0$.*

If the p above is a quaternary quartic, then the discriminant D will be a ternary sextic, for which we can use the results $B_{3,6}=10$, $B'_{3,6}=9$. This idea leads us to the following estimates on $B_{4,4}$ and $B'_{4,4}$:

Proposition 4.13. $8\leqq B'_{4,4}\leqq10\leqq B_{4,4}\leqq11$.

Proof. Consider $p\in P_{4,4}$ with $0<|\mathfrak{Z}(p)|<\infty$. After "putting" one zero at $(1,0,0,0)$, we can write p as $x^2f+2xg+h$ where f,g,h are forms in $\{y,z,w\}$ with degrees, respectively, 2, 3 and 4. Let $D=fh-g^2\in P_{3,6}$. If $f\equiv0$, (4.10) gives $\mathfrak{Z}(p)=\{(1,0,0,0)\}$, so we may assume $f\not\equiv0$. If rank $f\geqq2$ (as a quadratic form), then f has at most one zero so $|\mathfrak{Z}(D)|<\infty$. From this, we get $|\mathfrak{Z}(p)|\leqq1+|\mathfrak{Z}(D)|\leqq1+B_{3,6}=11$. For the final case, assume rank $f=1$, so after a linear change, $f=y^2$. We can write $g=yg_1$, and $D=y^2h_1$ where $h_1:=h-g_1^2\in P_{3,4}$. Moreover, $p=(xy+g_1)^2+h_1$. Consider any zero $(\bar{x},\bar{y},\bar{z},\bar{w})\in\mathfrak{Z}(p)$ distinct from $(1,0,0,0)$. By (4.12), we have $\bar{y}\neq0$; also, $h_1(\bar{y},\bar{z},\bar{w})=0$. If $|\mathfrak{Z}(h_1)|<\infty$, then $|\mathfrak{Z}(p)|\leqq1+|\mathfrak{Z}(h_1)|\leqq1+B_{3,4}=5$, so assume $|\mathfrak{Z}(h_1)|=\infty$. Only finitely many zeros of h_1 have y-coordinate $\neq0$, so $h_1(0,z,w)$ has infinitely many zeros. This says that $h_1(0,z,w)\equiv0$, so $h_1=y^2h_2$. For $(\bar{x},\bar{y},\bar{z},\bar{w})$ above, we have $h_2(\bar{y},\bar{z},\bar{w})=0$. If $|\mathfrak{Z}(h_2)|<\infty$, then, as above, $|\mathfrak{Z}(p)|\leqq1+B_{3,2}=2$. If $|\mathfrak{Z}(h_2)|=\infty$, then again $h_2=\varepsilon y^2$, so $h_1=\varepsilon y^4$. In this case, clearly $|\mathfrak{Z}(p)|=1$. We have thus shown that $B_{4,4}\leqq11$.

If, in the above argument, $p\in\Sigma_{4,4}$, then (2.7) gives $D\in\Sigma_{3,6}$, so using $B'_{3,6}=9$ instead of $B_{3,6}=10$, the same arguments yield $B'_{4,4}\leqq1+B'_{3,6}=10$. For a lower bound, (4.1) gives $B'_{4,4}\geqq2^3=8$. The lower bound on $B_{4,4}$ depends on our forthcoming work [6] on psd symmetric quartics. In this work, it is shown that, for any β between 1 and 2, the symmetric quartic $\Sigma x^2y^2+\beta\Sigma x^2yz+(4\beta^2-4\beta-2)xyzw$ is psd, and has exactly *ten* zeros. Therefore, $B_{4,4}\geqq10$. Q.E.D.

Our results on ternary forms and quaternary quartics presented above seem to have certain connections with Harnack's Theorem on real curves, and Hilbert's 16th Problem for algebraic curves and surfaces [15]. According to Harnack's Theorem (see. e.g. [22, p. 337]), a complete non-singular plane curve C of degree m has at most $g+1$ real components (or "ovals") where $g=(m-1)\cdot(m-2)/2$ is the genus of C. This theorem does not apply directly to curves defined by psd ternary forms p since these curves may have singularities. However, by analogy, the zero points in $\mathfrak{Z}(p)$ may be viewed as "degenerate" ovals. For $m=6$, Harnack's Theorem predicts a maximum of 11 ovals, but for $p\in P_{3,6}$ with finitely many zeros, we have the better bound $|\mathfrak{Z}(p)|\leqq10$. Heuristically, this may be reconciled by the fact that the 11 ovals of a sextic plane curve cannot lie externally to one another – a theorem of Hilbert [14]. In generalization to Hilbert's Theorem, Petrovskii has shown (see [22, p. 341], [12]) that a non-singular plane curve of degree $m=2r$ has at most $3r(r-1)/2+1$ ovals not containing each other. Assuming this result, it is possible to improve our upper bound $B_{3,m}\leqq\max(m^2/4,(m-1)(m-2)/2)$ to $B_{3,m}\leqq3m(m-2)/8+1$. This implies that the limit $\beta=\lim B_{3,m}/m^2$ studied in (4.3)(5) will fall within the closed interval [5/18, 3/8] (instead of [5/18, 1/2]).

In general, the Hilbert 16th Problem of determining all possible configurations of the maximal number of ovals of curves of arbitrary degree m seems to have remained unsolved. There is also an analogous problem for the configuration of the sheets of algebraic surfaces in 3-space. Recently, Petrovskii and Oleinik have proved that a fourth order (nonsingular) surface which consists entirely of ovals can contain at most ten ovals (see the Introduction to [12]). Assuming this, one can show that $B_{4,4}=10$. Also, by applying Bezout's Theorem to quadric surfaces in projective 3-space, it seems possible to show that $B'_{4,4}=8$, though the details are too tedious to present here.

For further modern work concerning the number of components of real algebraic varieties, see e.g. [9] and the references therein.

5. Quaternary Quartics

This and the next section will be devoted to the study of psd quaternary quartics p with a large number of zeros. By the latter, we shall mean $|\mathfrak{Z}(p)|>11$; since $B_{4,4}\leqq 11$ (see (4.13)), this has the same effect as $|\mathfrak{Z}(p)|=\infty$. Unlike the case of ternary forms, the hypothesis $|\mathfrak{Z}(p)|=\infty$ for $p\in P_{4,4}$ will *not* imply the existence of a square factor for p. Nevertheless, we can draw some interesting conclusions:

Theorem 5.1. *If $p\in P_{4,4}$ and $|\mathfrak{Z}(p)|=\infty$, then $p\in\Sigma_{4,4}$. Indeed, p is a sum of six squares of quadratics.*

The proof of this occupies two sections, and will involve several cases whose analysis requires substantially different techniques. We start with a lemma:

Lemma 5.2. *Suppose $p\in P_{n,m}$ and $(1,0,\ldots,0)$ is an accumulation point for $\mathfrak{Z}(p)$ (viewed projectively). Expand p as a polynomial in x_1:*

$$p(x_1,\ldots,x_n)=\sum_{i=0}^{m} h_i(x_2,\ldots,x_n)x_1^{m-i}$$

where h_i is a form of degree i. Then $h_0=h_1=0$, and h_2 is a psd quadratic form of rank $<n-1$.

Proof. Since $p(1,0,\ldots,0)=h_0(0,\ldots,0)=h_0=0$, x_1^m does not occur in p. Since p is psd as a polynomial in x_1, $p(x_1,\bar{x}_2,\ldots,\bar{x}_n)$ has even degree, and has a non-negative leading coefficient. Thus, $h_1\equiv 0$ and h_2 is a psd quadratic form; by a linear change, $h_2=x_2^2+\ldots+x_r^2$. Assume $r=n$; let

$$M_i=\operatorname{Max}\{|h_i(x_2,\ldots,x_n)|: x_2^2+\ldots+x_n^2=1\}.$$

Then, for $x_2^2+\ldots+x_n^2=\varepsilon^2$, we have

$$\begin{aligned}p(1,x_2,\ldots,x_n)&=\sum_{i=2}^{m} h_i(x_2,\ldots,x_n)\\&\geqq \varepsilon^2-\sum_{i=3}^{m} M_i\varepsilon^i\\&=\varepsilon^2\left(1-\sum_{i=3}^{m} M_i\varepsilon^{i-2}\right).\end{aligned}$$

For a sufficiently small $\varepsilon_0>0$, we have therefore $p(1,x_2,\ldots,x_n)>0$ whenever $0<x_2^2+\ldots+x_n^2<\varepsilon_0^2$. This says that $(1,0,\ldots,0)$ is (projectively) an isolated zero for p, a contradiction. Q.E.D.

Remark. Similarly, if $h_0=\ldots=h_{s-1}\equiv 0$ and $h_s\neq 0$, then s is even, and $h_s(x_2,\ldots,x_n)$ must have a nontrivial zero.

Consider the form $p(x,y,z,w)$ in Theorem 5.1. View $\mathfrak{Z}(p)$ as a set lying on the unit sphere $x^2+y^2+z^2+w^2=1$. Since $|\mathfrak{Z}(p)|=\infty$, the set $\mathfrak{Z}(p)$ must have an accumulation point. By a linear change, make this accumulation point $(1,0,0,0)$. By the lemma above, we can write p as

$$p(x,y,z,w)=f(y,z,w)x^2+2g(y,z,w)x+h(y,z,w), \tag{5.3}$$

where f is a psd quadratic form of rank $\leqq 2$. In the following, we shall write $D(y,z,w)=fh-g^2$; this discriminant is a psd ternary sextic.

Proposition 5.4. *For $p=fx^2+2gx+h\in P_{4,4}$, if $f(y,z,w)$ has rank $\leqq 1$, then p is a sum of four squares of quadratics.*

Proof. If $f\equiv 0$, then clearly $g\equiv 0$ also so $p(x,y,z,w)=h(y,z,w)$. By Hilbert [13], this $h\in P_{3,4}$ is a sum of *three* squares of quadratics. Now assume rank $f=1$; then after a linear change, $f(y,z,w)=y^2$. From $y^2h\geqq g^2$, we can write $g=y\bar{g}(y,z,w)$ so now $\bar{D}(y,z,w):=h-\bar{g}^2\in P_{3,4}$. By Hilbert [13] again, $\bar{D}$ is a sum of three squares of quadratics, so $p=(xy+\bar{g})^2+\bar{D}$ is a sum of four squares of quadratics. Q.E.D.

In (5.3), it thus remains to treat the case rank $f=2$. *Henceforth, by a linear change, assume that $f(y,z,w)=y^2+z^2$.* The analysis now divides into the following two subcases involving the discriminant D: *Case A.* $|\mathfrak{Z}(D)|<\infty$. *Case B.* $|\mathfrak{Z}(D)|=\infty$. (Of course, $\mathfrak{Z}(D)$ is understood to be in projective 2-space rather than projective 3-space). Using a certain theorem which is "essentially" known in the literature (see (5.6) below), we shall first dispose of Case A:

Proposition 5.5. *Suppose $p=fx^2+2gx+h\in P_{4,4}$, where $f=y^2+z^2$. If $|\mathfrak{Z}(p)|=\infty$ but $|\mathfrak{Z}(D)|<\infty$, then p is a sum of six squares of quadratics.*

Proof. Recall that $(\bar{x},\bar{y},\bar{z},\bar{w})\in\mathfrak{Z}(p)$ implies $(\bar{y},\bar{z},\bar{w})\in\mathfrak{Z}(D)$, and if $(\bar{y},\bar{z},\bar{w})\in\mathfrak{Z}(D)$, $f(\bar{y},\bar{z})\neq 0$, then there is a unique $\bar{x}$ with $(\bar{x},\bar{y},\bar{z},\bar{w})\in\mathfrak{Z}(p)$ (see (4.10)). Since $|\mathfrak{Z}(p)|=\infty$ and $|\mathfrak{Z}(D)|<\infty$, there must be infinitely many $(\bar{x},\bar{y},\bar{z},\bar{w})\in\mathfrak{Z}(p)$ with $f(\bar{y},\bar{z})=\bar{y}^2+\bar{z}^2=0$. Hence $p(x,0,0,w)\equiv 0$ i.e. $g(0,0,w)=h(0,0,w)=0$, which means that $\deg_w g\leqq 2$ and $\deg_w h\leqq 3$. But $h\geqq 0$, so $\deg_w h\leqq 2$, and from $(y^2+z^2)h-g^2\geqq 0$, we deduce further that $\deg_w g\leqq 1$. Thus, p is a quadratic polynomial in $\{x,w\}$ whose "coefficients" are forms in $\{y,z\}$. We are done by the following theorem:

Theorem 5.6. *Suppose $q(y,z;x_1,\ldots,x_n)\in P_{n+2,m}$ is a (not necessarily homogeneous) quadratic polynomial in $\{x_1,\ldots,x_n\}$ with "coefficients" which are forms in $\{y,z\}$. Then q is a sum of $2(n+1)$ squares in $\mathbb{R}[y,z;x_1,\ldots,x_n]$.*

This can be easily deduced from results of [18, 17, 21, 10]. The derivation will be given in Sect. 7; in fact, we shall prove a slightly more general result, (7.9), in which q need not be homogeneous.

We shall now begin the analysis of Case B, when the discriminant $D=fh-g^2\in P_{3,6}$ has infinitely many zeros. By Theorem 3.5, D can be factored into k^2q for suitable k and q. The remaining arguments will be carried out in the following three subcases: (B_1) $\deg k=1$, (B_2) $\deg k=2$, and (B_3) $\deg k=3$.

We first deal with (B_3), which depends on a result in [8] on sums of two squares. For convenience, we shall state the result we need:

Theorem 5.7 [8]. *Let r, s and p be polynomials in $\mathbb{R}[x_1,\ldots,x_n]$. If $(r^2+s^2)p$ is a nonzero sum of two squares in $\mathbb{R}[x_1,\ldots,x_n]$, then so is p.*

This has the following remarkable consequence which certainly subsumes our case (B_3):

Theorem 5.8. *Let $p(x_1,\ldots,x_n,x)=x^2f+2xg+h$ where $f,g,h\in\mathbb{R}[x_1,\ldots,x_n,x]$. If $D=fh-g^2$ is a perfect square, and f is a nonzero sum of two squares in $\mathbb{R}[x_1,\ldots,x_n,x]$, then p is also a sum of two squares in $\mathbb{R}[x_1,\ldots,x_n,x]$.*

Proof. Say $D=k^2$ and $f=r^2+s^2$, where $k,r,s\in\mathbb{R}[x_1,\dots,x_n,x]$. As in (4.11), we have

$$f\cdot p=(r^2+s^2)p=(xf+g)^2+D=(xf+g)^2+k^2 \tag{5.9}$$

so the desired conclusion follows from (5.7). Q.E.D.

We remark that the constructive methods of [8] will actually give an *explicit* expression of p as a sum of two squares of polynomials, starting from the Eq. (5.9).

Having disposed of Case (B_3), we shall now proceed to Case (B_2). For this case, we need a crucial lemma:

Lemma 5.10. *For $q(x_1,\dots,x_n)\in\mathbb{R}[x_1,\dots,x_n]$ $(n\geqq 2)$, the following are equivalent:*

(1) *$q(ix_2,x_2,x_3,\dots,x_n)$ is a perfect square in $\mathbb{C}[x_2,\dots,x_n]$ $(i=\sqrt{-1})$;*

(2) *There exist polynomials μ and λ in $\mathbb{R}[x_1,\dots,x_n]$ such that $q=\pm\lambda^2+(x_1^2+x_2^2)\mu$.*

Proof. (2) ⇒ (1) is obvious. For the converse, consider a polynomial q satisfying (1). By first working modulo $x_1^2+x_2^2$ and then lifting, we can write

$$q(x_1,\dots,x_n)=(x_1^2+x_2^2)a(x_1,\dots,x_n)+x_1b(x_2,\dots,x_n)+c(x_2,\dots,x_n). \tag{5.11}$$

Suppose $q(ix_2,x_2,\dots,x_n)=(s+it)^2$ where $s,t\in\mathbb{R}[x_2,\dots,x_n]$. Then $ix_2b+c=(s+it)^2=s^2-t^2+2ist$, so $c=s^2-t^2$ and $x_2b=2st$. We have the following two (not mutually exclusive) case:

Case (i) $x_2|t$. Say $t=x_2u$ $(u\in\mathbb{R}[x_2,\dots,x_n])$. Then $b=2su$ and $q=(x_1^2+x_2^2)a+2x_1su+s^2-x_2^2u^2=(x_1^2+x_2^2)(a-u^2)+(x_1u+s)^2$.

Case (ii) $x_2|s$. Say $s=x_2v$ $(v\in\mathbb{R}[x_2,\dots,x_n])$. Then $b=2tv$ and $q=(x_1^2+x_2^2)a+2x_1tv+x_2^2v^2-t^2=(x_1^2+x_2^2)(a+v^2)-(x_1v-t)^2$. Q.E.D.

Corollary 5.12. *Suppose $q(x_1,\dots,x_n)$ $(n\geqq 2)$ is a psd polynomial such that $q(ix_2,x_2,\dots,x_n)$ is a perfect square in $\mathbb{C}[x_2,\dots,x_n]$. Then there exist polynomials μ and λ in $\mathbb{R}[x_1,\dots,x_n]$ such that $q=\lambda^2+(x_1^2+x_2^2)\mu$. If, moreover, q is a quadratic form, then λ may be chosen to be a linear form, and μ is a non-negative real number.*

Proof. Keep the notations in the above proof. Suppose we are in Case (ii) so we have $q=-\lambda_1^2+(x_1^2+x_2^2)\mu_1$. Then $0\leqq q(0,0,x_3,\dots,x_n)=-\lambda_1(0,0,x_3,\dots,x_n)^2$ implies that $q(0,0,x_3,\dots,x_n)\equiv 0$. From (5.11), we see that $x_2|c$. But $c=s^2-t^2$ so $x_2|s\Rightarrow x_2|t$. Thus, we are back to Case (i) and have the desired equation $q=\lambda^2+(x_1^2+x_2^2)\mu$. Now suppose q is a quadratic form. Then clearly $\lambda(0,\dots,0)=0$. We may assume λ is a linear form and $\mu\in\mathbb{R}$ by replacing λ by its linear part and μ by its constant part. Now take any $\bar{x}=(\bar{x}_1,\dots,\bar{x}_n)\in\mathfrak{Z}(\lambda)$ such that $(\bar{x}_1,\bar{x}_2)\neq(0,0)$. Evaluation of q at $\bar{x}$ clearly shows that $\mu\geqq 0$. Q.E.D.

We now prove the following theorem which is analogous to (5.8), and which, in particular, settles the Case (B_2):

Theorem 5.13. *Let $p(x_1,\dots,x_n,x)=x^2f+2xg+h$ where $f,g,h\in A:=\mathbb{R}[x_1,\dots,x_n,x]$. Assume that $f=x_1^2+x_2^2$ and that $D:=fh-g^2$ factors into $k^2q(k,q\in A)$ where q is a psd quadratic form. Then, p is a sum of $\gamma(n)$ squares in A where $\gamma(n)=n+2$ if n is even, and $\gamma(n)=n+3$ if n is odd.*

Proof. We shall try to make a reduction to the situation in (5.8). From $(x_1^2+x_2^2)h-g^2=k^2q$, we have

$$-g(ix_2,x_2,\dots,x_n,x)^2=k(ix_2,x_2,\dots,x)^2q(ix_2,x_2,\dots,x_n,x).$$

Case 1. $k(ix_2,x_2,\dots,x_n,x)\not\equiv 0$. In this case, $q(ix_2,x_2,\dots,x_n,x)$ is a perfect square in $\mathbb{C}(x_2,\dots,x_n,x)$, hence a perfect square in $\mathbb{C}[x_2,\dots,x_n,x]$. By the last part of (5.12), we can write $q=\mu(x_1^2+x_2^2)+\lambda^2$, where $\lambda\in A$, and $\mu\in\mathbb{R}$, $\mu\geqq 0$. Using this expression for q, we have $fh=g^2+k^2(\mu f+\lambda^2)$, i.e. $f(h-\mu k^2)=g^2+(k\lambda)^2$. Set $\tilde{p}=x^2f+2xg+\tilde{h}$ where $\tilde{h}=h-\mu k^2$. For this new polynomial, the corresponding $\tilde{D}:=f\tilde{h}-g^2$ is a perfect square, $(k\lambda)^2$. By Theorem 5.8, $\tilde{p}$ is a sum of two squares in A. Since $\mu\geqq 0$, it follows that $p=\tilde{p}+(\sqrt{\mu}k)^2$ is a sum of three squares in A.

Case 2. $k(ix_2,x_2,\dots,x_n,x)\equiv 0$. This means that $f=x_1^2+x_2^2$ divides k in A. Writing $k=fk_0$, we have $fh=g^2+f^2k_0^2q$. Since f is irreducible over the reals, this implies that we can write $g=fg_0$ and hence also $h=fh_0$, with a relation $h_0=g_0^2+k_0^2q$. Now we have $p=f(x^2+2xg_0+h_0)=f[(x+g_0)^2+k_0^2q]$. Since $q(x_1,\dots,x_n,x)$ is a psd quadratic form, it is a sum of $n+1$ squares of linear forms. Using the two-square identity, we see that p is a sum of squares of $n+2$ polynomials if n is even, and $n+3$ polynomials if n is odd. Q.E.D.

Remark. In applications, the polynomials g and h will be free of x, so $q=q(x_1,\dots,x_n)$ will require only n squares. Thus, p will require $n+2$ squares if n is even (no change here), and $n+1$ squares if n is odd (better bound here).

Summing up the information implied by Theorems 5.8 and 5.13 for quaternary quartics, we have now proved:

Theorem 5.14. *Let $p=x^2f+2xg+h\in P_{4,4}$ where $g,h\in\mathbb{R}[y,z,w]$ and $f=y^2+z^2$. If $D=fh-g^2$ factors into k^2q where $\deg k=2$ or 3, then p is a sum of four squares of quadratics.*

6. The Final Case B_1

The only case left to be considered now is when the quaternary quartic $p(x,y,z,w)$ has the shape $x^2f+2xg+h(f=y^2+z^2, g,h\in\mathbb{R}[y,z,w])$ with discriminant $D=(\text{linear})^2$ (quartic). This "Case B_1" will be completely analysed in the present section (see (6.5)). We start with an important special case.

Proposition 6.1. *Let $p(x,y,z,w)=x^2f+2xg+h\in P_{4,4}$ with $f=y^2+z^2$. Suppose that $p(x,0,z,w)=(xz+w^2)^2$. Then p is a sum of five squares in $\mathbb{R}[x,y,z,w]$.*

Proof. $D(0,z,w)$ can be computed as the discriminant of $p(x,0,z,w)$ (as a quadratic polynomial in x). Thus $D(0,z,w)=z^2w^4-(zw^2)^2=0$, so we can write

$D(y, z, w) = y^2 q(y, z, w)$ (see (2.3)). Viewing p as a polynomial in y, write

$$(6.2) \qquad p = (xz + w^2)^2 + 2a_3(x, z, w)\, y + a_2(x, z, w)\, y^2 + a_1(x, z, w)\, y^3 + a_0 y^4.$$

Note that if $\bar{x}\bar{z} + \bar{w}^2 = 0$, then $p(\bar{x}, y, \bar{z}, \bar{w})$ is divisible by y and hence by y^2, i.e. $a_3(\bar{x}, \bar{z}, \bar{w}) = 0$. Thus, $\mathfrak{Z}(xz + w^2) \subset \mathfrak{Z}(a_3)$. From this, it is easy to see that a_3 is divisible by $xz + w^2$, say $a_3(x, z, w) = (xz + w^2)\,\ell(x, z, w)$.

First, suppose p has a zero $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ with $\bar{y} = 1$. Consider the linear change: $x' = x - \bar{x}y$, $y' = y$, $z' = z - \bar{z}y$, $w' = w - \bar{w}y$, which clearly preserves the property that $p(x, 0, z, w) = (xz + w^2)^2$. (This change *does* alter $f(y, z)$, so we shall give up the restriction $f(y, z) = y^2 + z^2$ *in this paragraph.*) After the linear change, we have $(0, 1, 0, 0) \in \mathfrak{Z}(p)$. Thus, $a_0 = 0$ and $a_1(x, z, w) = 0$ (since p is psd), so

$$(6.3) \qquad \begin{aligned} p(x, y, z, w) &= (xz + w^2)^2 + 2y(xz + w^2)\ell + a_2 y^2 \\ &= [(xz + w^2) + y\ell]^2 + y^2(a_2 - \ell^2). \end{aligned}$$

As a polynomial in y, the discriminant $(xz + w^2)^2(a_2 - \ell^2)$ is psd. The ternary quadratic form $a_2 - \ell^2$ is therefore psd, and so a sum of three squares. From (6.3), p is a sum of four squares.

In the discriminant $D(y, z, w) = y^2 q(y, z, w)$, assume that $\mathfrak{Z}(q) = \varnothing$. Let $\lambda = \min q(y, z, w)/y^2(y^2 + z^2)$ on the unit sphere $y^2 + z^2 + w^2 = 1$, and let $p' = p - \lambda y^4$. This is psd, since it has discriminant

$$D' = D - \lambda y^4(y^2 + z^2) = y^2(q - \lambda y^2(y^2 + z^2)) \geqq 0.$$

Moreover, $q' := q - \lambda y^2(y^2 + z^2)$ *has a non-trivial zero*. If p' is a sum of r squares, then p is a sum of $r + 1$ squares. Replacing p by p', *we may thus assume that* $\mathfrak{Z}(q) \neq \varnothing$. (Note, of course, that $p'(x, 0, z, w) = p(x, 0, z, w)$.)

Fix a zero $(\bar{y}, \bar{z}, \bar{w}) \in \mathfrak{Z}(q)$. If $\bar{y} \neq 0$, then by (4.10), there exists $\bar{x} \in \mathbb{R}$ such that $(\bar{x}, \bar{y}, \bar{z}, \bar{w}) \in \mathfrak{Z}(p)$. In this case, p is a sum of four squares by the earlier argument. Thus, we may assume $\bar{y} = 0$, i.e. $(0, \bar{z}, \bar{w}) \in \mathfrak{Z}(q)$.

Before we proceed further, let us first compute $q(0, 0, 1)$ and $q(0, 1, 0)$. These are, respectively, the coefficients of y^2w^4 and y^2z^4 in $D(y, z, w)$. Let us write

$$\begin{aligned} a_3(x, z, w) &= (xz + w^2)(c_0 x + c_1 z + c_2 w) \\ a_2(x, z, w) &= c_3 z^2 + \ldots. \end{aligned}$$

Then $2a_3 y$ contains the term $2c_0 x^2 yz$. Since we do assume $f(y, z) = y^2 + z^2$, we have $c_0 = 0$. By inspection, we read off

$$\begin{aligned} g(y, z, w) &\equiv zw^2 + z(c_1 z + c_2 w)\, y & (\mathrm{mod}\ y^2) \\ h(y, z, w) &\equiv w^4 + 2w^2(c_1 z + c_2 w)\, y + (c_3 z^2 + \ldots)\, y^2 & (\mathrm{mod}\ y^3) \end{aligned}$$

Thus, in $g(y, z, w)^2$, y^2w^4 does not appear, and y^2z^4 has coefficient c_1^2. On the other hand, in $fh = (y^2 + z^2)h(y, z, w)$, y^2w^4 has coefficient 1 and y^2z^4 has coefficient c_3. Therefore, $D(y, z, w) = fh - g^2 = y^2w^4 + (c_3 - c_1^2)\, y^2z^4 + \ldots$, i.e. $q(0, 0, 1) = 1$, $q(0, 1, 0) = c_3 - c_1^2$.

We now return to the zero $(0, \bar{z}, \bar{w}) \in \mathfrak{Z}(q)$. Since $q(0, 0, 1) = 1$, we must have $\bar{z} \neq 0$, so assume $\bar{z} = 1$. Now use a linear change $x = x' - \lambda(z', w')$, $y = y'$, $z = z'$, $w = w' + \bar{w}z'$ where $\lambda(z', w')$ is a linear form to be determined. We have a new form $p'(x', y', z', w') = p(x' - \lambda(z', w'), y', z', w' + \bar{w}z')$ with

$$\begin{aligned} p'(x', 0, z', w') &= p(x' - \lambda(z', w'), 0, z', w' + \bar{w}z') \\ &= [(x' - \lambda(z', w'))\, z' + (w' + \bar{w}z')^2]^2. \end{aligned}$$

Choosing $\lambda(z', w') = 2\bar{w}w' + \bar{w}^2 z'$, we can guarantee that $p'(x', 0, z', w') = (x'z' + w'^2)^2$. Of course, the coefficient of x'^2 in p' is still $y'^2 + z'^2$. By (2.6), the discriminant of p' with respect to x' is:

$$\begin{aligned} D'(y', z', w') &= D(y', z', w' + \bar{w}z') \\ &= y'^2 q(y', z', w' + \bar{w}z). \end{aligned}$$

For $(y', z', w') = (0, 1, 0)$, the second factor becomes $q(0, 1, \bar{w}) = 0$. Thus, dropping the "primes" altogether, *we may assume that* $q(0, 1, 0) = 0$, i.e. $c_3 = c_1^2$. Now, write $p(x, y, z, w)$ as a polynomial in z; we have, by inspection

$$p(x, y, z, w) = (x^2 + 2c_1 y + c_3 y^2)\, z^2 + 2g^*(x, y, w)\, z + h^*(x, y, w).$$

Since $c_3 = c_1^2$, the coefficient of z^2 is now a perfect square. By (5.4), p is a sum of four squares in $\mathbb{R}[x, y, z, w]$. Q.E.D.

The complete proof of the Case B_1 will ultimately depend on a reduction to the important special case considered in the Proposition above. To achieve this reduction, one more lemma is needed.

Lemma 6.4. *Let* $\bar{p}(x_1, x_2, x_3) = \bar{f}(x_2, x_3)\, x_1^2 + 2\bar{g}(x_2, x_3)\, x_1 + \bar{h}(x_2, x_3) \in P_{3,4}$. *Suppose that for every* $(y_2, y_3) \in \mathbb{R}^2$ *with* $y_2 \neq 0$, *there exists* $y_1 \in \mathbb{R}$ *with* $\bar{p}(y_1, y_2, y_3) = 0$. *If* $\bar{f}$ *has rank* 1, *then* $\bar{p}(x_1, x_2, x_3) = c(x_1, x_2, x_3)^2$. *If* $\bar{f}$ *has rank* 2, *then* $\bar{p} = \bar{f} \cdot (x_1 + \alpha_2 x_2 + \alpha_3 x_3)^2$ *for some* $\alpha_2, \alpha_3 \in \mathbb{R}$.

Proof. The hypothesis implies $|\mathfrak{Z}(\bar{p})| = \infty$, so by (3.5) $\bar{p}$ has a factorization $c^2 d$ $(c, d \in \mathbb{R}[x_1, x_2, x_3])$. First assume c is quadratic; say $d = 1$. Then $c = x_1 r(x_2, x_3) + s(x_2, x_3)$, so $\bar{f} = r(x_2, x_3)^2$ has rank $\leqq 1$. Now assume c is linear:

$$\bar{p}(x_1, x_2, x_3) = (\alpha x_1 + \beta x_2 + \gamma x_3)^2\, d(x_1, x_2, x_3).$$

If $\alpha = 0$, then for every (y_2, y_3) not proportional to $(\gamma, -\beta)$, with $y_2 \neq 0$, there exists y_1 with $d(y_1, y_2, y_3) = 0$, so $|\mathfrak{Z}(d)| = \infty$. Since $d \in P_{3,2}$, this implies that d is a square, so we are back to the first case. If $\alpha \neq 0$, then, since $\bar{p}$ is at most quadratic in x_1, $d = d(x_2, x_3)$. We have $\alpha^2 d = \bar{f}$ so $\bar{p} = \bar{f} \cdot (x_1 + \beta x_2/\alpha + \gamma x_3/\alpha)^2$. Q.E.D.

We are now in a position to settle completely the outstanding "Case B_1":

Theorem 6.5. *Let* $p(x, y, z, w) = f(y, z)\, x^2 + 2g(y, z, w)\, x + h(y, z, w) \in P_{4,4}$ *where* $f = y^2 + z^2$. *Suppose the discriminant* $D = fh - g^2$ *factors into* $k^2 q$ *where* $k = \alpha y + \beta z + \gamma w \not\equiv 0$ $(\alpha, \beta, \gamma \in \mathbb{R})$. *Then p is a sum of six squares of quadratics.*

Proof. There are two cases, depending on whether $\gamma = 0$ or $\gamma \neq 0$.

Case 1. $\gamma \neq 0$. After the linear change $w' = \alpha y + \beta z + \gamma w$ (with x, y, z fixed), we may assume that $D(y, z, w) = w^2 q$. By (4.10), for every $(\bar{y}, \bar{z}) \neq (0, 0)$, there is an $\bar{x}$ such that $(\bar{x}, \bar{y}, \bar{z}, 0) \in \mathfrak{Z}(p)$. Let

$$p_1(x, y, z) = p(x, y, z, 0) = f(y, z)x^2 + 2g(y, z, 0)x + h(y, z, 0).$$

By the preceding lemma, we have $p_1(x, y, z) = f(y, z)(x + \alpha_2 y + \alpha_3 z)^2$. Using the change $x' = x + \alpha_2 y + \alpha_3 z$ (with y, z, w fixed), we may thus assume $p(x, y, z, 0) = x^2 f(y, z)$. Hence $g(y, z, 0) = h(y, z, 0) = 0$, so $w | g$ and $w^2 | h$ (see (2.3)). Now

$$p(x, y, z, w) = f(y, z)x^2 + 2xwg_1(y, z, w) + w^2 h_1(y, z, w)$$

may be viewed as a quadratic polynomial in $\{y, z\}$ with "coefficients" which are forms in $\{x, w\}$. By a theorem cited earlier (see (5.6)), p is a sum of six squares in $\mathbb{R}[x, y, z, w]$.

Case 2. $\gamma = 0$, i.e. $k = \alpha y + \beta z$. After an orthogonal change $y' = \alpha y + \beta z$, $z' = \beta y - \alpha z$ (and scaling x if necessary), we may assume that $k = y$ without perturbing $f = y^2 + z^2$. We now have $D(0, z, w) = 0$, so for every $(\bar{z}, \bar{w})$ with $\bar{z} \neq 0$, there exists $\bar{x}$ so that $(\bar{x}, 0, \bar{z}, \bar{w}) \in \mathfrak{Z}(p)$. Thus, Lemma 6.4 can be applied to

$$p_2(x, z, w) = p(x, 0, z, w) = z^2 x^2 + 2g(0, z, w)x + h(0, z, w).$$

The conclusion drawn from the Lemma is that p_2 is a perfect square, so by inspection, it is $(xz + \delta_1 z^2 + \delta_2 zw + \delta_3 w^2)^2$ for suitable $\delta_i \in \mathbb{R}$. After a further change $x' = x + \delta_1 z + \delta_2 w$ (fixing y, z, w), we may assume that $p_2(x, z, w) = (xz + \delta_3 w^2)^2$. Two (final) cases may arise, depending on whether $\delta_3 = 0$ or $\delta_3 \neq 0$.

Subcase (i). $\delta_3 = 0$. Here, $p(x, 0, z, w) = z^2 x^2$, so $g(0, z, w) = h(0, z, w) = 0$. As before, we can write

$$p(x, y, z, w) = x^2(y^2 + z^2) + 2xyg_2(y, z, w) + y^2 h_2(y, z, w).$$

Since h_2 is at most quadratic in w, and $(y^2 + z^2)h_2 \geqq g_2(y, z, w)^2$, g_2 is at most linear in w. Therefore, p is a quadratic polynomial in $\{x, w\}$ with coefficients which are forms in $\{y, z\}$. Appealing once again to (5.6), p is a sum of six squares in $\mathbb{R}[x, y, z, w]$.

Subcase (ii). $\delta_3 \neq 0$. Scaling w and taking $x' = -x$ if necessary, we may assume that $p_2(x, z, w) = (xz + w^2)^2$. Now we are reduced to the situation investigated in Proposition 6.1. Q.E.D.

The proof of the Main Theorem stated in the Introduction is now complete.

7. A Theorem on Biforms

In the last two sections, we have repeatedly used Theorem 5.6. We shall now furnish a proof of this (and in fact a somewhat more refined) result. The proof will be based on a certain known result on biforms, stated in (7.1) below. Recall that a *biform of bidegree* (m_1, m_2) in the two sets of variables $\{y_1, \ldots, y_r\}$, $\{x_1, \ldots, x_n\}$ is a polynomial in $\mathbb{R}[\underline{y}, \underline{x}]$ which is an m_1-ic in $\underline{y}$ when $\underline{x}$ is viewed as constant, and an m_2-ic in $\underline{x}$ when $\underline{y}$ is viewed as constant. The following theorem concerns the situation when $r = 2$ and $m_2 = 2$.

Theorem 7.1. *Let $p(y, z; x_1, \ldots, x_n) = \sum_{i,j} a_{ij}(y, z)x_i x_j$ be a biform of bidegree $(m, 2)$. If p is psd, then p is a sum of $2n$ squares of biforms.*

This result has appeared before in [10, 17, 18, 21]. For the sake of completeness, a new proof will be given here. We believe this new proof will be useful since it is shorter and completely constructive, which seems to be an advantage considering that the result has applications to diverse areas such as the theory of nonlinear regularization, optimal control, and differential games. Also, the very special role played by (7.1) in the study of psd "multi-forms" will be apparent in Sect. 8.

We shall first prove two useful lemmas.

Lemma 7.2. *Let D_i, E_i be vectors in $\mathbb{R}^{2d}$ $(1 \leqq i \leqq n)$ such that*

(7.3) $$D_i \cdot D_j = E_i \cdot E_j, \quad D_i \cdot E_j + D_j \cdot E_i = 0 \quad \text{for all } i, j.$$

Then, after an orthonormal change of basis, we can arrange that

(7.4) $$D_i = (s_{i1}, t_{i1}, \ldots, s_{id}, t_{id})$$
$$E_i = (-t_{i1}, s_{i1}, \ldots, -t_{id}, s_{id}).$$

Proof. We induct on n. The case $n = 0$ is vacuous, so assume $n \geqq 1$. By hypothesis, D_1 and E_1 are orthogonal with the same length, say s. If $s = 0$, we are done by induction, so assume $s \neq 0$. Using $\frac{1}{s} D_1$ and $\frac{1}{s} E_1$ as part of an orthonormal basis, we can arrange that

$$D_1 = (s, 0, 0, \ldots, 0), \; D_i = (s_{i1}, t_{i1}, \bar{D}_i) \quad (i \geqq 2),$$
$$E_1 = (0, s, 0, \ldots, 0), \; E_i = (s'_{i1}, t'_{i1}, \bar{E}_i) \quad (i \geqq 2),$$

where $\bar{D}_i, \bar{E}_i \in \mathbb{R}^{2(d-1)}$. Putting $j = 1$ in (7.3), we see that $t'_{i1} = s_{i1}$ and $s'_{i1} = -t_{i1}$ for $i \geqq 2$. With this information, (7.3) yields the same inner product equations for $\bar{D}_i, \bar{E}_i$ $(2 \leqq i \leqq n)$ so the induction proceeds. Q.E.D.

Lemma 7.5. *Let $a_{ij}(y, z)$ be forms of degree m, with $a_{ij} = a_{ji}$ $(1 \leqq i, j \leqq n)$ and let $a(y, z)$ be a nonzero psd form. Suppose there exist vectors of forms $A_i(y, z) = (a_i^{(1)}(y, z), \ldots, a_i^{(2d)}(y, z))$ such that $a(y, z)a_{ij}(y, z) = A_i(y, z) \cdot A_j(y, z)$ for all i, j. Then there exist vectors of forms $B_i(y, z) = (b_i^{(1)}(y, z), \ldots, b_i^{(2d)}(y, z))$ such that $a_{ij}(y, z) = B_i(y, z) \cdot B_j(y, z)$ for all i, j.*

Proof. The strategy of the proof is to "peel off" the psd factors of $a(y, z)$ one at a time. Thus, we need only treat the following two cases: (1) $a(y, z) = (\alpha y + \beta z)^2$, (2) $a(y, z)$ is an irreducible psd quadratic form. The first case is easily handled by Proposition 2.3: taking $i = j$, the hypothesis implies that each coordinate of $A_i(y, z)$ is divisible by $\alpha y + \beta z$. In the second case, we may assume, by a linear change, that $a(y,z)=y^2+z^2$. From $a(y,z)\,a_{ii}(y,z)=\sum_k a_i^{(k)}(y,z)^2$, we see that $a_i^{(k)}(y,z)$ are forms of degree $r=\frac{1}{2}(m+2)$. Following the idea used in (5.11), write

$$A_i(y,z)=(y^2+z^2)\,C_i(y,z)+y^{r-1}zD_i+y^rE_i \tag{7.6}$$

where D_i, $E_i\in\mathbb{R}^{2d}$, and $C_i(y,z)$ is a vector of forms of degree $r-2$. For $y=1$, $z=\sqrt{-1}$, we have $A_i(1,\sqrt{-1})=E_i+\sqrt{-1}\,D_i$ and from $(y^2+z^2)\,a_{ij}(y,z)=A_i(y,z)\cdot A_j(y,z)$, we get $(E_i+\sqrt{-1}\,D_i)\cdot(E_j+\sqrt{-1}\,D_j)=0$. Comparing the real and imaginary parts, we have the equations (7.3), so after an orthonormal change, we may assume that the vectors D_i, E_i are as in (7.4). Note that an orthonormal change preserves the equations $a\cdot a_{ij}=A_i\cdot A_j$. Substituting (7.6) into these equations and cancelling y^2+z^2, we get

$$a_{ij}=(y^2+z^2)\,C_i\cdot C_j+y^{r-1}z(C_i\cdot D_j+C_j\cdot D_i)$$
$$+y^r(C_i\cdot E_j+C_j\cdot E_i)+y^{2r-2}D_i\cdot D_j.$$

Define vectors of forms $F_i=F_i(y,z)$ by

$$F_i=(y\,C_i^{(2)}+z\,C_i^{(1)},\,z\,C_i^{(2)}-y\,C_i^{(1)},\,\dots,\,y\,C_i^{(2d)}+z\,C_i^{(2d-1)},\,z\,C_i^{(2d)}-y\,C_i^{(2d-1)})$$

so $F_i\cdot F_j=(y^2+z^2)\,C_i\cdot C_j$. Now set $B_i(y,z)=F_i(y,z)+y^{r-1}\cdot D_i$. These B_i's will satisfy $a_{ij}=B_i\cdot B_j$ provided that

$$F_i\cdot D_j=C_i\cdot(yE_j+zD_j).$$

To check this, it suffices to show, by symmetry, that $C_i^{(1)}$ and $C_i^{(2)}$ occur with the same coefficient in the two sides of the equation. This is checked since, by inspection,

$$(y\,C_i^{(2)}+z\,C_i^{(1)})\,s_{i1}+(z\,C_i^{(2)}-y\,C_i^{(1)})\,t_{i1}$$
$$=C_i^{(1)}(-yt_{i1}+zs_{i1})+C_i^{(2)}(zt_{i1}+ys_{i1}).\qquad \text{Q.E.D.}$$

It is now easy to give the

Proof of (7.1). We induct on n. If $n=1$, we have $p(y,z;x_1)=a_{11}(y,z)\,x_1^2$; since $a_{11}(y,z)$ is psd, it is a sum of two squares of forms and so is p. For $n>1$, we may assume that $a_{ij}=a_{ji}$ and write

$$p=a_{11}(y,z)\,x_1^2+2\Big(\sum_{j\geqq 2}a_{1j}x_j\Big)x_1+\bar{p}(y,z;x_2,\dots,x_n).$$

We may assume that $a_{11}(y,z)\neq 0$, for otherwise $p=\bar{p}(y,z;x_2,\dots,x_n)$ and we are done by induction. Now $a_{11}(y,z)$ is psd, and so is the following discriminant (w.r.t. x_1):

$$D=a_{11}(y,z)\,\bar{p}(y,z;x_2,\dots,x_n)-\Big(\sum_{j\geqq 2}a_{1j}x_j\Big)^2.$$

By the inductive hypothesis, D is a sum of $2n-2$ squares of biforms in $\{y,z;x_2,\dots,x_n\}$. As in (4.11), we have

$$a_{11}(y,z)\cdot p=\Big(a_{11}(y,z)\,x_1+\sum_{j\geqq 2}a_{1j}x_j\Big)^2+D(y,z;x_2,\dots,x_n)$$

which is a sum of $2n-1$, in particular $2n$, squares of biforms. Say

$$a_{11}(y,z)\cdot p=\sum_{k=1}^{2n}\Big(\sum_{i=1}^{n}a_i^{(k)}(y,z)\,x_i\Big)^2. \tag{7.7}$$

Let $A_i(y,z)$ be the vector of forms $(a_i^{(1)}(y,z),\dots,a_i^{(2n)}(y,z))$. Then a comparison of the two sides of (7.7) yields $a_{11}\cdot a_{ij}=A_i\cdot A_j$. By (7.5), we have $a_{ij}=B_i\cdot B_j$ for suitable vectors of forms $B_i(y,z)=(b_i^{(1)}(y,z),\dots,b_i^{(2n)}(y,z))$. Thus, we have $p=\sum_{k=1}^{2n}\Big(\sum_{i=1}^{n}b_i^{(k)}(y,z)\,x_i\Big)^2$. Q.E.D.

In general, we cannot improve the bound $2n$ for the number of squares needed to express p. However, in the case $m=2$ (i.e. when $p(y,z;x_1,\dots,x_n)$ is a *biquadratic form*), the bound can be improved to $\left[\sqrt{3}\,n+\frac{\sqrt{3}-1}{2}\right]$: see [7].

Our next goal is to derive a version of (7.1) which allows non-homogeneity in the variables $\{x_1,\dots,x_n\}$. First we make an elementary observation:

Lemma 7.8. *If $p(x_1,\dots,x_n)=p_d(x_1,\dots,x_n)+\dots+p_e(x_1,\dots,x_n)$ is a psd polynomial arranged in homogeneous components of increasing degree, then p_d and p_e are psd forms.*

Proof. Note that $p_d(x_1,\dots,x_n)=\lim_{\lambda\to 0}p(\lambda x_1,\dots,\lambda x_n)/\lambda^d$ and $p_e(x_1,\dots,x_n)=\lim_{\lambda\to\infty}p(\lambda x_1,\dots,\lambda x_n)/\lambda^e$.

We can now prove the following result which was stated in part as (5.6):

Theorem 7.9. *Suppose $p(y,z;x_1,\dots,x_n)=\sum_{i,j}a_{ij}(y,z)\,x_ix_j+2\sum a_i(y,z)\,x_i+a(y,z)$ is psd, where $a_{ij}(y,z)$, $a_i(y,z)$ and $a(y,z)$ are forms of degrees r,t and s. Then $r+s=2t$, and p is a sum of $2(n+1)$ squares in $\mathbb{R}[y,z;x_1,\dots,x_n]$.*

Proof. Fix $(y,z;x_1,\dots,x_n)$ and consider $p(y,z;\lambda x_1,\dots,\lambda x_n)$. From $0\leqq p(y,z;\lambda x_1,\dots,\lambda x_n)=\lambda^2\sum a_{ij}x_ix_j+2\lambda\sum a_ix_i+a$, we have

$$D(y,z;x_1,\dots,x_n):=a(y,z)\sum a_{ij}(y,z)\,x_ix_j-\Big(\sum a_i(y,z)\,x_i\Big)^2\geqq 0.$$

This is a polynomial with homogeneous components of degree $r+s+2$ and $2(t+1)$. If *all* the $a_i(y,z)$ are zero, we may agree that their degrees are $(r+s)/2$ (r and s are both even since $a_{ij}\geqq 0$, $a\geqq 0$). Now assume $a_i(y,z)$ are *not* all zero. Then we must have $r+s=2t$ for otherwise $-(\sum a_i(y,z)\,x_i)^2$ would emerge as one of the "end" forms of $D(y,z;x_1,\dots,x_n)$, contradicting Lemma 7.8. We have now two cases, depending on whether $s\geqq r$ or $r\geqq s$.

Case 1. $s\geqq r$. Then $d:=t-r\geqq 0$. Introduce a new variable x_{n+1} and let

$$q(y,z;x_1,\dots,x_{n+1})=x_{n+1}^2\,p\left(y,z;\frac{x_1y^d}{x_{n+1}},\dots,\frac{x_ny^d}{x_{n+1}}\right)$$
$$=y^{2d}\sum^{n}a_{ij}(y,z)\,x_ix_j+y^d\sum^{n}a_i(y,z)\,x_ix_{n+1}+a(y,z)\,x_{n+1}^2$$

which is a biform of bidegree $(d+t, 2)$ in $\{y, z; x_1, \ldots, x_{n+1}\}$. This biform is $\geqq 0$ for all $\{y, z; x_1, \ldots, x_{n+1}\}$ with $x_{n+1} \neq 0$, so it is psd by continuity. By Theorem 7.1, q is a sum of $2(n+1)$ squares, and therefore so is

$$q(y, z; x_1, \ldots, x_n, y^d) = y^{2d} p(y, z; x_1, \ldots, x_n).$$

By repeated use of Proposition 2.3, it follows that p itself is a sum of $2(n+1)$ squares.

Case 2. $s \leqq r$. Then $e := t - s \geqq 0$, and we can apply a similar argument to the psd biform

$$q'(y, z; x_1, \ldots, x_{n+1}) = y^{2e} x_{n+1}^2 p\left(y, z; \frac{x_1}{y^e x_{n+1}}, \ldots, \frac{x_n}{y^e x_{n+1}}\right)$$
$$= \sum^n a_{ij}(y, z) x_i x_j + y^e \sum^{n} a_i(y, z) x_i x_{n+1} + y^{2e} a(y, z) x_{n+1}^2$$

with bidegree $(e+t, 2)$. Here q' is a sum of $2(n+1)$ squares, and, therefore, so is $y^{2e} q'\left(y, z; x_1, \ldots, x_n, \frac{1}{y^e}\right) = y^{2e} p$, and so is p. Q.E.D.

Remark. When we used Theorem 7.9 in Sect. 5 and 6, $p(y, z; x_1, \ldots, x_n)$ is itself a *form* in $\{y, z, x_1, \ldots, x_n\}$ (in fact of degree 4). This corresponds to the situation $t = r+1$, $s = r+2$; in particular, we are in Case 1 with $d = 1$.

8. Multiforms as Sums of Squares

The notion of a biform can be easily generalized to that of a multi-form: Let $\underline{x} = (x_1, \ldots, x_{n_1})$, $\underline{y} = (y_1, \ldots, y_{n_2})$, $\ldots$, $\underline{z} = (z_1, \ldots, z_{n_r})$ be independent sets of variables. By a multiform of type $(n_1, \ldots, n_r; m_1, \ldots, m_r)$, we shall mean a polynomial $p = p(\underline{x}, \underline{y}, \ldots, \underline{z})$ which is an m_1-ic in $\underline{x}$ when $\underline{y}, \ldots, \underline{z}$ are viewed as constants, $\ldots$, and an m_r-ic in $\underline{z}$ when $\underline{x}, \underline{y}, \ldots$ are viewed as constants. Just as for forms, we can define $\Delta(n_1, \ldots, n_r; m_1, \ldots, m_r)$ to be the set of psd multiforms of type $(n_1, \ldots, n_r; m_1, \ldots, m_r)$ which are *not* sums of squares of polynomials (or, equivalently, multiforms). The main question studied in Hilbert [13] has an obvious analogue for multiforms: *For which tuples* $(n_1, \ldots, n_r; m_1, \ldots, m_r)$ *is it true that* $\Delta(n_1, \ldots, n_r; m_1, \ldots, m_r) = \varnothing$?

Henceforth, we shall impose two mild restrictions on the type $(n_1, \ldots, n_r; m_1, \ldots, m_r)$ of multiforms we study. We shall always assume $m_1, \ldots, m_r$ are even, and $n_1, \ldots, n_r$ are $\geqq 2$. The first restriction is harmless because if $p \neq 0$ is a psd multiform of type $(n_1, \ldots, n_r; m_1, \ldots, m_r)$, then $m_1, \ldots, m_r$ are necessarily even. On the other hand, if one of the n_i is 1, say $n_1 = 1$, then p is just $x_1^{m_1}$ times a psd multiform of type $(n_2, \ldots, n_r; m_2, \ldots, m_r)$ and it suffices to study the latter form.

Lemma 8.1. *If* $\Delta(n_1, \ldots, n_r; m_1, \ldots, m_r) = \varnothing$, *then for any subset* $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}$, *we have also* $\Delta(n_{i_1}, \ldots, n_{i_s}; m_{i_1}, \ldots, m_{i_s}) = \varnothing$.

Lemma 8.2. *Suppose* $(n'_1, \ldots, n'_r; m'_1, \ldots, m'_r) \geqq (n_1, \ldots, n_r; m_1, \ldots, m_r)$ (i.e. $n'_i \geqq n_i$ *and* $m'_i \geqq m_i$ *for all* i). *Then* $\Delta(n_1, \ldots, n_r; m_1, \ldots, m_r) \neq \varnothing$ *implies* $\Delta(n'_1, \ldots, n'_r; m'_1, \ldots, m'_r) \neq \varnothing$.

The proofs are easy (cf. the arguments for (2.2) and (3.9)), and will be left to the reader.

We shall now examine a few basic cases for which $\Delta(n_1, \ldots, n_r; m_1, \ldots, m_r)$ is non-empty. By the second lemma above, one basic case will generate many other new cases.

Lemma 8.3. *There exist forms in* $\Delta(3,3;\ 2,2)$, $\Delta(2,2;\ 4,4)$, $\Delta(2,3;\ 2,4)$ *and* $\Delta(2,2,2;\ 2,2,2)$.

Proof. The first case is already in [C], where it was shown that the biquadratic form

$$p(x_1, x_2, x_3; y_1, y_2, y_3) = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2$$
$$- 2x_1 x_2 y_1 y_2 - 2x_1 x_3 y_1 y_3 - 2x_2 x_3 y_2 y_3$$

lies in $\Delta(3,3;2,2)$. For the remaining cases, define the following "variations" of the Q and the S in (2.1):

$$p_1(x_1, x_2; y_1, y_2) = x_1^4 y_1^2 y_2^2 + x_1^2 x_2^2 y_2^4 + x_2^4 y_1^4 - 3x_1^2 x_2^2 y_1^2 y_2^2,$$
$$p_2(x_1 x_2; y_1, y_2, y_3) = x_1^2 y_1^4 + x_1^2 y_1^2 y_2^2 + x_2^2 y_1^2 y_2^2 + x_2^2 y_3^4 - 4x_1 x_2 y_1^2 y_2 y_3,$$
$$p_3(x_1, x_2; y_1, y_2; z_1, z_2) = x_1^2 y_1^2 z_1^2 + x_1^2 y_2^2 z_2^2 + x_2^2 y_1^2 z_2^2 + x_2^2 y_2^2 z_1^2$$
$$- 4x_1 x_2 y_1 y_2 z_1 z_2.$$

Each is psd by the arithmetic-geometric inequality. We claim that *none of these is a sum of squares of polynomials*. In fact, say $p_1 = \sum h_i^2$. By letting $y_1 = x_1$, we have

$$p_1(x_1, x_2; x_1, y_2) = \sum h_i(x_1, x_2; x_1, y_2)^2 = x_1^2 S(x_1, y_2, x_2)$$

which implies that the form S in (2.1) is a sum of squares of polynomials, a contradiction. Thus, $p_1 \in \Delta(2,2;4,4)$.

The proofs that $p_2 \in \Delta(2,3;2,4)$ and $p_3 \in \Delta(2,2,2;2,2,2)$ are similar, upon noting that $p_2(y_3, y_2; y_1, y_2, y_3) = S(y_1, y_3, y_2)$ and that $p_3(x_1, y_1; y_1, z_1;\ z_1, x_1) = S(x_1, z_1, y_1)$. Q.E.D.

In spite of the above examples, we did know about one positive case: by (7.1), *a psd biform* $\sum_{i,j}^{n} a_{ij}(y,z) x_i x_j$ *is always a sum of squares of polynomials*, i.e. $\Delta(2, n; m, 2) = \varnothing$. It turns out, moreover, that this is essentially the only "good" case, i.e. for any other type of multiforms, psd is a weaker condition than being a sum of squares of polynomials. Thus, even Hilbert's beautiful discovery [13] that $\Delta_{3,4} = \varnothing$ has no legacy for multiforms.

Theorem 8.4. *Suppose* $r \geqq 2$, $n_i \geqq 2$ *and* $m_i = even \geqq 2$. *Then* $\Delta(n_1, \ldots, n_r; m_1, \ldots, m_r) = \varnothing$ *iff* $r = 2$, *and* $(n_1, \ldots, n_r; m_1, \ldots, m_r)$ *equals* $(2, n_2; m_1, 2)$ *or* $(n_1, 2; 2, m_2)$.

Proof. As we have observed above, the "if" part is Theorem 7.1. For the converse, *assume that* $\Delta(n_1, \ldots, n_r; m_1, \ldots, m_r) = \varnothing$. If $r \geqq 3$, this would contradict $\Delta(2,2,2;\ 2,2,2) \neq \varnothing$ by repeated use of (8.1) and (8.2). Thus, we must have $r = 2$. Assume, without loss of generality, that $n_1 \leqq n_2$. If $n_1 \geqq 3$, then $(n_1, n_2; m_1, m_2) \geqq (3,3;2,2)$; this contradicts $\Delta(3,3;2,2) \neq \varnothing$ by (8.2), so we must have $n_1 = 2$. By (8.1), we have $\Delta(n_2; m_2)$ $(= \Delta_{n_2, m_2}) = \varnothing$, so (n_2, m_2) is $(3,4)$, $(n_2, 2)$ or $(2, m_2)$. In the first case, we have $(n_1, n_2; m_1, m_2) = (2,3; m_1, 4) \geqq (2,3;2,4)$, which contradicts $\Delta(2,3;2,4) \neq \varnothing$. In the second case, we have $(n_1, n_2; m_1, m_2) = (2, n_2; m_1, 2)$, which is as predicted. In the third case, we have $\Delta(2,2;m_1,m_2) = \varnothing$. Since $\Delta(2,2;4,4) \neq \varnothing$, we must have $m_1 = 2$ or $m_2 = 2$, which are also as predicted. Q.E.D.

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