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EXTREMAL PSD FORMS WITH FEW TERMS

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1. Introduction

A psd form is a homogeneous polynomial p for which $p(x_1, \cdots, x_n) \geq 0$. Let $P_{n,2m}$ denote the convex cone of all psd forms in n variables with degree $2m$ and $\Sigma_{n,2m}$ denote the convex cone of all such forms which can be written as a sum of squares of forms. (It is clear that a sum of squares is psd.)

Hilbert [7] showed in 1888 that $\Sigma_{n,2m} = P_{n,2m}$ if and only if $(n, 2m)$ is $(n, 2)$, $(2, 2m)$ or $(3, 4)$ and that $\Sigma_{n,2m} \subset P_{n,2m}$ otherwise. He gave a method for constructing psd forms which are not a sum of squares, but did not carry it out. In fact, no explicit form in $P_{n,2m} - \Sigma_{n,2m}$ was exhibited until 1967.

Motzkin [9] demonstrated that

$$M(x_1, x_2, x_3) = x_1^6 + x_2^4x_3^2 + x_2^2x_3^4 - 3x_1^2x_2^2x_3^2$$

is such a form; the simplicity of M contrasts with the complexity of Hilbert's construction. Robinson [11] simplified Hilbert's method and provided several more such forms. Very recently Choi and Lam [1], [2], [3] have looked at $P_{n,2m}$ as a cone and searched for extremal elements. They proved that M, a number of Robinson's forms, and

$$S(x_1, x_2, x_3) = x_1^4x_2^2 + x_2^4x_3^2 + x_1^2x_3^4 - 3x_1^2x_2^2x_3^2$$

are all extremal psd forms in this sense.

The simplicity of M and S motivate this paper, in which all extremal psd forms with four or fewer terms (which are not sums of squares) will be described.

2. Preliminaries

Identify a form in n variables of degree m with the N-tuple of its coefficients ordered in any predetermined manner, where $N(n, m) = \binom{n+m-1}{n-1}$, and pull back the ordinary topology on $\mathbb{R}^N$. Then $P_{n,2m}$ is a closed cone. Ellison [5] has shown that $\Sigma_{n,2m}$ is also a closed cone. If f is extremal in $P_{n,2m}$ as a cone and $f = g_1 + g_2$, g_i psd, then $g_i = \lambda_i f$; if f is extremal in $\Sigma_{n,2m}$, then f is a perfect square. Let $E_{n,2m}$ consists of the extremal forms in $P_{n,2m}$ which are not perfect squares. We shall include the condition "not a perfect square" in any further use of the word "extremal". If $h = x_1^{a_1} \cdots x_n^{a_n}$, $\Sigma a_i = k$, and f is in $E_{n,2m}$ then h^2f is in $E_{n,2m+2k}$: if $x_j^{2a_j}$ divides $g_1 + g_2$, g_i psd, then $x_j^{2a_j}$ divides each g_i.

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Consider any change of variables $x_i = x_i(y_1, \cdots, y_s)$ in which each x_i is a form in $y_1, \cdots, y_s$. If $f(x_1, \cdots, x_n)$ is psd or a sum of squares then the induced form $f(y_1, \cdots, y_s)$ will also be psd or a sum of squares. Not being a sum of squares, however, is not necessarily preserved. For example,

$$S(x_1x_2^2, x_2x_3^2, x_1^2x_3) = x_1^4x_2^4x_3^4(x_1^6 + x_2^6 + x_3^6 - 3x_1^2x_2^2x_3^2),$$

which is a sum of squares—see Hardy, Littlewood and Polya [6] p. 55. In the special case that $s = n$ and the change of variables is linear and invertible, $P_{n,2m}$ and $\Sigma_{n,2m}$ are left invariant. So, therefore, is $E_{n,2m}$ and we shall frequently use this fact; two forms will be considered the same if they are related by an invertible change of variables.

A typical monomial $x_1^{r_1} \cdots x_n^{r_n}$, $\Sigma r_i = 2m$, will be written x^r with the understanding that t denotes a single real variable. A typical form is then $\Sigma a_i x^{r_i}$ where $r_i = (r_{i1}, \cdots, r_{in})$.

A lattice point in $\mathbb{R}^n$ is a point all of whose coordinates are integers; the set of lattice points in $\mathbb{R}^n$ is written $\mathbb{Z}^n$. The triangle with vertices p_1, p_2, p_3 will be written $T(p_1, p_2, p_3)$. Suppose the p_i's are lattice points and there are j lattice points (other than the vertices) on the edges of T and k lattice points in the open interior. Then by Pick's Theorem (see Coxeter [4] p. 208), the area of T is $(j + 2k + 1)/2$.

Suppose now that P is a plane which lies in $\mathbb{R}^n$ and $L = P \cap \mathbb{Z}^n$; L could be vacuous, one point or a lattice of one or two dimensions. We are interested in this last case. Let $x_0 \in L$ be arbitrarily selected. In an infinite number of ways x_1 and x_2 may be chosen in $\mathbb{Z}^n$ so that x is in L if and only if $x = x_0 + a_1x_1 + a_2x_2$ for $a_i \in \mathbb{Z}$. This induces an isomorphism ϕ between L and $\mathbb{Z}^2$, $\phi(x) = (a_1, a_2)$; ϕ depends on the choice of x_1 and x_2 and is affine so that convex combinations are preserved. For a triangle $T(p_1, p_2, p_3)$, $p_i \in L$, define $A(T)$ to be the area of the triangle $T(\phi(p_1), \phi(p_2), \phi(p_3))$ in $\mathbb{R}^2$. Since the area of any fundamental parallelogram in $\mathbb{Z}^2$ is 1 (see Coxeter [4], p. 208), $A(T)$ does not depend on the choice of x_1 and x_2. For any set X in $\mathbb{R}^n$ define $\lambda X = \{\lambda x\colon x \in X\}$, so $A(\lambda T) = \lambda^2 A(T)$. We shall use this along with $A(T)$ to enumerate the lattice points in T.

Finally, the arithmetic-geometric inequality (AGI) is well known. We shall use the following version of it: if $\Sigma\lambda_i = 1$, $\lambda_i \geq 0$, $x_i \geq 0$ then

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \geq x_1^{\lambda_1} \cdots x_n^{\lambda_n}$$

with equality only if $x_1 = \cdots = x_n$.

3. Cages and frames

Let $p(x_1, \cdots, x_n) = \Sigma a_i x^{r_i}$ be a form with degree $2m$; we assume that $a_i \neq 0$ and that the r_i's are distinct n-tuples. The cage of p, $C(p)$, is the convex hull of the r_i's, viewed as vectors in $\mathbb{R}^n$ lying in the hyperplane $u_1 + \cdots + u_n = 2m$. The frame of p, $F(p)$, consists of the lattice points in $C(p)$; $F(p) = C(p) \cap \mathbb{Z}^n$. The reduced frame of p consists of all lattice points in $C(p)$ with even coordinates and is written $2R(p)$; $R(p)$ consists of the lattice points in $\frac{1}{2}C(p)$. The extreme points of $C(p)$, $E(p)$, form a subset of the r_i's and so $E(p) \subseteq F(p)$. If $C(p)$ is a k-dimensional object in $\mathbb{R}^n$ then $F(p)$ will be a "k-dimensional" subset of a lattice.

LEMMA. *For any form* $p = \Sigma a_i x^{r_i}$, *$C(p)$ lies in the half-space* $b \cdot u = b_1 u_1 + \cdots + b_n u_n \leq d$ *if and only if* $\lim_{t \to \infty} |t^{-d} p(t)| < \infty$ *for any substitution* $x_j = c_j t^{b_j}$. *For any form* p, $C(p^2) = 2C(p)$. *If p is* psd *and r_i is in $E(p)$, then $a_i > 0$ and r_i is an even vector.*

Proof. Fix a substitution and let $b = (b_1, \cdots, b_n)$ and $c = (c_1, \cdots, c_n)$; then $p(t) = \Sigma a_i c^{r_i} t^{b \cdot r_i}$. If $C(p)$ lies in the given half-space then $b \cdot r_i \leq d$ so $t^{-d} p(t)$ is bounded as $t \to \infty$. Conversely, suppose $t^{-d} p(t)$ is bounded but $b \cdot r_i > d$. Assume $d' = b \cdot r_1 = \cdots = b \cdot r_s > b \cdot r_j$ for $j > s$ where $d' > d$. Then $0 = \lim_{t \to \infty} t^{-d'} p(t) - \Sigma' a_i c^{r_i}$, a form in the c's which vanishes. Since $a_i \neq 0$ and the r_i's are distinct, this is a contradiction.

Since a closed convex set in $\mathbb{R}^n$ is the intersection of the closed half-spaces which contain it, and

$$|t^{-2d} p^2(t)| = |t^{-d} p(t)|^2, \; C(p^2) = 2C(p).$$

Finally, suppose p is psd and r_i is extremal; choose b so that $b \cdot r_i = d > b \cdot r_j$ for $j \neq i$. Let $x_j = \epsilon_j t^{b_j}$, $\epsilon_j = \pm 1$; then $0 \leq \lim_{t \to \infty} t^{-d} p(t) = a_i \epsilon^{r_i}$, hence $a_i > 0$ and every r_{ij} is even. ∎

THEOREM 1. *For any* psd *forms f and g, $C(f + g) \supseteq C(f)$; if $f = \Sigma g_j^2$ then* $C(g_j) \subseteq \frac{1}{2} C(f)$.

Proof. Write $h = f + g$; since $h(x) \geq f(x) \geq 0$, absolute values are unnecessary and

$$\lim_{t \to \infty} t^{-d} h(t) < \infty \quad \text{implies} \quad \lim_{t \to \infty} t^{-d} f(t) < \infty,$$

so every half-space containing $C(f + g)$ contains $C(f)$. The theorem follows upon taking intersections. If $f = \Sigma g_j^2$ then $2C(g_j) = C(g_j^2) \subseteq C(f)$. ∎

Since the extreme points of $C(f)$ are in $F(f)$, all inclusion results for cages also apply to frames. Cages are really a fancy way of viewing the degree of vanishing at the unit vectors; for example, if $(2m, 0, \cdots, 0)$ is not in $C(f)$, then f vanishes at $(1, 0, \cdots, 0)$.

4. Finding the simplest case

We wish to determine the simplest elements in $E_{n,2m}$; that is, the extremal forms with the fewest number of terms. Suppose

$$p(x) = \sum_{i=1}^{k} a_i x^{r_i}$$

and every r_i is in $E(p)$. Then by Theorem 1, each r_i is even and $a_i > 0$ so that p, as it stands, is a sum of squares. Thus an extremal p must have at least one non-extremal r_i. Suppose $F(p)$ is one-dimensional, $E(p) = \{r_1, r_k\}$ (by re-indexing if necessary) and that $r_k - r_1 = 2ds$ where $2d$ is the greatest common divisor of the $(r_{kj} - r_{1j})$'s. If $r_j = r_1 + c_j s$, then $0 \le c_j \le 2d$ and $p(x) = x^{r_1} \Sigma a_i y^{c_i}$ where $y = x^s$. By the choice of $2d$, at least one s_j is odd. Let x vary over all n-tuples with $x_i \neq 0$. Then y ranges over all non-zero reals (since s_j is odd). As $x^{r_i} > 0$, $\bar{p}(t) = \Sigma a_i t^{c_i}$ is non-negative for all $t \neq 0$, and so for all t by continuity. Thus $\bar{p}(t)$ is a sum of squares, from which a representation of p as a sum of squares can be derived.

The simplest forms in $E_{n,2m}$, therefore, must have two-dimensional cages and at least one non-extremal r_i and so at least four terms. Both M and S satisfy these criteria. Suppose that

$$p(x) = \sum_{i=1}^{4} a_i x^{r_i}, \; E(p) = \{r_1, r_2, r_3\} \quad \text{and} \quad p \in E_{n,2m}.$$

It is possible that r_4 lies on an edge of $C(p)$, say $\overline{r_1 r_2}$. If so, there is a vector b so that

$$b \cdot r_1 = b \cdot r_2 = b \cdot r_4 = d > b \cdot r_3.$$

Under the substitution $x_i = c_i t^{b_i}$,

$$0 \le \lim_{t \to \infty} t^{-d} p(t) = a_1 c^{r_1} + a_2 c^{r_2} + a_4 c^{r_4} = q(c)$$

for all c. Thus $q(x)$ is psd and $C(q)$ is one dimensional so that $q(x)$ is a sum of squares, as is $p(x) = q(x) + a_3 x^{r_3}$.

Henceforth assume that r_4 is strictly interior to $C(p) = T(r_1, r_2, r_3)$. If r_4 is even and $a_4 > 0$ then p is once again, as it stands, a sum of squares. Otherwise, by taking the invertible change of variables $x_j \to -x_j$, if necessary, assume that $a_4 < 0$. The barycentric coordinates of r_4 are determined by the equations

$$\sum_{i=1}^{3} \lambda_i = 1, \; \sum_{i=1}^{3} \lambda_i r_i = r_4;$$

each λ_i is positive.

LEMMA. If $r_1, \cdots, r_k$ are linearly independent vectors in $\mathbb{R}^n$ then for every positive k-tuple $(y_1, \cdots, y_k)$ there exists a positive n-tuple $(v_1, \cdots, v_n)$ so that $v^{r_i} = y_i$.

Proof. The logarithm of the system $\{v^{r_i} = y_i\}$ is the system

$$\left\{ \sum_{j=1}^{n} r_{ij} \log v_j = \log y_i \right\}.$$

As the r_i's are linearly independent, the rank of this system is k and so a (not necessarily unique) set of log v_j's can be found. ∎

THEOREM 2. *If p is an extremal form with four terms then, up to a change of variables $x_i \to v_i x_i$, $v_i \neq 0$,*

$$p(x) = \sum_{i=1}^{3} \lambda_i x^{r_i} - x^{r_4},$$

where

$$r_4 = \sum_{i=1}^{3} \lambda_i r_i, \quad \sum_{i=1}^{3} \lambda_1 = 1 \quad \text{and} \quad \lambda_j > 0.$$

Proof. We have already established that r_4 is interior to $C(p)$. As r_1, r_2 and r_3 are not collinear points in $\mathbb{R}^n$, they are linearly independent vectors. Assume now that $p(x)$ is extremal and

$$p(x) = a_1 x^{r_1} + a_2 x^{r_2} + a_3 x^{r_3} - a_4 x^{r_4},$$

$a_i > 0$. Find $\{v_j\}$ by the last lemma so that $v^{r_i} = \lambda_i / a_i$ for $i = 1, 2, 3$, and make the change of variables $x_j = v_j y_j$. Then

$$p(y) = \sum_{i=1}^{3} \lambda_i y^{r_i} - a_4' y^{r_4} = q(y, a_4').$$

Of course q depends on the r_i's. By the AGI, $q(y, 1)$ is psd. If each $y_j = 1$, then $q(y, a_4') = 1 - a_4'$ so $a_4' \leq 1$. But

$$q(y, a_4') = a_4' q(y, 1) + (1 - a_4') q(y, 0),$$

each of which is psd. So $p(y)$ is extremal only if $a_4' = 1$ and p is, up to a change of variables, of the type described. ∎

5. Yoyos

Suppose

$$\lambda_i \geq 0, \quad \sum_{i=1}^{3} \lambda_i = 1 \quad \text{and} \quad r_4 = \sum_{i=1}^{3} \lambda_i r_i,$$

where r_1, r_2 and r_3 are even and not collinear (so that r_4 determines the λ_i's). The form

$$Y(r_1, r_2, r_3, r_4; x) = \sum_{i=1}^{3} \lambda_i x^{r_i} - x^{r_4}$$

will be called a yoyo. (The term "yoyo" is used to avoid confusion with other fields of research.) Every yoyo is psd by the AGI and Theorem 2 says that every extremal form with four terms is a yoyo, as

$$x_1^6 + x_2^4 x_3^2 + x_2^2 x_3^4 - 3x_1^2 x_2^2 x_3^2$$

demonstrates. If one of the λ_i's is 0 then r_4 lies on an edge, if two λ_i's are 0, then r_4 is one of the vertices and the yoyo is identically 0. Felicitously, yoyos have a natural additive property, leading to a necessary condition for extremality which proves to be sufficient.

LEMMA. *Suppose*

$$r_4 \in T(r_1', r_2', r_3') \subseteq T(r_1, r_2, r_3),$$

r_i, r_i' *even for* $1 \le i \le 3$ *and*

$$r_4 = \sum_{j=1}^{3} \mu_j r_j',$$

then

$$Y(r_1, r_2, r_3, r_4) = Y(r_1', r_2', r_3', r_4) + \sum_{j=1}^{3} \mu_j Y(r_1, r_2, r_3, r_j'). \quad \blacksquare$$

Proof. Let

since

$$r_j' = \sum_{i=1}^{3} \lambda_{ij} r_i, \quad \text{since} \quad T(r_1', r_2', r_3') \subseteq T(r_1, r_2, r_3), \qquad \lambda_{ij} \ge 0.$$

Then

$$r_4 = \sum_{j=1}^{3} \mu_j \left(\sum_{i=1}^{3} \lambda_{ij} r_i \right) = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \lambda_{ij} \mu_j \right) r_i.$$

Thus

$$\begin{aligned} &Y(r_1', r_2', r_3', r_4; x) + \sum_{j=1}^{3} \mu_j Y(r_1, r_2, r_3, r_j'; x) \\ &= \sum_{j=1}^{3} \mu_j x^{r_j'} - x^{r_4} + \sum_{j=1}^{3} \mu_j \left(\sum_{i=1}^{3} \lambda_{ij} x^{r_i} - x^{r_j'} \right) \\ &= Y(r_1, r_2, r_3, r_4; x). \end{aligned}$$

THEOREM 3. *If* $Y(r_1, r_2, r_3, r_4; x)$ *is extremal, then*

$$2R(Y) = \{r_1, r_2, r_3, r_4\} \quad \textit{and} \quad r_4 = \frac{1}{3}(r_1 + r_2 + r_3).$$

Proof. Recall that $2R(Y)$ consists of all even points in $C(Y)$. Suppose $r \in 2R(Y)$ is not one of the r_i's. The triangle $T(r_1, r_2, r_3)$ is divided into two or three triangles by connecting r and each of the vertices (depending on whether r is on an edge or in the interior.) In either case, r_4 is contained in some triangle $T(r_i, r_j, r) \subseteq T(r_1, r_2, r_3)$. By the last lemma, $Y(r_1, r_2, r_3, r_4)$ can be written as a sum of yoyos. (Actually, two of the new yoyos vanish identically and the other two are not multiples of the original.) For example, suppose $r_1 = (12, 0, 0)$, $r_2 =$

$(0, 12, 0)$, $r_3 = (0, 0, 12)$ and $r_4 = (4, 4, 4)$. (This is a sum of squares by [6].) Let $r = (6, 4, 2)$ then $r_4 \in T(r_2, r_3, r)$ and

$$(x_1^{12} + x_2^{12} + x_3^{12} - 3x_1^4x_2^4x_3^4)/3$$
$$= (6x_1^6x_2^4x_3^2 + x_2^{12} + 2x_3^{12} - 9x_1^4x_2^4x_3^4)/9 + (2/3)(3x_1^{12} + 2x_2^{12} + x_3^{12} - 6x_1^6x_2^4x_3^2)/6$$

and so is not extremal.

Suppose $2R(Y) = \{r_1, r_2, r_3\}$ and let $r_i = 2s_i$ then

$$\frac{1}{2} C(Y) = T(s_1, s_2, s_3)$$

and so by Pick's theorem,

$$A\left(\frac{1}{2} C(Y)\right) = \frac{1}{2}.$$

Thus $A(C(Y)) = 2$. Since $s_i + s_j$ is an edge lattice point for $1 \le i < j \le 3$, there are at least 3 edge points in $C(Y)$ and so, by Pick's Theorem, no interior points. But r_4 is an interior point, a contradiction.

Therefore, if Y is extremal then $r_4 = 2s_4$ is an even point, $R(Y) = \{s_1, s_2, s_3, s_4\}$ so $A(C(Y)) = 3/2$. However, $T(s_1, s_2, s_3)$ is decomposed into three triangles, $T(s_i, s_j, s_4)$. By Pick's Theorem, each $T(\phi(s_i), \phi(s_j), \phi(s_4))$ has area $\frac{1}{2}$ hence $\phi(s_4)$ is the unique point in $T(\phi(s_1), \phi(s_2), \phi(s_3))$ which divides it into three equal subtriangles—the median. Since ϕ is affine,

$$s_4 = \frac{1}{3}(s_1 + s_2 + s_3) \quad \text{and so} \quad r_4 = \frac{1}{3}(r_1 + r_2 + r_3). \quad \blacksquare$$

Call Y an optimal yoyo if it satisfies the conclusions of Theorem 3. Observe that M and S are both optimal yoyos. For example,

$$\frac{1}{2} C(S) = T((2, 1, 0), (0, 2, 1), (1, 0, 2)),$$

which has exactly one interior point: $(1, 1, 1)$.

For any optimal yoyo Y, $A(C(Y)) = 6$; $s_i + s_j$ for $1 \le i < j \le 3$ are 3 edge points and $s_i + s_4$ for $1 \le i \le 4$ are 4 interior points. By Pick's Theorem, there can be no other points in $F(Y)$.

6. Extremality

The results of this section were first shown by Choi and Lam, in a slightly different fashion, for the forms $M(x)$ and $S(x)$ in [3].

THEOREM 4. *An optimal yoyo is not a sum of squares.*

Proof. If $Y = \Sigma g_i^2$ then, by Theorem 1, $F(g_i) \subset \{s_1, s_2, s_3, s_4\}$ so $g_i(x) = \Sigma b_{ij} x^{s_j}$. Since $s_i + s_j = s_{i'} + s_{j'}$ implies $\{s_i, s_j\} = \{s_{i'}, s_{j'}\}$, the coefficient of x^{r_4} in Σg_i^2 is b_{i4}^2 hence $-1 = \Sigma b_{i4}^2$, a contradiction. $\blacksquare$

The question "Which yoyos are sums of squares?" remains open.

THEOREM 5. *An optimal yoyo is an extremal form.*

Proof. Suppose $Y(r_1, r_2, r_3, r_4; x)$ is an optimal yoyo and $Y(x) = g(x) + h(x)$ where g and h are psd. Then $C(Y) \supseteq C(g)$ so

$$g(x) = \sum_{i \le j} b_{ij} x^{s_i + s_j}.$$

If $x^0 = (x_1^0, \cdots, x_n^0)$ and $Y(x^0) = 0$, then $g(x^0) = 0$ and $\frac{\partial g}{\partial x_i}(x^0) = 0$ since $Y \ge g \ge 0$. It follows that

$$x_i^0 \frac{\partial g}{\partial x_i}(x^0) = 0$$

for every zero of Y. Since r_i is even, $x^{r_i} \ge 0$ and since

$$x^{r_4} = (x^{r_1} x^{r_2} x^{r_3})^{1/3},$$

by the AGI $Y(x) = 0$ precisely when $x^{r_1} = x^{r_2} = x^{r_3}$. By the homogeneity of the forms we take this common value to be 1.

Let $s_i = (s_{i1}, \cdots, s_{in})$, then

$$x_k \frac{\partial g}{\partial x_k}(x) = \sum_{i \le j} b_{ij}(s_{ik} + s_{jk}) x^{s_i + s_j}.$$

If $x_j = \pm 1$, then $x^{r_i} = 1$ so that g vanishes at all 2^n points $(\epsilon_1, \cdots, \epsilon_n)$ where $\epsilon_i = \pm 1$. (There might be other zeroes of g, which we ignore.) The values of $x_k \frac{\partial g}{\partial x_k}$ at these points thus depends on the values that the x^{s_i}'s achieve. Let $\delta_i = x^{s_i}$, for short, if $x = (\epsilon_1, \cdots, \epsilon_n)$; clearly $\delta_i = \pm 1$. Since $\delta_4^3 = \delta_1\delta_2\delta_3$, $\delta_1\delta_2\delta_3 = \delta_4$. As $\delta_i = \epsilon^{s_i}$, the mapping $(\epsilon_1, \cdots, \epsilon_n) \to (\delta_1, \cdots, \delta_4)$ preserves component-wise multiplication and the set of achieved δ's is closed under this operation. Finally suppose $s_{ik} \equiv s_{jk} \bmod 2$ for every k and some $1 \le i < j \le 4$. Then $(s_i + s_j)/2$ is also a lattice point, it is in $R(Y)$, a contradiction. For any $i \ne j$, let k be such that $s_{ik} \not\equiv s_{jk} \bmod 2$ and let $\epsilon_k = -1$, $\epsilon_l = 1$, $l \ne k$. The δ defined at this point will have $\delta_i \ne \delta_j$. Since the set of attained δ's "separates" δ_i and δ_j, is closed under multiplication and satisfies $\delta_1\delta_2\delta_3\delta_4 = 1$, it is not hard to show that, up to a permutation of indices, the quadruples in (1) (at least) are attained and the equation (2) holds at every point listed in (1).

(1) $(x^{s_1}, x^{s_2}, x^{s_3}, x^{s_4}) = (1, 1, 1, 1), (1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1)$

(2) $\sum_{i \le j} b_{ij}(s_{ik} + s_{jk})\, x^{s_i + s_j} = 0 \quad \text{for} \quad 1 \le k \le n$

Since $\delta_i\delta_j = \delta_{i'}\delta_{j'}$ for $\{i, j, i', j'\} = \{1, 2, 3, 4\}$, it is convenient to make the following abbreviations:

$$c_{1k} = \sum_{i=1}^{4} 2b_{ii}s_{ik},$$

$$c_{2k} = b_{12}(s_{1k} + s_{2k}) + b_{34}(s_{3k} + s_{4k}),$$

$$c_{3k} = b_{13}(s_{1k} + s_{3k}) + b_{24}(s_{2k} + s_{4k}),$$

$$c_{4k} = b_{14}(s_{1k} + s_{4k}) + b_{23}(s_{2k} + s_{3k}).$$

Upon evaluating (2) at each of the points given in (1), the following equations are found:

$$\begin{aligned} c_{1k} + c_{2k} + c_{3k} + c_{4k} &= c_{1k} + c_{2k} - c_{3k} - c_{4k} \\ &= c_{1k} - c_{2k} + c_{3k} - c_{4k} \\ &= c_{1k} - c_{2k} - c_{3k} + c_{4k} = 0. \end{aligned}$$

Thus each $c_{ik} = 0$, and the permutation of indices made in the choice of δ's is rendered harmless. Since $c_{2k} = 0$ for each k, $b_{12}(s_1 + s_2) + b_{34}(s_3 + s_4) = 0$; as $3s_4 = s_1 + s_2 + s_3$, $b_{12} = b_{34} = 0$. Similarly, $b_{13} = b_{24} = b_{14} = b_{23} = 0$. Since $c_{1k} = 0$,

$$b_{11}s_1 + b_{22}s_2 + b_{33}s_3 + b_{44}s_4 = 0$$

so that $b_{11} = b_{22} = b_{33} = \lambda$, $b_{44} = -3\lambda$; and, since g is psd, $\lambda > 0$. Hence g is a multiple of Y and Y is shown to be extremal. ∎

7. Examples

The question of extremal four-term forms is now reduced to a question about triangles in the plane. Suppose T is a triangle of the desired type: the vertices and median of T are lattice points and there are no other lattice points in T. Then T corresponds to infinitely many extremal forms, one form for each lattice plane in $\mathbb{R}^n$ and particular map onto $\mathbb{Z}^2$. We choose one representative form from each such family by selecting the canonical plane $u_1 + u_2 + u_3 = m$ and map $\phi_0(u_1, u_2, u_3) = (u_1, u_2)$, giving a form p in $E_{3,2m}$. Since every ϕ factors through ϕ_0, all extremal four-term forms which correspond to T may be written $p(y_1, y_2, y_3)$, where y_j is a monomial in the x_i's. Further, as noted in Section 2, h^2p is extremal if h is a monomial and p is extremal. It is reasonable, then, to restrict ourselves to an enumeration of extremal four-term forms with no common factor. Such a form in $E_{3,2m}$ as described is associated with a triangle which contains at least one vertex on each edge (vertices included) of the triangle $T((0, 0), (0, m), (m, 0))$, since multiplication by h^2 corresponds to a translation of T.

In this scheme of "natural" reductions, the problem becomes finite for fixed m, and, having used a variety of ad hoc methods, we present the following list of all extremal four-term forms. The list is complete for $m \leq 12$ up to a permutation of variables; for graphic reasons, the yoyos have been multiplied by 3. It can be shown that $S(x)$ and the form for $m = 4$ are the only yoyos which do not contain x_i^{2m} as one term. A duality relating two triangles which are halves of a parallelogram can also be defined.

$m = 3$ $\qquad x_1^4x_2^2 + x_2^4x_3^2 + x_1^2x_3^4 - 3x_1^2x_2^2x_3^2 \qquad (= S(x))$

	$x_1^6 + x_2^4x_3^2 + x_2^2x_3^4 - 3x_1^2x_2^2x_3^2$	$(= M(x))$
$m = 4$	$x_1^2x_3^6 + x_2^2x_3^6 + x_1^4x_2^4 - 3x_1^2x_2^2x_3^4$	
$m = 5$	$x_1^{10} + x_1^8x_2^2 + x_2^4x_3^6 - 3x_1^6x_2^2x_3^2$	
	$x_1^{10} + x_1^2x_3^2x_3^6 + x_2^4x_3^6 - 3x_1^4x_2^2x_3^4$	
$m = 7$	$x_1^{14} + x_1^{10}x_2^2x_3^2 + x_2^4x_3^{10} - 3x_1^8x_2^2x_3^4$	
	$x_1^{14} + x_1^4x_2^2x_3^8 + x_2^4x_3^{10} - 3x_1^6x_2^2x_3^6$	
$m = 8$	$x_1^{16} + x_1^{14}x_2^2 + x_2^{10}x_3^6 - 3x_1^{10}x_2^4x_3^2$	
	$x_1^{16} + x_1^2x_2^8x_3^6 + x_2^{10}x_3^6 - 3x_1^6x_2^6x_3^4$	
$m = 9$	$x_1^{18} + x_1^6x_2^2x_3^{10} + x_2^4x_3^{14} - 3x_1^8x_2^2x_3^8$	
	$x_1^{18} + x_1^{12}x_2^2x_3^4 + x_2^4x_3^{14} - 3x_1^{10}x_2^2x_3^6$	
$m = 11$	$x_1^{22} + x_2^4x_3^{18} + x_1^8x_2^2x_3^{12} - 3x_1^{10}x_2^2x_3^{10}$	
	$x_1^{22} + x_2^4x_3^{18} + x_1^{14}x_2^2x_3^6 - 3x_1^{12}x_2^2x_3^8$	
	$x_1^{22} + x_2^6x_3^{16} + x_1^{20}x_3^2 - 3x_1^{14}x_2^2x_3^6$	
	$x_1^{22} + x_2^6x_3^{16} + x_1^2x_2^6x_3^{14} - 3x_1^8x_2^4x_3^{10}$	
$m = 12$	$x_1^{24} + x_2^{10}x_3^{14} + x_1^{18}x_2^2x_3^4 - 3x_1^{14}x_2^4x_3^6$	
	$x_1^{24} + x_2^{10}x_3^{14} + x_1^6x_2^8x_3^{10} - 3x_1^{10}x_2^6x_3^8$	

The area of a triangle is unaltered if one vertex is translated in a direction parallel to its opposite side. The triangles which generate these yoyos all have area 3/2 so that families of yoyos are generated. We list a few of these families below.

$x_1^{4m+2} + x_2^4x_3^{4m-2} + x_1^{2m-2}x_2^2x_3^{2m+2} - 3x_1^{2m}x_2^2x_3^{2m}$	$m \geq 1$
$x_1^{4m+2} + x_2^4x_3^{4m-2} + x_1^{2m+4}x_2^2x_3^{2m-4} - 3x_1^{2m+2}x_2^2x_3^{2m-2}$	$m \geq 2$
$x_1^{6m+4} + x_2^6x_3^{6m-2} + x_1^{6m+2}x_3^2 - 3x_1^{4m+2}x_2^2x_3^{2m}$	$m \geq 1$
$x_1^{6m+4} + x_2^6x_3^{6m-2} + x_1^2x_2^6x_3^{6m-4} - 3x_1^{2m+2}x_2^4x_3^{4m-2}$	$m \geq 1$
$x_1^{10m+4} + x_2^{10}x_3^{10m-6} + x_1^{8m+2}x_2^2x_3^{2m} - 3x_1^{6m+2}x_2^4x_3^{4m-2}$	$m \geq 1$
$x_1^{10m+4} + x_2^{10}x_3^{10m-6} + x_1^{2m+2}x_2^8x_3^{8m-6} - 3x_1^{4m+2}x_2^6x_3^{6m-4}$	$m \geq 1$
$x_1^{10m+6} + x_2^{10}x_3^{10m-4} + x_1^{8m+6}x_2^2x_3^{2m-2} - 3x_1^{6m+4}x_2^4x_3^{4m-2}$	$m \geq 1$
$x_1^{10m+6} + x_2^{10}x_3^{10m-4} + x_1^{2m}x_2^8x_3^{8m-2} - 3x_1^{4m+2}x_2^6x_3^{6m-2}$	$m \geq 1$

Although there are no "primitive" yoyos for $m = 6$ and 10, through the use of these and other formulas, one can find "primitive" yoyos for every other $m \leq 64$, at least.

8. Extremal five-term forms

By arguments analogous to those in section 4, an extremal five-term form would either have a tetrahedral cage with one interior point or a planar cage, quadrilateral or triangular. In the first case, a difficulty lies in the fact that reduced frames with no interior points may belong to frames with arbitrarily many interior points (See Reeve [10] and MacDonald [8] for the correct generalizations of Pick's Theorem to n dimensions). For example, the tetrahedron on vertices $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(0, 0, m)$ has no interior points, but $(2, 0, 0)$, $(0, 2, 0)$, $(2, 2, 0)$ and $(0, 0, 2m)$ has $(1, 1, k)$ as an interior point for $1 \le k \le m - 1$. The obvious derived yoyo for $m = 2$,

$$\frac{1}{4}\,(x_1^2x_4^2 + x_2^2x_4^2 + x_1^2x_2^2 + x_3^4) - x_1x_2x_3x_4$$

was, in fact, shown to be extremal in [3]. But, for $m = 3$, neither of the two derived yoyos (one for each interior point) is extremal, and the decomposition into extremal forms does not follow the lines of Theorem 3. We have that

$$2x_1^2x_4^4 + 2x_2^2x_4^4 + x_1^2x_2^2x_4^2 + x_3^6 - 6x_1x_2x_3x_4^3$$
$$= (x_1x_2x_4 - x_3^3)^2/2 + (4x_1^2x_4^4 + 4x_2^2x_4^4 + x_1^2x_2^2x_4^2 + x_3^6 + 2x_1x_2x_3^3x_4 - 12x_1x_2x_3x_4^3)/2.$$

It is non-trivial to show that the last form is psd, but it is actually extremal.

If the frame is planar, then no obvious analogue to Theorem 2 seems to exist, and a discussion of the forms seems to depend on the configuration of the interior points. It also seems to entail a generalization of the AGI to the case of two "interior" points. No extremal five-term forms of this kind have yet been determined.

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